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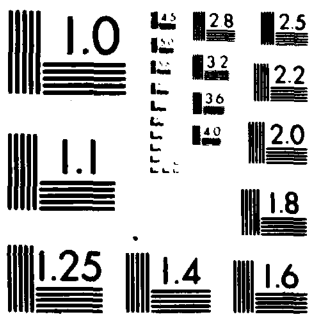

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DAVIDSON LABORATORY

Report SIT-DL-80-9-2124

May 1980

A PROCEDURE FOR CALCULATION OF THE PRESSURE ALONG THE AXIS OF A TIP VORTEX

by

T. V. Davies

and

J. P. Breslin

Research was sponsored by the
Naval Sea Systems Command
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The generation of vortical flow abreast the lateral edge and downstream of a lifting surface is examined from the basic equations of fluid mechanics using the Oseen approximation. Coupled equations which arise from matching inner and outer expansions about a conical region are derived. Their numerical solution (not attempted here) is expected to provide a more accurate prediction of the pressure distribution along the vortex axis and, hence, to a reliable prediction of inception of tip cavitation as a function of Reynolds number, and the loading on the lifting surface in the vicinity of the tip.

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Approved:
J. P. Breslin
J. P. Breslin
Director

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INTRODUCTION

This report presents the mathematical development of coupled equations whose solutions are expected to give the variation in pressure along the axis of tip vortices commonly generated by lifting surfaces (hydrofoils, propeller blades, etc.). This work was motivated by the long-observed lack of agreement of cavitation inception speeds measured with models in water tunnels and those determined at ship scale. Generally, the model prediction is highly non-conservative, yielding cavitation inception speeds which are roughly twice those observed during prototype operation.

This lack of scaling of tip vortex inception speed is countered through the use of empirical correction formulae which are based on the 1954 work of McCormick[1] who resorted to a semi-empirical digest of data which was ultimately published in 1962[2]. Somewhat more recent (1961) theoretical work is that of M.G. Hall[3] who developed a theory stemming from fundamentals which shows quite good correlation with pressure distributions in delta wing tip vortices in air.

A highly significant outcome of Hall's analysis is the large contribution to the pressure reduction along the vortex axis arising from the induced *axial* velocity component. McCormick's empiricism does not include this component. However, significant differences between Hall's theory and measurements remain. The current analysis is directed at overcoming a deficiency of Hall's theory which employed an outer solution in which the velocity and pressure are only functions of the angular variable and independent of the radial coordinate. As the solution must tend to uniform flow for large radial distances, it is not possible to reconcile Hall's assumption. Hence, the need for a more general formulation seems apparent.

The theoretical model developed herein regards the flow as viscous everywhere in contrast to many previous studies which assume at the outset that the flow is potential except for a viscous core region. Inner and outer flow regimes are encountered giving rise to a matching procedure along a conical surface having a very small apex angle. The swirling motion is generated by an input transverse velocity which is obtainable from the solution for the flow about the lifting surface of interest. The resulting coupled equations are complicated, requiring extensive computational efforts to produce numerical results.

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1. *Statement of Problem*

We assume that there exists a conical vortex of extremely small apex angle whose vertex is at 0 and whose axis of symmetry is along the z-direction, which is also the direction of a uniform stream at infinity. We take cylindrical coordinates (r,θ,z) and the complete velocity field will be

$$u\hat{r} + v\hat{\theta} + (W+w)\hat{z} \quad , \tag{1.1}$$

W being the uniform stream at infinity. The flow is axially symmetric and, on the basis of "Oseen's approximation", the equations of motion and continuity will be taken in the following form:

$$W\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\nabla^2 u - \frac{u}{r^2}) \quad , \tag{1.2}$$

$$W\frac{\partial v}{\partial z} = \nu(\nabla^2 v - \frac{v}{r^2}) \quad , \tag{1.3}$$

$$W\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\nabla^2 w) \quad , \tag{1.4}$$

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0 \quad , \tag{1.5}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial z} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{1.6}$$

is the Laplacian and ν is the kinematic viscosity.

Let η be the vorticity about the $\hat{\theta}$ direction, so that

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \quad . \tag{1.7}$$

If we eliminate the pressure p between (1.2) and (1.4), we obtain

$$W \frac{\partial \eta}{\partial z} - \frac{2}{r} v v_z = \nu(\nabla^2 \eta - \frac{\eta}{r^2}) \tag{1.8}$$

From (1.5) we can introduce the stream function ψ such that

$$ru = \frac{\partial \psi}{\partial z} \quad , \quad rw = -\frac{\partial \psi}{\partial r} \tag{1.9}$$

and, from (1.7), we have

$$r\eta = \psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r \tag{1.10}$$

If, in place of ψ, we use the function Ψ defined by

$$\psi = r\Psi \tag{1.11}$$

then

$$\eta = \nabla^2 \Psi - \frac{\Psi}{r^2} \tag{1.12}$$

We note now that Equation (1.3) is a p.d.e. for v, Equation (1.8) connects η and v, Equation (1.12) connects the function Ψ (and, therefore, the stream function ψ) with η, and the problem consists in solving these equations successively.

We are assuming that there is no vorticity in the oncoming stream $W\hat{z}$ and in solving (1.8) we shall impose the condition

$$\eta = 0 \quad , \quad z = 0 \tag{1.13}$$

and we shall solve for η in $z > 0$. Likewise, in solving (1.12) for Ψ, we shall impose the condition

$$\Psi = 0 \quad , \quad z = 0 \tag{1.14}$$

and we shall solve for Ψ in $z > 0$. The conditions which will be imposed upon v will be discussed later, but we envisage that the vortex is driven by an imposed transverse inflow which is known along a generator of the cone $r = z \tan \theta_o$.

2. *Inner and Outer Solutions for the Transverse Velocity* v

It is convenient to introduce the parameter k defined by

$$2k = \frac{W}{\nu} \tag{2.1}$$

and (1.3) can then be written in the form

$$\nabla^2 v - \frac{v}{r^2} - 2k \frac{\partial v}{\partial z} = 0 \tag{2.2}$$

If we write

$$v = \frac{\partial \chi}{\partial r} \tag{2.3}$$

then, using (1.6)

$$\frac{\partial^3 \chi}{\partial z^2 \partial r} + \frac{\partial^3 \chi}{\partial r^3} + \frac{1}{r} \frac{\partial^2 \chi}{\partial r^2} - \frac{1}{r^2} \frac{\partial \chi}{\partial r} - 2k \frac{\partial^2 \chi}{\partial z \partial r} = 0$$

that is

$$\frac{\partial}{\partial r} \{\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} - 2k \frac{\partial \chi}{\partial z}\} = 0$$

and we take χ to satisfy the equation

$$\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial r^2} + \frac{1}{r} \frac{\partial \chi}{\partial r} - 2k \frac{\partial \chi}{\partial z} = 0 \tag{2.4}$$

If we write

$$\chi = e^{kz}\Phi \tag{2.5}$$

then Φ will satisfy the p.d.e.

$$\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} - k^2 \Phi = 0$$

or

$$(\nabla^2 - k^2)\Phi = 0 \tag{2.6}$$

The functions v and Φ will be related as follows:

$$v = e^{kz} \frac{\partial \Phi}{\partial r} \tag{2.7}$$

We can now build up solutions for Φ using the method of separation of variables. If we look for a solution for Φ of the form

$$\Phi = e^{-mz} f(r) \tag{2.8}$$

then

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} + (m^2 - k^2)f = 0 \tag{2.9}$$

There are two cases to discuss according as $m \lesseqgtr k$.

Case 1 $m > k$: Inner Solution

In this case, the acceptable solution of (2.9) which is finite at $r = 0$ is

$$f(r) = J_o\{r\sqrt{m^2-k^2}\} \tag{2.10}$$

and thus we can build up a general solution for Φ of the form

$$\Phi_i(r,z) = -\int_{m=k}^{\infty} \mu(m)e^{-mz}J_o\{r\sqrt{m^2 - k^2}\}dm \tag{2.11}$$

where $\mu(m)$ is an arbitrary function of m.

Case 2 $0 < m < k$: Outer solution

In this case, we use the solution of

$$\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - (k^2 - m^2)f = 0 \tag{2.12}$$

which tends to zero as $r \to \infty$, namely

$$f(r) = K_o\{r\sqrt{k^2 - m^2}\} \tag{2.13}$$

We can now build up the general solution for Φ in the form

$$\Phi_e(r,z) = -\int_{m=o}^{k} \nu(m)e^{-mz}\, K_o\{r\sqrt{k^2 - m^2}\}dm \tag{2.14}$$

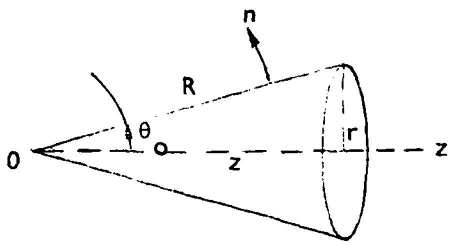
The formulae for v now follow from (2.7) and, using $J_o'(\psi) = -J_1(\psi)$, $K_o'(\psi) = -K_1(\psi)$, we obtain

$$v_i(r,z) = e^{kz}\int_{m=k}^{\infty} \mu(m)\sqrt{m^2 - k^2}\; e^{-mz}\; J_1\{r\sqrt{m^2 - k^2}\}dm \tag{2.15}$$

$$v_e(r,z) = e^{kz} \int_{m=o}^{k} \nu(m)\sqrt{k^2 - m^2}\ e^{-mz}\ K_1\{r\sqrt{k^2 - m^2}\}dm \qquad (2.16)$$

In (2.15) and (2.16), $\mu(m)$ and $\nu(m)$ are arbitrary functions of the parameter m.

The functions $\mu(m)$ and $\nu(m)$ in (2.15) and (2.16) will be determined as follows. Let Γ be the cone of semi-vertical angle θ_o with its vortex at 0 and its axis along the positive z-axis; then on Γ we can write



$$z = R\cos\theta_o \quad , \quad r = R\sin\theta_o \qquad (2.17)$$

where R measures distance from 0. We shall assume that the transverse velocity is prescribed as a function of R on the cone Γ, that is

$$v = V_o(R) \quad \text{on} \quad \Gamma \qquad (2.18)$$

From (2.15), we then have

$$\int_{m=k}^{\infty} \mu(m)\sqrt{m^2 - k^2}e^{-mR\cos\theta_o}\ J_1\{R\sin\theta_o\sqrt{m^2 - k^2}\}dn = V_o(R)e^{-kR\cos\theta_o} \qquad (2.19)$$

This is an integral equation for the unknown function $\mu(m)$. It will be noted that if we write

$$m = k\cosh\theta \qquad (2.20)$$

and

$$A(\theta) = k^2\sinh^2\theta\ \mu(k\cosh\theta) \qquad (2.21)$$

then the integral equation for $A(\theta)$ takes the form

$$\int_{o}^{\infty} A(\theta)\ e^{-kR\cos\theta\cos\theta_o}\ J_1\{kR\sinh\theta\sin\theta_o\}d\theta = V_o(R)e^{-kR\cos\theta_o} \qquad (2.22)$$

In a similar way, the integral equation for $\nu(m)$ will be

$$\int_{m=o}^{k} \nu(m)\sqrt{k^2 - m^2}\ e^{-mR\cos\theta_o}\ K_1\{R\sin\theta_o\sqrt{k^2 - m^2}\}dm = V_o(R)e^{-kR\cos\theta_o} \qquad (2.23)$$

If we write

$$m = k\cos\theta \qquad (2.24)$$

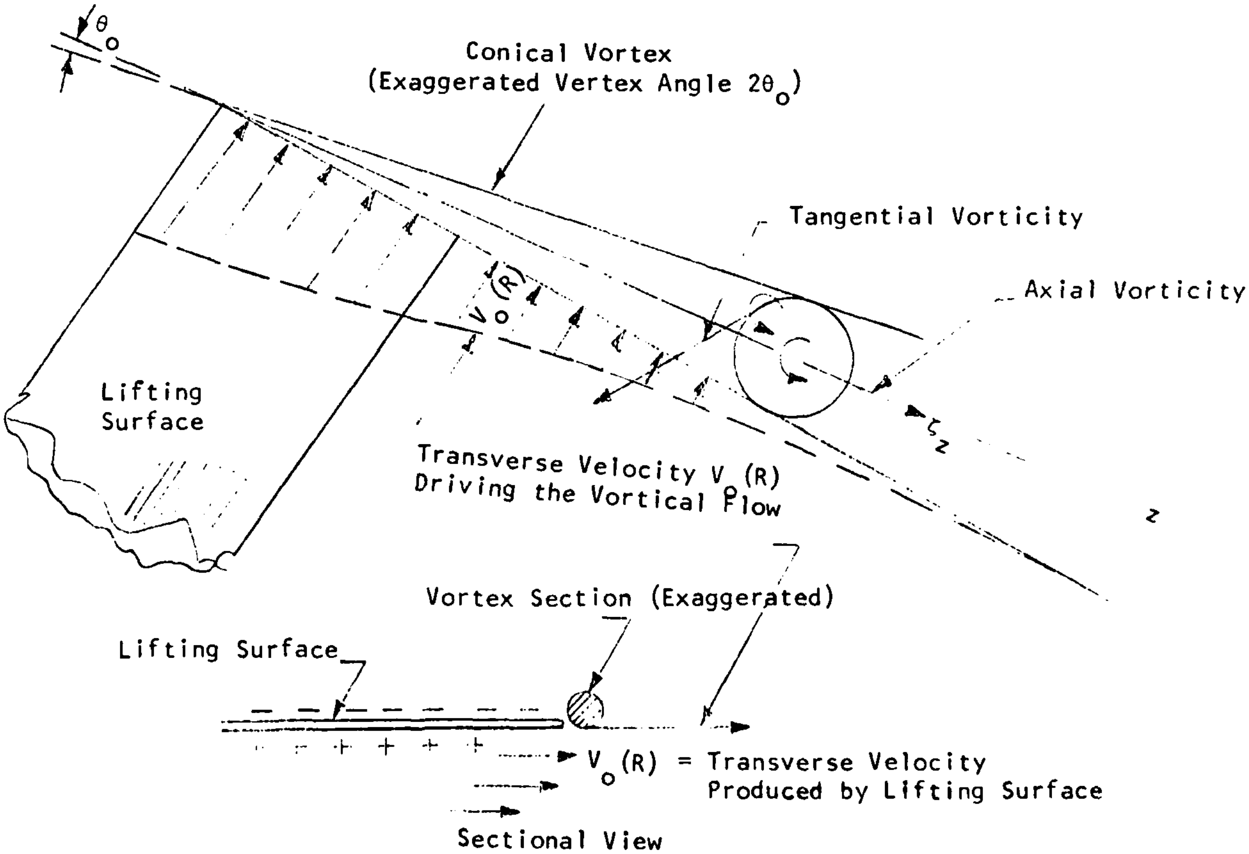
and

$$B(\theta) = k^2\sin^2\theta\ \nu(k\cos\theta) \qquad (2.25)$$

then the integral equation for $B(\theta)$ will be

$$\int_0^{\pi/2} B(\theta)\, e^{-kR\cos\theta\cos\theta_o} K_1\{kR\sin\theta\sin\theta_o\}d\theta = V_o(R)e^{-kR\cos\theta_o} \qquad (2.26)$$

It is clear that, when we proceed to calculate $v_i(r,z)$ and $v_e(r,z)$ in the above way, these two solutions will be continuous on the cone $\theta = \theta_o$; however, the normal derivatives $\partial v_i/\partial n$ and $\partial v_e/\partial n$ will be discontinuous on $\theta = \theta_o$. The sketch below defines the role of the driving transverse velocity produced by the lifting surface along the generator of the matching cone from the leading edge to downstream infinity.



3. *The Solutions for the Vorticity* η *and the Stream Function* ψ

We can write (1.8) in the form

$$\nabla^2\eta - \frac{\eta}{r^2} - 2k\,\frac{\partial\eta}{\partial z} = -\,\frac{2}{\nu r}\,vv_z \tag{3.1}$$

If in (3.1) we write

$$\eta = e^{kz}\,\frac{\partial\alpha}{\partial r} \tag{3.2}$$

we obtain

$$\frac{\partial}{\partial r}\,\{\alpha_{zz} + \alpha_{rr} + \frac{1}{r}\,\alpha_r - k^2\alpha\} = -\,\frac{2}{\nu r}\,e^{-kz}vv_z \quad , \tag{3.3}$$

Thus we can take α to satisfy the equation

$$(\nabla^2 - k^2)\alpha = f(r,z) \tag{3.4}$$

where

$$f(r,z) = -\,\frac{2}{\nu}\int_o^r \frac{1}{\sigma}\;e^{-kz}\;v(\sigma,z)\;\frac{\partial v(\sigma,z)}{\partial z}\,d\sigma \quad . \tag{3.5}$$

We can always add to α an arbitrary function of z, say A(z), but this will leave η unchanged and thus we can take A(z) = 0. The principal requirement upon η is that it should tend to zero at infinity and thus we can look for the solution for α which satisfies (3.4) and which tends to zero at infinity. The solution of this problem is obtained in the Appendix and is given by

$$4\pi\alpha(r_o,z_o) = -\int_{z=-\infty}^{\infty}\int_{r=o}^{\infty}\int_{\phi=o}^{2\pi}\frac{e^{-kR}}{R}\;f(r,z)r\;dr\;dz\;d\phi \tag{3.6}$$

where

$$R^2 = r^2 + r_o^2 - 2rr_o\cos\phi + (z-z_o)^2 \tag{3.7}$$

It follows from (3.2) that the function η is given by

$$4\pi\eta(r_o,z_o) = -e^{kz_o}\,\frac{\partial}{\partial r_o}\int_{-\infty}^{\infty}\int_o^{\infty}\int_o^{2\pi}\frac{e^{-kR}}{R}\;f(r,z)r\;dr\;d\phi\;dz \tag{3.8}$$

We can simplify the r.h.s. of (3.8) as follows; we write

$$J = \int_0^\infty \int_0^{2\pi} \frac{e^{-kR}}{R} f(r,z) r\, dr\, d\phi \tag{3.9}$$

Then we have

$$\frac{\partial J}{\partial r_o} = \int_0^\infty \int_0^{2\pi} \frac{\partial}{\partial r_o}\left(\frac{e^{-kR}}{R}\right) f(r,z)\; r\, dr\, d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{\partial}{\partial R}\left(\frac{e^{-kR}}{R}\right) \frac{(r_o - r\cos\phi)}{R}\, r f(r,z) dr\, d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{d}{dR}\left(\frac{e^{-kR}}{R}\right) \frac{\{\cos\phi(r_o\cos\phi - r) + r_o \sin^2\phi\}}{R}\, r f(r,z) dr\, d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{d}{dR}\left(\frac{e^{-kR}}{R}\right) \left\{-\cos\phi \frac{\partial R}{\partial r} + \frac{\sin\phi}{r}\frac{\partial R}{\partial \phi}\right\} r\, f(r,z) dr\, d\phi$$

Hence

$$\frac{\partial J}{\partial r_o} = \int_0^\infty \int_0^{2\pi} \left\{-\cos\phi \frac{\partial}{\partial r}\left(\frac{e^{-kR}}{R}\right) + \frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\frac{e^{-kR}}{R}\right)\right\} r\, f(r,z) dr\, d\phi \tag{3.10}$$

We can now integrate by parts the terms on the r.h.s. of (3.10) to give

$$\frac{\partial J}{\partial r_o} = -\int_0^{2\pi} \left[r \cos\phi\, f(r,z) \frac{e^{-kR}}{R}\right]_0^\infty d\phi + \int_0^\infty \int_0^{2\pi} \cos\, \frac{e^{-kR}}{R} \frac{\partial}{\partial r}\{r f(r,z)\} dr\, d\phi$$

$$+ \int_0^\infty \left[\sin\phi\, f(r,z) \frac{e^{-kR}}{R}\right]_{\phi=0}^{2\pi} dr - \int_0^\infty \int_0^{2\pi} \frac{e^{-kR}}{R} f(r,z) \frac{\partial}{\partial \phi}(\sin\phi) dr\, d\phi$$

Since there is no contribution from the square bracket terms, we have

$$\frac{\partial J}{\partial r_o} = \int_0^\infty \int_0^{2\pi} \cos\phi \frac{e^{-kR}}{R} \left\{\frac{\partial}{\partial r}(rf) - f\right\} dr\, d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \cos\phi \frac{e^{-kR}}{R} \frac{\partial f(r,z)}{\partial r} r\, dr\, d\phi \tag{3.11}$$

It then follows from (3.11) and (3.8) that

$$4\pi\eta(r_o,z_o) = -e^{kz_o} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{e^{-kR}}{R} \cos\phi \, \frac{\partial f(r,z)}{\partial r} \, r \, dr \, dz \, d\phi \tag{3.12}$$

and we can now substitute for $\partial f/\partial r$ using (3.5), namely

$$\frac{\partial f}{\partial r} = - \frac{2}{\nu r} e^{-kz} v(r,z) \frac{\partial v(r,z)}{\partial z} \tag{3.13}$$

so that we have

$$2\pi\nu\eta(r_o,z_o) = e^{kz_o} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{e^{-kz-kR}}{R} \cos\phi \; v(r,z) \frac{\partial v(r,z)}{\partial z} \, dr \, dz \, d\phi \tag{3.14}$$

and we note that the formula (3.14) relates η directly with the transverse velocity v.

We consider next the formula for the stream function in terms of η, namely the equation (1.12). If in (1.12) we write

$$\Psi = \frac{\partial \beta}{\partial r} \tag{3.15}$$

we obtain

$$\frac{\partial}{\partial r} \{\beta_{zz} + \beta_{rr} + \frac{1}{r} \beta_r\} = \eta(r,z) \tag{3.16}$$

so that

$$\nabla^2 \beta = F(r,z) \tag{3.17}$$

where

$$F(r,z) = \int_0^r \eta(\sigma,z) d\sigma \tag{3.18}$$

The solution for β which vanishes at infinity is given by

$$4\pi\beta(r_o,z_o) = - \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{F(r,z) \, r \, dr \, d\phi \, dz}{R} \tag{3.19}$$

where R is defined in (3.7). It follows from (3.19) and (3.15) that

$$4\pi\Psi(r_o,z_o) = -\frac{\partial}{\partial r_o}\int_{-\infty}^{\infty}\int_{o}^{\infty}\int_{o}^{2\pi}\frac{1}{R}F(r,z)r\,dr\,d\phi\,dz \tag{3.20}$$

If we consider

$$I = \int_o^{\infty}\int_o^{2\pi}\frac{1}{R}F(r,z)r\,dr\,d\phi \tag{3.21}$$

we have

$$\frac{\partial I}{\partial r_o} = \int_o^{\infty}\int_o^{2\pi}\frac{\partial}{\partial r_o}\left(\frac{1}{R}\right)F(r,z)r\,dr\,d\phi$$

$$= \int_o^{\infty}\int_o^{2\pi}\frac{(r\cos\phi - r_o)}{R^3}F(r,z)r\,dr\,d\phi$$

$$= \int_o^{\infty}\int_o^{2\pi}\frac{\cos\phi(r - r_o\cos\phi) - r_o\sin^2\phi}{R^3}F(r,z)r\,dr\,d\phi$$

$$\frac{\partial I}{\partial r_o} = \int_o^{\infty}\int_o^{2\pi}\left\{-\cos\phi\,\frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{\sin\phi}{r}\,\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)\right\}F(r,z)\,r\,dr\,d\phi \tag{3.22}$$

We integrate by parts the two portions of the integral (3.22) and we obtain

$$\frac{\partial I}{\partial r_o} = \int_o^{2\pi}\left[-\frac{\cos\phi}{R}\,r\,F(r,z)\right]_{r=o}^{r=\infty}d\phi + \int_o^{\infty}\int_o^{2\pi}\frac{\cos\phi}{R}\,\frac{\partial}{\partial r}\{r\,F(r,z)\}dr\,d\phi$$

$$+ \int_o^{\infty}\left[\frac{\sin\phi}{rR}\cdot r\,F(r,z)\right]_{\phi=o}^{2\pi}dr - \int_o^{\infty}\int_o^{2\pi}\frac{F(r,z)}{R}\,\frac{\partial}{\partial\phi}(\sin\phi)dr\,d\phi$$

Provided $F(r,z)$ tends to zero at $r \to \infty$, which will be satisified in the present case, there is no contribution from the square brackets; hence we obtain

$$\frac{\partial I}{\partial r_o} = \int_o^{\infty}\int_o^{2\pi}\frac{\cos\phi}{R}\left\{\frac{\partial}{\partial r}(rF) - F\right\}dr\,d\phi$$

$$= \int_o^{\infty}\int_o^{2\pi}\frac{r\cos\phi}{R}\,\frac{\partial F(r,z)}{\partial r}\,dr\,d\phi \tag{3.23}$$

It now follows from (3.20) and (3.23) that

$$4\pi\Psi(r_o,z_o) = -\int_{-\infty}^{\infty}\int_{o}^{\infty}\int_{o}^{2\pi} \frac{1}{R}\frac{\partial F(r,z)}{\partial r}\cos\phi \; r\,dr\,d\phi\,dz \tag{3.24}$$

and since, from (3.18), we have

$$\frac{\partial F}{\partial r} = \eta(r,z) \tag{3.25}$$

we obtain finally

$$4\pi\Psi(r_o,z_o) = -\int_{-\infty}^{\infty}\int_{o}^{\infty}\int_{o}^{2\pi} \frac{1}{R}\,\eta(r,z)\cos\phi \; r\,dr\,d\phi\,dz \quad , \tag{3.26}$$

a formula relating the vorticity function $\eta(r,z)$ and the stream function $\psi = r\Psi$.

4. *Properties of the Stream Function ψ and a Formula for the Axial Flow Along Vortex Axis*

We note from the definition of R in (3.7) that

$$\lim_{r_o \to o} R = \sqrt{r^2 + (z-z_o)^2} \tag{4.1}$$

and, since this limiting value is independent of ϕ, it follows from (3.26) that

$$\lim_{r_o \to o} \Psi(r_o,z_o) = 0 \tag{4.2}$$

Since $\psi(r_o,z_o) = r_o\,\Psi(r_o,z_o)$, we have

$$\lim_{r_o \to o} \psi(r_o,z_o) = 0 \tag{4.3}$$

We consider next the function $\partial\Psi/\partial r_o$. From (3.26), we obtain

$$4\pi\frac{\partial\Psi(r_o,z_o)}{\partial r_o} = -\int_{-\infty}^{\infty}\int_{o}^{\infty}\int_{o}^{2\pi} \frac{\partial}{\partial r_o}\left(\frac{1}{R}\right) \eta(r,z)\cos\phi \; r\,dr\,d\phi\,dz \tag{4.4}$$

If we take

$$I_1 = \int_0^\infty \int_0^{2\pi} \frac{1}{R}\, \eta(r,z)\cos\phi \; r\,dr\,d\phi \tag{4.5}$$

then

$$\frac{\partial I_1}{\partial r_o} = \int_0^\infty \int_0^{2\pi} \frac{\partial}{\partial r_o}\left(\frac{1}{R}\right) \eta(r,z)\cos\phi \; r\,dr\,d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{r\cos\phi - r_o}{R^3}\, \eta\cos\phi \; r\,dr\,d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{\cos\phi(r - r_o\cos\phi) - r_o\sin^2\phi}{R^3}\, \eta\cos\phi \; r\,dr\,d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \left\{-\cos\phi \frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)\right\} \eta\cos\phi \; r\,dr\,d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \left\{-\cos^2\phi \frac{\partial}{\partial r}\left(\frac{1}{R}\right) + \sin\phi\cos\phi\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)\right\} \eta(r,z) \; r\,dr\,d\phi \tag{4.6}$$

We can now integrate by parts the two portions of the integral in (4.6) to give

$$\frac{\partial I_1}{\partial r_o} = \int_0^{2\pi} \left[-r\eta\cos^2\phi \cdot \frac{1}{R}\right]_{r=o}^{\infty} d\phi + \int_0^\infty \int_0^{2\pi} \frac{\cos^2\phi}{R}\frac{\partial}{\partial r}(r\eta)\,dr\,d\phi$$

$$+ \int_0^\infty \left[\eta\sin\phi\cos\phi \cdot \frac{1}{R}\right]_{\phi=o}^{2\pi} dr - \int_0^\infty \int_0^{2\pi} \frac{\eta}{R}\frac{\partial}{\partial\phi}(\sin\phi\cos\phi)\,dr\,d\phi$$

and, since there is no contribution from the square brackets, we have

$$\frac{\partial I_1}{\partial r_o} = \int_0^\infty \int_0^{2\pi} \left\{\frac{\cos^2\phi}{R}\frac{\partial}{\partial r}(r\eta) - \frac{\eta}{R}(\cos^2\phi - \sin^2\phi)\right\} dr\,d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{1}{R}\left\{\cos^2\phi\left(r\frac{\partial\eta}{\partial r} + \eta\right) - \eta(\cos^2\phi - \sin^2\phi)\right\} dr\,d\phi$$

$$= \int_0^\infty \int_0^{2\pi} \frac{1}{R} \{r \cos^2\phi \frac{\partial n}{\partial r} + n \sin^2\phi\} dr\, d\phi \tag{4.7}$$

It now follows from (4.4) and (4.7) that

$$4\pi \frac{\partial\Psi(r_o,z_o)}{\partial r_o} = - \int_{-\infty}^\infty \int_0^\infty \int_0^{2\pi} \frac{1}{R} \{r \cos^2\phi \frac{\partial n}{\partial r} + n \sin^2\phi\} dr\, dz\, d\phi \tag{4.8}$$

this result being valid for all r_o and z_o.

We can deduce from (4.8) the limiting value of $\partial\Psi/\partial r_o$ as $r_o \to 0$; using (4.1), we have

$$4\pi \lim_{r_o \to o} \frac{\partial\Psi(r_o,z_o)}{\partial r_o} = - \int_{-\infty}^\infty \int_0^\infty \int_0^{2\pi} \frac{r \cos^2\phi\ \partial n/\partial r + n \sin^2\phi}{\sqrt{r^2 + (z-z_o)^2}} dr\, dz\, d\phi$$

Hence

$$4\pi \lim_{r_o \to o} \frac{\partial\Psi(r_o,z_o)}{\partial r_o} = - \pi \int_{-\infty}^\infty \int_0^\infty \frac{r\ \partial n/\partial r + n}{\sqrt{r^2 + (z-z_o)^2}} dr\, dz \tag{4.9}$$

Using (4.2), we also have the result

$$\lim_{r_o \to o} \frac{\partial\Psi(r_o,z_o)}{\partial r_o} = \lim_{r_o \to o} \frac{\Psi(r_o,z_o)}{r_o} \tag{4.10}$$

so that from (4.9) we deduce that

$$\lim_{r_o \to o} \{\frac{\partial\Psi(r_o,z_o)}{\partial r_o} + \frac{\Psi(r_o,z_o)}{r_o}\} = - \frac{1}{2} \int_{-\infty}^\infty \int_0^\infty \frac{\partial/\partial r(rn) dr dz}{\sqrt{r^2 + (z-z_o)^2}} \tag{4.11}$$

Using the stream function ψ and the results (1.9), (1.11), we have the general formula for the velocity component $w(r_o,z_o)$ in the z-direction,

$$w(r_o,z_o) = - \frac{\partial\Psi(r_o,z_o)}{\partial z_o} - \frac{\Psi(r_o,z_o)}{r_o} \tag{4.12}$$

From (4.11) and (4.12), we obtain the result

$$w(0,z_o) = \lim_{r_o \to o} w(r_o,z_o) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{o}^{\infty} \frac{\partial/\partial r(r\eta)drdz}{\sqrt{r^2 + (z-z_o)^2}} \tag{4.13}$$

We note from (4.13) that $w(0,z_o)$ vanishes as $z_o \to +\infty$. We can obtain an alternative formula for $w(0,z_o)$ if we use integration by parts in (4.13). This gives

$$w(0,z_o) = \frac{1}{2} \int_{-\infty}^{\infty} \left[\frac{r\eta(r,z)}{\sqrt{r^2 + (z-z_o)^2}}\right]_{r=o}^{r=\infty} dz - \frac{1}{2} \int_{-\infty}^{\infty} \int_{o}^{\infty} r\,\eta \frac{\partial}{\partial r} \left\{\frac{1}{\sqrt{r^2 + (z-z_o)^2}}\right\} drdz$$

Hence

$$w(0,z_o) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{o}^{\infty} \frac{r^2\eta(r,z)drdz}{\{r^2 + (z-z_o)^2\}3/2} \tag{4.14}$$

and this result makes it clear that $w(0,z_o) \sim A/z_o^3$ when z_o is large.

By making use of (3.14) and (4.14), we can express $w(0,z_o)$ in terms of the transverse velocity v. With a slight change of notation, we have

$$w(0,\xi) = \frac{1}{2} \int_{z_o=-\infty}^{+\infty} \int_{r_o=o}^{\infty} \frac{r_o^2\, \eta(r_o,z_o)dr_o dz_o}{\{r_o^2 + (z_o-\xi)^2\}^{3/2}}$$

so that

$$4\pi\nu\, w(0,\xi) = \int_{z_o=-\infty}^{+\infty} \int_{r_o=o}^{\infty} \frac{r_o^2\, e^{kz_o} dr_o dz_o}{\{r_o^2 + (z_o-\xi)^2\}^{3/2}} \int_{-\infty}^{\infty} \int_{o}^{\infty} \int_{o}^{2\pi} \frac{e^{-kz-kR}}{R} \cos\phi\, v(r,z) \frac{\partial v(r,z)}{\partial z} dr\, d\phi\, dz$$

and by interchanging orders of integration, we can write this in the form

$$4\pi\nu\, w(0,\xi) = \int_{-\infty}^{\infty} \int_{o}^{\infty} \int_{o}^{2\pi} e^{-kz} \cos\phi\, v(r,z) \frac{\partial v(r,z)}{\partial z} \tilde{\omega}(r,z,\phi,\xi) dr\, dz\, d\phi \tag{4.15}$$

where

$$\tilde{\omega}(r,z,\phi,\xi) = \int_{z_o=-\infty}^{+\infty} \int_{r_o=o}^{\infty} \frac{r_o\, e^{kz_o-kR} dr_o dz_o}{R\{r_o^2 + (z_o-\xi)^2\}^{3/2}} \quad ; \tag{4.16}$$

In (4.16) R is defined by (3.7), namely

$$R^2 = r^2 + r_o^2 - 2r_o r \cos\phi + (z-z_o)^2 \quad . \tag{4.17}$$

5. *The Determination of the Pressure Function*

It is easily shown that the equations (1.2) and (1.4) can be written in the form

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = W\frac{\partial u}{\partial z} - \frac{v^2}{r} - \frac{\nu}{r}\frac{\partial}{\partial z}(r\eta) \quad , \tag{5.1}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} = W\frac{\partial w}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}(r\eta) \quad . \tag{5.2}$$

Elimination of p between (5.1) and (5.2) gives rise to the vorticity equation (1.8); accordingly the right hand side of the equation

$$-\frac{1}{\rho}dp = -\frac{1}{\rho}\left(\frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz\right) = \{W\frac{\partial u}{\partial z} - \frac{v^2}{r} - \frac{\nu}{r}\frac{\partial}{\partial z}(r\eta)\}dr + \{W\frac{\partial w}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}(r\eta)\}dz \tag{5.3}$$

is necessarily a perfect differential and, in theory, we should be able to determine p from the line integral

$$-\frac{p}{\rho} = \int_c \{W\frac{\partial u}{\partial z} - \frac{v^2}{r} - \frac{\nu}{r}\frac{\partial}{\partial z}(r\eta)\}dr + \{W\frac{\partial w}{\partial z} + \frac{\nu}{r}\frac{\partial}{\partial r}(r\eta)\}dz \tag{5.4}$$

As an alternative approach, we note that (5.1) and (5.2) can be written in the form

$$-\frac{r}{\rho}\frac{\partial p}{\partial r} = W\frac{\partial}{\partial z}(ru) - v^2 - \nu\frac{\partial}{\partial z}(r\eta) \tag{5.5}$$

$$-\frac{r}{\rho}\frac{\partial p}{\partial z} = W\frac{\partial}{\partial z}(rw) + \nu\frac{\partial}{\partial r}(r\eta) \tag{5.6}$$

Differentiating (5.5) w.r.t. r, (5.6) w.r.t. z and adding, we obtain

$$-\frac{1}{\rho}\{\frac{\partial}{\partial r}(r\frac{\partial p}{\partial r}) + r\frac{\partial^2 p}{\partial z^2}\} = W\frac{\partial}{\partial z}\{\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw)\} - 2v\frac{\partial v}{\partial z} \quad ,$$

and, using (1.5), this becomes

$$\frac{r}{\rho}\{\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2}\} = 2v\frac{\partial v}{\partial z}$$

or

$$\frac{1}{\rho}\nabla^2 p = \frac{2}{r}v\frac{\partial v}{\partial z} \tag{5.7}$$

A particular solution for p of (5.7) is given by

$$p^*(r_o,z_o) = -\frac{\rho}{4\pi} \int_{z=-\infty}^{\infty} \int_{r=o}^{\infty} \int_{o}^{2\pi} \frac{2/r\ v(r,z)\ \partial v(r,z)/\partial z\ r dr\, dz\, d\phi}{\sqrt{r^2 + r_o^2 - 2rr_o\cos\phi\ + (z-z_o)^2}} \qquad (5.8)$$

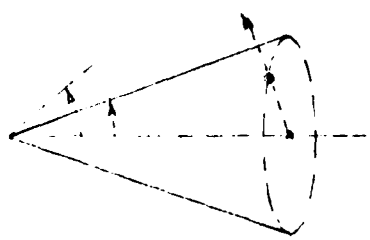
If we write

$$p = p^* + \underline{P} \qquad (5.9)$$

then $\underline{P}$ will be a harmonic function, i.e., satisfying

$$\nabla^2 \underline{P} = 0 \qquad (5.9)$$

The pressure function p is continuous throughout the space occupied by the liquid, but at the surface of the cone $\theta = \theta_o$ (see Section 2) there will be a discontinuity in $\partial p/\partial n$ where $\hat{n}$ is the normal to the surface of the cone.



Accordingly, in order to determine $\underline{P}$, we must consider an inner solution $\underline{P}_i$ valid in $0 < \theta < \theta_o$ and also an outer solution $\underline{P}_e$ valid in $\theta_o < \theta < \pi$, with

$$\underline{P}_i = \underline{P}_e \quad \text{at} \quad \theta = \theta_o \quad , \qquad (5.11)$$

$$\frac{\partial \underline{P}_e}{\partial n} - \frac{\partial \underline{P}_i}{\partial n} = \alpha \neq 0 \quad \text{at} \quad \theta = \theta_o \quad , \qquad (5.12)$$

This pattern of behavior of $\underline{P}$ can be achieved by distributing pressure sources over the cone $\theta = \theta_o$. The function α entering into (5.12) will be determined from the discontinuity in $\partial v/\partial n$ at $\theta = \theta_o$, v being the transverse velocity. Thus, the determination of p using (5.7) will lead eventually to an integral equation for the unknown pressure source distribution on $\theta = \theta_o$.

If we are interested solely in the pressure distribution along the vortex axis $r_o = 0$, $z_o > 0$, it is possible to avoid the solution of an integral equation by returning to (5.3). If we take $r_o = 0$ and $dr_o = 0$ in (5.3), we obtain

$$-\frac{1}{\rho}\, dp = \left\{W \frac{\partial w(r_o,z_o)}{\partial z_o} + \nu \left(\frac{\partial n(r_o,z_o)}{\partial r_o} + \frac{n(r_o,z_o)}{\partial r_o}\right)\right\}_{r_o = o} dz_o \qquad (5.13)$$

Thus $p(0,z_o)$ will be given by

$$-\frac{p(0,z_o)}{\rho} + \frac{p_\infty}{\rho} = W\, w(0,z_o) + \nu \int_\infty^{z_o} \left\{\frac{\partial n(r_o,z_o)}{\partial r_o} + \frac{n(r_o z_o)}{r_o}\right\}_{r_o=o} dz_o \qquad (5.14)$$

In order to determine the integrand of the integral in (5.14), we return to Eq. (3.8) from which we obtain

$$4\pi \left(\frac{\partial n}{\partial r_o} + \frac{n}{r_o}\right) = -e^{kz_o} \left(\frac{\partial^2}{\partial r_o^2} + \frac{1}{r_o}\frac{\partial}{\partial r_o}\right) \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{e^{-kR}}{R}\, f(r,z) r\, dr\, d\phi\, dz\ , \qquad (5.15)$$

$f(r,z)$ being defined in (3.5). From the Appendix, we see that if

$$4\pi\Phi(r_o,z_o) = -\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{e^{-kR}}{R}\, f(r,z)\, r\, dr\, d\phi\, dz \qquad (5.16)$$

then

$$\frac{\partial^2\Phi}{\partial r_o^2} + \frac{1}{r_o}\frac{\partial\Phi}{\partial r_o} + \frac{\partial^2\Phi}{\partial z_o^2} - k^2\Phi = f(r_o,z_o) \qquad (5.17)$$

If we apply this result to the right-hand side of (5.15), we obtain

$$4\pi \left(\frac{\partial n}{\partial r_o} + \frac{n}{r_o}\right) = +e^{kz_o} \left(\frac{\partial^2}{\partial z_o^2} - k^2\right) \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{e^{-kR}}{R}\, f(r,z) r\, dr\, d\phi\, dz + 4\pi e^{kz_o} f(r_o,z_o)$$
(5.18)

It is permissible to make $r_o \to 0$ in the result (5.18) and, if we make use of the definition of $f(r,z)$ in (3.5), we obtain

$$4\pi\left(\frac{\partial n}{\partial r_o} + \frac{n}{r_o}\right)_{r_o=o} = e^{kz_o}\left(\frac{\partial^2}{\partial z_o^2} - k^2\right)\int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{2\pi} \frac{e^{-k\sqrt{r^2+(z-z_o)^2}}}{\sqrt{r^2+(z-z_o)^2}}\, f(r,z) r\, dr\, d\phi\, dz$$

$$= 2\pi e^{kz_o}\left(\frac{\partial^2}{\partial z_o^2} - k^2\right)\int_{-\infty}^{\infty}\int_0^{\infty} \frac{e^{-k\sqrt{r^2+(z-z_o)^2}}}{\sqrt{r^2+(z-z_o)^2}}\, f(r,z) r\, dr\, dz \qquad (5.19)$$

Thus (5.14) can be written in the form

$$-\frac{p(0,z_o)}{\rho} + \frac{p_\infty}{\rho} = W\, w(0,z_o) + \frac{1}{2}\nu \int_\infty^{z_o} e^{kz_o}\left\{\left(\frac{\partial^2}{\partial z_o^2} - k^2\right)\int_{-\infty}^{\infty}\int_0^{\infty} \frac{e^{-k\sqrt{r^2+(z-z_o)^2}}}{\sqrt{r^2+(z-z_o)^2}}\, f(r,z) r\, dr\, dz\right\} dz_o$$
(5.20)

and $w(0,z_o)$ is given by (4.14) or (4.15). Since we have

$$\int_{\infty}^{z_o} e^{kz_o}\left(\frac{d^2\alpha}{dz_o^{\ 2}} - k^2\alpha\right)dz_o \equiv e^{kz_o}\left(\frac{d\alpha}{dz_o} - k\alpha\right) ,$$

provided $\alpha(z_o)$ and $\alpha'(z_o)$ tend to zero at infinity, it follows from (5.20) that

$$-\frac{p(0,z_o)}{\rho} + \frac{p_\infty}{\rho} = W w(0,z_o) + \frac{1}{2}\nu e^{kz_o}\left(\frac{d}{dz_o} - k\right)\int_{-\infty}^{\infty}\int_{0}^{\infty} \frac{e^{-k\sqrt{r^2+(z-z_o)^2}}}{\sqrt{r^2+(z-z_o)^2}} f(r,z) r\,dr\,dz \tag{5.21}$$

We have

$$\int_0^\infty \frac{re^{-k\sqrt{r^2+(z-z_o)^2}}}{\sqrt{r^2+(z-z_o)^2}} f(r,z)dr = -\frac{1}{k}\int_0^\infty \frac{\partial}{\partial r}\{e^{-k\sqrt{r^2+(z-z_o)^2}}\}f(r,z)dr$$

$$= -\frac{1}{k}\left[e^{-k\sqrt{r^2+(z-z_o)^2}} f(r,z)\right]_{r=o}^{\infty} + \frac{1}{k}\int_0^\infty e^{-k\sqrt{r^2+(z-z_o)^2}}\frac{\partial f(r,z)}{\partial r}dr ,$$

and, using the definition of $f(r,z)$ in (3.5), we see that the square bracket makes no contribution and thus we have

$$\int_0^\infty \frac{re^{-k\sqrt{r^2+(z-z_o)^2}}}{\sqrt{r^2+(z-z_o)^2}} f(r,z)dr = \frac{1}{k}\int_0^\infty e^{-k\sqrt{r^2+(z-z_o)^2}}\frac{\partial f(r,z)}{\partial r}dr \tag{5.22}$$

From (3.5), we have

$$\frac{\partial f(r,z)}{\partial r} = -\frac{2}{\nu}\frac{1}{r}e^{-kz}v(r,z)\frac{\partial v(r,z)}{\partial z} , \tag{5.23}$$

hence (5.21) can be written in the form

$$-\frac{p(0,z_o)}{\rho} + \frac{p_\infty}{\rho} = W w(0,z_o) - \frac{1}{k}e^{kz_o}\left(\frac{d}{dz_o} - k\right)\int_{-\infty}^{\infty}\int_0^\infty \frac{1}{r}e^{-k\sqrt{r^2+(z-z_o)^2}}e^{-kz}v(r,z)\frac{\partial v(r,z)}{\partial z}drdz \tag{5.24}$$

The result (5.24) is valid along the vortex axis $r_o = 0$, $z_o > 0$. A general equation for the pressure p can be derived from (5.4) as follows. We write (5.4) in the form

$$-\frac{p}{\rho} = \int_c (W\frac{\partial u}{\partial z} - \frac{v^2}{r} - \nu\frac{\partial \eta}{\partial z})dr + \{W\frac{\partial w}{\partial z} + \nu(\frac{\partial \eta}{\partial r} + \frac{\eta}{r})\}dz$$

$$= \int_c \{W(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r})dr - \frac{v^2}{r}dr - \nu\frac{\partial \eta}{\partial z}dr + W(\frac{\partial w}{\partial r}dr + \frac{\partial w}{\partial z}dz) + \nu(\frac{\partial \eta}{\partial r} + \frac{\eta}{r})dz\}$$

Hence

$$-\frac{p}{\rho} = Ww(r,z) + \int_c \{w\eta dr - \frac{v^2}{r}dr - \nu\frac{\partial \eta}{\partial z}dr + \nu(\frac{\partial \eta}{\partial r} + \frac{\eta}{r})dz\} \qquad (5.25)$$

It is convenient now to write $\eta = e^{kz}\,\partial\alpha/\partial r$ as in (3.2), where the function α satisfies (3.4). We then have

$$-\frac{p}{\rho} = Ww(r,z) + \int_c \{W e^{kz}\frac{\partial \alpha}{\partial r}dr - \frac{v^2}{r}dr - \nu\frac{\partial}{\partial z}(e^{kz}\frac{\partial \alpha}{\partial r})dr + \nu e^{kz}(\frac{\partial}{\partial r} + \frac{1}{r})\frac{\partial \alpha}{\partial r}dz\}$$

and since, by the definition (2.1), $W = 2k\nu$, this becomes

$$-\frac{p}{\rho} = Ww(r,z) + \int_c \{2k\nu\, e^{kz}\frac{\partial \alpha}{\partial r}dr - \frac{v^2}{r}dr - \nu\, e^{kz}(\frac{\partial^2 \alpha}{\partial z \partial r} + k\frac{\partial \alpha}{\partial r})dr + \nu e^{kz}(\frac{\partial}{\partial r} + \frac{1}{r})\frac{\partial \alpha}{\partial r}dz\}$$

$$= Ww(r,z) + \int_c \{\nu e^{kz}(\frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{r}\frac{\partial \alpha}{\partial r})dz - \nu e^{kz}(\frac{\partial^2 \alpha}{\partial z \partial r} - k\frac{\partial \alpha}{\partial r})dr - \frac{v^2}{r}dr\} \qquad (5.26)$$

The function $\alpha(r,z)$ satisfies the differential equation (3.4), namely

$$\frac{\partial^2 \alpha}{\partial r^2} + \frac{1}{r}\frac{\partial \alpha}{\partial r} + \frac{\partial^2 \alpha}{\partial z^2} - k^2\alpha = f(r,z) \qquad (5.27)$$

and we replace the coefficient of dz in (5.26) using (5.27) to give

$$-\frac{p}{\rho} = Ww(r,z) + \int_c \{e^{kz}[f(r,z) - \frac{\partial^2 \alpha}{\partial z^2} + k^2\alpha]dz - \nu e^{kz}\frac{\partial}{\partial r}(\frac{\partial \alpha}{\partial z} - k\alpha)dr - \frac{v^2}{r}dr\} \qquad (5.28)$$

We have

$$\frac{\partial}{\partial z}\{e^{kz}(\frac{\partial \alpha}{\partial z} - k\alpha) = e^{kz}(\frac{\partial^2 \alpha}{\partial z^2} - k^2\alpha)$$

Hence

$$-\frac{p}{\rho} = Ww(r,z) + \int_c \{\nu e^{kz} f(r,z)dz - \frac{v^2}{r}dr - \nu\frac{\partial}{\partial z}[e^{kz}(\frac{\partial\alpha}{\partial z} - k\alpha)]dz - \nu e^{kz}\frac{\partial}{\partial r}(\frac{\partial\alpha}{\partial z} - k\alpha)dr\}$$

$$= Ww(r,z) + \int_c \{\nu e^{kz} f(r,z)dz - \frac{v^2}{r}dr - \nu d[e^{kz}(\frac{\partial\alpha}{\partial z} - k\alpha)]\}$$

$$= Ww(r,z) - \nu e^{kz}(\frac{\partial\alpha}{\partial z} - k\alpha) + \int_c \{\nu e^{kz} f(r,z)dr - \frac{v^2}{r}dr\} \quad (5.29)$$

We now introduce a function $\gamma(r,z)$ having the property

$$e^{kz} f(r,z) = \frac{\partial\gamma}{\partial z} \quad (5.30)$$

where f is defined in (3.5). If we differentiate (5.30) with respect to r, we obtain

$$e^{kz}\frac{\partial f}{\partial r} = \frac{\partial^2\gamma}{\partial r\partial z} \quad ;$$

replacing $\partial f/\partial r$ from (5.23), we have

$$\frac{\partial^2\gamma}{\partial r\partial z} = -\frac{2}{\nu}\frac{1}{r}v(r,z)\frac{\partial v(r,z)}{\partial z}$$

Hence

$$\frac{\partial^2\gamma}{\partial r\partial z} = -\frac{1}{\nu}\frac{\partial}{\partial z}(\frac{v^2}{r})$$

and we take

$$\frac{\partial\gamma}{\partial r} = -\frac{1}{\nu}\frac{v^2}{r} \quad (5.31)$$

If we use (5.30) and (5.31) in (5.29), we obtain

$$-\frac{p}{\rho} = Ww(r,z) - \nu e^{kz}(\frac{\partial\alpha}{\partial z} - k\alpha) + \int\nu(\frac{\partial\gamma}{\partial z}dz + \frac{\partial\gamma}{\partial r}dr)$$

giving

$$-\frac{p}{\rho} = C_o + Ww(r,z) - \nu e^{kz}(\frac{\partial\alpha}{\partial z} - k\alpha) + \nu\gamma(r,z) \quad (5.32)$$

where C_o is a constant and the function $\gamma(r,z)$ will be given by the line integral

$$\gamma(r,z) = \int_C \{e^{kz} f(r,z)\,dz - \frac{1}{\nu}\frac{v^2}{r}\,dr\} \tag{5.33}$$

If we choose the initial point of the curve C to be at infinity so that

$$\gamma(r,z) = \int_{\infty}^{\underline{P}(r,z)} \{e^{kz} f(r,z)dz - \frac{1}{\nu}\frac{v^2}{r}\,dr\} \tag{5.34}$$

then the functions w, $e^{kz}(\alpha_z - k\alpha)$ and γ will tend to zero at infinity and the constant C_o will be given by

$$C_o = -p_\infty/\rho \tag{5.35}$$

It will be noted that, when the pressure along the vortex axis is determined from the general formula (5.32), we have r = 0 and dr = 0 in (5.34) and, since f ≡ 0 on r = 0, it follows that $\gamma(r,z) \equiv 0$; the formula (5.32) then gives the result (5.24). The result (5.24) is the only simple formula for the pressure; in all other cases it is necessary to determine the function $\gamma(r,z)$ defined in (5.34). When we substitute in (5.32) for the function $\alpha(r,z)$ defined in (3.6), we obtain the following general formula for the pressure

$$\frac{p_\infty - p}{\rho} = W w(r_o, z_o) + \frac{\nu e^{kz_o}}{4\pi}\left(\frac{\partial}{\partial z_o} - k\right) \int_{z=-\infty}^{+\infty} \int_0^{\infty} \int_0^{2\pi} \frac{e^{-kR}}{R} f(r,z)\, r\, dr\, dz\, d\phi + \nu\gamma(r,z) \tag{5.36}$$

SUMMARY

We summarize the many results presented in the previous sections by pointing out the numerical work required in order to have an understanding of the physics underlying the mathematics. The same numerical results can also, of course, be used to make comparisons with pressure measurements and also to yield predictions with regard to cavitation inception.

The first step is to solve the two integral equations for $A(\theta)$ and $B(\theta)$ as presented in Equations (2.22) and (2.26). In order to be in a position to solve these equations, it is necessary to specify the function $V_o(R)$ as well as the angle θ_o. These two inputs must come from an outside source, presumably from the solution of the inviscid lifting surface theory near the wing tip. Nevertheless, the schemata for solving the equations can be specified numerically even without any specific inputs available. Once $A(\theta)$ and $B(\theta)$ have been determined, the velocity $v(r,z)$ can be calculated using Equations (2.15) and (2.16)

The axial flow along the vortex axis can then be evaluated from Equation (4.15), while the pressure along the vortex axis is determined from Equation (5.24), and the pressure elsewhere is determined from Equation (5.36).

It would be of some interest to determine the circumstances under which the pressure is minimum along the vortex axis, for it is the minimum pressure that controls cavitation inception. If the minimum pressure can be shown to lie along the axis, then the relatively simple Equation (5.24) would control cavitation inception rather than the more complicated Equation (5.36).

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APPENDIX

The solution of the p.d.e. $(\nabla^2-k^2)\Psi = f(x,y,z)$, with $f(x,y,z)$ given and with the condition $\Psi \to 0$ at infinity, ∇^2 being the three-dimensional Laplacian.

The function $\Psi(x,y,z)$ satisfies

$$(\nabla^2-k^2)\Psi = f(x,y,z) \tag{1}$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and $\Psi \to 0$ at infinity.

Introduce a Green's Function G with with properties

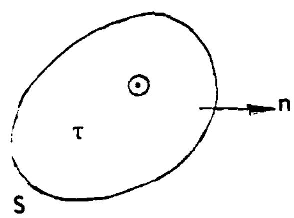
$$(\nabla^2-k^2)G = 0 \text{ at all points except } (x_o y_o z_o) \tag{2}$$

$$G = \frac{1}{R} \text{ near } (x_o y_o z_o),\ R^2 = (x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2 \tag{3}$$

$$G \to 0 \text{ at infinity} \tag{4}$$

We use the identity

$$\iiint_\tau (\chi\nabla^2\phi - \phi\nabla^2\chi)d\tau = \iint_S (\chi \frac{\partial\phi}{\partial n} - \phi\frac{\partial\chi}{\partial n})dS + \iint_\sigma (\chi \frac{\partial\phi}{\partial n} - \phi\frac{\partial\chi}{\partial n})d\sigma \quad , \tag{5}$$



$\hat{n}$ = outward normal from τ

σ = sphere radius center $(x_o y_o z_o)$

$\hat{n}$ being the outward normal, i.e., drawn away from the volume τ. We identify

$$\chi \equiv \Psi \quad , \quad \phi \equiv G$$

and from (1) and (2), we have, within τ

$$\begin{aligned}\chi\nabla^2\phi - \phi\nabla^2\chi &= \Psi\nabla^2 G - G\nabla^2\Psi \\ &= \Psi(\nabla^2 - k^2)G - G(\nabla^2 - k^2)\Psi \\ &\equiv - Gf\end{aligned}$$

Thus the identity (5) becomes

$$\iint_S (\Psi \frac{\partial G}{\partial n} - G \frac{\partial\Psi}{\partial n})dS + \iint_\sigma (\Psi \frac{\partial G}{\partial n} - G \frac{\partial\Psi}{\partial n})d\sigma = - \iiint_\tau Gf\, d\tau \tag{6}$$

For the integral over σ, we have, using (3),

$$\Psi \frac{\partial G}{\partial n} - G \frac{\partial \Psi}{\partial n} = - \Psi \frac{\partial G}{\partial R} + G \frac{\partial \Psi}{\partial R}$$

$$= -\Psi \frac{\partial}{\partial R}\left(\frac{1}{R}\right) + \frac{1}{R} \frac{\partial \Psi}{\partial R}$$

$$= \frac{\Psi}{R^2} + \frac{1}{R} \frac{\partial \Psi}{\partial R}$$

Hence

$$\lim_{\varepsilon \to 0} \iint_\sigma \left(\Psi \frac{\partial G}{\partial n} - G \frac{\partial \Psi}{\partial n}\right) d\sigma = \lim_{\varepsilon \to 0} \iint \left(\frac{\Psi}{\varepsilon^2} + \frac{1}{\varepsilon} \frac{\partial \Psi}{\partial R}\right) \varepsilon^2 d\omega = 4\pi\Psi(x_o y_o z_o) \qquad (7)$$

From (6) and (7) we have

$$4\pi\Psi(x_o y_o z_o) + \iint_S \left(\Psi \frac{\partial G}{\partial n} - G \frac{\partial \Psi}{\partial n}\right) dS = -\iiint_\tau Gf \, d\tau$$

With appropriate safeguards upon the behavior of Ψ and G at infinity, the integral over S will tend to zero as S moves off to infinity giving the result

$$4\pi\Psi(x_o, y_o, z_o) = \iiint_\tau Gf \, d\tau \quad . \qquad (8)$$

In order to determine G, we change the origin to $(x_o y_o z_o)$ by writing

$$x = x_o + X, \; y = y_o + Y, \; z = z_o + Z \qquad (9)$$

Then Eq. (2) becomes

$$G_{XX} + G_{YY} + G_{ZZ} - k^2 G = 0 \qquad (10)$$

If we look for the spherically symmetric solution of Eq. (10), then G will satisfy

$$\frac{\partial^2 G}{\partial R^2} + \frac{2}{R} \frac{\partial G}{\partial R} - k^2 G = 0 \qquad (11)$$

where $R^2 = X^2 + Y^2 + Z^2$. If, in (11), we write

$$G = \frac{g}{R} \qquad (12)$$

it follows that g satisfies

$$\frac{\partial^2 g}{\partial R^2} - k^2 g = 0 \tag{13}$$

so that

$$g(R) = Ae^{kR} + Be^{-kR}$$

Since we require $G \to 0$ as $R \to \infty$, we choose $A = 0$ and then, to satisfy the condition (3), we choose $B = 1$; hence

$$G = \frac{e^{-kR}}{R} \tag{14}$$

Accordingly, we can write (8) in the form

$$4\pi\Psi(x_o y_o z_o) = -\iiint_\tau \frac{e^{-kR}}{R} \, f \, d\tau$$

or

$$4\pi\Psi(x_o y_o z_o) = -\iiint_\tau \frac{\exp\{-k\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}\}}{\sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}} f(x,y,z)\,dx\,dy\,dz \tag{15}$$

In problems with axial symmetry where $x = r\cos\theta$, $y = r\sin\theta$, $x_o = r_o\cos\theta_o$, $y_o = r_o\sin\theta_o$, we can write the solution for Ψ in the form

$$4\pi\Psi(r_o,z_o) = -\int_{z=-\infty}^{+\infty} \int_0^\infty \int_0^{2\pi} \frac{e^{-kR}}{R} \, f(r,z)\; r\,dr\,dz\,d\phi \tag{16}$$

where

$$R^2 = r^2 + r_o^2 - 2rr_o\cos(\theta-\theta_o) + (z-z_o)^2 \tag{17}$$

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