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Research Report CCS 372

ON THE UNIMODALITY OF HIGH CONVOLUTIONS

by

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On the Unimodality of High Convolutions

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Abstract. It has been conjectured, for any discrete density function $\{p_j\}$ on the integers, that there exists an n_0 such that the n-fold convolution $\{p_j\}^{*n}$ is unimodal for all $n \geq n_0$. A similar conjecture has been stated for continuous densities. We present several counterexamples to both of these conjectures.

As a positive result, it is shown for a discrete density with a conn cted 3-point integer support that its n-fold convolution is fully unimodal for all sufficiently large n.

1. Introduction

The limiting distributions of properly normalized sums of independent random variables are called class L distributions, (Gnedenko and Kolmogorov 1954). If the random variables are identically distributed then the limit law is stable. The problem of unimodality of the class L distributions, and the stable laws in particular, has only recently been decided after numerous false proofs. Yamazato (1978) established that all class L distributions are unimodal.

In view of the apparent unimodality of such limiting distributions, A. Renyi conjectured for a discrete distribution $\{p_j\}$ on the integers that there is a number n_0 such that the n-fold convolution $\{p_j\}^{*n}$ is unimodal for all $n \geq n_0$. In a similar vein, P. Medgyessy conjectured that for any continuous probability density f there exists a number n_0 such that the n-fold convolution f^{*n} is unimodal for $n \geq n_0$. See Medgyessy (1977) for both these conjectures.

If these conjectures were true then it would be easy to deduce the unimodality of the stable laws. Both are in fact false. We present two counterexamples to the conjecture of Medgyessy. The first is a bounded and infinitely differentiable density with the property that none of its convolutions is unimodal. We also exhibit a density on $[0,2\pi]$ which is continuous and such that each of its convolutions is nowhere differentiable. Since unimodality implies differentiability almost everywhere, this example disproves the conjecture of Medgyessy.

As to discrete distributions, we construct one which has the nonnegative integers as its support and such that none of its convolutions is unimodal.

As a positive result, we show in the final section that for any measure μ with 3-point support $\{0,1,2\}$ the n-fold convolution μ^n is unimodal over the full range $\{0,1,\ldots,2n\}$, for all sufficiently large n. We conjecture the corresponding result for any measure with a finite and connected support $\{0,1,\ldots,N\}$. Already for $N = 3$ this is an open problem.

2. Discrete Unimodality

In the present section, we will be concerned with probability measure μ on the integers $\mathbb{Z}$. Let $p_j = \mu(\{j\})$ denote its mass at j. Such a discrete distribution is called unimodal if the sequence $\{p_{j+1} - p_j\}_{-\infty}^{+\infty}$ has exactly one change of sign. A discrete distribution $\{p_j\}$ is said to be strongly unimodal if $\{p_j\} * \{q_j\}$ is unimodal for any unimodal discrete distribution $\{q_j\}$ with connected lattice support. As was shown by Keilson and Gerber (1971), this happens if and only if $p_j^2 \geq p_{j-1}p_{j+1}$ for all $j \in \mathbb{Z}$. For instance, the Poisson, geometric and binomial densities are all strongly unimodal.

Counterexample I.

Consider integers

$$a_0 = 0 < a_1 < a_2 < \dots \quad \text{with} \quad a_{h+1}/a_h \to \infty . \tag{2.1}$$

Let X be a random variable with range $A = \{a_0, a_1, a_2, \dots\}$ and put $p_h = P[X = a_h]$, thus, $p_h > 0$ and $\Sigma_h p_h = 1$. Let $X_1, X_2, \dots$ be independent copies of X and $S_n = X_1 + X_2 + \dots + X_n$. Observe that the range of S_n includes $a_0 = 0$ as well as arbitrarily large positive integers. Hence, S_n can have a discrete unimodal distribution only if $P[S_n = x] > 0$ for all integers $x \geq 0$.

Claim 0. Let h and x be fixed positive integers such that $a_{h+1} > na_h$. Then

$$P[S_n = x] = 0 \quad \text{whenever} \quad na_h < x < a_{h+1} . \tag{2.2}$$

Moreover,

$$P[S_n = na_h] = p_h^n \quad \text{and} \quad P[S_n = a_{h+1}] = np_0^{n-1}p_{h+1} . \tag{2.3}$$

Proof. Suppose $P[S_n = x] > 0$. Then x must be of the form

$$x = \sum_{j=0}^{\infty} m_j a_j \quad \text{with} \quad m_j \in \mathbb{Z}^+ \quad \text{and} \quad \sum_{j=0}^{\infty} m_j = n .$$

If $m_j = 0$ for all $j > h$ then $x \leq a_h \sum_{j=0}^{h} m_j = na_h$. If $m_j > 0$ for some $j > h$ then $x \geq m_j a_j \geq a_j \geq a_{h+1}$. This proves (2.2). A similar proof yields (2.3).

To complete the counterexample, one only needs to note that there exists a positive integer x with $na_h < x < a_{h+1}$ as soon as $a_{h+1} > na_h + 1$. For each fixed n this is true for all large h. Therefore, S_n is not unimodal for any n.

One may wonder whether S_n with n large might be unimodal at least over its support. The answer is negative in general since one can arrange that for each large but fixed n one has

$$P[S_n = a_{h+1}] = np_0^{n-1} p_{h+1} > P[S_n = na_h] = p_h^n ,$$

for infinitely many h. For instance, $p_{h+1}/p_0 > \frac{1}{n}(p_h/p_0)^n$ for all large h as soon as $p_h = c/(h^2+1)$.

One may further wonder whether assuming $P[X = j] > 0$ for all $j \in \mathbb{Z}^+$ (connected lattice support) might be sufficient. The following example shows that also here the answer is negative.

Counterexample II.

Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be independent random variables such that the X_i are i.i.d. with their common distribution the same as in Counterexample I, while the Y_i are Poisson variables with mean λ. Put $S_n = \sum_{i=1}^{n} X_i$ and $T_n = \sum_{i=1}^{n} Y_i$. Let further $Z_i = X_i + Y_i$ and $S_n' = \sum_{i=1}^{n} Z_i = S_n + T_n$. Observe that Z_i has the connected lattice support $\mathbb{Z}^+ = \{0,1,2,\ldots\}$.

Let n be fixed and choose h so large that $a_{h+1} > na_h$. We have from Claim 0 of Counterexample I that

$$P[S_n' = a_{h+1} - 1] \leq P[S_n \leq na_h]\varepsilon_h \leq \varepsilon_h ,$$

where

$$\varepsilon_h = \varepsilon_{n,h} = \max\{P[T_n = j]: j \geq a_{h+1} - 1 - na_h\} .$$

Let $b_h = b_{n,h}$ denote the smallest integer $\geq a_{h+1} - 1 - na_h$. If h is sufficiently large then b_h exceeds the mode $[n\lambda]$ of the Poisson variable T_n, hence, $\varepsilon_h = e^{-n\lambda}(n\lambda)^{b_h}/b_h!$. On the other hand,

$$P[S_n' = a_{h+1}] \geq P[S_n = a_{h+1}]P[T_n = 0] = np_0^{n-1}p_{h+1}e^{-n\lambda} .$$

Therefore,

$$P[S_n' = a_{h+1} - 1] < P[S_n' = a_{h+1}] \text{ for all large } h , \tag{2.4}$$

as soon as $\varepsilon_h = o(p_{h+1})$ as $h \to \infty$, which is the same as

$$(n\lambda)^{b_h}/b_h! = o(p_{h+1}) \text{ as } h \to \infty .$$

For large h, $b_n \geq \frac{1}{2} a_{h+1}$ where $\{a_h\}$ and thus $\{b_h\}$ increases faster than exponentially. It follows that (2.4) holds as soon as $\{p_h\}$ does not decrease too fast, for instance, only at an algebraic or exponential rate. In such a situation, $P[S_n' = j]$ increases infinitely often. This clearly rules out unimodality.

3. Continuous Unimodality

One can easily carry over the examples of Section 2 to the absolutely continuous case. A continuous density function f is called unimodal if there exists a value x_0 such that f is non-decreasing over $(-\infty, x_0)$ and non-increasing over (x_0, ∞). The following is an analogue of the above Counterexamples I and II.

Counterexample III.

Let $\{p_h\}_0^\infty$ satisfy $p_h > 0$, $\Sigma p_h = 1$ and let $\{a_h\}_0^\infty$ be as in (2.1). Let $f(x)$ denote the density function which is obtained by distributing the mass p_h

uniformly over $(a_h - 1/2,\ a_h + 1/2)$. Thus, $f(x) = \sum_0^\infty p_h \psi(x-a_h)$ with ψ as the characteristic function of the interval $(-1/2, 1/2)$.

Let $S_n = \sum_{i=1}^{n} X_i$ where $X_1, X_2, \ldots$ are i.i.d. with common density f. The density f^{*n} of S_n satisfies

$$(3.1) \qquad f^{*n}(x) = 0 \quad \text{for} \quad na_h + n/2 < x < a_{h+1} - n/2 ,$$

(compare the proof of (2.2)). Since $a_{h+1} > na_h + n$ for all large h, it follows that f^{*n} can never be unimodal.

If a strictly positive density is desired such that f^{*n} is never unimodal, one may start with Counterexample II and afterwards spread the mass $q_j = P[Z = j]$ uniformly over the interval $[j - 1/2,\ j + 1/2]$.

Or one can start with $\{X_i\}$ as the i.i.d. sequence of Counterexample I. Let further $Z_i = X_i + Y_i$ with the X_i and Y_j independent, each Y_j having density g. Thus, the Z_i have density

$$f(x) = \sum_{h=0}^{\infty} p_h g(x-a_h) .$$

This density f is infinitely differentiable as soon as each derivative of g exists and is bounded. Let us take g as the standard normal density $\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}$. Then $S_n' = \sum_{i=1}^{n} Z_i$ has density

$$(3.2) \qquad f_n(x) = n^{-1/2} \sum_{j=0}^{\infty} P[S_n = j] \varphi((x-j)/n^{1/2}) ,$$

where $S_n = \sum_{i=1}^{n} X_i$.

Let n be fixed and h so large that $\Delta_h > 0$, where $\Delta_h = (a_{h+1} - na_h)/2$. It follows from (2.2) and (3.2) that

$$f_n(a_{h+1} - \Delta_h) \le n^{-1/2} \varphi(\Delta_h / n^{1/2}) .$$

On the other hand, using (2.3),

$$f_n(a_{h+1}) \geq n^{-1/2} P[S_n = a_{h+1}]\varphi(0) = (2\pi n)^{-1/2} n p_0^{n-1} p_{h+1} .$$

One has $\varphi(\Delta_h/n^{1/2}) = o(p_{h+1})$ as $h \to \infty$, provided $\{p_h\}$ decreases only at an exponential or algebraic rate. In this case, $f_n(a_{h+1}-\Delta_h) < f_n(a_{h+1})$ for all large h, showing that f_n is not unimodal.

Counterexample IV.

Observe that a unimodal density function is necessarily differentiable almost everywhere. In this example we exhibit a continuous density function f supported on $[0,2\pi]$ such that the n-fold convolution f_n of f is nowhere differentiable for every n. Actually, Bogdanowicz (1965) already showed that nearly each continuous function f on $[0,2\pi]$ has this property. More precisely, in the space $C[0,2\pi]$ with supremum norm the collection of $f \in C[0,2\pi]$ with each convolution f_n nowhere differentiable is the complement of a set of first category.

Let us now exhibit an explicit density with this property. It is based on the following result due to Freud (1962). A short proof may be found in Kahane (1964).

Theorem (Freud). Let $g(x) = \sum_{k=1}^{\infty} c_k \cos(b_k x)$, where $\{b_k\}$ is a sequence of positive integers satisfying Hadamard's lacunarity condition $b_{k+1}/b_k \geq q > 1$. Then g being differentiable at some point implies that $c_k = o(b_k^{-1})$ as $k \to \infty$.

To construct our example, let f be a probability density on $[0,2\pi]$ and let f_n be its n-fold convolution. Note that f_n is carried by the interval $[0,2\pi n]$. Consider further the essentially finite sum

$$g_n(x) = \sum_{h=-\infty}^{+\infty} f_n(x+2\pi h) \ .$$

Since $g_n(x+2\pi) = g_n(x)$, one may regard g_n also as a function on the circle group T of the reals modulo 2π. Relative to the additive group T, the function g_n is a probability density equal to the n-fold convolution of g_1. The easiest way to calculate g_n is to calculate its characteristic function. For each integer m,

$$\int_T e^{imx} g_n(x)dx = \{\int_T e^{imx} g_1(x)dx\}^n = \{\int_R e^{imx} f(x)dx\}^n \ .$$

In other words,

$$g_n(x) \sim \sum_m (\gamma_m)^n e^{-imx} \tag{3.3}$$

as soon as

$$f(x) \sim \sum_m \gamma_m e^{-imx} \quad \text{for} \quad 0 < x < 2\pi \ . \tag{3.4}$$

Let us take

$$f(x) = c_0 - \sum_{k=1}^{\infty} c_k \cos b_k x \quad \text{for} \quad 0 \leq x \leq 2\pi \ ,$$

while $f(x) = 0$, otherwise. Here, the b_k are as in the above Theorem. Further $c_0 = 1/(2\pi)$ and $c_k > 0$ $(k \geq 1)$ such that $\sum_{k=1}^{\infty} c_k = c_0$. Thus f is a continuous probability density with $g(0) = g(2\pi) = 0$. It is of the form (3.4) with $\gamma_0 = c_0$, $\gamma_{b_k} = \gamma_{-b_k} = -c_k/2$ when $k > 0$, while $\gamma_m = 0$, otherwise. Thus (3.3) yields that

$$g_n(x) = (c_0)^n + 2^{-n+1} \sum_{k=1}^{\infty} (-c_k)^n \cos b_k x \ . \tag{3.5}$$

Suppose f_n were unimodal. Then f_n is differentiable almost everywhere. The function $g_n(x)$ restricted to $(0, 2\pi)$ is the superposition of the finitely many translates $f_n(x+2\pi k)$, $(k=0,1,\ldots,n-1)$, and thus would also have a derivative almost everywhere. By (3.5) and Freud's theorem, this would imply that $(c_k)^n = o(b_k^{-1})$ as $k \to \infty$.

Consequently, if we choose $b_k = 2^k$ and $c_k = c/k^2$ $(k \geq 1)$ we have an example of a continuous density f such that none of its convolutions f_n is unimodal. (In fact, no convolution is anywhere differentiable.)

It is interesting to note that, for any $\varepsilon > 0$, one can find a density f_ε uniformly closer than ε to the standard normal density, and such that the n-fold convolution f_ε^{*n} of f_ε is not unimodal and nowhere differentiable for any n. To construct such an f_ε, we start with a random variable X with a density f as outlined in Counterexample IV. Since X has a compact support, it follows from well-known local limit theorems (e.g., Petrov (1975), Theorems 7 or 15 of Ch. VII) that the density h_N of $\sum_{i=1}^{N} (X_i - N\mu)/(\sigma\sqrt{N})$ is *uniformly* within ε of the standard normal density as soon as N is sufficiently large. Since h_N is linearly related to the previous f_N, this density h_N is not unimodal or differentiable and neither is any of its convolutions h_N^{*n}. Thus, the reasoning behind the original conjectures of Rényi and Medgyessy is faulty. The central limit effect is much too weak for the property of <u>exact</u> unimodality of high convolutions.

4. Positive Results and Conjectures

Let us now investigate what positive results can be obtained concerning the eventual unimodality of sums of independent random variables. Also in view of the counterexamples, we shall restrict our attention to an integer valued random variable X having a finite support A with $A \subset \{0,1,\ldots,N\}$. Let $p_j = P[X=j]$ thus $A = \{j \in \mathbb{Z}: p_j > 0\}$. We will assume that $p_0 > 0$ and $p_N > 0$ and further that the members of A have their greatest common divisor equal to 1.

Let

$$f_n(j) = P[\sum_{i=1}^{n} X_i = j]$$

be the n-fold convolution of $\{p_j\}$. The support of f_n is precisely the n-fold sum $A_n = A + A + \ldots + A$ of the support A of $\{p_j\}$. One has $A_n \subset \{0,1,\ldots,nN\}$ and $0 \in A_n$; $nN \in A_n$. In order that f_n be unimodal it is at least necessary that A_n be connected, that is, $A_n = \{0,1,\ldots,nN\}$. One has $A_n \subset A_{n+1}$ while $G = \bigcup_n A_n$ is precisely the semigroup generated by A. Hence, in order that f_n be unimodal for n sufficiently large it is necessary that G be connected, that is, $G = \mathbb{Z}^+ = \{0,1,2,\ldots\}$. This rules out a situation like $A = \{0,6,10,15\}$ since in this case G has the holes $\{1,2,3,4,5\}$, $\{7,8,9\}$, $\{11\}$, $\{13,14\}$, $\{17\}$, $\{19\}$, $\{23\}$ and $\{29\}$.

Since G contains all sufficiently large integers, one easily shows that, for large n, the support A_n of f_n has only a few holes all located at the very beginning and very end of A_n. One might therefore conjecture that, for n large, $f_n(.)$ is unimodal at least over the "solid" part of A_n. However, even this can be shown to be false. Because of that, we will restrict our attention to the case of a connected lattice support

$$A = \{0,1,\ldots,N\} .$$

That is, $p_j > 0$ for $j = 0,1,\ldots,N$ and $p_j = 0$, otherwise. The results so far, as well as certain numerical calculations, lead us to make the following conjecture.

Conjecture. Let $\{p_0, p_1, \ldots, p_N\}$ be a finite discrete distribution with $p_i > 0$ $(i = 0,1,\ldots,N)$. Then the n-fold convolution $f_n(.)$ of $\{p_j\}$ is unimodal for all sufficiently large n.

If $N = 1$ then this convolution is a binomial law and thus always unimodal. Nothing seems to be known for the case $N \geq 2$, not even in the symmetric case.

The conjecture, if true, would have further consequences. For instance, applying it to the discrete density $\tilde{p}_j = c_t p_j t^j$ with $t > 0$ fixed, it would follow that also $\tilde{f}_n(j) = f_n(j) t^j$ becomes unimodal in j for $n \geq n_0(t)$.

For density functions we make an analogous conjecture: if f is an analytic density on the finite interval (a,b) with only finitely many modes, then for n sufficiently large, the n-fold convolution f_n of f is unimodal over the full range (na, nb).

The major result of the present section is that the first conjecture is true when $N = 2$.

Theorem. Let X_i $(i = 1,2,\ldots)$ be a sequence of i.i.d. discrete random variables with connected lattice support $\{0,1,2\}$. Then $S_n = \sum_{i=1}^{n} X_i$ is unimodal for all n sufficiently large.

Proof. Let $p_i = P[X = i]$ thus $p_i > 0$ for $i = 0,1,2$ and $p_i = 0$, otherwise. Let $f_n(j) = P[S_n = j]$ thus

$$f_n(j) > 0 \quad \text{for} \quad j = 0,1,\ldots,2n\,; \quad f_n(j) = 0\,, \quad \text{otherwise}\,. \tag{4.1}$$

Moreover,

$$f_{n+1}(j) = \sum_{i=0}^{2} p_i f_n(j-i)\,. \tag{4.2}$$

Let us introduce the ratios

$$\rho_n(j) = f_n(j+1)/f_n(j)\,, \tag{4.3}$$

letting $\rho_n(j) = \infty$ for $j < 0$ and $\rho_n(j) = 0$ for $j \geq 2n$. Note that unimodality of S_n above an integer mode $m_0(n)$ is equivalent to $\rho_n(j) \geq 1$ for $j < m_0(n)$ together with $\rho_n(j) \leq 1$ for $j \geq m_0(n)$.

Lemma 1. We have $\rho_n(j+2) \leq \rho_n(j)$ for all j.

Proof. We shall proceed by induction. The stated result is equivalent to

$$f_n(h)f_n(h+3) \leq f_n(h+1)f_n(h+2) \quad \text{for all } h . \tag{4.4}$$

Since $f_n(h) = 0$ if $h < 0$ or $h > 2n$, (4.4) is obviously true if $h \leq -1$ or $h \geq 2n-2$, and therefore for $n = 1$.

In view of (4.2), inequality (4.4) with n resplaced by $n+1$ is equivalent to showing for all j

$$\sum_{r=0}^{2} p_r f_n(j-r) \sum_{s=0}^{2} p_s f_n(j+3-s) \leq \sum_{r=0}^{2} p_r f_n(j+1-r) \sum_{s=0}^{2} p_s f_n(j+2-s) . \tag{4.5}$$

We must show that (4.4) implies (4.5). Rearranging terms, we see that (4.5) is equivalent to

$$0 \leq \sum_{r=0}^{2} p_r^2 [f_n(j+1-r)f_n(j+2-r) - f_n(j-r)f_n(j+3-r)] + \sum_{r<s} p_r p_s F_{j-r,j-s} , \tag{4.6}$$

where

$$F_{j,k} = f_n(j+1)f_n(k+2) + f_n(j+2)f_n(k+1) - f_n(j)f_n(k+3) - f_n(j+3)f_n(k) . \tag{4.7}$$

Applying (4.4) with $h = j - r$, we see that the first sum in (4.6) is non-negative. Hence, it suffices to show that $F_{j,j-1} \geq 0$, $F_{j,j-2} \geq 0$ and $F_{j-1,j-2} \geq 0$.

From (4.7), condition $F_{j,j-1} \geq 0$ reduces to

$$f_n^2(j+1) \geq f_n(j-1)f_n(j+3) .$$

One may as well assume that $1 \leq j \leq 2n-3$ so that $f_n(j)f_n(j+2) > 0$. Now, applying (4.4) with $h = j$ and $h = j-1$ and then multiplying the results, one obtains

$$f_n(j-1)f_n(j+2)f_n(j)f_n(j+3) \leq f_n(j)f_n(j+1)f_n(j+1)f_n(j+2) \ .$$

Dividing by $f_n(j)f_n(j+2)$, one obtains the desired result. Replacing j by $j-1$, one also has $F_{j-1,j-2} \geq 0$.

From (4.7), condition $F_{j,j-2} \geq 0$ reduces to

$$f_n(j-2)f_n(j+3) \leq f_n(j-1)f_n(j+2) \ .$$

One may as well assume that $2 \leq j \leq 2n-3$ so that $f_n(j)f_n(j+1) > 0$. Applying (4.4) with $h=j-2$ and $h=j$ and then multiplying, one obtains that

$$f_n(j-2)f_n(j+1)f_n(j)f_n(j+3) \leq f_n(j-1)f_n(j)f_n(j+1)f_n(j+2) \ .$$

Dividing by $f_n(j)f_n(j+1)$, one obtains the desired result. This completes the proof of Lemma 1.

Remark. Lemma 1 is related to the paper "A Hurwitz matrix is totally positive", by J.H.B. Kemperman, which has been submitted for publication.

It follows from Lemma 1 that $\rho_n(j)$ is monotonically decreasing if j runs through the even integers and also when j runs through the odd integers.

As to the Theorem, it suffices to prove that, for all sufficiently large n, there exists an integer $k=k(n)$ such that

$$(4.8) \qquad \rho_n(k-2) \geq 1 \ ; \quad \rho_n(k-1) \geq 1 \ ; \quad \rho_n(k) \leq 1 \ ; \quad \rho_n(k+1) \leq 1 \ .$$

For afterwards we have from Lemma 1 that $\rho_n(j) \geq 1$ for all $j \leq k-1$ and $\rho_n(j) \leq 1$ for all $j \geq k$, implying that $f_n(.)$ is unimodal about k.

In other words, it only remains to show that, for n sufficiently large, the restriction of $f_n(.)$ to some 5-point set

$$\{k-2, k-1, k, k+1, k+2\}$$

is nonzero and unimodal about the central value k. We will do this by using a local limit theorem. In the sequel, $\mu = EX$, $\sigma^2 = \mathrm{Var}\ X$ while γ_j denotes the j-th cumulant of X. Further, B denotes positive constant with $B > |\gamma_3|/(2\sigma^2) + 1$.

Lemma 2 For n sufficiently large, $n \geq n_0(B)$, the restriction of $f_n(\cdot)$ to the interval $|j-n\mu| \leq B$ is unimodal about one of the two integers neighboring the value $n\mu - \gamma_3/(2\sigma^2)$.

Proof. Let n be large but fixed and let j satisfy $|j-n\mu| \leq B$. Further put

$$\Delta = 1/(\sigma\sqrt{n}) \ ; \quad x = (j-n\mu)/(\sigma\sqrt{n}) = (j-n\mu)\Delta \ .$$

By a local limit theorem for discrete distributions, (Petrov 1975, pages 207 and 139), one has

$$\sigma(2\pi n)^{1/2} f_n(j) = g_n(x)e^{-x^2/2} + R_n(x) \ ,$$

where

$$|R_n(x)| \leq Cn^{-3/2} \ ,$$

with C as a constant independent of x or n. Moreover,

$$g_n(x) = 1 + [(x^3-3x)\gamma_3/(6\sigma^3)]n^{-1/2} + [(x^4-6x^2+3)\gamma_4/(24\sigma^4) + (x^6-15x^4+45x^2-15)(\gamma_3)^2/(72\sigma^6)]n^{-1} \ .$$

Since $x = 0(n^{-1/2})$ and $e^{-x^2/2} = 1 - x^2/2 + 0(n^{-2})$, it follows from an easy calculation that

$$\sigma(2\pi n)^{1/2} f_n(j) = 1 - x^2/2 - x\Delta\gamma_3/(2\sigma^2) + \beta n^{-1} + 0(n^{-3/2}) \ ,$$

where $\beta = \gamma_4/(8\sigma^4) - 5(\gamma_3)^2/(24\sigma^6)$ is a constant.

Replacing j by $j+1$ amounts to replacing x by $x+\Delta$. Therefore,

$$(2\pi n)^{1/2}[f_n(j+1)-f_n(j)] = -\Delta[x+\Delta/2+\Delta\,\gamma_3/(2\sigma^2) + O(n^{-1})] .$$

Here, $x = (j-n\mu)\Delta$. It follows that

$$f_n(j+1) < f_n(j) \quad \text{as soon as} \quad j > n\mu - 1/2 - \gamma_3/(2\sigma^2) + O(n^{-1/2}) .$$

Similarly,

$$f_n(j+1) > f_n(j) \quad \text{as soon as} \quad j < n\mu - 1/2 - \gamma_3(2\sigma^2) - O(n^{-1/2}) .$$

We conclude that the restriction of $f_n(.)$ to the interval $|j-n\mu| \le B$ is strictly positive and unimodal about the integer $k(n)$ closest to the value $n\mu - \gamma_3/(2\sigma^2)$, with one exception. Namely, there is a constant $K > 0$ such that for integers n satisfying

$$|n\mu-\gamma_3/(2\sigma^2) - 1/2 - k(n)| \le Kn^{-1/2} ,$$

with $k(n)$ as a (unique) integer, one can only say that the above restriction is unimodal about one of the two values $k(n)$ or $k(n)+1$.

Remark. It should be noted that, under mild side conditions, the Theorem and its proof carry over to the case of independent random variables $X_1, X_2, \ldots$ which are not necessarily identically distributed, each having either $\{0,1\}$ or $\{0,1,2\}$ as its support. In fact, all that is needed is that for n sufficiently large there exists an integer $k = k(n)$ such that the distribution of $S_n = X_1 + \ldots + X_n$ restricted to the 5-point set $\{k-2, k-1, k, k+1, k+2\}$ is unimidal about k. Conditions for this may be derived from local limit theorems for sums of independent, non-identically distributed random variables (cf. Petrov (1975)).

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It has been conjectured, for any discrete density function $\{p_j\}$ on the integers, that there exists an n_0 such that the n-fold convolution $\{p_j\}^{*n}$ is unimodal for all $n \geq n_0$. A similar conjecture has been stated for continuous densities. We present several counterexamples to both of these conjectures.

As a positive result, it is shown for a discrete density with a connected 3-point integer support that its n-fold convolution is fully unimodal for all sufficiently large n.

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