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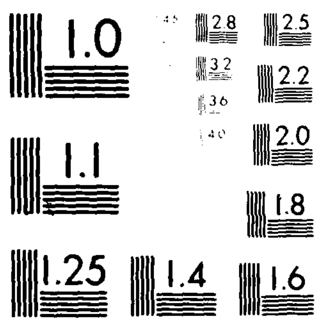

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QUALITATIVE MATRICES : STRONG SIGN-SOLVABILITY AND WEAK SATISFIABILITY.

by

Victor Klee ~~and~~ Richard Ladner

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13. ABSTRACT

The study of "qualitative" linear systems was suggested by Paul Samuelson in 1947 in his influential book on Foundations of Economic Analysis. The present paper deals with a rather general form of qualitative solvability, of which strong sign-solvability is a special case, and with a closely related notion, weak satisfiability, from propositional logic. The study of strong sign-solvability is reduced to that of S-matrices and a conjecture of T. Gorman on the structure of S-matrices is elucidated. An algorithm is described which uses linear programming methods to recognize S-matrices (and much more general systems) and thus solves this recognition problem in polynomial time. However, this is close to the boundary of NP-completeness, for the closely related problem of recognizing weak satisfiability is shown to be NP-complete.

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linear programming						
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NW-matrix						
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S-matrix						
sign-pattern						
sign-solvable						
simplex						
simplex algorithm						
strongly sign-solvable						
weak satisfiability						

QUALITATIVE MATRICES:

STRONG SIGN-SOLVABILITY AND WEAK SATISFIABILITY*

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QUALITATIVE MATRICES:
STRONG SIGN-SOLVABILITY AND WEAK SATISFIABILITY

VICTOR KLEE and RICHARD LADNER

Introduction

The study of "qualitative" linear systems was suggested by Paul Samuelson in 1947 in his influential book on Foundations of Economic Analysis [29]. Since then it has been carried on by economists, mathematicians, ecologists and chemists, with emphasis on questions of qualitative solvability and qualitative stability. Survey articles have been written, and references collected, by Maybee and Quirk [24], Maybee [23] and Quirk [28]. The present paper is devoted to a rather general form of qualitative solvability, of which strong sign-solvability is a special case, and to a closely related notion, weak satisfiability, from propositional logic.

The simplest qualitative conditions are those involving sign-patterns. For a (real) $m \times n$ matrix $A = (a_{ij})$, let $Q(A)$ denote the convex cone consisting of all matrices that have the same sign-pattern as A. Thus a matrix $B = (b_{ij})$ belongs to $Q(A)$ if and only if B is an $m \times n$ matrix with $\operatorname{sgn} b_{ij} = \operatorname{sgn} a_{ij}$ for all i and j. Here $\operatorname{sgn} \tau$ is $-$, 0 or + according to whether the real number τ is < 0, $= 0$, or > 0.

Points of R^m are taken as column matrices unless the contrary is specified. For $c \in R^m$, $(A:c)$ will denote the linear system whose coefficient matrix is A and "constants column" is c. The system $(A:c)$ is said to be *sign-solvable* if A is square, the system is solvable, and both its solvability and the sign-pattern of its solution depend only on the sign-patterns of A and c. That is, there exists x with $Ax = c$ such that for each $B \in Q(A)$ and $d \in Q(c)$ it is true that (i) there exists y with $By = d$ and (ii) for each such y, $Q(y) = Q(x)$. The system $(A:c)$ is *strongly sign-solvable* if there exists x as described with all coordinates of x different from 0.

As is shown in Section 1, the study of strong sign-solvability is easily reduced to the study of what we call S-matrices. An S-*matrix* is an $m \times (m+1)$ matrix A such that for each $B \in Q(A)$ and $x \in R^{m+1}$ with $Bx = 0$, all coordinates of x are of the same sign; equivalently, each such B has as its nullspace a line that intersects the open positive orthant of R^{m+1}. In 1962 Lancaster [18] suggested a general form for S-matrices; more inclusive forms were then described by Gorman [12] and Lancaster [19]. In 1965 Lancaster [20] initiated the algorithmic approach to S-matrices, based on duality properties of convex cones. Two central problems have been those of finding:

(a) a small collection of "standard forms" of S-matrices to which all such matrices can be reduced by certain elementary transformations;

(b) a fast algorithm for the recognition of S-matrices.

(The algorithms of Lancaster [20, 21] are of exponential complexity.)

It is not clear to us that one should expect a truly useful solution of (a). In any case, Section 2 describes some ways of constructing S-matrices and the closely related NW-matrices. A conjecture of Gorman [12] is elucidated but not settled.

In Section 4 the goal (b) is attained by reducing the S-matrix recognition problem to a sequence of $m^2 + 2m + 2$ linear feasibility tests. For the recognition of S-matrices, per se, it is probably more efficient to use the graph-theoretic characterization of sign-solvability established by Bassett, Maybee and Quirk [2] and discussed further by Maybee [22, 23]. However, our method goes considerably beyond the case of S-matrices, applying whenever the columns of the $m \times (m + 1)$ matrix A are restricted only by membership in given polyhedral cones. If the feasibility tests are made by means of the recent Shor-Khachian method [15, 31], the worst-case complexity of our algorithm is bounded by a polynomial in the length of the binary encoding of the description of the cones. Though the Shor-Khachian procedure is not well-suited to actual computation (see the comments in Section 4), this result is of theoretical interest because the S-matrix recognition problem is in a sense close to the boundary of NP-completeness.

In the language of propositional logic, recognizing that a given $m \times (m + 1)$ matrix is not an S-matrix amounts to recognizing that an associated Boolean formula, consisting of the conjunction of $n + 1$ disjunctive clauses in n propositional variables, is weakly satisfiable. Although this problem can be solved in polynomial time, we show in Section 3 that for each $k > 0$ the problem of recognizing the weak satisfiability of $n + \lfloor n^{1/k} \rfloor$ clauses in n variables in NP-complete.

The section headings are as follows: §1. Strong sign-solvability, S-matrices and weak satisfiability; §2. Constructions of NW-matrices; §3. The recognition of weak satisfiability; §4. The recognition of S-matrices; §5. Open problems.

1. Strong Sign-solvability, S-matrices and Weak Satisfiability

The following remark is straightforward.

THEOREM 1 *If* c *is a column of an* S-*matrix* A *and* B *is the square matrix formed by* A*'s other columns then the system* $(B:c)$ *is strongly sign-solvable. Now suppose, conversely, that* B *is an* $m \times m$ *matrix, the system* $(B:c)$ *is strongly sign-solvable, and* $Bx = c$ *. Let* A *be the* $m \times (m+1)$ *matrix whose* $(m+1)$*th column is* $-c$ *and whose* j*th column is (for* $1 \le j \le m$*) the* j*th column of* B *or the negative of that column according as* $x_j > 0$ *or* $x_j < 0$ *. Then* A *is an* S-*matrix.*

In view of condition (ii) of the next result, we may regard S as standing for "simplex" as well as for "strongly sign-solvable". We could avoid (ii), and establish the equivalence of (i) and (iii) directly, by appealing to standard results on convex polyhedral cones [9, 11]. In fact, that was done by Lancaster [20]. However, we want to include (ii) because it provides useful geometric insight. A more geometrical formulation of (iii) is as follows: for each open halfspace H in R^m whose bounding hyperplane passes through the origin there is a column v_j of A such that the cone $Q(v_j)$ is contained in H .

THEOREM 2 *Suppose that* A *is an* $m \times (m+1)$ *matrix and* $v_1, \dots, v_{m+1}$ *are the columns of* A *. Then the following three conditions are equivalent:*

(i) A *is an* S-*matrix;*

(ii) *for each choice of* $c_1 \in Q(v_1), \dots, c_{m+1} \in Q(v_{m+1})$*, the* c_j*'s are the vertices of an* m-*simplex whose interior contains the origin;*

(iii) *for each nonzero* $z \in R^m$ *there exists* j *such that* $c^t z > 0$ *for all* $c \in Q(v_j)$.

Proof. Suppose first that (i) holds and let $c_j \in Q(v_j)$ for $1 \le j \le m + 1$. Let B be the $m \times (m + 1)$ matrix whose jth column is c_j. Since the rank of B is at most m, there exists a nonzero x such that $Bx = 0$. By (i), each such x has exclusively negative or exclusively positive coordinates, whence it follows readily that the set $\{c_1,\ldots,c_{m+1}\}$ is affinely independent and the origin is a strictly positive convex combination of the c_j's. But that is the content of (ii). Hence (i) implies (ii), and the argument is easily reversed to show (ii) implies (i).

If (iii) fails there exists $z \neq 0$ and there exist $c_1 \in Q(v_1),\ldots,$ $c_{m+1} \in Q(v_{m+1})$ such that $c_j^t z \le 0$ for all j. Plainly that contradicts (ii), and hence (ii) implies (iii). Suppose, finally, that (iii) holds and $c_j \in Q(v_j)$ for $1 \le j \le m + 1$. Let K denote the convex hull of the c_j's. It follows from (iii) that K intersects every open halfspace whose bounding hyperplane passes through the origin. But then the origin must be interior to K, and since K has at most $m + 1$ vertices it must in fact be an m-simplex. That shows (iii) implies (ii) and completes the proof of Theorem 1. $\square$

Now let us define a *W-matrix* as an $m \times n$ matrix A for which there exists $B \in Q(A)$ and nonzero $x \in R^n$ with $Bx \ge 0$ (that is, each coordinate of Bx is ≥ 0). An *NW-matrix* is one that is not a W-matrix. Condition (iii) above asserts the transpose A^t is an NW-matrix. In terms of the notions defined in the next paragraph, W may be regarded as standing for "weakly satisfiable". The connection with satisfiability in propositional logic is clarified in Section 3.

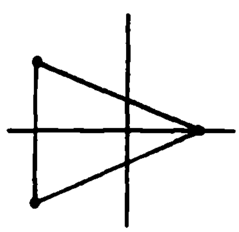
When $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$ are ordered n-tuples of real numbers, we say that x *hits* ⟨resp. *satisfies*⟩ y if there exists j such that $x_j y_j \neq 0$ ⟨resp. > 0⟩; and x *misses* y if x does not hit y. Further x *weakly satisfies* y if x satisfies y or x is nonzero but misses y. And x *weakly satisfies* the $m \times n$ matrix A if x weakly satisfies each row of A. Note that A is weakly satisfiable if and only if A is a W-matrix.

2. Construction of NW-matrices

In constructing $m \times n$ NW-matrices, we are especially interested in those for which $m = n + 1$, since they are the transposes of S-matrices. When $m \leq n$ there are no $m \times n$ NW-matrices because each set of n points in R^n lies in a closed halfspace whose bounding hyperplane passes through the origin.

If A is an NW-matrix then so is every matrix obtained from A by permuting rows, permuting columns, replacing columns by their negatives, and changing the magnitudes (but not the signs) of individual entries. These operations provide the natural equivalence relation for NW-matrices. Since only sign-patterns are involved, each equivalence class may be represented by various arrays of the symbols $-$, 0 and $+$. We take as the *canonical representative* the one which is lexicographically first when each matrix is considered as the sequence formed by writing down its successive rows. The lexicographic ordering is based on the ordering $- < 0 < +$ of the sign-symbols. Thus, for example, $\mp$ and $\pm$ are the two 2×1 NW-matrices and $\mp$ is the canonical representative because $-+$ precedes $+-$ in the lexicographic ordering. Fig. 1 shows the two canonical representatives of the 3×2 NW-matrices.

$$\begin{matrix} - & - \\ - & + \\ + & 0 \end{matrix}$$



$$\begin{matrix} - & - \\ 0 & + \\ + & 0 \end{matrix}$$

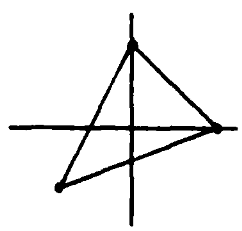


Fig. 1: The two canonical 3×2 NW-matrices and associated 2-simplices

By means of a computer search we have verified that there are precisely ten equivalence classes of 4×3 NW-matrices. Their canonical representatives are shown in Fig. 2.

$$
\begin{matrix} - & - & - \\ - & - & + \\ - & + & 0 \\ + & 0 & 0 \end{matrix}
\qquad
\begin{matrix} - & - & - \\ - & - & + \\ 0 & + & 0 \\ + & 0 & 0 \end{matrix}
\qquad
\begin{matrix} - & - & - \\ - & 0 & + \\ - & + & 0 \\ + & 0 & 0 \end{matrix}
\qquad
\begin{matrix} - & - & - \\ - & 0 & + \\ 0 & + & - \\ + & 0 & 0 \end{matrix}
\qquad
\begin{matrix} - & - & - \\ - & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{matrix}
$$

$$
\begin{matrix} - & - & - \\ 0 & 0 & + \\ 0 & + & 0 \\ + & 0 & 0 \end{matrix}
\qquad
\begin{matrix} - & - & 0 \\ - & 0 & - \\ 0 & + & + \\ + & 0 & 0 \end{matrix}
\qquad
\begin{matrix} - & - & 0 \\ - & + & 0 \\ 0 & 0 & - \\ + & 0 & + \end{matrix}
\qquad
\begin{matrix} - & - & 0 \\ - & + & 0 \\ + & 0 & - \\ + & 0 & + \end{matrix}
\qquad
\begin{matrix} - & - & 0 \\ 0 & 0 & - \\ 0 & + & 0 \\ + & 0 & + \end{matrix}
$$

Fig. 2: The ten canonical 4×3 NW-matrices

Each of the 3×2 classes can be constructed from the 2×1 class in the manner shown in Fig. 3, and all but the next-to-last 4×3 class arises from a 3×2 class in the same way. In Fig. 3, the new row has all entries 0 except for a single nonzero $s(-$ or $+)$ in the new column has s in the new row, $\neg s$ in some other position, and its remaining entries are all 0 or $\neg s$. (We agree that $\neg -$ is $+$ and $\neg +$ is $-$.)

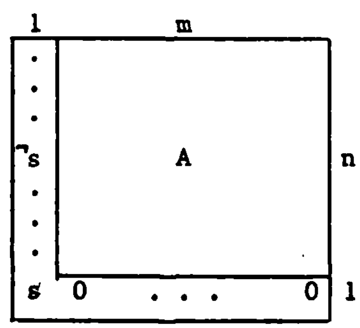


Fig. 3: Forming an $(m + 1) \times (n + 1)$ NW-matrix from an $m \times n$ NW-matrix A

With $m_1 = m_2 = k = 2$ and $n_1 = n_2 = \ell = 1$, the construction shown in Fig. 4 yields a 4×3 NW-matrix that is equivalent to the next-to-last one of Fig. 2. In general, one starts from a $k \times \ell$ NW-matrix C and

$m_i \times n_i$ NW-matrices A_i for $1 \le i \le k$ to obtain an $(m_1 + \ldots + m_k) \times (\ell + n_1 + \ldots + n_k)$ NW-matrix. Here the entries not in the small boxes are all 0 and for $1 \le i \le k$ each of the m_i rows of C_i is identical to the ith row of C. (It actually suffices to have each row of C_i either zero or identical to the ith row of C, with at least one row of the latter sort for each i.)

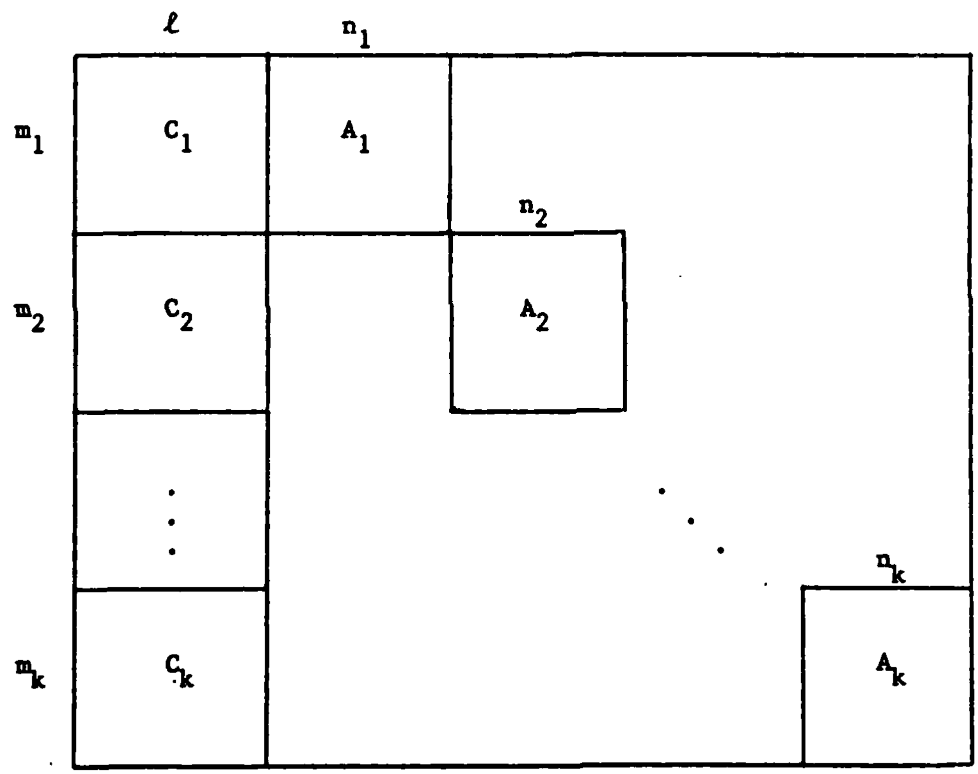


Fig. 4: Forming an NW-matrix from NW-matrices $C, A_1, \ldots, A_k$

The constructions of Figs. 3-4 are very special cases of a general construction based on partition-trees, which we now describe. A *partition-tree* for a set X is an ordered pair (T, Y) that satisfies the following conditions:

(a) T is a rooted tree in which each internal node has at least two sons;

(b) for each node j of T, $Y(j)$ is a nonempty subset of X ;

(c) for the root r of T, $Y(r) = X$;

(d) if i is an internal node of T and $S(i)$ is the set of all sons of i in T then $\{Y(j) : j \in S(i)\}$ is a partition of $Y(i)$ -- that is, $Y(j) \cap Y(j') = \emptyset$ for each choice of distinct $j, j' \in S(i)$, and $\cup_{j \in S(i)} Y(j) = Y(i)$.

For our purposes, no generality is lost by adding the requirement that X is a finite linearly ordered set and for each node j the set $Y(j)$ is an interval with respect to the ordering. Fig. 5 depicts a tree with root $r = 1$ and node-set $\{1,\ldots,8\}$, Fig. 6 an associated partition-tree for the set $X = \{1,\ldots,11\}$.

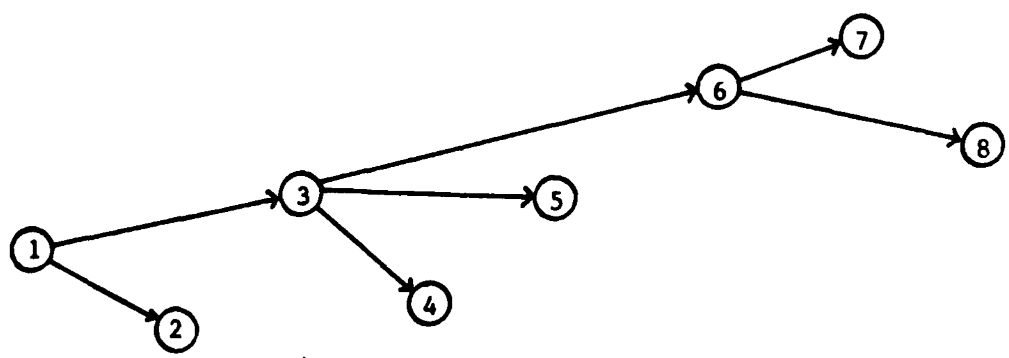


Fig. 5: A tree with root 1 and node-set $\{1,\ldots,8\}$

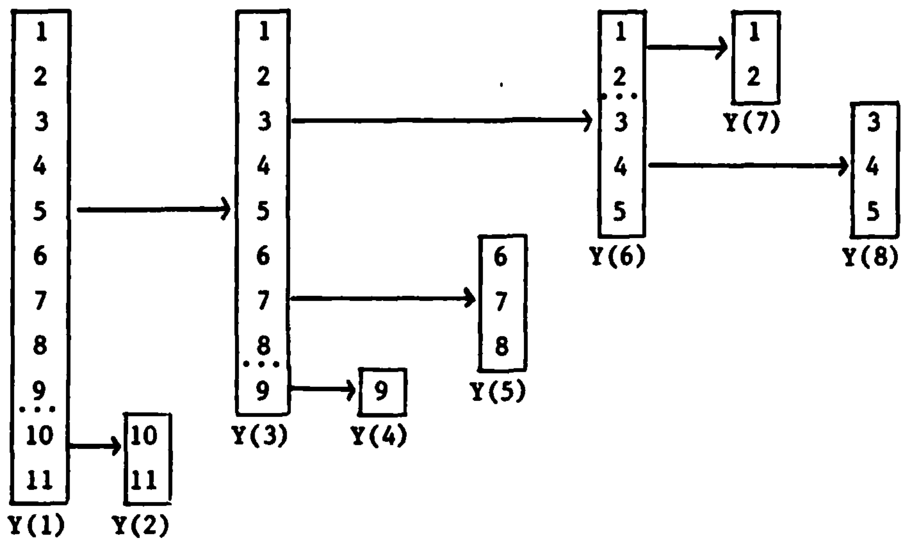


Fig. 6: A partition-tree for $\{1,\ldots,11\}$ associated with the tree of Fig. 5

THEOREM 3 *Suppose that* (T, Y) *is a partition-tree for the set* $\{1,\ldots,m\}$, I *is the set of all internal nodes of* T , *and* L *is the set of all leaves* ℓ *of* T *for which* $|Y(\ell)| \geq 2$. *For each* $i \in I$ *let* A_i *be an* $|S(i)| \times n_i$ *NW-matrix and for each* $\ell \in L$ *let* A_ℓ *be a* $|Y(\ell)| \times n_\ell$ *NW-matrix. Then the following construction yields an* $m \times (\Sigma_{j \in I \cup L} n_j)$ *NW-matrix* B .

(i) *For each* $\ell \in L$ *let* B *have* n_ℓ *columns which, in the rows corresponding to* $\{1,\ldots,m\} \sim Y(\ell)$ $\langle$*resp.* $Y(\ell)\rangle$ *have all entries* 0 $\langle$*resp. a copy of* $A_\ell\rangle$;

(ii) *For each* $i \in I$ *let* $R_1,\ldots,R_{|S(i)|}$ *be the sets* $Y(j)$ *for* $j \in S(i)$. *Let* B *have* n_i *columns which have all entries* 0 *in the rows corresponding to* $\{1,\ldots,m\} \sim Y(i)$ *and which satisfy the following condition for* $1 \leq k \leq S(i)$: *each row corresponding* *to* R_k *has all entries* 0 *or is a copy of the* k*th row of* A_i *, and there is (for each* k *) at least one such copy.*

Before proving Theorem 2, let us illustrate its use in the construction of NW-matrices. To obtain the construction of Fig. 3 from Theorem 2, let T and Y be as in Fig. 7, let A_1 be $\mp$ or $\pm$, and let A_2 be the A of Fig. 3. There is no A_3 because $|Y(3)| = 1$.

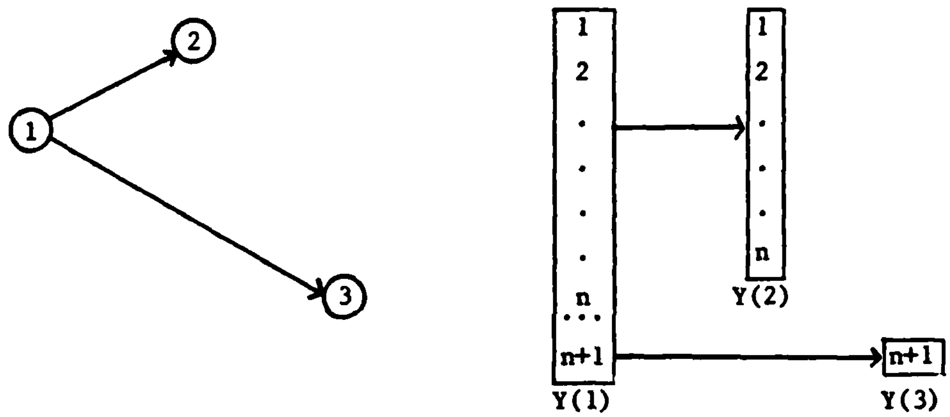


Fig. 7: Obtaining Fig. 3's construction as a special case of Theorem 2

To obtain the construction of Fig. 4 from Theorem 2, let T consist of a root 1 directly joined to nodes $2,\ldots,k+1$, let $\sigma_j = \sum_1^j m_i$ for $1 \le j \le k$, and let $Y(1) = \{1,\ldots,\sigma_k\}$, $Y(2) = \{1,\ldots,\sigma_1\},\ldots,$ $Y(k+1) = \{\sigma_{k-1}+1,\ldots,\sigma_k\}$. Let the $A_1, A_2, \ldots, A_k$ of Theorem 2 be respectively the $C, A_1, \ldots, A_k$ of Fig. 4.

For one more illustration, let (T, Y) be as in Figs. 5-6 and let $A_1 = A_2 = \mp$, $A_6 = A_7 = \pm$, $A_3 = A_5 = \begin{matrix} - & - \\ - & + \\ + & 0 \end{matrix}$, $A_8 = \begin{matrix} - & - \\ 0 & + \\ + & 0 \end{matrix}$.

Because of the freedom allowed in (ii) in forming the columns of B corresponding to internal nodes of T , the number of 11×10 NW-matrices that can be constructed in the manner of Theorem 2 with the specified T,

Y and A_i's is

$$(2^9 - 1)\ (2^2 - 1)\ (2^5 - 1)\ (2^3 - 1)\ (2^2 - 1)\ (2^3 - 1) = 6{,}999{,}552.$$

In particular, if no use is made of the freedom allowed in (ii) (that is, if each row corresponding to R_k is a copy of the kth row of A_i), the matrix of Fig. 8 is obtained.

$$\begin{array}{cccccccccc}
- & 0 & - & - & 0 & 0 & + & + & 0 & 0 \\
- & 0 & - & - & 0 & 0 & + & - & - & - \\
- & 0 & - & - & 0 & 0 & - & 0 & 0 & + \\
- & 0 & - & - & 0 & 0 & - & 0 & + & 0 \\
- & 0 & - & - & 0 & 0 & - & 0 & 0 & 0 \\
- & 0 & - & + & - & - & 0 & 0 & 0 & 0 \\
- & 0 & - & + & - & + & 0 & 0 & 0 & 0 \\
- & 0 & - & + & + & 0 & 0 & 0 & 0 & 0 \\
- & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Fig. 8: An 11×10 NW-matrix constructed by the method of Theorem 2

Proof of Theorem 2. Suppose there exists a matrix B which is weakly satisfiable even though it is generated by a partition-tree (T, Y) and by NW-matrices A_j in the manner of Theorem 2. Among all such B, consider one for which the number of nodes of an associated T is a minimum. With $n = \sum_{j\in I\cup L} n_j$, the number of columns of B, let $x = (x_1,\ldots,x_n)$ be an n-tuple that weakly satisfies B. Let H denote the set of all nodes h of T such that x hits some row of B in the portion of B that is generated by the node h and the NW-matrix A_h. Then H is nonempty, for all nonzero entries of B are generated in this way and no column of B is zero. Among all members of H, choose h_0 closest to the root r of T in the sense that the path from r to h_0 in T intersects H only at h_0. Let T_0 be the subtree of T that is rooted at h_0 and let B_0 be the submatrix of B that is associated with T_0. Then B_0 is weakly satisfied by the relevant portion of x, and since B_0 is also generated in the manner of Theorem 2 it follows from the minimality of T that $T_0 = T$ and $h_0 = r$. If r is a leaf of T then B is the NW-matrix A_r and an immediate contradiction ensues.

Now suppose that the root r is an internal node of T, and let x^* denote the restriction of x to the column associated with r. Since $x^* \neq 0$ and A_r is an NW-matrix, there exist r' and k such that (in the notation of (ii) of Theorem 2) $r' \in S(r)$, $1 \leq k \leq |S(r)|$, $R_k = Y(r')$, and x^* hits but does not satisfy the k^{th} row of A_r. Let T' denote the subtree of T that is rooted at r' and let B' denote the submatrix of B that is formed from the rows of B corresponding to R_k and the columns of B associated with nodes of T'. Let x' denote the restriction of x to these columns. From the fact that x weakly satisfies B while x^* hits but does not satisfy the k^{th} row of A_r it can be deduced that x' weakly satisfies B'. But B' is formed from T' and its associated $Y(j)$'s and A_j's in the manner of Theorem 2, contradicting the minimality of T and completing the proof. □

Gorman's 1964 conjecture [12] on the construction of S-matrices was rephrased in [18, 23]. In our terms, it amounts to saying that if $\underset{\sim}{M}$ is the class of all NW-matrices that have exactly one more row than columns, then each member of $\underset{\sim}{M}$ is equivalent to one that is generated in the manner of Theorem 2 by a partition-tree (T, Y) and a collection of NW-matrices A_j such that T is binary (each internal node has exactly two sons) and each A_j is $\mp$ or $\pm$. We have not settled this conjecture. More generally, let P denote any one of the following three construction procedures: (a) combining the method of Fig. 2 with that of Fig. 3; (b) restricting the method of Theorem 2 to binary trees; (c) using the method of Theorem 2 without restriction. Then we are unable to say whether, for an arbitrary member M of $\underset{\sim}{M}$ that has more than two rows, a member equivalent to M can be generated by applying P to members of $\underset{\sim}{M}$ that are smaller than M.

3. The Recognition of W-matrices

For an $m \times n$ matrix A, weak satisfiability can be tested in $O(3^n mn)$ steps by generating each of the $3^n - 1$ possibilities for the sign-pattern of nonzero $y \in R^n$ and, as each one is generated, testing to see whether it weakly satisfies each row of A. Similar procedures were suggested by Lancaster [20, 21] for similar purposes. Although these algorithms are finite and very easy to program, they are practical only for small values of n. However, we show below that for general m and n, and even for the cases in which $m = n + \lfloor n^{1/k} \rfloor$ for an arbitrary fixed $k > 0$, the problem of recognizing weak satisfiability is NP-complete and hence algorithms requiring an exponential number of steps may be unavoidable.

It will be convenient, for the rest of this section, to change the language from that of matrix theory to that of propositional logic. The n columns of an $m \times n$ matrix $A = (a_{ij})$ correspond to n propositional variables $u_1, \ldots, u_n$ and the m rows of A correspond to clauses $C_1, \ldots, C_m$ in the literals $u_1, \ldots, u_n, \bar{u}_1, \ldots, \bar{u}_n$, where $\bar{u}_j$ stands for $\neg u_j$. The matrix A is represented by a Boolean formula

$$B = C_1 \wedge C_2 \wedge \ldots \wedge C_m$$

in conjunctive normal form, where the clause C_i is the disjunction of literals obtained from the i^{th} row of A as follows:

> u_j appears in C_i if and only if $a_{ij} > 0$; $\bar{u}_j$ appears in C_i if and only if $a_{ij} < 0$; if $a_{ij} = 0$ then neither u_j nor $\bar{u}_j$ appears in C_i .

An *assignment* for the variables $u_1,\ldots,u_n$ is a function $\alpha : \{u_1,\ldots, u_n\} \to \{F, T\}$. A clause C_i is *satisfied* by α if there exists j such that u_j appears in C_i and $\alpha(u_j) = T$ or $\bar{u}_j$ appears in C_i and $\alpha(u_j) = F$. The formula B is *satisfied* by α if each clause C_i in B is satisfied by α. The *satisfiability problem* is that of recognizing formulas B that are satisfiable. NP-completeness of this problem was established by Cook [4], and that was the fundamental result from which the theory of NP-completeness was developed by Karp [14] and others (see Garey and Johnson [10] for an extensive survey). It seems very unlikely that the general satisfiability problem admits a polynomially bounded algorithm.

A *weak assignment* for the variables $u_1,\ldots,u_n$ is a function $\alpha : \{u_1, \ldots,u_n\} \to \{F, Z, T\}$ (where Z may be regarded as standing for "zero") such that for at least one j, $\alpha(u_j) \neq Z$. A clause C_i is *weakly satisfied* by α if C_i is satisfied by α or α misses C_i in the sense that $\alpha(u_j) = Z$ for all j such that u_j or $\bar{u}_j$ appears in C_i. The formula B is *weakly satisfied* by α if each C_i is weakly satisfied. The *weak satisfiability* problem is that of recognizing formulas B that are weakly satisfiable. Plainly a matrix A is weakly satisfiable (that is, A is a W-matrix) if and only if the corresponding Boolean formula B is weakly satisfiable.

Three examples serve to illustrate the above notions.

3×2 matrix	Boolean Formula	Properties
+ 0 + - - -	$(u_1) \wedge (u_1 \vee \bar{u}_2) \wedge (\bar{u}_1 \vee \bar{u}_2)$	Satisfied by $\alpha(u_1) = T$, $\alpha(u_2) = F$. Weakly satisfied (but not satisfied) by $\alpha(u_1) = Z$, $\alpha(u_2) = F$.

3 × 2 matrix	Boolean Formula	Properties
+ 0 0 + - 0	$(u_1) \wedge (u_2) \wedge (\bar{u}_1)$	Not satisfiable. Weakly satisfied by $\alpha(u_1) = Z$, $\alpha(u_2) = T$.
+ + - 0 0 -	$(u_1 \vee u_2) \wedge (\bar{u}_1) \vee (\bar{u}_2)$	Not weakly satisfiable.

We now show how to reduce the satisfiability problem to the weak satisfiability problem.

THEOREM 4 *Suppose that* $C_1,\ldots,C_m$ *are clauses in the* n *variables* $u_0,\ldots,u_{n-1}$ *and* $D_1,\ldots,D_m$ *are clauses in the* $2n$ *variables* $u_0,\ldots,u_{n-1}, v_0,\ldots,v_{n-1}$, *where* D_i *is obtained from* C_i *by replacing each occurrence of* $\bar{u}_j$ *with* v_j. *Form* $4n$ *additional clauses* E_j, F_j, G_j *and* H_j *as follows, where the subscript* j *ranges from* 0 *to* $m - 1$ *and* $j + 1$ *is reduced modulo* m :

$$E_j \text{ is } u_j \vee v_j \,,\; F_j \text{ is } \bar{u}_j \vee \bar{v}_j \,,$$

$$G_j \text{ is } \bar{u}_j \vee u_{j+1} \vee v_{j+1} \,,\; H_j \text{ is } \bar{v}_j \vee u_{j+1} \vee v_{j+1} \,.$$

For the formula $B = C_1 \wedge \ldots \wedge C_m$ *and the formula*

$$B' = D_1 \wedge \ldots \wedge D_m \wedge E_1 \wedge \ldots \wedge E_n \wedge F_1 \wedge \ldots \wedge F_n \wedge G_1 \wedge \ldots \wedge G_n \wedge H_1 \wedge \ldots \wedge H_n \,,$$

the following three conditions are equivalent: B *is satisfiable;* B' *is satisfiable;* B' *is weakly satisfiable.*

Proof. If B is satisfied by an assignment $\alpha : \{u_1,\ldots,u_n\} \to \{F, T\}$ then B' is satisfied by the assignment $\alpha' : \{u_1,\ldots,u_n, v_1,\ldots,v_n\} \to \{F, T\}$, where $\alpha'(u_j) = \alpha(u_j)$ and $\alpha'(v_j) = \neg\alpha(u_j)$. Conversely, if α'

is any assignment that satisfies B' then the restriction of α' to $\{u_1,\ldots,u_n\}$ satisfies B because from the satisfaction of E_j and F_j if follows that α' assigns complementary values to u_j and v_j.

To complete the proof we show that if $\beta : \{u_1,\ldots,u_n, v_1,\ldots,v_n\} \to \{F, Z, T\}$ is a weak assignment that weakly satisfies B' then Z is missing from the range of β, whence β is in fact an assignment that satisfies B'.

By the definition of weak assignment, there exists k such that $\beta(u_k) \neq Z$ or $\beta(v_k) \neq Z$. For any such k, if $\beta(u_k) = T$ then $\beta(v_k) = F$ by F_k, while if $\beta(u_k) = F$ then $\beta(v_k) = T$ by E_k. Similarly, if $\beta(v_k) = T$ then $\beta(u_k) = F$ by F_k, while if $\beta(v_k) = F$ then $\beta(u_k) = T$ by E_k.

It remains only to show, by induction on j, that if $\beta(u_k) \in \{T, F\}$ then $\beta(u_{k+j}) \in \{T, F\}$ for all j. But if $\beta(u_{k+j}) = T$ then by G_{k+j} one of u_{k+j+1} and v_{k+j+1} must be assigned T, whence the other is assigned F by the observation of the preceding paragraph. And if $\beta(u_{k+j}) = F$ then $\beta(v_{k+j}) = T$, whence by H_{k+j} one of u_{k+j+1} and v_{k+j+1} must be assigned T; hence the other is assigned F. That completes the induction and the proof. $\square$

For each $k > 0$, let the decision problem $WSAT_k$ be as follows:

Instance: A Boolean formula B in conjunctive normal form, involving a total of n propositional variables and consisting of the conjunction of $n + \lfloor n^{1/k} \rfloor$ disjunctive clauses;

Question: Is B weakly satisfiable?

THEOREM 5 *For each* $k > 0$ *the problem* $WSAT_k$ *is* NP-*complete.*

Proof. Plainly $WSAT_k$ belongs to NP , so it suffices to describe a polynomial reduction of SAT (the usual satisfiability problem) to $WSAT_k$. Consider an instance of SAT that concerns a Boolean formula B in conjunctive normal form, involving a total of n propositional variables and consisting of the conjunction of m disjunctive clauses. Let the variables be $u_0,\dots,u_{n-1}$, introduce additional variables $v_0,\dots,v_{n-1}$, and let the formula B' be as in Theorem 4. Let $r = (m + 2n + 1)^k$, introduce additional variables $w_0,\dots,w_{r-1}$, and let B^* be a not-weakly-satisfiable formula consisting of the conjunction of $r + 1$ disjunctive clauses in these variables (easily constructed with the aid of Theorem 1). Finally, let B'' denote the conjunction of B' and B^* . Then B'' consists of $m + 4n + r + 1$ clauses and involves a total of $2n + r$ variables. Since the inequality

$$m + 4n + r + 1 < (2n + r) + (2n + r)^{1/k}$$

is equivalent to the valid inequality

$$(m + 2n + 1)^k < 2n + r ,$$

some of the clauses of B'' can be repeated if necessary to produce an instance B''' of $WSAT_k$. From Theorem 4 and the fact that B^* is not weakly satisfiable, it follows that B is satisfiable if and only if B''' is weakly satisfiable. □

As can be seen from the references in Garey and Johnson [10], there are several other relatives of the satisfiability problem for which NP-completeness has been established. See especially Schaefer [30].

4. The Recognition of S-systems

As the term is used here, a *cone* is a nonempty subset C of R^m such that

$$0 \notin C = C + C =]0,\infty[C .$$

Thus C omits the origin and is closed under vector addition and multiplication by positive scalars; in particular, C is convex. A cone is *polyhedral* if it is defined by a finite system of linear inequalities. An S-*system* is a sequence $(C_1,\ldots,C_{m+1})$ of cones in R^m such that for each $m \times (m+1)$ matrix B whose k^{th} column belongs to C_k $(1 \le k \le m+1)$ and for each $x \in R^{m+1}$ with $Bx = 0$, all coordinates of x are of the same sign. Thus an $m \times (m+1)$ matrix with columns $V_1,\ldots,V_{m+1}$ is an S-matrix if and only if the sequence $(Q(V_1),\ldots,Q(V_{m+1}))$ is an S-system.

Despite the NP-completeness result of Section 3 and the close relationship between S-matrices and weak satisfiability, it can be determined in polynomial time whether a given sequence of polyhedral cones is an S-system. The algorithm is based on the following characterization.

THEOREM 6 *If* $C_1,\ldots,C_{m+1}$ *are cones in* R^m *and* $p_k \in C_k$ *for each* k , *then the following four conditions are equivalent:*

(i) $(C_1,\ldots,C_{m+1})$ *is an* S-*system*;

(ii) *for each choice of* $c_1 \in C_1,\ldots,c_{m+1} \in C_{m+1}$, *the* c_k*'s are the vertices of an* m-*simplex whose interior contains the origin;*

(iii) *for each nonzero* $z \in R^m$ *there exists* k *such that* $c^t z > 0$ *for all* $c \in C_k$;

(iv) *for each choice of distinct* r , $s \in \{1,\ldots,m+1\}$ *the linear hull of the set* $\{p_k : r \neq k \neq s\}$ *is a hyperplane that strictly separates* C_r *from* C_s .

Proof. Conditions (i)-(iii) are merely the extensions, to general cones, of the corresponding conditions of Theorem 2. The reasoning of Theorem 2 applies here as well. Our algorithm is based on the equivalence between (ii) and (iv), which will now be established.

If (ii) holds and $c_k \in C_k$ for each k then the points c_k are affinely independent and the affine hull of any m of them is a hyperplane in R^m that misses the origin. This implies that each m of the c_k's form a linearly independent set, whence the linear hull of any $m - 1$ of them is a hyperplane through the origin. The remaining two c_k's are on opposite sides of that hyperplane, for the origin is interior to the convex hull of the c_k's . It follows that (ii) implies (iv).

Now suppose that (iv) holds, let Γ denote the set of all $(m+1)$-tuples $(c_1,\ldots,c_{m+1})$ such that $c_k \in C_k$ for all k , and let Γ^* denote the set of all $(m + 1)$-tuples in Γ such that for each choice of distinct r , $s \in \{1,\ldots,m + 1\}$ the linear hull of the set $\{c_k : r \neq k \neq s\}$ is a hyperplane that strictly separates C_r from C_s . Then $(p_1,\ldots,p_{m+1}) \in \Gamma^*$ by (iv), and we claim that in fact $\Gamma^* = \Gamma$. To prove this it suffices to show that if $(c_1,\ldots,c_{m+1}) \in \Gamma^*$, $1 \leq \ell \leq m + 1$, and D_ℓ is the set of all $d \in C_\ell$ such that Γ^* includes the $(m + 1)$-tuple obtained from $(c_1,\ldots,c_{m+1})$ by replacing c_ℓ with d , then $D_\ell = C_\ell$.

For each $e \in C_\ell$ and each $r \in \{1,\ldots,m + 1\} \sim \{\ell\}$ let

$$X_r(e) = \{e\} \cup \{c_k : k \notin \{\ell, r\}\},$$

and for each $s \in \{1,\ldots,m + 1\} \sim \{r, s\}$ let

$$X_{rs}(e) = \{e\} \cup \{c_k : k \notin \{\ell, r, s\}\} .$$

The set $X_r(e)$ is linearly independent because the relevant c_k's are linearly independent and their linear hull misses e . Hence the set $X_{rs}(e)$ is linearly independent and its linear hull is a hyperplane $H_{rs}(e)$ through the origin. To show $e \in D_\ell$ it remains to show that $H_{rs}(e)$ strictly separates C_r from C_s. Since $c_\ell \in D_\ell$, such separation does occur when $e \in]0,\infty[c_\ell$. Hence if the desired separation fails to occur for the e in question, there exists $d \in]c_\ell, e[\sim]0,\infty[c_\ell$ such that $H_{rs}(d)$ intersects C_r or C_s. We suppose without loss of generality that $H_{rs}(d)$ intersects C_r. Then there exists $b \in C_r$ and there exist scalars λ_k such that

$$b = \lambda_\ell d + \sum_{k \notin \{\ell, r, s\}} \lambda_k c_k .$$

Also, since the linear hull of $\{c_k : r \neq k \neq s\}$ is a hyperplane that strictly separates C_r from C_s, there are scalars μ_k such that

$$b = \mu_\ell c_\ell + \sum_{k \notin \{r,s\}} \mu_k c_k .$$

Plainly $\lambda_\ell < 0$ and $\mu_\ell < 0$. Since the set $X_r(b)$ is linearly independent it follows from the two equations that $d \in]0,\infty[c_\ell$, a contradiction showing that $D_\ell = C_\ell$. It follows, then, that $\Gamma^* = \Gamma$.

Now consider an arbitrary $(m + 1)$-tuple $(c_1,\ldots,c_{m+1}) \in \Gamma = \Gamma^*$ and an arbitrary choice of scalars γ_k such that $\sum_1^{m+1} \gamma_k c_k = 0$. If there exist r and s such that $\gamma_r \leq 0 \leq \gamma_s$ then all γ_k's are 0, as follows from the equation

$$\gamma_s c_s = (-\gamma_r) c_r - \sum_{r \neq k \neq s} \gamma_k c_k$$

in conjunction with the fact that the linear hull of $\{c_k : r \neq k \neq s\}$ is a hyperplane that strictly separates c_r from c_s. This condition on the γ_k's implies the c_k's are affinely independent and are in fact the vertices of an m-simplex whose interior contains the origin. Hence (iv) implies (ii). $\square$

The following theorem applies to arbitrary cones but may be difficult to implement if the cones are not polyhedral.

THEOREM 7 *The following procedure decides correctly whether a given sequence* $(C_1,\dots,C_{m+1})$ *of cones in* R^m *is an* S-*system:*

(1) *Select points* $p_1 \in C_1,\dots,p_{m+1} \in C_{m+1}$.

(2) *Decide whether there exists a nonzero* $z \in R^m$ *such that* $p_k^t z \geq 0$ *for all* k . *If so, stop;* $(C_1,\dots,C_{m+1})$ *is not an* S-*system.*

(3) *For each* r,s *with* $1 \leq r < s \leq m+1$, *form the linear hull* H_{rs} *of* $\{p_k : r \neq k \neq s\}$. *Decide whether* H_{rs} *intersects* C_r *and whether* H_{rs} *intersects* C_s . *If a nonempty intersection is found, stop;* $(C_1,\dots,C_{m+1})$ *is not an* S-*system* .

(4) *If* $C_r \cap H_{rs} = \emptyset = C_s \cap H_{rs}$ *for all* $r < s$ *then* $(C_1,\dots,C_{m+1})$ *is an* S-*system* .

Proof. If there exists a z as in (2) then $\{p_1,\dots,p_{m+1}\}$ lies in a closed halfspace whose bounding hyperplane passes through the origin. Hence the origin is not interior to the convex hull of the p_k's and $(C_1,\dots,C_{m+1})$ is not an S-system. If no such z exists then each m of the points $p_1,\dots,p_{m+1}$ are linearly independent and each H_{rs} is a hyperplane. If H_{rs} intersects C_r or C_s, we see as in the proof of Theorem 6 that $(C_1,\dots,C_{m+1})$ is not an S-system. But if, in step (3), the hyperplane H_{rs} misses both C_r and C_s then it strictly separates them; for otherwise C_r and C_s lie together in a closed halfspace bounded by H_{rs} , and a z would have been found (and the computation halted) in step (2). Thus if the computation progresses past step (3), it follows from Theorem 6 that $(C_1,\dots,C_{m+1})$ is an S-system. $\square$

When the cones C_k are polyhedral, the selection of the p_k's in step (1) and the consistency tests in steps (2) and (3) can be carried out by means of various linear programming methods. In particular, Phase I of Dantzig's simplex method [5] is applicable, as in the recent Shor-Khachian method [15, 31]. In view of numerous reports of computational experience with these two methods [3, 5, 6, 25, 32], we may say that each leads to a "good" algorithm for recognizing polyhedral S-systems. The recognition algorithm based on the simplex method is good in the sense that it is usually very efficient in practice, even though its worst-case behavior is not polynomially bounded [16]. The recognition algorithm based on the Shor-Khachian method is good in the sense that it is polynomially bounded (see Theorem 8), even though it is usually very inefficient in practice. It would be very nice, for the recognition of polyhedral S-systems as well as for solving linear programming problems, to have an algorithm that is good in both senses!

THEOREM 8 *By using the procedure of Theorem 7 in conjunction with the Russian method for linear inequalities, the problem of recognizing polyhedral S-systems can be solved in polynomial time and space on a deterministic Turing machine. Specifically, when each of the cones* $C_1,\ldots,C_{m+1}$ *in* R^m *is given by a system of linear inequalities with integer coefficients and* E *is the length of the binary encoding of the input data, the algorithm decides in time* $O(m^5E)$ *whether* $(C_1,\ldots,C_{m+1})$ *is an S-system.*

Proof. It follows from Khachian's analysis [15] of the Shor-Khachian algorithm [15, 31] that a polynomially bounded algorithm for recognizing polyhedral S-systems is obtained if one implements the procedure of Theorem 7 by suitable use of the SK-algorithm to find the points p_k and to make the required consistency tests. Khachian's paper [15] is an extended abstract, without proofs, but proofs and various improvements have appeared in technical reports issued subsequently by many operations researchers, computer scientists and mathematicians. We mention Gács and Lovász [8], Aspvall and Stone [1], and especially Padberg and Rao [26, 27] who have an improvement of the SK-algorithm that they call the *Russian method for linear inequalities*. Our discussion is based on that of Padberg and Rao.

In order to handle both strong and weak linear inequalities, let us assume that for each k there are finite sets Y_k and Z_k of nonzero lattice points in R^m such that

$$C_k = \{x \in R^m : y^t x > 0 \text{ for all } y \in Y_k \text{ and } z^t x \geq 0 \text{ for all } z \in Z_k\} .$$

The set Z_k may be empty, but we may assume Y_k is nonempty for otherwise $0 \in C_k$ and $(C_1,\ldots,C_{n+1})$ is not an S-system. Equality constraints $w^t x = 0$ are handled, as usual, by including both w and $-w$ in the set Z_k. In particular, if C_k is of the form $Q(V_k)$, $\{u_1,\ldots,u_m\}$ is the standard basis for R^m, and ℓ of V_k's coordinates are 0, then Y_k includes u_i or $-u_i$ for $m - \ell$ values of i and Z_k includes both u_i and $-u_i$ for the remaining ℓ values of i.

For each k let $n_k = |Y_k| + |Z_k|$, let b^k be the point of R^{n_k} whose first $|Y_k|$ coordinates are 1 and last $|Z_k|$ coordinates are 0, let $A^k = (a^k_{ij})$ be the $n_k \times m$ matrix whose first $|Y_k|$ rows are the vectors y^t for $y \in Y_k$ and last $|Z_k|$ rows are the vectors z^t for $z \in Z_k$, and let

$$D_k = \{x \in R^m : A^k x \geq b^k\} \subset C_k .$$

Note that a linear subspace of R^m intersects C_k if and only if it intersects D_k. Thus it is permissible, in following the procedure described in Theorem 7, to work with the D_k's rather than the C_k's.

If we adopt the special convention that $\log_2 0 = 0$, the length of the binary encoding of the description of D_k (or of C_k) may be taken as

$$E_k = 1 + \lfloor \log_2 m \rfloor + L_k ,$$

where

$$L_k = 2 + \lfloor \log_2 |Y_k| \rfloor + \lfloor \log_2 n_k \rfloor + 2mn_k + \sum_{i=1}^{n_k} \sum_{j=1}^{m} \lfloor \log_2 |a_{ij}^k| \rfloor .$$

Here $|Y_k|$ denotes cardinality and $|a_{ij}^k|$ denotes absolute value. We assume that for each integer x, $1 + \log_2 \lfloor x \rfloor$ bits are used to represent $|x|$ and an additional bit represents the sign of x if the problem's formulation does not automatically require $x > 0$ (as it does when x is m or n_k). The length of the encoding of the description of the system $(C_1, \ldots, C_{m+1})$ is defined as

$$E = 1 + \lfloor \log_2 m \rfloor + \sum_{k=1}^{m+1} L_k .$$

It is easy to quibble in minor ways with these definitions of the E_k's and of E, but such quibbles do not affect the validity of our theorem.

Let

$$F_k = 1 + \lfloor \log_2 m \rfloor + \Delta_k ,$$

where Δ_k is the maximum of the absolute values of the determinants of the square submatrices of the matrix $[A^k, b^k]$, and let

$$D_k' = \{x \in D_k : |x_i| \le \frac{1}{2m} 2^{F_k} \text{ for all } i \} .$$

It follows from the discussion of Padberg and Rao [26, 27] that $F_k \le E_k$, nonemptiness of D_k implies nonemptiness of D_k' , and the Russian method yields a point $p_k \in D_k'$ in time (and space) $O(m^3 F_k)$. Thus the complexity of step (1) in the procedure of Theorem 7 is

$$O(m^3 \textstyle\sum_{k=1}^{m+1} F_k) \le O(m^3 E) .$$

With $p_k \in D_k'$ for $1 \le k \le m + 1$, the consistency problem of step (2) admits a binary encoding of length $O(E)$ and hence step (2) is handled in time $O(m^3 E)$ by the Russian method.

To form the linear hull H_{rs} of $\{p_k : r \ne k \ne s\}$, as required by step (3), let M_{rs} be the $(m - 1) \times m$ matrix whose columns are the p_k's for $r \ne k \ne s$; then use the Russian method (or elementary column operations) to find a nonzero $q_{rs} \in R^m$ such that $q_{rs}^t M_{rs} = 0$ and all coordinates of q_{rs} are less than E in absolute value. This can be done in time $O(m^3 E)$, and then

$$H_{rs} = \{x \in R^m : q_{rs}^t x = 0\} .$$

Testing whether H_{rs} intersects C_r or C_s is also of complexity $O(m^3 E)$ if the Russian method is used. Since there are $m(m + 1)/2$ pairs (r, s) to contend with, the overall complexity of step (3) and of the entire algorithm is $O(m^5 E)$. □

5. Open Problems

If each $(n + 1) \times n$ NW-matrix can be generated from smaller ones by the method of Theorem 2, then each S-matrix has at least one row in which there are at most two nonzero entries. Gorman has conjectured that this is so and can be used as the basis of a fast algorithm for recognizing S-matrices.

Let $\Phi(m, n)$ denote the number of equivalence classes of $m \times n$ NW-matrices. What can be said about the asymptotic behavior of the function Φ, or about its values for small (m, n) ? There is special interest in $\Phi(n + 1, n)$, the number of equivalence classes of $n \times (n + 1)$ S-matrices. We have seen that $\Phi(2,1) = 1$, $\Phi(3,2) = 2$, $\Phi(4,3) = 10$. What is $\Phi(5,4)$?

Consider the problem of recognizing whether a given $m \times n$ matrix is an NW-matrix. As we have seen, this problem can be solved in polynomial time when $m = n + 1$ but is NP-complete when $m = n + \lfloor n^{1/k} \rfloor$ for an arbitrary fixed $k > 0$. What happens when $m = n + k$ for a fixed $k > 1$?

For each n let $h(n)$ denote the smallest integer r that has the following property: whenever a Boolean formula is in conjunctive normal form, involves a total of n propositional variables, and is not weakly satisfiable, then some subformula, consisting of the conjunction of $h(n)$ or fewer of the clauses of the original formula, is not weakly satisfiable. It is easy to see that $2n \le h(n) \le 3^n - 1$. Plainly $h(1) = 2$ and it can be verified that $h(2) = 4$. Is $h(n)$ always equal to $2n$?

For each $x \in R^n \sim \{0\}$, let $U(x)$ denote the set of all unit vectors $u \in R^n$ such that u weakly satisfies c. Since $U(x)$ depends only $Q(x)$, there are only $3^n - 1$ sets of the form $U(x)$ for $x \in R^n \sim \{0\}$. For example, when $n = 2$ there are four sets like each of the samples shown in Fig. 9.

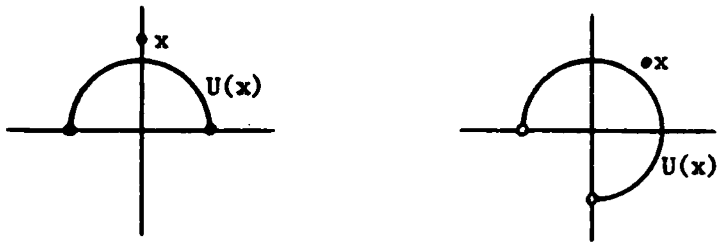


Fig. 9: The two types of sets $U(x)$ for $x \in R^2 \sim \{0\}$

Let us use the term U_n-*set* to denote a set of the form $U(x)$ for $x \in R^n \sim \{0\}$. Then $h(n)$ is the smallest k such that whenever each k members of a family of U_n-sets have a common point, the entire family has nonempty intersection. Thus $h(n)$ is a Helly number in the sense of Danzer, Grünbaum and Klee [7]. It seems possible that methods similar to those of Grünbaum and Motzkin [13] would be useful in determining $h(n)$.

For each cone K in R^n , let $W(K)$ denote the set of all $w \in R^n \sim \{0\}$ such that $w^t k \geq 0$ for each $k \in K$. For each family $\underset{\sim}{K}$ of cones, let $\underset{\sim}{W}(\underset{\sim}{K}) = \{W(K) : K \in \underset{\sim}{K}\}$ and let $H(\underset{\sim}{K})$ denote the Helly number of $\underset{\sim}{W}(\underset{\sim}{K})$. Then $h(n)$ is $H(\{Q(x) : x \in R^n \sim \{0\}\})$. What can be said about $H(\underset{\sim}{K})$ for other interesting choices of $\underset{\sim}{K}$?

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