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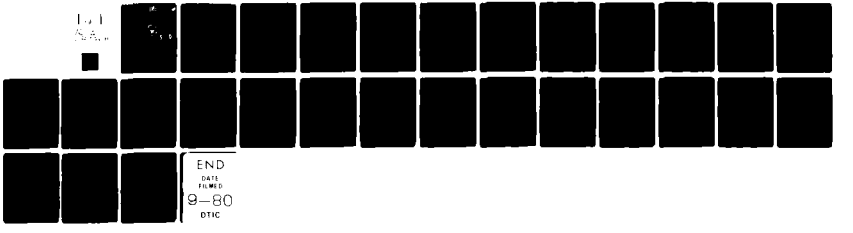

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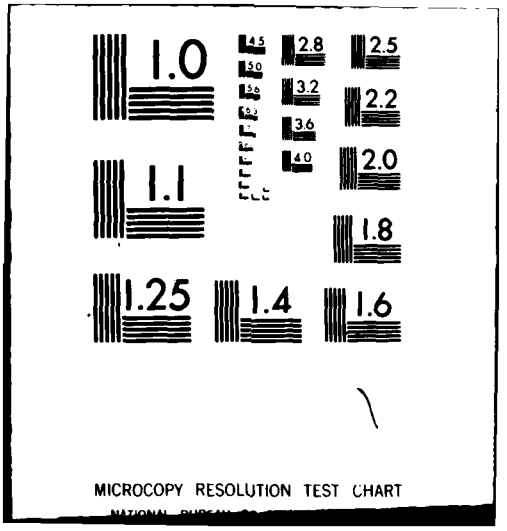
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SIMULATION OF SOME AUTOREGRESSIVE
MARKOVIAN SEQUENCES
OF POSITIVE RANDOM VARIABLES

by

A. J. Lawrance

and

P. A. W. Lewis

October 1979

Approved for public release; distribution unlimited.

Prepared for:

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Arlington, Virginia 22217

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20. Abstract Cont.

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ABSTRACT

Methods for simulating dependent sequences of continuous positive-valued random variables with exponential, Gamma, mixed exponential and uniform marginal distributions are given. In most cases the sequences are first-order, linear autoregressive, Markovian processes. A two-parameter family of this type with exponential marginals is defined and its transformation to a similar multiplicative process with uniform marginals is given. It is shown that for a subclass of this two-parameter family extension to mixed exponential marginals is possible, giving a model of broad applicability for analyzing data and modelling stochastic systems. Efficient simulation of some of these schemes is discussed.

1. INTRODUCTION

In a recent series of papers [1,2,3,4,5,6,7,8] some simple models have been derived for stationary dependent sequences of positive, continuous random variables with given first-order marginal distributions. In general the dependency structure, as measured by second-order joint moments (serial correlations) mimics that of the usual linear mixed auto-regressive-moving average (ARMA) models which have been used for so long in time-series analysis. In the ARMA models, which are defined quite generally, there is in usage an implicit assumption of marginal normality of the random variables. This is clearly not the case if the random variables are positive, say the times between events in a series of events [9] or the successive response times at a computer terminal. Thus the new models are derived to accommodate situations in which the dependent random variables have, for instance, exponential, Gamma, Laplace and mixed exponential marginal distributions. The exponential case is the most highly developed, with the nomenclature [4] EARMA(p,q) (exponential process with mixed moving average-autoregressive structures of orders p and q respectively) and NEARMA(p,q) (new EARMA(p,q)).

The development of the probabilistic properties of these processes is given in the referenced papers, applications to queueing models and computer system modelling in [10,11] while development of estimation and testing procedures has just begun.

The object of the present paper is to define and discuss the simulation of the processes on digital computers, though for the sake of brevity only the first-order Markovian, autoregressive case is considered. The simplicity of structure of these models--in general they are linear additive mixtures of random variables--makes them ideal for this purpose. However stationarity conditions are sometimes difficult to derive analytically and in some cases it is not simple to generate the innovation random variables in the processes. A striking example of this is the case of the Gamma first-order autoregressive process for which an efficient means of simulation has only recently been found [7] for some parametric values. It is shown that a simple transformation of the exponential sequences gives a direct multiplicative method for generating dependent processes with uniform marginals. These could be the basis in simulations for many other types of dependent sequences.

2. EXPONENTIAL AUTOREGRESSIVE MARKOVIAN SEQUENCES

We give here three methods of generating first-order autoregressive, Markovian sequences with exponential marginal distributions. The first two are defective in terms of their sample path properties (the first more so than the second) while the third, NEAR(1), is satisfactory in this respect. The defect of the first two models is also highlighted by the simulation procedures used; they can be generated from one sequence of exponential variables.

Note that autoregression in the context of a stochastic sequence $\{X_n\}$ is vaguely used. In the first place linear, additive autoregression is usually implied. In the second place autoregression can mean that in the defining equation for X_n the previous value enters explicitly, but more particularly it means that the conditional expectation of X_n, given $X_{n-1} = x_{n-1}$, is an additive linear function of x_{n-1};

$$E(X_n | X_{n-1} = x_{n-1}) = a + bx_{n-1}. \tag{1}$$

The Markovian property (first-order) means that the probability structure of $X_n, X_{n+1}, \ldots$, given $X_{n-1} = x_n$ is independent of $X_{n-2}, X_{n-3}, \ldots$.

2A. The Exponential DAR(1) Process

A very simple exponential autoregressive Markovian sequence is generated by the equation

$$X_n = V_n X_{n-1} + (1 - V_n)E_n \ , \tag{2}$$

where $P\{V_n=1\} = 1 - P\{V_n=0\} = \rho$ and E_n, $n = 1,2,\ldots$ are independent exponential random variables with parameter λ;

$$\begin{aligned} P\{E_n \leq x\} &= 1 - e^{-\lambda x}, \quad x \geq 0, \ \lambda > 0 \\ &= 0 \qquad , \quad x < 0. \end{aligned} \tag{3}$$

For this process the serial correlations $\rho_k = \mathrm{corr}(X_n, X_{n+k})$ are

$$\rho_k = \rho^k \tag{4}$$

and

$$E(X_n | X_{n-1} = x_{n-1}) = \rho_1 x_{n-1} + (1 - \rho_1)/\lambda. \tag{5}$$

This process is an exponential version of the DAR(1) process [12,13] but is rather useless for modelling real data because simulations of the process show runs of X_n's with the same value. These occur when X_{n-1} is picked successively in (2),

rather than the innovation E_n. Moreover the lengths of the runs of similar values are geometrically distributed.

2B. The Exponential EAR(1) Process

Another model is derived from the usual linear model

$$X_n = \rho X_{n-1} + \varepsilon_n \tag{6}$$

in which the i.i.d. innovation process $\{\varepsilon_n\}$ is chosen so that the X_n's are marginally exponential(λ). Gaver and Lewis [1] show that for this to be true, one must have $0 \leq \rho < 1$ and

$$\begin{aligned} \varepsilon_n &= E_n \qquad \text{w.p. } 1-\rho, \\ &= 0 \qquad \text{w.p. } \rho\ , \end{aligned} \tag{7}$$

where $\{E_n\}$, as throughout the paper, are i.i.d. exponential(λ). Again $\rho_k = \rho^k$ and $E(X_n | X_{n-1} = x_{n-1}) = \rho_1 x_{n-1} + (1-\rho_1)\lambda$, as at (4) and (5) for the exponential DAR(1) model. The difference is in the sample paths; for the EAR(1) process simulations show runs of X_n's decreasing geometrically. These occur when only ρX_{n-1} is picked in (6). Again the lengths of these runs are geometrically distributed.

The Markov property of the two sequences implies that if X_0 is chosen to be E_0, an exponential(λ) random variable independent of $E_1, E_2, \ldots$, then $X_1, X_2, \ldots$ forms a stationary sequence.

Naive inspection of the defining equations (2), (6) and (7) suggest that to generate a stationary sequence of length N, $X_1, \ldots, X_N$, (N+1) i.i.d. exponential deviates and N uniform variates (for the selection process) are needed. However, the sequences can be generated from only one exponential sequence; this is possibly related to the degeneracy in the processes. This method uses the memoryless property of exponential(λ) variables, namely that if E_n is given to be greater than a constant γ, then $E_n - \gamma$ is again exponential(λ).

Thus the algorithm is to initialize by setting $X_0 = E_0$; subsequently set $X_n = \rho X_{n-1}$ if $E_n \leq x_\rho = -\ell n(1-\rho)/\lambda$; otherwise set $X_n = \rho X_{n-1} + (E_n - x_\rho)$. This uses the fact that, from (3), $P\{E_n \leq x_\rho\} = \rho$.

Even greater efficiency can be obtained, though this must be qualified by considerations as to whether the X_n's are to be generated one at a time or in an array; whether a subroutine is available to generate exponential random variables faster than can be done by taking logarithms of uniform deviates, and the relative speed of division and generation of uniform deviates.

The more efficient scheme recycles uniform variables, i.e. if U is given to be between constants a and b, where $0 < a < b \leq 1$, then $(U-a)/(b-a)$ is a uniform random variable. (Note that its value is not given, only that it is in (a,b).) Thus to generate an array $X_1, \ldots, X_N$, of dependent exponential variates with mean 1 from the EAR(1) process we have

Algorithm 1 (EAR(1) process)

1. Generate U and set $X_0 \leftarrow -\ell n\ U$
2. Generate U and set $Y \leftarrow U$
3. $n \leftarrow 1$
4. If $Y \leq \rho$ go to 7
5. $X_n \leftarrow \rho X_{n-1} - \ell n(Y-\rho) + \ell n(1-\rho)$
6. Generate U, set $Y \leftarrow U$ and go to 8
7. $X_n \leftarrow \rho X_{n-1}$; $Y \leftarrow Y/\rho$
8. Set $n \leftarrow n+1$
9. If $n \leq N$ go to 4
10. Otherwise exit.

The expected number of uniform deviates required in this algorithm is $1 + (1-\rho)N$, which is less than the number N required to generate an i.i.d. exponential(λ) sequence.

2C. The Exponential NEAR(1) Process

A broader two-parameter exponential sequence which is a first-order autoregressive, Markovian process and an additive linear mixture of random variables is given by Lawrance [7] and developed by Lawrance and Lewis [5]. Called NEAR(1), the sequence is defined as

$$X_n = \varepsilon_n + \begin{cases} \beta X_{n-1} & \text{w.p. } \alpha \\ 0 & \text{w.p. } 1-\alpha \end{cases} \qquad n = 1,2,\ldots, \qquad (8)$$

where $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. It can be shown that for the X_n to be marginally exponential(λ) the innovation variable ε_n must be generated from an E_n by the exponential mixture

$$\varepsilon_n = \begin{cases} E_n & \text{w.p. } \dfrac{1-\beta}{1-(1-\alpha)\beta} \\ (1-\alpha)\beta E_n & \text{w.p. } \dfrac{\alpha\beta}{1-(1-\alpha)\beta} \end{cases} \qquad n = 1,2,\ldots \qquad (9)$$

providing α and β are not both equal to one. When $\alpha = 0$ or $\beta = 0$ the $\{X_n\}$ are exponential i.i.d., whereas when $\alpha = 1$ the EAR(1) model given at (6) and (7) is obtained. In fact fixing either α or β and varying the other parameter gives an exponential model with a full positive range of serial correlation of order one, since it is easily shown that

$$\rho_k = (\alpha\beta)^k \ . \tag{10}$$

Again

$$E(X_n | X_{n-1} = x_{n-1}) = \alpha\beta x_{n-1} + (1-\alpha\beta)/\lambda$$

$$= \rho_1 x_{n-1} + (1-\rho_1)/\lambda \tag{11}$$

and $X_0 = E_0$ gives a stationary sequence. The NEAR(1) process allows one to model a broader class of exponential sequence as measured either by sample path behavior or higher-order joint moments; see [5] for details.

A particularly simple case occurs when $\beta = 1$; this model, called TEAR(1), is very tractable analytically and, as will be shown below, extends easily to the case of mixed exponential distributions for the X_n.

Note that in the NEAR(1) process the innovation ε_n is always present unless $\alpha = 1$ and it is therefore not possible to simulate the stationary process with less than N+1 uniform variates. The simplest method to generate dependent exponential variates with mean 1 from the NEAR(1) process seems to be the following:

Algorithm 2 (NEAR(1) process)

1. Generate U; set $X_0 \leftarrow -\ell n\ U$; $\gamma \leftarrow 1-\alpha$; $\delta \leftarrow (1-\beta)/[1 - (1-\alpha)\beta]$.
2. Generate U; set $n \leftarrow 1$.
3. If $U \leq \gamma$ set $Y \leftarrow U/\gamma$ and go to 7.

4. Otherwise $Y \leftarrow (U - \gamma)/(1 - \gamma)$.

5. If $Y \leq \delta$ set $X_n \leftarrow \beta X_{n-1} - \ell n\ Y + \ell n\ \delta$ and go to 9.

6. Otherwise set $X_n \leftarrow \beta X_{n-1} - \gamma\beta\ \ell n[(Y-\delta)/(1-\delta)]$ and go to 9.

7. If $Y \leq \delta$ set $X_n \leftarrow -\ell n\ Y + \ell n\ \delta$ and go to 9.

8. Otherwise set $X_n \leftarrow -\gamma\beta\ \ell n[(Y-\delta)/(1-\delta)]$.

9. Set $n \leftarrow n+1$.

10. If $n \leq N$ generate U and go to 3. Otherwise exit.

Note that for stationary array of N X_n's, exactly $N+1$ uniforms are required and therefore it could be advantageous to generate these in an array which would be replaced one at a time by the X_n's. Care must be taken with the recycling of the uniform variates U if $\gamma = 1-\alpha$ is close to one or zero. In that case it is probably better for computational reasons to use $2(N+1)$ uniform variates. Note that $\gamma = 1$ gives the EAR(1) process.

3. UNIFORM MARKOVIAN SEQUENCES

It is convenient to have dependent sequences of random variables with marginal distributions other than exponential. Before discussing other solutions to the Equation (8) we show that a simple transformation of the NEAR(1) process gives a two-parameter family of Markovian random variables with

uniform marginal distributions. It is well-known that an exponential transformation of a unit exponential random variable gives a uniformly distributed random variable. Thus we have from (8) and (9) the multiplicative model for a uniform Markovian sequence $\{X_n\}$, $n = 1,2,\ldots$;

$$X_n = \varepsilon_n \cdot X_{n-1}^{\beta} \qquad \text{w.p.} \quad \alpha$$

$$= \varepsilon_n \qquad \text{w.p.} \quad (1-\alpha) \qquad (12)$$

$$n = 1,2,\ldots ,$$

where

$$\varepsilon_n = U_n \qquad \text{w.p.} \quad \frac{1-\beta}{1-(1+\alpha)\beta} \qquad (13)$$

$$= U_n^{(1-\alpha)\beta} \qquad \text{w.p.} \quad \frac{\alpha\beta}{1-(1-\alpha)\beta} \qquad (14)$$

$$n = 1,2,\ldots$$

for U_n, $n = 1,2,\ldots$, i.i.d. uniformly distributed, providing that α and β are not both equal to one. Again if X_0 is uniformly distributed and independent of $U_1, U_2, \ldots$ the sequence is stationary.

An algorithm for generating this two-parameter uniform sequence is easily adapted from Algorithm 2. It remains to find the correlation structure and the regression of X_n on X_{n-1}.

To do the former, let X_n^* be a NEAR(1) sequence with $\lambda = 1$, so that the sequence X_n at (12) is given by

$X_n = \exp\{-X_n^*\}$. Now the joint Laplace-Stieltjes transform of X_n^*, X_{n-k}^* is given by Lawrance and Lewis [5] as

$$\phi_{X_n^*, X_{n-k}^*}(s,t)$$

$$= E\{\exp[-sX_n^* - tX_{n-k}^*]\} \tag{15}$$

$$= \alpha^k \prod_{i=0}^{k} \phi_\varepsilon(\beta^i s)\, \phi_{X^*}(\beta^k s + t) + \sum_{j=0}^{k-1} (1-\alpha)\alpha^j \prod_{i=0}^{j} \phi_\varepsilon(\beta^i s)\phi_{X^*}(t) ,$$

where $\phi_\varepsilon(s)$ and $\phi_{X^*}(s)$ are respectively Laplace-Stieltjes transforms of ε defined at (9) and the exponential variable X^*. Setting $s = t = 1$ in (15) gives

$$\phi_{X_n^*, X_{n-k}^*}(1,1) = E\{\exp(-X_n^*)\exp(-X_{n-1}^*)\}$$

$$= E(X_n X_{n-k}) . \tag{16}$$

Then using the fact that for a uniform random variable $E(X) = 1/2$ and $\text{var}(X) = 1/12$, we have from (15) and (16), after simplification

$$\rho_k = \text{corr}(X_n, X_{n-k}) = \frac{3}{2 + \beta^k} \sum_{i=1}^{k} \left(\frac{\alpha\beta}{1 + (1-\alpha)\beta^i} \right)$$

$$k = 1,2,\ldots . \tag{17}$$

Note that this is not simply a geometrically decaying correlation sequence, as for the NEAR(1) process. However, for the important special case when $\beta = 1$ we get

$$\rho_k = \left(\frac{\alpha}{2-\alpha}\right)^k , \qquad k = 1,2,\ldots , \tag{18}$$

and thus the serial correlations ρ_k are the kth power of ρ_1, which takes on any value between 0 and 1. Thus we have a particularly simple uniform Markovian sequence.

A similar analysis given in Lawrance and Lewis [5] shows that

$$E(X_n | X_{n-1} = u) = \frac{1}{2} \frac{1 + \beta}{\{(1 + (1-\alpha)\beta\}} \{1 - \alpha + \alpha u^{\beta}\} \tag{19}$$

so that the regression is not linear.

This uniform sequence could form the basis, via a probability integral transform, of many other sequences with given marginals. However, marginal transformations do not preserve correlation structure, as shown at (17), and it is therefore useful to see whether sequences with marginals other than exponential can be generated from (8) for special cases with a suitable choice of innovation sequence ε_n. This will result in a simple process with autoregressive Markovian structure.

4. MARKOVIAN SEQUENCES WITH SOME OTHER MARGINALS

Although an exponential distribution is a common assumption for positive random variables met with in problems in operations research, it is too narrow an assumption to encompass real situations. Therefore parametric distribution models are invoked which include the exponential as a special case and which allow for the modelling of data which has greater or lesser dispersion than exponentially distributed data. Two commonly used models are

(i) the Gamma(k,λ) distribution whose probability density function is

$$f(x) = \frac{\lambda(\lambda x)^{k-1}e^{-\lambda x}}{\Gamma(k)}, \quad k > 0;\ \lambda > 0;\ x \geq 0, \tag{20}$$

where $\Gamma(k)$ is the complete gamma function, and

(ii) the (convex) mixture of exponential random variables

$$f(x) = \pi_1\lambda_1 e^{-\lambda_1 x} + (1-\pi_1)e^{-\lambda_2 x},$$

$$0 < \lambda_1 < \lambda_2;\ x \geq 0,\ 0 \leq \pi_1 \leq 1. \tag{21}$$

The Gamma distribution has dispersion, measured by the coefficient of variation $C(X) = \sigma(X/E(X))$, which is greater than the exponential value of 1 if $k < 0$ and less than 1 if $k > 1$. The mixed exponential always has $C(X) \geq 1$, the equality occurring when the special case of an exponential random variable with parameters λ_1 or λ_2 holds.

4A. The Gamma GAR(1) process

Direct solution of equation (6) using Laplace-Stieltjes transforms gives [1] that, in the stationary sequence, for the X_n to be Gamma(k,λ) we must have

$$\phi_\varepsilon(s) = E(e^{-s\varepsilon}) = \left\{\rho + (1-\rho)\,\frac{\lambda}{\lambda+s}\right\}^k . \qquad (22)$$

For k integer this has an explicit inverse. For example, for $k = 2$ the innovation ε is zero with probability ρ^2, is exponential(λ) with probability $2\rho(1-\rho)$ and is Gamma$(2,\lambda)$ with probability $(1-\rho)^2$. It is easy to show in general that ε is zero with probability ρ^k, so that the "zero defect" is not serious for large k. A method of simulating a random variable whose Laplace-Stieltjes transform is equation (22) was derived by Lawrance [7], using the fact that this sequence arises in a particular type of shot noise process. Thus we have the

Gamma Innovation Theorem

Let N be a Poisson random variable with parameter $\theta = -k\,\ell n(\rho)$. Let $U_1, U_2, \ldots, U_N$ be uniformly distributed over $(0,1)$ and independent. Let $Y_1, \ldots, Y_N$ be exponential(λ) and independent. Then ε can be simulated using

$$\varepsilon = \sum_{m=1}^{N} Y_m \rho^{U_m} \quad \text{if} \quad N > 0 ,$$

$$= 0 \qquad\qquad \text{if} \quad N = 0 . \qquad (23)$$

A proof is not given here. Note that ε is zero with probability $\exp\{-k\ \ell n(\rho)\} = \rho^k$. Also the Poisson number N of uniform and exponential random variables which must be generated for each ε has expected value $\theta = -k\ \ell n(\rho)$. This will be prohibitively large, and the simulation will be very inefficient, if k is large and/or ρ is close to zero. Neither of these cases is serious, however. If k is large, say greater than 50, the sequence is almost normal and the usual normally distributed, AR(1) linear process can be used. If ρ is as small as 0.001 then $E(N)$ is still only $k \times (6.9078)$ which is still reasonable. However, for ρ this small the sequence is approximately i.i.d. Gamma and acceptance-rejection techniques for simulating Gamma variables are known.

It is quite simple to adapt Algorithm 1 to the GAR(1) case. It would pay to have a built-in routine for generating the Poisson variable which will bypass further calculations if $N = 0$. In other words routines for generating Poisson variates which start by searching at the median of a table of cumulative Poisson probabilities will be inefficient.

Unfortunately the NEAR(1) process does not appear to extend to the Gamma case; it can be shown explicitly that there is no innovation ε_n in equations (8) and (9) which will make X_n have a Gamma distribution with $k = 2$.

4B. Mixed Exponential Markovian Process

Fortunately first-order autoregressive Markovian processes with mixed exponential marginal distributions can be obtained from equations (8) and (9) in two special cases, and these sequences should be widely useful in modelling stochastic systems.

(i) The case $\alpha = 1$; MEAR(1).

In (1) it is shown that the solution to the Laplace transform of ε_n for the linear model (6) is a constant ρ plus a (generally) non-convex mixture of three exponential functions. This can be shown to be a proper density function if $\rho \leq \lambda_1/\lambda_2$, but it can also be shown that it is not a density function for all ρ less than one and greater than or equal to zero. However, Lawrance [6] showed that unless λ_1 is much smaller than λ_2 (and thus the X_n are very over-dispersed relative to an exponential random variable) a solution exists for ε_n for all ρ. Thus we have a useful process, although again the zero-defect of order ρ is a problem.

(ii) The case $\beta = 1$; MEAR(1).

When $\beta = 1$ in equation (8), a mixed exponential process TMEAR(1) is obtained which is extremely simple to simulate since the innovation ε_n is just the mixture of two exponentials for all $0 \leq \rho < 1$. Moreover, the process has no zero-defect.

As discussed above, the sample paths will tend to "run up," but this is no great problem unless ρ is fairly large. Thus we have the following Theorem which we state without proof:

TMEAR(1) Theorem

Let the first-order autoregressive, Markovian sequence $\{X_n\}$ be defined by

$$X_n = \varepsilon_n + V_n X_{n-1} , \qquad n = 1,2,3,...$$

where $P\{V_n=1\} = 1 - P\{V_n=0\} = \alpha$ for $0 \leq \alpha < 1$. Then the sequence $\{X_n\}$ is stationary and has a (convex) mixed exponential marginal distribution with probability density function

$$f_X(x) = \pi_1\lambda_1 e^{-\lambda_1 x} + (1-\pi_1)e^{-\lambda_2 x} ,$$

$$0 < \lambda_1 < \lambda_2; \; 0 < \pi_1 < 1; \quad x \geq 0 , \qquad (24)$$

if ε_n is i.i.d. and has a mixed exponential distribution given by

$$f_\varepsilon(x) = \eta_1\gamma_1 e^{-\gamma_1 x} + \eta_2\gamma_2 e^{-\gamma_2 x} ,$$

$$\gamma_1 > \gamma_2 > 0; \quad \eta_1 \geq 0; \quad \eta_2 = 1-\eta_1 , \qquad (25)$$

where

$$\eta_1 = \frac{1}{a}\,(\beta - 1/\gamma_1)/(\gamma_1 - \gamma_2);$$

$$\gamma_{1,2} = \{b \pm (b^2 - 4ac)^{1/2}\}/2a\ ;$$

$$a = (1-\alpha)\mu_1\mu_2;$$

$$\mu = \pi_2\mu_2 + \pi_1\mu_1 = E(X)\ ;$$

$$b = \mu_2 + \mu_1 - \alpha\mu\ ;$$

$$\beta = \mu_2 + \mu_1 - \mu; \quad \mu_1 = 1/\gamma_1; \quad \mu_2 = 1/\gamma_2$$

and X_0 is independent of $\varepsilon_1, \varepsilon_2, \ldots$ and has probability density function (24).

Note that the special cases where $\pi_1 = 0$ or $\pi_2 = 1$ give NEAR(1) exponential processes with parameters λ_2 and λ_1 respectively. Thus they should be handled by Algorithm 2 since they will cause computational problems. The case $\lambda_1 = \lambda_2$ also gives a NEAR(1) process and is excluded for similar reasons.

Another computational problem arises from the fact that the probability density function (25) for ε is not a convex mixture if, as is possible, η_1 is greater than one. Of course, if $\eta_1 < 1$ then ε is generated as an exponential(γ_1) random variable with probability η_1 and as an exponential(γ_2) with probability $(1-\eta_1)$. In the other case we use the following theorem (see e.g. (1)):

Simulation of Nonconvex Mixed Exponential

Let $\delta = \eta_1(1-\gamma_2/\gamma_1)$. Then with probability $1-\delta$ the innovation ε is an exponential(γ_1) random variable and with probability δ the innovation ε is the sum of an exponential(γ_1) random variable and an exponential(γ_2) random variable.

It would be useful to have a mixed exponential solution for the sequence (8) for all β for convex mixed exponential marginal distributions, but this seems difficult to prove because of the algebra involved.

5. GENERALIZATIONS

In all of the processes discussed here the correlations are non-negative and geometrically decreasing. A scheme for obtaining alternating correlations which are possibly negative is given in [1] and [5]. Another problem is that different types of dependence and higher-order Markovian dependence might be encountered in data. Schemes for obtaining mixed autoregressive moving average exponential sequences where the autoregression has order p and the moving average has order q are given in (4). The mixed exponential process TMEAR(1) is easily extended to give a process with this type of extended correlation structure. This will be discussed elsewhere.

6. ACKNOWLEDMENTS

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