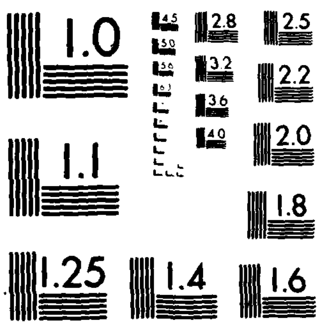
AD-A087 359 CLAREMONT MEN'S COLL CA INST OF DECISION SCIENCE F/G 12/1
A COMPARISON BETWEEN MLE AND BLUE FOR A DECREASING FAILURE RATE--ETC(U)
MAR 80 J LUCKE, J MYHRE, S SAUNDERS N00014-78-C-0213
UNCLASSIFIED 80-3 NL

1 of 1
AD A
087359

END
DATE
FILMED
9-80
DTIC



MICROCOPY RESOLUTION TEST CHART

INSTITUTE OF DECISION SCIENCE
FOR BUSINESS & PUBLIC POLICY

A COMPARISON BETWEEN MLE AND BLUE FOR A DECREASING FAILURE RATE DISTRIBUTION WHEN USING CENSORED DATA

by

James Lucke, Janet Myhre, Patrick Williams
Claremont Men's College
Claremont, CA 91711

Sam Saunders
Washington State University
Pullman, WA 99163

Research supported by
Office of Naval Research Contract
N00014-78-C-0213

Claremont Men's College
Claremont, California

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 80-3	2. GOVT ACCESSION NO. AD-A087359	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Comparison Between MLE and BLUE for a Decreasing Failure Rate Distribution When Using Censored Data.		5. TYPE OF REPORT & PERIOD COVERED Technical rept.
		6. PERFORMING ORG. REPORT NUMBER 80-3
7. AUTHOR(s) James Lucke, Janet Myhre, Sam Saunders, Patrick Williams		8. CONTRACT OR GRANT NUMBER(s) N00014-78-C-0213
9. PERFORMING ORGANIZATION NAME AND ADDRESS Institute of Decision Science Claremont Men's College Claremont, California 91711		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Reliability
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research 800 N. Quincy Arlington, AV 22217		12. REPORT DATE March 1980
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

N/A

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Best linear unbiased estimator
Maximum likelihood estimator
Mixed exponential distribution
Simulated comparisons

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Estimation procedures for the shape and scale parameters of a mixed exponential, decreasing failure rate distribution are compared and the more advantageous methods are determined for various situations. The maximum likelihood estimate, MLE, is slightly more accurate than a Best Linear Unbiased Estimator of the scale parameter based on the first K order statistics. In the joint case the MLE's are much superior to other possible estimates of the distribution.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

KEY WORDS

Best linear unbiased estimator
Maximum likelihood estimator
Mixed exponential distribution
Simulated comparisons

ABSTRACT

Estimation procedures for the shape and scale parameters of a mixed exponential, decreasing failure rate distribution, are compared and the more advantageous methods are determined for various situations.

When the scale parameter is known and the shape parameter is to be estimated, a transformation of the data reduces the problem to that of estimating the failure rate of an exponential distribution, in which case the Maximum Likelihood Estimate, MLE, involves the total life statistic, the properties of which are well known.

A Best Linear Unbiased Estimator of the scale parameter, based on the first k order statistics, BLUE-k, is more accurate than the MLE in the case of type I or type II censoring for small sample sizes but is shown to be generally less accurate for random censoring of a type often found in industrial applications.

In the case both parameters are unknown, the BLUE methods are not available. As is usually the case, the joint MLEs, despite being only implicitly defined, are much superior in performance to estimates made using the method of moments. In fact, the distribution function estimated using the joint MLE's of the parameters is surprisingly closer to the true distribution for regions of interest in reliability theory, than is the estimated distribution function using BLUE-k for the scale parameter and a known shape parameter.

1. INTRODUCTION

A mixed exponential distribution, sometimes called the Lomax or Pareto distribution of the second kind has for parameters $\alpha, \beta > 0$, the representation

$$F(x) = 1 - (1+x/\beta)^{-\alpha} \qquad \text{for} \quad x > 0. \tag{1}$$

This distribution has proved to be useful for representing the failure times of electronic packages, Myhre and Saunders [1979].

The estimation of the parameters of this distribution has been discussed in numerous papers. Kulldorff and Vännman [1973] have derived the best linear unbiased estimate (BLUE) of the scale parameter, β, for known values of the shape parameter, $\alpha > 2$. For known $\alpha < 2$ an asymptotic best linear unbiased estimator (ABLUE) for β was also derived. Both of these procedures require a complete sample. Vännman [1976] has obtained, for known α, a BLUE estimate of β from censored data (BLUE-k). The only restriction on the procedure is that $k < n + 1 - 2/\alpha$ where n is the sample size and k is the number of failures. Each of these estimation procedures assumes a knowledge of the shape parameter, $\alpha > 0$.

Whenever the scale parameter β is known the transformation $\ln(1+x/\beta)$ makes the life observed exponentially distributed with unknown failure rate α, in which case the optimum property of the total life statistic for type I or type II censoring is well known. Consequently we do not discuss the case further.

Harris and Singpurwalla [1968] suggested the method of moments as an estimation procedure wherever β is, or α and β are, unknown. This method, however, requires not only a complete sample but the restriction $\alpha > 2$, for otherwise the moments fail to exist. The only point in favor of this method of moments seems to be that it gives simple closed expressions for the estimates of both α and β. Unfortunately, we find that they fail to exist so often in practice as to be virtually unusable. For examqle, in our simulations using complete samples as large as $n = 25$ with $\alpha = 4$, the estimates fail to exist nearly 30% of the time.

Harris and Singpurwalla [1969] first exhibited the maximum likelihood equations of this family for complete samples. In Myhre and Saunders [1979] maximum likelihood equations are derived for both the shape parameter α, and the scale parameter β, for censored or complete samples.

While the maximum likelihood estimates are only implicitly defined as the solutions to certain equations, this causes little practical difficulty since they can be calculated with a programmable hand calculator.

For given α, the MLE for β, (say $\hat{\beta}$), is shown by Myhre and Saunders [1979] to be the solution of the equation $A(1/\hat{\beta}) = 0$, where

$$A(x) = \alpha \sum_{j=1}^{n} (1+t_j x)^{-1} + \sum_{1}^{k} (1+t_j x)^{-1} - n\alpha ,$$

with $t_1,\ldots,t_k$ denoting the k failure times and $t_{k+1},\ldots,t_n$ denoting the n-k success times in the sample $(t_1,\ldots,t_n)$.

Since $A(o) = k$, $A(\infty) = -n\alpha$ and $A(x)$ is a decreasing function in x, a unique solution always exists for any positive α or any type of censoring. This is in contrast with the BLUE which requires a complete sample and $\alpha > 2$, the ABLUE which needs a complete sample and the BLUE-k for which $k < n+1 - 2/\alpha$ must hold. (Note that for modest n and small α, k is very restricted.)

In this paper the MLE procedure of Myhre and Saunders [1979] is compared with the BLUE and ABLUE procedures described in Kulldorff and Vannman [1973]. For complete samples with $\alpha > 2$ we know that the BLUE and the MLE have an asymptotic relative efficiency of one. However, in reliability applications, the sample is often censored and some data from industry indicates that $\alpha \leq 2$ can arise. For these cases the variance of the MLE has not been represented in a closed form and hence no relative efficiency with the BLUE-k can be computed. But, the asymptotic variance of the MLE is known in these cases, as was shown by Myhre and Saunders, [1980]. In the appendix it is shown that for type II censoring with large n and k, the variance of the MLE is less than the variance of the BLUE-k. For small n and k, realistic comparisons are made using simulation techniques with particular attention given to censored data for the case with $\alpha \leq 2$.

2. SIMULATION MODEL

To compare the BLUE and the MLE of β for fixed α, ordered failure times are generated from the distribution in equation (1) with $\beta = 1.0$ and the two estimates are computed. These simulations were run for $n = 25, 50, 100, 200$; $\alpha = 2.2, 2.4, 2.6, 2.8, 3.0, 4.0$. For a fixed α and one sample size, one hundred simulations were run and from these date, the averages of each of the two estimates of β and their sample standard deviations about $\beta = 1.0$ were computed. It was seen that the comparisons did not change significantly if more simulations were run for each case, so 100 was used to keep the computer time low. For complete samples, with $\alpha > 2$, the BLUE was seen, Table 1, to be no better than the MLE.

The ABLUE introduced for the case $\alpha < 2$, is a more difficult procedure to use than the BLUE. In simulations comparing the ABLUE and the MLE, ordered observations are generated for a known α, and both ABLUE and MLE estimates of β are computed from this complete sample and compared. The ABLUE is based on ten order statistics, the largest number for which computational results are available in Kulldorff and Vännman [1973]. These simulations are run for $n = 25, 50, 100, 200$; $\alpha = .5, 1.0, 1.5, 2.0, 3.0, 4.0$. Again for a known α, and one sample size, one hundred simulations were run and from these simulations the averages of each of the two estimates of β and their sample standard deviations about $\beta = 1.0$

were computed. The MLE was a closer estimator of $\beta = 1.0$ as is seen in Table 2.

The estimation procedures developed in Vännman [1976], (referred to as BLUE-k), produce a more versatile estimator than the BLUE and ABLUE developed earlier (loc. cit.), and therefore we subjected it to a more thorough investigation. In order to reflect practical applications in which tests are not only truncated on time (Type I censoring) or number of failures (Type II censoring), but are also randomly truncated, simulations were conducted for all these types of censoring. The BLUE-k is based on k order statistics, where k is known. It is possible, however, to compute an estimate of β based on the BLUE-k procedure using randomly censored data by letting k be the observed number of failures; while this procedure is not theoretically justified, it is a natural way to attempt to use the estimator in such cases.

The simulation comparing the BLUE-k and the MLE of β was again run for Type I, Type II and random censoring with four sample sizes, $n = 25, 50, 100$ and 200. Samples that illustrate the results for Types I and II censoring are grouped in Tables 3 and 4. For these samples, α runs through a selected set of values from 0.1 to 4.0. The pattern that emerges does not depend on the type of censoring: the averages and sample standard deviations for all methods improve with sample size, especially for the MLE; the BLUE-k is best for small α, $\alpha = .1$, for all sample sizes checked but the difference is small when $n = 200$. For slightly larger α,

α = .2, .3, .4, the BLUE-k does better than the MLE, i.e., has less bias and smaller sample standard deviation, when $n \le 50$, but the results are very close when $n = 100$ and $n = 200$. For larger α the results for $n = 100, 200$ are very close with a slight edge to the BLUE-k.

Often in industrial applications, on doesn't find Type I or Type II censoring but instead encounters random censoring. Because of this, simulations with random truncation times were run. The censoring distribution function

$$Q(\gamma) = \begin{cases} \dfrac{(\gamma+D-T)^2}{(m+1)D^2} & \text{when} \quad T-D \le \gamma \le T \\[2ex] \dfrac{1-(T+mD-\gamma)^2}{m(m+1)D^2} & \text{when} \quad T \le \gamma \le T+mD \end{cases}$$

for positive m, D and T, was chosen for its versatility and simplicity. Here $m > 1$ makes the distribution skewed to the right and $m < 1$ makes it skewed to the left. The spread of the truncated times is given by $(m+1)D$. Using this censoring distribution for a fixed T, D, and m, a truncation time is generated for each item on test. If the generated failure (observation) time, t_i, is greater than γ_i, then the item is a success and t_i is set to γ_i, while if $t_1 \le \gamma_i$ it is counted as a failure and is unchanged.

The simulations using random censoring, compared the BLUE-k with the MLE of β, and were run with sample sizes of 25, 50, 100 and 200. For each specifie. α, different values for D, m and

T were specified. D ranged from .5T to 0.8T, m = 1,2,3, and T from $F^{-1}(0.5)$ to $F^{-1}(0.8)$ where F is given in equation (1). These simulations showed the MLE to be better than the BLUE-k as an estimator of β except for small α coupled with small sample sizes. (For $\alpha < 1.0$, $n = 25$ the BLUE-k has less bias than the MLE.) These are the cases where the random censoring had little chance to show its effect. But for $50 \leq n \leq 200$ and $\alpha > 0.2$ the MLE showed itself to be more accurate. These results follow in Table 5.

3. JOINT ESTIMATES

The comparisons so far have been made for estimates of β when α is known. For many applications in reliability the parameter α must also be estimated. The joint MLE of α and β for a sample consisting of k failure times $t_1,\ldots,t_k$, and n-k success times $t_{k+1},\ldots,t_n$ requires a numerical solution for x in $H(x) = 0$ where

$$H(x) = 1/k \sum_{i=1}^{k} (1+t_i x)^{-1} \sum_{j=1}^{n} \ln(1+t_j x) - \sum_{j=1}^{n} \frac{t_j x}{1+t_j x} .$$

We obtain $\hat{\beta} = 1/x$ while $\hat{\alpha} = 1/k \sum_{i=1}^{n} \ln(1+t_i x)$. A necessary and sufficient condition that the solution exists, viz., $H(x) = 0$ for some $x > 0$, is that

$$(2/k) \sum_{i=1}^{k} t_i \sum_{j=1}^{n} t_j < \sum_{j=1}^{n} t_j^2 .$$

If this condition does not hold, the distribution, F, is assumed to be exponential with estimated failure rate, $\hat{\lambda} = (1/k) \sum_{i=1}^{n} t_i$.

No BLUE methods exist to estimate α and β jointly. Harris and Singpuwalla [1968] used the method of moments assuming $\alpha > 2$ as an estimation procedure for α and/or β but only for complete samples.

The joint MLE will first be compared with the BLUE-k with α assumed known. Then the joint MLE is compared with the joint estimation by method of moments.

These comparisons will be made using the distance of the estimated distribution from the true distribution in the region of interest, not by using a measure of the estimated $\hat{\alpha}$, $\hat{\beta}$ from the true. This is thought to be the more natural method. Each estimated distribution $\tilde{F}$ is compared to the true, F, at ten evenly spaced values from the interval $(0,F^{-1}(.2))$ by the sum

$$S = \sum_{i=1}^{10} |\tilde{F}(x_i) - F(x_i)| .$$

This is computationally easier than finding the maximum distances between the functions over the same interval. Tables 6 and 7 contain the results comparing the estimated distribution $\tilde{F}_M$ using MLE estimates of α and β, with the estimated distribution $\tilde{F}_B$ with the true α and the BLUE-k estimate of β. When the data dictates, $\tilde{F}_M$ is an estimated exponential distribution.

From Table 6 (Type I censoring) it is seen that for the cases examined, the accuracy of the estimate of the Mixed Exponential distribution function using joint MLE's for unknown α and β is surprisingly better than the accuracy of the estimated distribution obtained with α fixed and using the BLUE-k estimate of β. Under Type II censoring the accuracy of the two estimation procedures appears to be approximately the same even though α is assumed known when using the BLUE-k estimate of β.

To get a fairer comparison of the behavior of the joint estimates of α, β, the parameter α must also be estimated, and the only alternative procedure is the method of moments which restricts one to complete samples with $\alpha > 2$. Even for complete samples of size, say $n = 25$ or 50, the method of moment estimator fails to be admissable a significant number of times. In Table 8, some samples are given comparing the joint MLE with joint method of moments estimates.

Thus for a very important case in applications, namely censored data with both parameters unknown, only the maximum likelihood estimates are available and they are seen to provide reasonably accurate estimators of the distribution in the lower percentiles, the region of our interest.

4. CONCLUSION

For the case where the shape parameter $\alpha \geq 2$ is known, with complete samples, the BLUE of the scale parameter β, was no better than the MLE. For known $\alpha > 2$, with complete samples the comparison between the ABLUE and the MLE showed the MLE to be the better estimator. For censored samples, moreover simulations show that the MLE is as accurate an estimate of the scale parameter β as the BLUE-k except for small sample sizes and for small values of the shape parameter, α.

In practice, both parameters must be estimated jointly. Such joint estimation is not possible with BLUE, ABLUE, or BLUE-k methods, and the method of moments fails too often. Therefore, in applications the joint MLE procedure would be used even if the MLE of β were not more accurate. Moreover, the estimate of the distribution function using joint MLE's of the scale and shape parameters compares favorably or looks superior, in the regions of interest in reliability theory, to the estimate of the distribution function obtained using the BLUE-k for the scale parameter and a <u>known</u> shape parameter.

APPENDIX

Let $\alpha > 0$ be given and denote $\tilde{\beta}$ and $\hat{\beta}$ as the BLUE-k and the MLE, respectively, of β. Then from Vannman [1976], we have

$$\mathrm{VAR}(\tilde{\beta}) = \frac{\beta^2(\alpha+2)}{n\alpha - 2 - [(n-k)\alpha - 2]R}$$

where R is the ratio of gamma functions, namely

$$R = \Gamma(n+1-\omega)\Gamma(n-k+1)/\Gamma(n+1-k-\omega)\Gamma(n+1)$$

with $\omega = 2/\alpha$.

From Halperin [1954], the asymptotic variance of $\hat{\beta}$ is given by

$$\mathrm{AVAR}(\hat{\beta}) = \frac{\beta^2(\alpha+2)}{n\alpha[1-(1-\frac{k}{n})^{1+\omega}]}$$

from n large.

If $k = qn$ for some $q \in (0,1)$ and $n - k > \omega$, we will show that the ratio of variances satisfies

$$\frac{\mathrm{AVAR}(\hat{\beta})}{\mathrm{VAR}(\tilde{\beta})} = \frac{1 - \frac{\omega}{n} - (1-q-\frac{\omega}{n})R}{1-(1-q)^{1+\omega}} \leq 1$$

with equality of $\omega = 0$ or in the limit as $n \to \infty$.

Using Sterling's approximation, we find for the ratio R

$$R \cong \sqrt{\frac{(n-\omega)(n-k)}{(n-\omega-k)n}}\,\frac{(n-\omega)^{n-\omega}(n-k)^{n-k}}{(n-\omega-k)^{n-\omega-k}n^{n}}$$

The square root obviously approaches unity as $n \to \infty$, while the second can be rewritten

$$(1-\frac{\omega}{n})^n\,[1-\frac{\omega}{(1-q)n}]^{-n}\left(\frac{1-\frac{\omega}{n}-q}{1-\frac{\omega}{n}}\right)^{\omega}[1-\frac{\omega}{n(1-q)}]^k$$

which can be seen to approach $(1-q)^{\omega}$ since $k = qn$. Thus, in the limit as $n \to \infty$, we see quality is attained.

We want to examine the derivative of the ratio of variances as a function of $n > \omega + k$. Since the denominator is independent of n we find the derivative, call it D, to be proportional to

$$D = \frac{\omega}{n^2}(1-R) - (1-q-\frac{\omega}{n})R(\ln R)'.$$

Since $0 < R < 1$, it is sufficient to show only that $(\ln R)' < 0$, to conclude $D > 0$ from which it follows that the ratio of the variances increases to unity. We check that, setting $p - 1-q$,

$$(\ln R)' = \psi(n+1-\omega) - \psi(n+1) + p[\psi(np+1-\omega) - \psi(np+1)]$$

where the psi (digamma) function is the derivative of the logarithm of the gamma function, see Abramowitz and Stegun [1964]. And since the psi function is increasing for non-negative argument, loc. cit., the result is proved.

REFERENCES

[1] Abramowitz, M., and Stegun, Irene A. (1964), Handbook of Mathematical Functions, National Bureau of Standards, 258.

[2] Halperin, M. (1952), "Maximum Likelihood Estimation in Truncated Samples," Annals of Mathematical Statistics, 23, 226-238.

[3] Harris, C.M., and Singpurwalla, N.D. (1968), "Life Distributions Derived from Stochastic Hazard Functions," IEEE Transactions in Reliability, R-17, 70-79.

[4] Harris, C.M., and Singpurwalla, N.D. (1968), "On Estimation in Weibull Distributions with Random Scale Parameters," Naval Research Logistics Quarterly, 16, 405-410.

[5] Kulldorff, Gunnor, and Vannman, Kerstin (1973), "Estimation of the Location and Scale Parameters of a Pareto Distribution by Linear Functions of Order Statistics," Journal of the American Statistical Association, 68, 218-227.

[6] Myhre, Janet and Saunders, Sam C. (1979), "Problems of Estimation for a Decreasing Failure Rate Distribution Applied to Reliability," submitted for publication to Technometrics.

[7] Myhre, Janet and Saunders, Sam C. (1980), "Asymptotic Normality of Maximum Likelihood Estimates under Random Censoring," submitted for publication to Journal of the American Statistical Association.

[8] Vannman, Kerstin (1976), "Estimators Based on Order Statistics from a Pareto Distribution," Journal of the American Statistical Association, 71, 704-708.

Table 1. Simulation experiments (100 realizations) comparing MLE and BLUE estimates of scale parameter $\beta = 1.0$.

Average of Estimates

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE	MLE	BLUE	MLE	BLUE	MLE	BLUE
2.2	1.02	1.00	1.01	1.00	1.01	1.00	1.00	1.00
2.4	1.03	1.00	1.04	1.02	1.01	1.00	1.00	.99
2.6	1.07	1.05	.99	.98	1.01	1.00	1.00	1.00
2.8	1.02	1.00	1.02	1.00	.99	.99	1.00	1.00
3.0	1.02	1.00	1.01	.99	1.00	.99	1.01	1.01
4.0	1.06	1.04	.99	.98	.99	.99	.99	.99

Sample Standard Deviations

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE	MLE	BLUE	MLE	BLUE	MLE	BLUE
2.2	.28	.27	.20	.19	.15	.15	.09	.09
2.4	.33	.32	.19	.18	.16	.16	.10	.09
2.6	.33	.32	.18	.18	.13	.13	.10	.10
2.8	.27	.27	.17	.16	.12	.12	.10	.10
3.0	.26	.25	.16	.16	.13	.13	.10	.10
4.0	.26	.26	.18	.18	.12	.12	.08	.08

Table 2. Simulation experiments (100 realizations) comparing MLE and ABLUE estimates of the scale parameter $\beta = 1.0$

Average of Estimates

	n= 25		n = 50		n = 100		n = 200	
α	MLE	ABLUE	MLE	ABLUE	MLE	ABLUE	MLE	ABLUE
.5	1.12	1.22	1.01	1.06	1.03	1.06	1.02	1.04
1.0	1.08	1.13	1.02	1.04	1.00	1.01	1.00	1.01
1.5	1.09	1.13	1.01	1.02	1.00	1.01	1.00	1.01
2.0	1.00	1.08	1.00	1.01	1.01	1.02	1.00	1.01
3.0	1.00	1.01	1.02	1.04	1.04	1.04	.99	.99
4.0	1.07	1.08	.99	1.01	1.01	1.01	1.00	1.00

Sample Standard Deviations

	n = 25		n = 50		n = 100		n = 200	
α	MLE	ABLUE	MLE	ABLUE	MLE	ABLUE	MLE	ABLUE
.5	.49	.58	.29	.32	.26	.27	.18	.19
1.0	.38	.42	.25	.27	.18	.19	.12	.13
1.5	.31	.34	.25	.26	.14	.14	.10	.11
2.0	.28	.33	.20	.20	.13	.13	.10	.10
3.0	.25	.26	.16	.18	.14	.15	.09	.09
4.0	.27	.27	.16	.15	.14	.14	.09	.10

Table 3. Simulation experiments (100 realizations) comparing MLE and BLUE-k estimates of the scale parameter $\beta = 1.0$ under Type I censoring of data.

Average of Estimates

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
.1	2.26	.91	1.23	.90	1.08	.91	1.00	.92
.2	1.22	.78	1.23	.95	1.12	.99	1.03	.98
.3	1.26	.80	1.22	1.00	1.09	.99	1.04	1.00
.4	1.30	.93	1.17	.98	1.06	.98	1.01	.97
.5	1.49	1.02	1.21	1.02	1.02	.95	1.06	1.02
.6	1.32	.92	1.00	.87	1.10	1.00	1.04	1.00
.8	1.24	.90	1.18	1.01	1.10	1.01	1.00	.98
1.0	1.38	.98	1.20	1.02	1.08	1.02	1.05	1.01
1.2	1.31	.92	1.17	1.02	1.08	1.01	1.03	1.00
1.6	1.43	.94	1.11	.98	1.10	1.02	1.04	1.01
2.0	1.15	.88	1.12	.99	1.03	.97	1.02	1.00
2.5	1.35	1.04	1.18	1.05	1.05	.98	1.01	.98
3.0	1.21	.92	1.06	.94	1.10	1.03	1.02	.99

Sample Standard Deviations

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
.1	3.67	1.63	1.04	.75	.49	.41	.31	.39
.2	1.00	.64	.80	.51	.35	.28	.26	.25
.3	1.09	.49	.71	.50	.39	.33	.20	.19
.4	.88	.65	.60	.49	.35	.32	.21	.20
.5	1.35	.79	.65	.48	.29	.28	.21	.20
.6	1.12	.57	.38	.35	.42	.27	.20	.18
.8	.92	.63	.56	.40	.31	.28	.18	.17
1.0	1.23	.67	.62	.38	.32	.28	.18	.17
1.2	1.16	.68	.51	.44	.30	.26	.19	.19
1.6	1.42	.67	.58	.44	.31	.26	.18	.17
2.0	.71	.44	.40	.32	.26	.24	.18	.17
2.5	.95	.67	.53	.41	.29	.26	.19	.19
3.0	.77	.43	.48	.37	.34	.25	.17	.16

Table 4. Simulation experiments (100 realizations) comparing MLE and BLUE-k estimates of the scale parameter $\beta = 1.0$ under Type II censoring with $k = .2n$.

Average of Estimates

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
0.1	1.96	1.01	1.40	1.04	1.14	1.00	1.03	.96
0.2	1.40	1.03	1.18	1.03	1.07	1.00	1.04	1.00
0.3	1.19	1.01	1.06	.97	1.03	.99	1.01	.99
0.4	1.21	1.07	1.14	1.07	1.02	.99	1.00	.99
0.5	1.08	.98	1.02	.97	1.02	.99	1.02	1.00
0.6	1.09	1.01	.98	.94	1.02	1.00	1.01	1.00
0.8	1.10	1.03	1.06	1.03	1.02	1.00	1.01	1.00
1.0	1.01	.96	1.01	.99	1.02	1.01	1.01	1.00
1.2	1.04	1.00	1.00	.99	1.02	1.01	1.01	1.01
1.6	1.06	1.03	.93	.92	.99	.98	1.01	1.01
2.0	1.02	1.00	1.02	1.01	.99	.99	1.01	1.01
2.5	.98	.97	.97	.96	.96	.96	1.03	1.03
3.0	1.02	1.00	1.06	1.05	1.03	1.03	1.01	1.00
4.0	1.04	1.03	1.00	.99	1.00	1.00	1.02	1.02
10.0	.98	.97	.95	.95	.99	.99	1.02	1.02

Sample Standard Deviations

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
0.1	3.57	1.50	1.30	.93	.59	.51	.33	.31
0.2	1.27	.91	.66	.55	.34	.32	.26	.24
0.3	.91	.77	.42	.38	.34	.32	.22	.22
0.4	.87	.75	.50	.46	.31	.30	.19	.19
0.5	.59	.52	.37	.36	.31	.30	.20	.20
0.6	.63	.58	.33	.33	.27	.27	.19	.18
0.8	.58	.54	.37	.36	.24	.24	.19	.19
1.0	.50	.48	.37	.36	.28	.27	.19	.19
1.2	.48	.46	.34	.34	.27	.27	.19	.18
1.6	.46	.44	.33	.33	.24	.24	.16	.15
2.0	.50	.49	.36	.35	.22	.22	.18	.18
2.5	.47	.47	.29	.29	.23	.23	.17	.17
3.0	.49	.48	.35	.35	.22	.22	.17	.16
4.0	.48	.48	.34	.33	.25	.25	.17	.17
10.0	.44	.44	.28	.28	.22	.22	.16	.16

Table 5. Simulation experiments (100 realizations) comparing MLE and BLUE-k estimates of the scale parameter $\beta = 1.0$ under random censoring.

Average of Estimates

	n = 25		n = 50		n = 100		n = 200	
	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
0.1	1.53	.52	1.32	1.02	1.19	1.10	1.08	1.09
0.2	1.46	.95	1.26	1.17	1.13	1.24	1.08	1.30
0.3	1.35	1.15	1.16	1.22	1.08	1.28	1.03	1.35
0.4	1.28	1.09	1.15	1.24	1.03	1.30	1.03	1.43
0.5	1.28	1.18	1.05	1.20	1.04	1.36	1.03	1.45
0.6	1.16	1.10	1.11	1.30	1.01	1.38	1.02	1.49
0.8	1.29	1.22	1.11	1.32	.99	1.36	1.02	1.52
1.0	1.26	1.23	1.09	1.32	1.06	1.43	1.00	1.54
1.2	1.17	1.24	1.08	1.35	1.04	1.45	1.00	1.53
1.6	1.22	1.31	1.01	1.29	1.04	1.52	1.00	1.56
2.0	1.08	1.17	1.11	1.46	1.06	1.52	.99	1.54

Sample Standard Deviations

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
0.1	2.00	1.22	.99	.78	.58	.55	.40	.42
0.2	1.55	.68	.80	.81	.44	.55	.30	.50
0.3	1.06	.86	.51	.63	.35	.52	.21	.48
0.4	.88	.68	.56	.67	.34	.53	.20	.56
0.5	.85	.15	.47	.54	.30	.62	.17	.54
0.6	.74	.81	.44	.58	.27	.58	.18	.58
0.8	.78	.66	.47	.64	.21	.52	.16	.61
1.0	1.18	.90	.42	.65	.28	.56	.17	.63
1.2	.63	.73	.39	.65	.23	.59	.14	.59
1.6	.75	.87	.28	.50	.22	.65	.16	.64
2.0	.48	.69	.47	.79	.23	.64	.13	.59

Table 6. Simulation experiments (100 realizations) comparing estimates of the mixed exponential distribution function by joint MLE estimates of parameters and BLUE-k estimate of β with known α . Type I censoring of data.

Average of Sum of Absolute Differences from true distributions.

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
0.2	.49	.62	.32	.34	.23	.24	.17	.16
0.4	.45	.59	.32	.38	.24	.23	.14	.13
0.6	.41	.55	.35	.37	.21	.21	.14	.15
0.8	.41	.46	.27	.27	.22	.23	.14	.14
1.0	.40	.47	.27	.31	.24	.24	.15	.15
1.5	.41	.48	.34	.32	.21	.21	.16	.15
2.0	.39	.54	.33	.34	.22	.22	.13	.13
2.5	.40	.48	.33	.37	.21	.20	.13	.12
3.0	.38	.51	.31	.31	.21	.19	.14	.13

Table 7. Simulation experiments (100 realizations) comparing estimates of the mixed exponential distribution function by joint MLE estimates of parameters and BLUE-k estimate of β with known α . Type I censoring of data.

Average of Sum of Absolute Differences from true distributions

	n = 25		n = 50		n = 100		n = 200	
α	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k	MLE	BLUE-k
0.2	.68	.54	.45	.38	.27	.25	.21	.18
0.4	.52	.48	.33	.31	.22	.19	.18	.16
0.6	.53	.49	.30	.29	.23	.22	.15	.15
0.8	.51	.51	.31	.30	.21	.21	.14	.13
1.0	.51	.49	.33	.32	.21	.20	.14	.12
1.5	.53	.51	.29	.29	.23	.22	.14	.14
2.0	.53	.53	.24	.24	.21	.20	.15	.14
2.5	.42	.41	.29	.28	.21	.20	.14	.13
3.0	.43	.43	.29	.29	.22	.20	.15	.15

Table 8. Simulation experiments (100 realizations) comparing estimates of the mixed exponential distribution functions by joint MLE estimates of the parameters and method of moments of the parameters. Complete samples.

Average of Sum of Absolute Differences

	n = 25		n = 50		n = 100		n = 200	
α	MLE	MofM	MLE	MofM	MLE	MofM	MLE	MofM
2.5	.27	2.17	.18	1.45	.14	.76	.10	.52
3.0	.26	1.38	.18	.83	.14	.67	.09	.33
3.5	.27	1.06	.19	.78	.13	.74	.09	.51
4.0	.22	.79	.18	.87	.14	.60	.09	.60