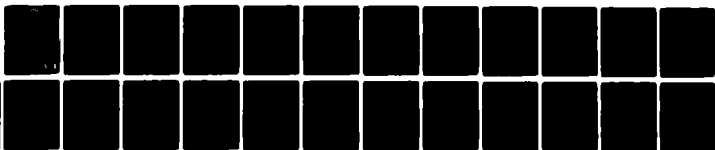AD-A087 424   OKLAHOMA UNIV NORMAN SCHOOL OF AEROSPACE MECHANICAL --ETC F/G 11/4
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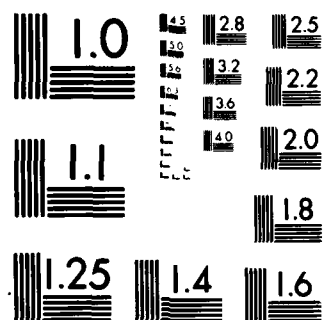UNCLASSIFIED   OU-AMNE-80-8   NL

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VIBRATION OF THICK RECTANGULAR PLATES OF

BIMODULUS COMPOSITE MATERIAL

by

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VIBRATION OF THICK RECTANGULAR PLATES

OF BIMODULUS COMPOSITE MATERIAL

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Abstract

A finite-element analysis is carried out for small-amplitude free vibration of laminated, anisotropic, rectangular plates having arbitrary boundary conditions, finite thickness-shear moduli, rotatory inertia, and bimodulus action (different elastic properties depending upon whether the fiber-direction strain is tensile or compressive). The element has five degrees of freedom, three displacements and two slope functions, per node. An exact closed-form solution is also presented for the special case of freely supported single-layer orthotropic and two-layer, cross-ply plates. This provides benchmarks to evaluate the validity of the finite-element analysis. Both solutions are compared with numerical results existing in the literature for special cases (all for ordinary, not bimodulus, materials) and good agreement is obtained.

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Introduction

Structural uses have been increasing for laminates consisting of multiple layers of fiber-reinforced composite materials. Consequently, there is an increasing need for more realistic mathematical modeling of the material behavior for incorporation into static and dynamic structural analyses. Certain fiber-reinforced materials have been found experimentally to exhibit quite different elastic behavior depending upon whether the fiber-direction strain (ε_f) is tensile or compressive [1-3]. Examples of such materials are tire cord-rubber, reinforced solid propellants, and some biological tissues. Although the stress-strain behavior of such materials is actually curvilinear, it is often approximated as being bilinear, with different slopes (elastic properties) depending upon the sign of ε_f. Thus, they are called bimodulus composite materials.

The limited number of previous analyses of bimodulus-material plates were reviewed in [4-6], and all were limited to static analyses. Thus, it is believed that the present work is the first vibrational analysis of such plates. The present work is not limited to just thin plates of isotropic bimodulus material, rather it is applicable to moderately thick plates laminated of orthotropic bimodulus material. Two formulations are presented and solved: one is a mixed finite-element formulation with five degrees of freedom per node, and the other is an exact closed-form solution.

Governing Equations

Mindlin's linear dynamic theory [7] of moderately thick plates was first extended to plates laminated of ordinary (not bimodulus) monoclinic elastic material by Yang, Norris, and Stavsky (YNS)[8]. Later, Wang and Chou [9]

showed that a slightly different version of the YNS theory, presented by Whitney and Pagano [10], is more accurate than the original version [8]. Here, this class of theory is extended to bimodulus-material laminates.

The origin of a Cartesian coordinate system is taken to be in the mid-plane (xy plane) of the plate with the z axis being normal to this plane and directed positive downward.

Using the fiber-governed symmetric material model introduced in [11], we take the generalized Hooke's law for the in-plane action in each layer (ℓ) to be of the following bimodular form:

$$\left\{\begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array}\right\} = \left[\begin{array}{ccc} Q_{11k\ell} & Q_{12k\ell} & Q_{16k\ell} \\ Q_{12k\ell} & Q_{22k\ell} & Q_{26k\ell} \\ Q_{16k\ell} & Q_{26k\ell} & Q_{66k\ell} \end{array}\right] \left\{\begin{array}{c} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{array}\right\} \tag{1}$$

Here the stresses ($\sigma_x, \sigma_y, \tau_{xy}$) and engineering strains ($\epsilon_x, \epsilon_y, \gamma_{xy}$) are denoted in the usual fashion, and the Q's are the plane-stress-reduced stiffnesses (symmetric array). The first two subscripts of the Q's are those classically used in anisotropic elasticity [12] and composite-material mechanics [13]. Here the third subscript (k) refers to the sign of the fiber-direction strain (k=1 for tension and k=2 for compression), and ℓ refers to the layer number (ℓ=1,2,...,n, where n is the total number of layers).

It is assumed that the thickness-shear behavior is unaffected by bimodular action, thus

$$\left\{\begin{array}{c} \tau_{yz} \\ \tau_{xz} \end{array}\right\} = \left[\begin{array}{cc} C_{44} & C_{45} \\ C_{45} & C_{55} \end{array}\right] \left\{\begin{array}{c} \gamma_{yz} \\ \gamma_{xz} \end{array}\right\} \tag{2}$$

The stress and moment resultants, each per unit length, are expressed in terms of stresses as

$$(N_x, N_y, N_{xy}, Q_x, Q_y) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}, \tau_{xz}, \tau_{yz})\, dz \tag{3}$$

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})z\, dz \tag{4}$$

The displacement components, u, v, and w in the x, y, and z directions, respectively, can be expressed in terms of mid-plane displacements u^0, v^0, w^0 and slope functions ψ_x and ψ_y as:

$$u = u^0(x,y,t) + z\psi_x(x,y,t) \quad ; \quad v = v^0(x,y,t) + z\psi_y(x,y,t)$$

$$w = w(x,y,t) \tag{5}$$

where t is time.

The constitutive equations for an unsymmetric cross-ply laminate are:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & 0 \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & B_{66} \\ B_{11} & B_{12} & 0 & D_{11} & D_{12} & 0 \\ B_{12} & B_{22} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & B_{66} & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} u^0_{,x} \\ v^0_{,y} \\ v^0_{,x} + u^0_{,y} \\ \psi_{x,x} \\ \psi_{y,y} \\ \psi_{y,x} + \psi_{x,y} \end{Bmatrix} \tag{6}$$

and

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} S_{44} & 0 \\ 0 & S_{55} \end{bmatrix} \begin{Bmatrix} w_{,y} + \psi_y \\ w_{,x} + \psi_x \end{Bmatrix} \tag{7}$$

Here differentiation is denoted by a comma, i.e., $( )_{,x} \equiv \partial( )/\partial x$, and the extensional, flexural-extensional coupling, and flexural stiffnesses of the laminate are defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (Q_{ij})(1, z, z^2) dz \qquad (8)$$
$$i,j=1,2,6$$

Also, the thickness-shear stiffnesses of the laminate are defined by

$$S_{ii} = K_i^2 C_{ii} \qquad i=4,5 \qquad (9)$$

where the K_i^2 are the thickness-shear correction coefficients, which can be determined by various approaches, cf. [12].

In addition to performing the integrations in a piecewise manner from layer to layer, one also has to take into consideration the possibility of different elastic properties (tension or compression) within a layer. This is explained in detail for a two-layer cross-ply laminate in Appendix A.

Taking into account the coupling and rotatory inertias, one can write the equations of motion as follows:

$$N_{x,x} + N_{xy,y} = Pu^0_{,tt} + R\psi_{x,tt}$$

$$N_{xy,x} + N_{y,y} = Pv^0_{,tt} + R\psi_{y,tt}$$

$$Q_{x,x} + Q_{y,y} = Pw_{,tt} \qquad (10)$$

$$M_{x,x} + M_{xy,y} - Q_x = Ru^0_{,tt} + I\psi_{x,tt}$$

$$M_{xy,x} + M_{y,y} - Q_y = Rv^0_{,tt} + I\psi_{y,tt}$$

Here P, R, and I are the normal, coupling, and rotatory inertia coefficients per unit mid-plane area and are defined by

$$(P,R,I) = \int_{-h/2}^{h/2} \rho(1,z,z^2)\, dz \tag{11}$$

where ρ is the material density.

Substituting equations (6) and (7) into equations (10), we obtain the equations of motion. In operator form, we have

$$[L_{k\ell}] \begin{Bmatrix} u^o \\ v^o \\ w \\ h\psi_y \\ h\psi_x \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{12}$$

$$k,\ell = 1,2,3,4,5$$

where $[L_{k\ell}]$ is a symmetric linear differential operator matrix with the following elements:

$$L_{11} \equiv A_{11}d_x^2 + A_{66}d_y^2 - Pd_t^2 \quad ; \quad L_{12} \equiv (A_{12} + A_{66})d_x d_y \quad ; \quad L_{13} \equiv 0$$

$$L_{14} \equiv [(B_{12} + B_{66})/h]d_x d_y \quad ; \quad L_{15} \equiv (B_{11}/h)d_x^2 + (B_{66}/h)d_y^2 - (R/h)d_t^2$$

$$L_{22} \equiv A_{66}d_x^2 + A_{22}d_y^2 - Pd_t^2 \quad ; \quad L_{23} \equiv 0 \quad ; \quad L_{24} \equiv (B_{66}/h)d_x^2 + (B_{22}/h)d_y^2 - (R/h)d_t^2$$

$$L_{25} \equiv L_{14} \quad ; \quad L_{33} \equiv -S_{55}d_x^2 - S_{44}d_y^2 + Pd_t^2$$

$$L_{34} \equiv -(S_{44}/h)d_y \quad ; \quad L_{35} \equiv -(S_{55}/h)d_x \tag{13}$$

$$L_{44} \equiv (D_{66}/h^2)d_x^2 + (D_{22}/h^2)d_y^2 - (S_{44}/h^2) - (I/h^2)d_t^2$$

$$L_{45} \equiv [(D_{12} + D_{66})/h^2]d_x d_y$$

$$L_{55} \equiv (D_{11}/h^2)d_x^2 + (D_{66}/h^2)d_y^2 - (S_{55}/h^2) - (I/h^2)d_t^2 \quad ; \quad d_x \equiv \partial/\partial x, \text{ etc.}$$

Application to Plate Freely Supported on all Edges

The boundary conditions on all edges are freely supported (simply supported without in-plane normal restraint).

Along the edges at $\quad x = 0$ and $x = a$,

$$w = \psi_y = M_x = 0$$
$$v^o = N_x = 0$$

$$(14)$$

Along the edges at $\quad y = 0$ and $y = b$,

$$w = \psi_x = M_y = 0$$
$$u^o = N_y = 0$$

Closed-Form Solution

The governing equations (12) and the boundary conditions (14) are exactly satisfied in closed form by the following set of functions:

$$u^o = U \cos \alpha x \sin \beta y\ e^{i\omega t}$$

$$v^o = V \sin \alpha x \cos \beta y\ e^{i\omega t}$$

$$w = W \sin \alpha x \sin \beta y\ e^{i\omega t} \qquad (15)$$

$$h\psi_y = Y \sin \alpha x \cos \beta y\ e^{i\omega t}$$

$$h\psi_x = X \cos \alpha x \sin \beta y\ e^{i\omega t}$$

Here, ω is the natural frequency associated with the mode having axial and transverse wave numbers m and n, and

$$\alpha \equiv m\pi/a \quad , \quad \beta \equiv n\pi/b \qquad (16)$$

where a and b are plate dimensions in the x and y directions, respectively.

Substituting solutions (15) into the governing equations (12), we obtain the following:

$$[C_{k\ell}] \left\{ \begin{array}{c} U \\ V \\ W \\ Y \\ X \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right\} \qquad (17)$$

$$k,\ell = 1,2,3,4,5$$

where $C_{k\ell}$ is a 5x5 symmetric determinant containing the following elements:

$$C_{11} \equiv - A_{11}\alpha^2 - A_{66}\beta^2 + P\omega^2 \quad ; \quad C_{12} \equiv - (A_{12} + A_{66})\alpha\beta \quad ; \quad C_{13} \equiv 0$$

$$C_{14} \equiv - [(B_{12} + B_{66})/h]\alpha\beta \quad ; \quad C_{15} \equiv - (B_{11}/h)\alpha^2 - (B_{66}/h)\beta^2 + (R/h)\omega^2$$

$$C_{22} \equiv - A_{66}\alpha^2 - A_{22}\beta^2 + P\omega^2 \quad ; \quad C_{23} \equiv 0$$

$$C_{24} \equiv - (B_{66}/h)\alpha^2 - (B_{22}/h)\beta^2 + (R/h)\omega^2 \quad ; \quad C_{25} \equiv C_{14}$$

$$C_{33} \equiv - (S_{55}\alpha^2 + S_{44}\beta^2 - P\omega^2) \quad ; \quad C_{34} \equiv - (S_{44}/h)\beta$$

$$C_{35} \equiv - (S_{55}/h)\alpha \quad ; \quad C_{44} \equiv - (D_{66}/h^2)\alpha^2 - (D_{22}/h^2)\beta^2 - (S_{44}/h^2) + (I/h^2)\omega^2$$

$$C_{45} \equiv - [(D_{12} + D_{66})/h^2]\alpha\beta$$

$$C_{55} \equiv - (D_{11}/h^2)\alpha^2 - (D_{66}/h^2)\beta^2 - (S_{55}/h^2) + (I/h^2)\omega^2$$

(18)

The frequency ω can be determined by setting $|C_{k\ell}| = 0$.

To determine the z-position of the fiber-direction neutral surface, one sets

$$\varepsilon_f = \varepsilon_f^0 + z\kappa_f = 0$$

or

$$z_{nf} = - \varepsilon_f^0/\kappa_f \quad . \qquad (19)$$

Thus, $z_{nx} = - hU/X$ and $z_{ny} = - hV/Y$. An iterative procedure is used to obtain the final displacement ratios and corresponding frequency.

Finite-Element Formulation

An exact closed-form solution to equations (12) can be obtained only under special conditions of geometry, edge conditions, loadings, and lamination. Here we present a simple finite-element formulation which does not have any limitations (except for those implied in the formulation of the governing equations)[15].

Suppose that the region R is subdivided into a finite number N of sub-regions: finite elements, R_e ($e=1,2,\ldots,N$). Over each element the general-ized displacements ($u^o, v^o, w, \psi_x, \psi_y$) are interpolated according to

$$u^o = \sum_i^r u_i \phi_i^1 \quad , \quad v^o = \sum_i^r v_i \phi_i^1 \quad , \quad w = \sum_i^s w_i \phi_i^2$$

$$\psi_x = \sum_i^p \beta_{x_i} \phi_i^3 \quad , \quad \psi_y = \sum_i^p \beta_{y_i} \phi_i^3 \tag{20}$$

where ϕ_i^α ($\alpha=1,2,3$) is the interpolation function corresponding to the i-th node in the element. Note that the in-plane displacements, the transverse displacement, and the slope functions are approximated by different sets of interpolation functions. While this generality is included in the formu-lation (to indicate the fact that such independent approximations are possible), we dispense with it in the interest of simplicity when the element is actually programmed and take $\phi_i^1 = \phi_i^2 = \phi_i^3$ ($r=s=p$). Here r,s, and p denote the number of degrees of freedom for each variable. That is, the total number of degrees of freedom per element is $2r + s + 2p$.

Substituting equations (20) into the Galerkin integrals associated with the operator equation (12), which must also hold in each element R_e,

$$\int_{R_e} [L]\{\delta\}\{\phi\}dxdy = 0 \tag{21}$$

and using integration by parts once (to distribute the differentiation equally between the terms in each expression), we obtain

$$
\begin{bmatrix}
[K^{11}][K^{12}][K^{13}][K^{14}][K^{15}] \\
\quad [K^{22}][K^{23}][K^{24}][K^{25}] \\
\quad\quad\quad [K^{33}][K^{34}][K^{35}] \\
\text{Symmetric} \quad [K^{44}][K^{45}] \\
\quad\quad\quad\quad\quad [K^{55}]
\end{bmatrix}_e
\begin{Bmatrix}
\{u\} \\
\{v\} \\
\{w\} \\
\{\psi_x\} \\
\{\psi_y\}
\end{Bmatrix}_e
=
\begin{Bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{Bmatrix}
\tag{22}
$$

where the $\{u\}$, $\{v\}$, etc. denote the columns of the nodal values of u,v, respectively, and the elements $K_{ij}^{\alpha\beta}$ ($\alpha,\beta=1,2,\ldots,5$) of the symmetric stiffness matrix are given by

$$K_{ij}^{11} = A_{11}G_{ij}^{x} + A_{66}G_{ij}^{y}$$

$$K_{ij}^{12} = A_{12}G_{ij}^{xy} + A_{66}G_{ji}^{xy}$$

$$K_{ij}^{13} = 0$$

$$K_{ij}^{14} = B_{11}H_{ij}^{x} + B_{66}H_{ij}^{y}$$

$$K_{ij}^{15} = B_{12}H_{ij}^{xy} + B_{66}H_{ji}^{xy}$$

$$K_{ij}^{22} = A_{22}G_{ij}^{y} + A_{66}G_{ij}^{x}$$

$$K_{ij}^{23} = 0$$

$$K_{ij}^{24} = B_{66}H_{ij}^{xy} + B_{12}H_{ji}^{xy}$$

$$K_{ij}^{25} = B_{66}H_{ij}^{x} + B_{22}H_{ij}^{y}$$

$$K_{ij}^{33} = S_{55}S_{ij}^{x} + S_{44}S_{ij}^{y}$$

$$K_{ij}^{34} = S_{55}R_{ij}^{xo}$$

$$K_{ij}^{35} = S_{44}R_{ij}^{yo}$$

$$K_{ij}^{44} = D_{11}T_{ij}^{x} + D_{66}T_{ij}^{y} + S_{55}T_{ij}^{0}$$

$$K_{ij}^{45} = D_{12}T_{ij}^{xy} + D_{66}T_{ji}^{xy}$$

$$K_{ij}^{55} = D_{66}T_{ij}^{x} + D_{22}T_{ij}^{y} + S_{44}T_{ij}^{0}$$

$$\tag{23}$$

where

$$G_{ij}^{\xi\eta} = \int_{R_e} \phi_{i,\xi}^1 \phi_{j,\eta}^1 \, dxdy \qquad (i,j=1,2,\ldots,r)$$

$$H_{ij}^{\xi\eta} = \int_{R_e} \phi_{i,\xi}^1 \phi_{j,\eta}^3 \, dxdy \qquad (i=1,2,\ldots,r \; ; \; j=1,2,\ldots,t)$$

$$M_{ij}^{\xi\eta} = \int_{R_e} \phi_{i,\xi}^1 \phi_{j,\eta}^2 \, dxdy \qquad (i=1,2,\ldots,r \; ; \; j=1,2,\ldots,s)$$

$$S_{ij}^{\xi\eta} = \int_{R_e} \phi_{i,\xi}^2 \phi_{j,\eta}^2 \, dxdy \qquad (i,j=1,2,\ldots,s) \qquad\qquad (24)$$

$$R_{ij}^{\xi\eta} = \int_{R_e} \phi_{i,\xi}^2 \phi_{j,\eta}^3 \, dxdy \qquad (i=1,2,\ldots,s \; ; \; j=1,2,\ldots,t)$$

$$T_{ij}^{\xi\eta} = \int_{R_e} \phi_{i,\xi}^3 \phi_{j,\eta}^3 \, dxdy \qquad (i,j=1,2,\ldots,s)$$

$$(\xi,\eta=0,x,y)$$

and $G_{ij}^{xx} = G_{ij}^x$, etc. In the special case in which $\phi_i^1 = \phi_i^2 = \phi_i^3$, all of the matrices in equations (24) coincide.

In the present study, elements of the serendipity family are employed with the same interpolation for all of the variables. The resulting stiffness matrices are 20 by 20 for this four-node element and 40 by 40 for the eight-node element. Reduced integration [16] must be used to evaluate the matrix coefficients in equations (23). That is, if the four-node rectangular element is used, the 1x1 Gauss rule must be used in place of the standard 2x2 Gauss rule to numerically evaluate the coefficients K_{ij}.

Substituting solution (22) into equations (19), we get

$$z_{nx}^e = - u_{,x}^e / \psi_{x,x}^e \quad ; \quad z_{ny}^e = - v_{,y}^e / \psi_{y,y}^e \qquad (25)$$

Numerical Results

Computations using the closed-form and finite-element solutions were carried out on an IBM 370 computer. Since there is no previous analysis for vibration of bimodulus plates, the present results could be compared only with those for rectangular plates laminated of ordinary materials. Comparisons with the fundamental-frequency results of Jones [17] for thin plates and Fortier and Rossettos [18] for thick and thin plates are presented in Tables 1 and 2. It can be seen that the agreement is good.

As examples of some actual bimodulus materials, two composites used in automobile tires, aramid-cord/rubber and polyester-cord/rubber, are selected. The material properties used are listed in Table 3. These are based on the experiments of Patel et al. [2] and are the same data used in [6] with the addition of the specific-gravity values, which were estimated on the basis of the volume fractions. The numerical results for single-layer 0° ortho-tropic and two-layer cross-ply plates are presented in Tables 4 and 5-6, respectively. As can be seen from these tables, again the agreement is good.

There may be a question regarding the effect of bimodulus action on plate stiffness in different portions of each cycle. For example, Fig. 1 represents a single-layer bimodulus-material plate at the two extremes of its deflection. Figure 1(a) depicts the initial half cycle, during which the top surface is in compression and the bottom in tension, thus causing the neutral surface for ε_x to be positive ($z_{nx} > 0$), i.e., below the plate midplane a certain distance. Figure 1(b) depicts depicts the latter half cycle, during which the top surface is now in tension and the bottom in compression, thus

causing z_{nx} to be negative, i.e., to fall above the plate midplane. However, the absolute value of z_{nx} is identical to its value in the first half cycle. Thus, it can be concluded that the the frequency associated with the second half cycle is identical to that of the first half cycle and either modal shape, Fig. 1(a) or 1(b), will give the same computational result for the natural frequencies.

Now consider a two-layer laminate with the bottom layer (layer $\ell=1$) oriented at 0 degrees and the top layer ($\ell=2$) at 90 degrees; see Fig. 2. Initially, as shown in Fig. 2(a), the neutral surface for ε_x falls below the interface, within the 0-degree layer, while the neutral surface for ε_y falls above the interface, completely within the 90-degree layer. In the latter portion of the cycle, Fig. 2(b), the ε_x neutral surface falls outside of the 0-degree layer, and the ε_y neutral surface falls outside of the 90-degree layer. Thus, compressive properties are used for the entire 0-degree layer, and tensile ones for the 90-degree layer.

From the above considerations for a two-layer cross-ply laminate, it is clear that the plate stiffnesses acting in the two portions of a cycle are different and thus the associated frequencies are also different, except in the case of a square plate. Denoting the frequencies associated with the two portions of a cycle by ω_1 and ω_2, it can be shown from the standpoint of energy conservation that the average frequency (ω) over the entire cycle must be given by

$$\omega^{-1} = (1/2)(\omega_1^{-1} + \omega_2^{-1}) \tag{26}$$

Thus, the computational procedure used for a cross-ply plate is to calculate

ω_1 and ω_2 associated with modal shapes shown in Figures 2(a) and 2(b), respectively, and then to apply equation (26).

Concluding Remarks

A finite element has been developed to analyze the small-deflection free vibration of laminated, anisotropic, rectangular thick plates of bimodulus material. The results obtained agree well with those of an exact, closed-form solution derived for such a plate freely supported on all four edges. Thus, it is concluded that the element has been validated and may be used for computations involving more complicated boundary conditions.

Acknowledgments

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APPENDIX A

DERIVATION OF THE PLATE STIFFNESSES FOR TWO-LAYER
CROSS-PLY LAMINATE OF BIMODULUS MATERIAL

In problems involving laminates comprised of bimodulus-material layers, it is necessary to evaluate the integral forms involved in the definitions of the plate stiffnesses, equation (8). The derivation presented here is for the case of a two-layer cross-ply laminate.

Each layer is assumed to be of the same thickness, $h/2$, and the same orthotropic elastic properties with respect to the fiber direction. Since each layer is oriented at either $0°$ or $90°$ to the x axis, the laminate is also orthotropic, i.e., there are no stiffnesses with subscripts 16 and 26. The bottom layer is denoted as layer 1, i.e., $\ell=1$ in $Q_{ijk\ell}$, and occupies the thickness space from $z=0$ to $z=h/2$, where z is measured positive downward from the midplane. The top layer is denoted as layer 2, i.e., $\ell=2$, and occupies the thickness space from $z=-h/2$ to $z=0$.

In the first case derived here, it is assumed that the upper portion of the top layer ($\ell=2$) is in compression ($k=2$ in $Q_{ijk\ell}$) in the fiber direction and that the lower portion of the top layer is in tension ($k=1$), while the inner portion of the bottom layer ($\ell=1$), from $z=0$ to $z=z_{nx}$, is in compression ($k=2$), while the outer portion (from z_{nx} to $h/2$) of layer 1 is in tension ($k=1$).

Thus, the general integral expression for A_{ij}, in equation (8), may be taken as the sum of the integrals for each of these regions:

Case 1
$(\overline{0.5 > } Z_x > 0, \; -0.5 < Z_y < 0)$

$$A_{ij} = \int_{-h/2}^{h/2} Q_{ijk\ell} \; dz$$

$$= \int_{-h/2}^{z_{ny}} Q_{ij22} \; dz + \int_{z_{ny}}^{0} Q_{ij12} \; dz + \int_{0}^{z_{nx}} Q_{ij21} \; dz + \int_{z_{nx}}^{h/2} Q_{ij11} \; dz \qquad (A-1)$$

Since the planar reduced stiffnesses $Q_{ijk\ell}$ are each respectively constant in the appropriate regions, equation (A-1) integrates to the following result:

$$A_{ij} = (Q_{ij22} + Q_{ij11})(h/2) + (Q_{ij21} - Q_{ij11})z_{nx} \\ + (Q_{ij22} - Q_{ij12})z_{ny} \qquad (A-2)$$

or

$$A_{ij} = (1/2)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})Z_x \\ + (Q_{ij22} - Q_{ij12})Z_y \qquad (A-3)$$

Similarly

$$B_{ij} = \int_{-h/2}^{h/2} zQ_{ijk\ell} \; dz$$

$$= \int_{-h/2}^{z_{ny}} zQ_{ij22} \; dz + \int_{z_{ny}}^{0} zQ_{ij12} \; dz + \int_{0}^{z_{nx}} zQ_{ij21} \; dz + \int_{z_{nx}}^{h/2} zQ_{ij11} \; dz \qquad (A-4)$$

$$= (- Q_{ij22} + Q_{ij11})(h^2/8) + (Q_{ij21} - Q_{ij11})(z_{nx}^2/2) \\ + (Q_{ij22} - Q_{ij12})(z_{ny}^2/2) \qquad (A-5)$$

or

$$B_{ij}/h^2 = (1/8)(- Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})(Z_x^2/2)$$

$$+ (Q_{ij22} - Q_{ij12})(Z_y^2/2) \qquad \text{(A-6)}$$

Also

$$D_{ij} = \int_{-h/2}^{h/2} z^2 Q_{ijk\ell}\, dz$$

$$= \int_{-h/2}^{z_{ny}} z^2 Q_{ij22}\, dz + \int_{z_{ny}}^{0} z^2 Q_{ij12}\, dz + \int_{0}^{z_{nx}} z^2 Q_{ij21}\, dz + \int_{z_{nx}}^{h/2} z^2 Q_{ij11}\, dz \qquad \text{(A-7)}$$

$$= (Q_{ij22} + Q_{ij11})(h^3/24) + (Q_{ij21} - Q_{ij11})(z_{nx}^3/3)$$

$$+ (Q_{ij22} - Q_{ij12})(z_{ny}^3/3) \qquad \text{(A-8)}$$

or

$$D_{ij}/h^3 = (1/24)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})(Z_x^3/3)$$

$$+ (Q_{ij22} - Q_{ij12})(Z_y^3/3) \qquad \text{(A-9)}$$

Similarly

Case 2
$$(-0.5 < Z_x < 0,\ 0.5 > Z_y > 0)$$

$$A_{ij}/h = (Q_{ij11} + Q_{ij22})/2 + (Q_{ij22} - Q_{ij12})Z_x + (Q_{ij21} - Q_{ij11})Z_y$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij22})/8 + (Q_{ij22} - Q_{ij12})(Z_x^2/2) + (Q_{ij21} - Q_{ij11})(Z_y^2/2) \qquad \text{(A-10)}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij22})/24 + (Q_{ij22} - Q_{ij12})(Z_x^3/3) + (Q_{ij21} - Q_{ij11})(Z_y^3/3)$$

Case 3
$(0.5 > Z_x > 0, 0.5 > Z_y > 0)$

$$A_{ij}/h = (Q_{ij11} + Q_{ij22})/2 + (Q_{ij21} - Q_{ij11})Z_x$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij22})/8 + (Q_{ij21} - Q_{ij11})(Z_x^2/2) \qquad \text{(A-11)}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij22})/24 + (Q_{ij21} - Q_{ij11})(Z_x^3/3)$$

Case 4
$(-0.5 < Z_x < 0, -0.5 < Z_y < 0)$

$$A_{ij}/h = (Q_{ij11} + Q_{ij22})/2 + (Q_{ij22} - Q_{ij12})Z_y$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij22})/8 + (Q_{ij22} - Q_{ij12})(Z_y^2/2) \qquad \text{(A-12)}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij22})/24 + (Q_{ij22} - Q_{ij12})(Z_y^3/3)$$

In the presence of excessively high in-plane loads, such as those due to excessive heating of the midplane or due to large deflections, the neutral surfaces can go outside of the thickness of the laminate and, thus, make it act as it were homogeneous. However, this does not occur for small-deflection free vibrations and thus the equations for these cases are not presented here.

Single 0° Layer
$(|Z_x| < 0.5)$

$$A_{ij}/h = (Q_{ij11} + Q_{ij21})/2 + (Q_{ij21} - Q_{ij11})Z_x$$

$$B_{ij}/h^2 = (Q_{ij11} - Q_{ij21})/8 + (Q_{ij21} - Q_{ij11})Z_x^2/2 \qquad \text{(A-13)}$$

$$D_{ij}/h^3 = (Q_{ij11} + Q_{ij21})/24 + (Q_{ij21} - Q_{ij11})Z_x^3/3$$

Table 1. Comparison of fundamental natural frequencies (m=n=1) of rectangular antisymmetric cross-ply plates at different aspect ratios and thicknesses (E_{11}/E_{22}=40, G_{12}/E_{22}=G_{13}/E_{22}=G_{23}/E_{22}=0.5, ν_{12}=0.25, K_4^2=K_5^2=5/6)[*]

Aspect Ratio a/b	Dimensionless frequency $\omega(b/\pi)^2(P/D_{22})^{\frac{1}{2}}$				
	Thin-plate theory [17]	b/h=50		b/h=10	
		C.F.	F.E.	C.F.	F.E.
0.5	2.24	2.400	2.421	1.942	1.946
1.0	0.865	0.858	0.877	0.794	0.799
1.5	0.65	0.656	0.668	0.612	0.615
2.0	0.606	0.604	0.617	0.565	0.569
2.5	0.59	0.590	0.599	0.548	0.552
3.0	0.580	0.578	0.591	0.541	0.544

[*]C.F. denotes the closed-form solution and F.E. denotes the finite-element solution.

Table 2. Comparison of fundamental natural frequencies of square anti-symmetric cross-ply plates at different thicknesses (E_{11}/E_{22}=40, G_{12}/E_{22}=G_{13}/E_{22}=1, G_{23}/E_{22}=0.5, ν_{12}=0.25, K_4^2=K_5^2=5/6)

b/h	Dimensionless frequency $\omega b^2(P/E_{22}h^3)^{\frac{1}{2}}$		
	Fortier & Rossettos [18]	C.F.	F.E.
10	10.80	11.11	11.15
50	11.65	11.82	12.06

Table 3. Material properties for two tire-cord/rubber, unidirectional, bimodulus composite materials[a]

	Aramid-Rubber		Polyester-Rubber	
	k=1	k=2	k=1	k=2
Longitudinal Young's modulus, GPa	3.58	0.0120	0.617	0.0369
Transverse Young's modulus, GPa	0.00909	0.0120	0.00800	0.0106
Major Poisson's ratio, dimensionless[b]	0.416	0.205	0.475	0.185
Longitudinal-transverse shear modulus, GPa[c]	0.00370	0.00370	0.00262	0.00267
Transverse-thickness shear modulus, GPa	0.00290	0.00499	0.00233	0.00475
Specific gravity, dimensionless	0.970		1.00	

[a]Fiber-direction tension is denoted by k=1, and fiber-direction compression by k=2.

[b]It is assumed that the minor Poisson's ratio is given by the reciprocal relation.

[c]It is assumed that the longitudinal-thickness shear modulus is equal to this one.

Table 4. Dimensionless fiber-direction neutral-surface locations and fundamental frequencies for single-layer 0° orthotropic plates having b/h=10 by two methods (closed form and finite element)

Aspect Ratio a/b	$Z_x \equiv z_{nx}/h$		$\omega b^2 (P/E_{22}^C h^3)^{1/2}$	
	C.F.	F.E.	C.F.	F.E.
Aramid-Rubber:				
0.5	0.4484	0.4484	19.065	19.255
0.6	0.4484	0.4475	14.339	14.564
0.7	0.4467	0.4468	11.324	11.515
0.8	0.4467	0.4458	9.304	9.553
0.9	0.4445	0.4450	7.893	8.019
1.0	0.4433	0.4435	6.877	7.062
1.2	0.4404	0.4413	5.554	5.782
1.4	0.4373	0.4370	4.766	4.968
1.6	0.4338	0.4340	4.263	4.443
1.8	0.4301	0.4338	3.925	4.092
2.0	0.4262	0.4302	3.688	3.856
Polyester-Rubber:				
0.5	0.3089	0.3083	25.134	23.136
0.6	0.3089	0.3076	19.110	18.046
0.7	0.3072	0.3071	15.058	14.421
0.8	0.3072	0.3064	12.226	11.955
0.9	0.3056	0.3056	10.180	10.023
1.0	0.3056	0.3049	8.668	8.648
1.2	0.3030	0.3031	6.647	6.698
1.4	0.3011	0.3013	5.421	5.533
1.6	0.2990	0.2997	4.643	4.796
1.8	0.2969	0.2977	4.128	4.265
2.0	0.2945	0.2950	3.777	3.918

Table 5. Dimensionless neutral-surface locations in the first and second portions of a cycle for two-layer, cross-ply plates having b/h=10 by closed-form and finite-element methods[*]

	$Z_x^{(1)}$		$Z_y^{(1)}$		$Z_x^{(2)}$		$Z_y^{(2)}$	
a/b	C.F.	F.E.	C.F.	F.E.	C.F.	F.E.	C.F.	F.E.
Aramid-Rubber:								
0.5	0.4457	0.4458	-0.0648	-0.0660	-0.0171	-0.0170	0.4247	0.4257
0.6	0.4446	0.4446	-0.0563	-0.0554	-0.0206	-0.0205	0.4303	0.4309
0.7	0.4434	0.4436	-0.0490	-0.0491	-0.0240	-0.0238	0.4338	0.4344
0.8	0.4421	0.4421	-0.0432	-0.0429	-0.0275	-0.0274	0.4363	0.4365
0.9	0.4408	0.4408	-0.0385	-0.0386	-0.0311	-0.0306	0.4381	0.4379
1.0	0.4394	0.4394	-0.0347	-0.0344	-0.0347	-0.0346	0.4394	0.4394
1.2	0.4366	0.4366	-0.0293	-0.0289	-0.0424	-0.0416	0.4412	0.4415
1.4	0.4335	0.4337	-0.0250	-0.0249	-0.0494	-0.0497	0.4423	0.4426
1.6	0.4301	0.4302	-0.0218	-0.0217	-0.0565	-0.0559	0.4423	0.4433
1.8	0.4265	0.4264	-0.0193	-0.0193	-0.0635	-0.0662	0.4437	0.4437
2.0	0.4228	0.4237	-0.0174	-0.0175	-0.0705	-0.0700	0.4437	0.4442
Polyester-Rubber:								
0.5	0.3687	0.3691	-0.1335	-0.1295	-0.0830	-0.0825	0.3569	0.357
0.6	0.3675	0.3677	-0.1213	-0.1203	-0.0844	-0.0844	0.3588	0.359
0.7	0.3664	0.3663	-0.1119	-0.1113	-0.0868	-0.0868	0.3603	0.360
0.8	0.3653	0.3653	-0.1050	-0.1049	-0.0895	-0.0895	0.3615	0.362
0.9	0.3642	0.3641	-0.0999	-0.0999	-0.0926	-0.0927	0.3615	0.362
1.0	0.3632	0.3633	-0.0960	-0.0960	-0.0959	-0.0959	0.3631	0.363
1.2	0.3611	0.3611	-0.0906	-0.0905	-0.1033	-0.103	0.3631	0.364
1.4	0.3589	0.3596	-0.0870	-0.0870	-0.1115	-0.112	0.3648	0.365
1.6	0.3565	0.3564	-0.0846	-0.0844	-0.1202	-0.120	0.3648	0.365
1.8	0.3540	0.3538	-0.0829	-0.0829	-0.1294	-0.130	0.3648	0.366
2.0	0.3514	0.3513	-0.0817	-0.0817	-0.1389	-0.139	0.3660	0.366

[*]Here $Z_x^{(1)} \equiv z_{nx}/h$ for the first portion of a cycle, etc.

Table 6. Dimensionless fundamental frequencies in the first partial cycle, second partial cycle and complete cycle of motion for two-layer, cross-ply plates having b/h=10 by closed-form and finite-element methods[†]

	$\omega_1 b^2 (P/E_{22}^C h^3)^{\frac{1}{2}}$		$\omega_2 b^2 (P/E_{22}^C h^3)^{\frac{1}{2}}$		$\omega b^2 (P/E_{22}^C h^3)^{\frac{1}{2}}$	
a/b	C.F.	F.E.	C.F.	F.E.	C.F.	F.E.
Aramid-Rubber:						
0.5	19.38	20.23	13.88	14.55	16.18	16.93
0.6	14.65	15.32	11.05	11.69	12.60	13.26
0.7	11.60	12.17	9.353	9.807	10.35	10.86
0.8	9.537	9.825	8.269	8.635	8.860	9.192
0.9	8.088	8.488	7.543	7.879	7.806	8.172
1.0	7.038	7.386	7.038	7.364	7.038	7.375
1.2	5.661	5.928	6.402	6.727	6.008	6.302
1.4	4.838	5.045	6.037	6.356	5.371	5.625
1.6	4.313	4.536	5.812	6.088	4.951	5.199
1.8	3.960	4.116	5.655	5.910	4.658	4.852
2.0	3.712	3.909	5.551	5.821	4.449	4.677
Polyester-Rubber:						
0.5	19.12	19.81	15.95	16.61	17.39	18.07
0.6	14.42	14.98	12.26	12.79	13.25	13.80
0.7	11.43	11.92	10.04	10.45	10.69	11.14
0.8	9.435	9.855	8.632	9.014	9.016	9.416
0.9	8.059	8.421	7.711	8.144	7.881	8.280
1.0	7.084	7.406	7.085	7.394	7.085	7.400
1.2	5.856	6.111	6.337	6.613	6.081	6.352
1.4	5.164	5.407	5.928	6.193	5.520	5.773
1.6	4.748	4.986	5.694	5.928	5.178	5.416
1.8	4.485	4.693	5.543	5.778	4.958	5.179
2.0	4.310	4.518	5.435	5.688	4.807	5.036

[†]Here ω_1 and ω_2 denote the frequencies corresponding to the first and second portions of a cycle, respectively, and ω denotes the effective frequency for an entire cycle.
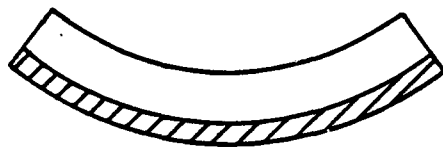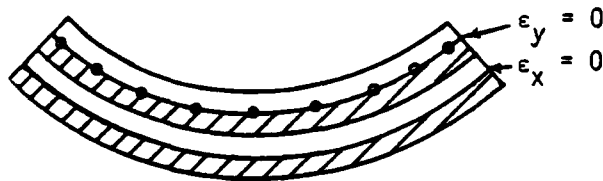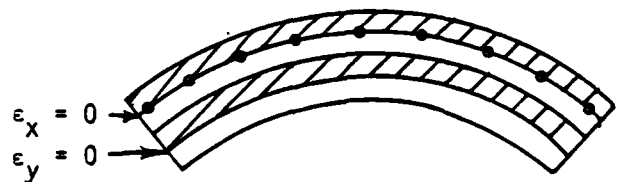
(a) First half cycle
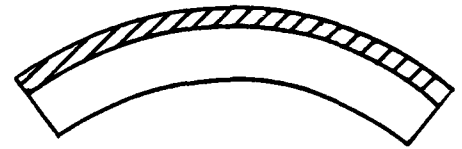
(b) Second half cycle

Fig. 1 Bimodulus action during the two half cycles of motion of a single-layer bimodulus plate. Shaded material is in longitudinal tension.

(a) First portion of cycle

(b) Second portion of cycle

Fig. 2 Bimodulus action during the two portions of motion of a two-layer plate in the fundamental mode of vibration. Bottom layer is in x direction (0°), top layer is in y (90°). Shaded portions are in tension in the respective fiber directions.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A finite-element analysis is carried out for small-amplitude free vibration of laminated, anisotropic, rectangular plates having arbitrary boundary conditions, finite thickness-shear moduli, rotatory inertia, and bimodulus action (different elastic properties depending upon whether the fiber-direction strain is tensile or compressive). The element has five degrees of freedom, three displacements and two slope functions, per node. An exact closed-form solution is also presented for the special case of

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20. Abstract - Cont'd