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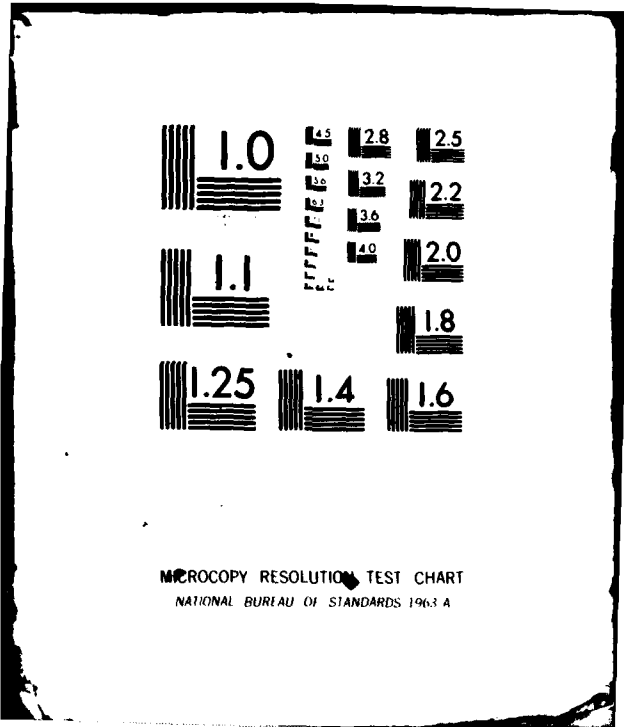
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ANOTHER GENERALIZATION OF CARATHÉODORY'S THEOREM

by

Victor Klee

Technical Report No. 67

May 1980

Contract N00014-67-0103-003

Project Number NR044 353

Department of Mathematics

University of Washington

Seattle, Washington 98195

**SDTIC
SELECTED
JUL 31 1980**

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

University of Washington

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

3. REPORT TITLE

6 ANOTHER GENERALIZATION OF CARATHEODORY'S THEOREM

(11) May 80

4. DESCRIPTIVE NOTES (Type of report and, inclusive dates)

9 Technical rept.

5. AUTHOR(S) (First name, middle initial, last name)

10 Victor/Klee

(14) TR-67

(12) 76

6. REPORT DATE

May 1980

7a. TOTAL NO. OF PAGES

4

7b. NO. OF REFS

6

8a. CONTRACT OR GRANT NO.

15 N00014-67-A-0103-0003

8b. PROJECT NO.

NR044 353

9a. ORIGINATOR'S REPORT NUMBER(S)

Technical Report No. 67

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Releasable without limitations on dissemination

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11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

13. ABSTRACT

When P is a d-dimensional convex polytope with vertex-set V, we use the term V-simplex to denote a d-simplex whose vertices all belong to V. A slight variant of Carathéodory's theorem asserts that for each v in V there is a collection S of V-simplices such that P = union S and v in intersection S. In connection with some constructions in ring theory, Kenneth Goodearl has conjectured there is a collection S of V-simplices such that P = con union S and dim intersection S = d. For 0 <= k < d the present note establishes a theorem concerning the generation of P by V-simplices in conjunction with the operation con_{k+1}, where con_n X is the set of all convex combinations of n or fewer points of X. When k = 0 the theorem is Carathéodory's and when k = d-1 it is a slight sharpening of Goodearl's conjecture.

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
<p>Caratheodory</p> <p>convex</p> <p>convex combination</p> <p>polytope</p> <p>simplex</p> <p>vertex</p>						

Accession For

NTIS GWA&I

DDC TAB

Unannounced

Justification _____

By _____

Distribution/ _____

Availability Codes

Dist	Avail and/or special
A	

ANOTHER GENERALIZATION OF CARATHÉODORY'S THEOREM

VICTOR KLEE

When P is a d -dimensional convex polytope with vertex-set V , we use the term V -simplex to denote a d -simplex whose vertices all belong to V . A slight variant of Carathéodory's theorem [2] asserts that for each $v \in V$ there is a collection \mathcal{S} of V -simplices such that $P = \cup \mathcal{S}$ and $v \in \cap \mathcal{S}$. In connection with some constructions in ring theory, Kenneth Goodearl has conjectured there is a collection \mathcal{S} of V -simplices such that $P = \text{con } \cup \mathcal{S}$ and $\dim \cap \mathcal{S} = d$. (This result is used in [4].) For $0 \leq k < d$ the present note establishes a theorem concerning the generation of P by V -simplices in conjunction with the operation con_{k+1} , where $\text{con}_n X$ is the set of all convex combinations of n or fewer points of X . When $k = 0$ the theorem is Carathéodory's and when $k = d-1$ it is a slight sharpening of Goodearl's conjecture.

THEOREM Suppose that P is a d -dimensional convex polytope with vertex-set V , $0 \leq k < d$, and F is a k -face of P . Then there is a collection \mathcal{S} of V -simplices such that

$$P = \text{con}_{k+1} \cup \mathcal{S} \quad \text{and} \quad \dim (F \cap (\cap \mathcal{S})) = k.$$

When $k = d-1$ the intersection $\cap \mathcal{S}$ is d -dimensional. If V is in general position then con_{k+1} may be replaced by $\text{con}_{\lfloor d/(d-k) \rfloor}$

Proof. Observe first that if H is a $(j-1)$ -flat in a j -flat G , Q is one of the two closed halfspaces into which H divides G , and \mathcal{B} is a finite collection of j -dimensional convex subsets of Q such that the set $C = \cap (\cap \mathcal{B})$ is $(j-1)$ -dimensional, then $\cap \mathcal{B}$ is j -dimensional. Indeed, choose points c and q in the relative interiors of C and Q respectively, and note that for each $B \in \mathcal{B}$ there exists $\lambda_B > 0$ such that $(1-\lambda_B)c + \lambda_B q \in B$. With $\epsilon = \min \{\lambda_B : B \in \mathcal{B}\} > 0$, $\cap \mathcal{B}$ contains the j -dimensional set

$$\text{con} (C \cup \{(1-\epsilon)c + \epsilon q\}).$$

Whenever P is a d -polytope with vertex-set V , $0 \leq k \leq d$, and

$F_0 \subset F_1 \subset \dots \subset F_k$ is a sequence of faces of P with $\dim F_i = i$ for each i , let $\mathcal{S}_P(F_0, \dots, F_k)$ denote the collection of all sets of the form $\text{con}\{v_0, \dots, v_d\}$ such that

- (i) for $0 \leq i \leq k$, $v_i \in F_i$
- (ii) for $1 \leq i \leq d$, $v_i \in V - \text{aff}\{v_0, \dots, v_{i-1}\}$.

Plainly each member of $\mathcal{S}_P(F_0, \dots, F_k)$ is a V -simplex. A straightforward induction on i , based on the observation of the preceding paragraph, shows that for $0 \leq i \leq k$,

$$\dim \bigcap_{F_i} (F_0, \dots, F_i) = i.$$

To construct the \mathcal{S} whose existence is claimed by the theorem, simply set $\mathcal{S} = \mathcal{S}_P(F_0, \dots, F_k)$ for an arbitrary sequence of faces $F_0 \subset F_1 \subset \dots \subset F_k$ with $F_k = F$ and $\dim F_i = i$ for all i . Plainly $\dim (F \cap (n\mathcal{S})) = k$, for $n\mathcal{S} \supset n\mathcal{S}_{F_k}(F_0, \dots, F_k)$. And since

$$\mathcal{S}_P(F_0, \dots, F_{d-1}) = \mathcal{S}_P(F_0, \dots, F_{d-1}, P),$$

$n\mathcal{S}$ is d -dimensional when $k = d-1$.

It remains to show that $P = \text{con}_r n\mathcal{S}$ with $r = k+1$ in general and $r = \lceil d/(d-k) \rceil$ (the smallest integer $\geq d/(d-k)$) when V is in general position. With $v_0 \in F_0$, consider an arbitrary point $p \in P - \{v_0\}$ and let q be the last point of the ray from v_0 through p that belongs to P . If $q \in \text{con}_r n\mathcal{S}$ then $p \in \text{con}_r n\mathcal{S}$ because $p \in [v_0, q]$ and each member of \mathcal{S} is a convex set that contains v_0 .

Let j denote the dimension of the smallest face G of P that contains q . By Carathéodory's theorem, $q \in \text{con } X$ for an affinely independent set X consisting of $j+1$ points of $V \cap G$. If $G \subset F_k$ then $j < k$ and for each $x \in X$ there is a member \mathcal{S}_x of \mathcal{S} which contains x . Hence $q \in \text{con}_k n\mathcal{S}$.

Suppose, on the other hand that $G \not\subset F_k$, and let W be the vertex-set of an arbitrary member of $\mathcal{S}_P(F_0, \dots, F_k)$. Let $m+1$ denote the cardinality of the

maximal affinely independent subsets of $W \cup X$. From the facts that $W \not\subset G$ and $X \not\subset F_k$ it follows that $m > k$ and $m < j$. Since W is affinely independent, there is a set $Y \subset X$ such that the set $W \cup Y$ is affinely independent and of cardinality $m+1$, whence $|Y| = m-k$. Plainly $W \cup Y$ lies in a member of \mathcal{S} , as does each of the $(j+1)-(m-k)$ remaining points of X . Hence $p \in \text{con}_{r+1} U\mathcal{S}$ with $r = (j+1)-(m-k) \leq k$.

Now suppose, finally, that the vertex-set V of P is in general position, meaning that each set of $d+1$ points of V is affinely independent. Then all proper faces of P are simplices, and \mathcal{S} consists merely of all V -simplices that contain F_k . Consider v_0, p, q, G, X, W as described earlier. Then $W \cup Y$ is affinely independent for each set $Y \subset X \setminus W$ with $|Y| \leq d-k$. Hence $X \setminus W$ can be covered by $\lceil (j+1)/(d-k) \rceil$ members of \mathcal{S} , and since $j < d$ it follows that q (and hence p) belongs to $\text{con}_{\lceil d/(d-k) \rceil} U\mathcal{S}$. That completes the proof.

To see that the theorem cannot be improved by reducing the subscripts $k+1$ and $\lceil d/(d-k) \rceil$, consider a d -polytope $P = \text{con } V$ where V is the union of the vertex-set W of a k -simplex F and the vertex-set X of a $(d-1)$ -simplex. Let \mathcal{S} be the collection of all V -simplices S such that $\dim(F \cap S) = k$ for each $S \in \mathcal{S}$, whence the centroid of $\text{con } X$ does not belong to $\text{con}_{\lceil d/(d-k) \rceil - 1} U\mathcal{S}$. If a translate W' of W is contained in X then $|W' \cap S| = 1$ for each $S \in \mathcal{S}$, whence the centroid of $\text{con } W'$ does not belong to $\text{con}_k U\mathcal{S}$.

Other generalizations of Carathéodory's theorem appear in the literature. Some of them can be found in the references below.

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