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SPECTRAL ANALYSIS OF SOUND PROPAGATION IN STRATIFIED FLUIDS.(U)
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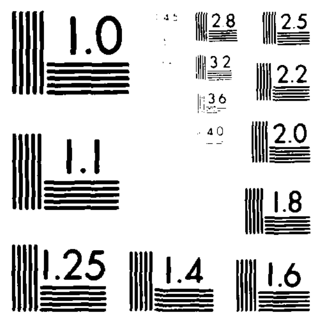

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SPECTRAL ANALYSIS OF SOUND PROPAGATION IN STRATIFIED FLUIDS

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Contents

Abstract.

A spectral analysis and normal mode expansions are developed for the acoustic propagator

$$Au = -c^2(y)\ \rho(y)\ \nabla \cdot (\rho^{-1}(y)\ \nabla u)$$

of a stratified fluid with sound speed $c(y)$ and density $\rho(y)$ at depth y. For an infinite fluid it is assumed that the (in general discontinuous) functions $c(y)$, $\rho(y)$ are uniformly positive and bounded and satisfy

$$\pm\int_0^{\pm\infty} |c(y) - c(\pm\infty)|\ dy < \infty,\ \pm\int_0^{\pm\infty} |\rho(y) - \rho(\pm\infty)|\ dy < \infty.$$

Semi-infinite and finite fluid layers are also treated. The work provides a basis for the analysis of transient and steady-state sound fields in such fluids.

§1. Introduction.

This paper presents a spectral analysis of the acoustic fields in stationary plane stratified fluids whose densities and sound speeds are functions of the depth. The analysis is based on families of normal mode fields that have simple physical interpretations.

The acoustic field in such a fluid may be described by an acoustic potential or by the excess pressure. Each of these is a real valued function $u(t,x,y)$ that satisfies the wave equation [4, 13]

$$\frac{\partial^2 u}{\partial t^2} - c^2(y)\,\rho(y)\,\nabla\cdot(\rho^{-1}(y)\,\nabla u) = 0 \tag{1.1}$$

where t is a time coordinate, $x = (x_1,x_2)$ are rectangular coordinates in a horizontal plane, y is a vertical depth coordinate and $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial y)$. $c(y)$ and $\rho(y)$ are the variable sound speed and density, respectively, and $\rho^{-1}(y) = 1/\rho(y)$.

The paper presents a spectral analysis of the acoustic propagator

$$Au = -c^2(y)\,\rho(y)\,\nabla\cdot(\rho^{-1}(y)\,\nabla u) \tag{1.2}$$

for the cases of an unlimited fluid ($(x_1,x_2,y) \in R^3$), a semi-infinite layer ($(x_1,x_2) \in R^2$, $0 < y < \infty$) and a finite layer ($(x_1,x_2) \in R^2$, $0 < y < h$). Only the first case is presented in detail. The modifications required in the second and third cases are described in §9 at the end of the paper.

In the case of an unlimited fluid $\rho(y)$ and $c(y)$ will be assumed to be Lebesgue measurable functions that satisfy

$$(1.3)\qquad 0 < \rho_m \le \rho(y) \le \rho_M < \infty,\ 0 < c_m \le c(y) \le c_M < \infty,$$

for all $y \in R$, and

$$(1.4)\qquad \pm\int_0^{\pm\infty} |\rho(y) - \rho(\pm\infty)|\ dy < \infty,\ \pm\int_0^{\pm\infty} |c(y) - c(\pm\infty)|\ dy < \infty,$$

where ρ_m, ρ_M, c_m, c_M, $\rho(\pm\infty)$ and $c(\pm\infty)$ are constants. It is clear that (1.3), (1.4) imply

$$(1.5)\qquad \rho_m \le \rho(\pm\infty) \le \rho_M,\ c_m \le c(\pm\infty) \le c_M.$$

The spectral analysis of A will be based on the observation that it is formally selfadjoint with respect to the scalar product defined by

$$(1.6)\qquad (u,v) = \int_{R^3} \overline{u(x,y)}\ v(x,y)\ c^{-2}(y)\ \rho^{-1}(y)\ dxdy\ ,$$

where $\overline{u}$ is the complex conjugate of u. This suggests the introduction of the Hilbert space

$$(1.7)\qquad \mathcal{H} = L_2(R^3, c^{-2}(y)\ \rho^{-1}(y)\ dxdy)$$

with scalar product (1.6). Note that (1.3) implies that $\mathcal{H}$ is equivalent as a normed space to the usual Lebesgue space $L_2(R^3)$, although they are distinct as Hilbert spaces.

A selfadjoint realization of A in $\mathcal{H}$ is obtained by defining the domain of A in $\mathcal{H}$ to be

$$(1.8)\qquad D(A) = L_2^1(R^3) \cap \{u \mid \nabla \cdot (\rho^{-1}(y)\ \nabla u) \in L_2(R^3)\}$$

where

$$L_2^1(R^3) = L_2(R^3) \cap \{u \mid \nabla u \in L_2(R^3)\} \tag{1.9}$$

is the usual first Sobolev space [1]. All the differential operations in (1.8), (1.9) are to be understood in the sense of the theory of distributions. The linear operator in $\mathcal{H}$ defined by (1.2), (1.8) satisfies

$$A = A^* \geq 0 \tag{1.10}$$

where A^* is the adjoint of A with respect to the scalar product (1.6). A proof of (1.10) may be given by the method employed in [17]. Alternatively, (1.10) may be verified by showing that A is the operator in $\mathcal{H}$ associated with the sesquilinear form $\mathcal{A}$ in $\mathcal{H}$ defined by

$$D(\mathcal{A}) = L_2^1(R^3) \subset \mathcal{H} \tag{1.11}$$

and

$$\mathcal{A}(u,v) = \int_{R^3} \nabla \bar{u} \cdot \nabla v \, \rho^{-1}(y) \, dxdy; \tag{1.12}$$

see [8, p. 322].

The principal result of this report is a construction of the spectral family of A by means of a normal mode expansion. The spectrum of A is continuous and contains no imbedded eigenvalues. This fact, which is verified below, implies that the normal mode functions of A must be generalized eigenfunctions; that is, solutions ψ of the differential equation

$$-c^2(y)\, \rho(y)\, \nabla \cdot (\rho^{-1}(y)\, \nabla\psi(x,y)) = \lambda\psi(x,y) \tag{1.13}$$

that are not in $\mathcal{H}$. Solution of (1.13) by separation of variables leads to solutions of the form

$$\psi(x,y) = e^{ip\cdot x}\,\psi(y),\; p = (p_1,p_2) \in R^2, \tag{1.14}$$

where $p \cdot x = p_1x_1 + p_2x_2$ and $\psi(y)$ is a solution of the equation

$$-c^2(y)\left[\rho(y)\,\frac{d}{dy}\left(\rho^{-1}(y)\,\frac{d\psi}{dy}\right) - |p|^2\psi\right] = \lambda\psi \tag{1.15}$$

with $|p|^2 = p_1^2 + p_2^2$.

The operator A_μ defined by

$$A_\mu\,\psi = -c^2(y)\left[\rho(y)\,\frac{d}{dy}\left(\rho^{-1}(y)\,\frac{d\psi}{dy}\right) - \mu^2\psi\right] \tag{1.16}$$

will be called the reduced acoustic propagator. For every $\mu \geq 0$, A_μ has a selfadjoint realization in the Hilbert space

$$\mathcal{H}(R) = L_2(R, c^{-2}(y)\,\rho^{-1}(y)\,dy). \tag{1.17}$$

The domain of A_μ is the set

$$D(A_\mu) = L_2^1(R) \cap \left\{\psi \,\middle|\, \frac{d}{dy}\left(\rho^{-1}(y)\,\frac{d\psi}{dy}\right) \in L_2(R)\right\}. \tag{1.18}$$

The properties

$$A_\mu = A_\mu^* \geq c_m^2\,\mu^2 \tag{1.19}$$

can be verified by showing that A_μ is the operator in $\mathcal{H}(R)$ associated with the sesquilinear form A_μ in $\mathcal{H}(R)$ defined by

$$D(A_\mu) = L_2^1(R) \subset \mathcal{H}(R) \tag{1.20}$$

$$A_\mu(\phi,\psi) = \int_R \left\{\frac{d\bar{\phi}}{dy}\frac{d\psi}{dy} + \mu^2\,\bar{\phi}\,\psi\right\} \rho^{-1}(y)\,dy. \tag{1.21}$$

The spectral analysis of A will be derived from that of A_μ. The main steps of the analysis are the following. First, hypotheses (1.3) and (1.4) are used to construct solutions of $A_\mu\,\phi = \lambda\phi$ that have prescribed asymptotic behaviors for $y \to \pm\infty$. Second, these solutions are used to construct an eigenfunction expansion for A_μ. The construction is based on the Weyl-Kodaira theory of singular Sturm-Liouville operators. Finally, the expansion for A_μ and Fourier analysis in the variables x_1, x_2 are used to construct a spectral representation for A. This method has been applied to the special cases of the Pekeris and Epstein profiles [5, 17] where explicit representations of the solutions of $A_\mu\,\phi = \lambda\phi$ by means of elementary functions are available. Thus the main technical advance in the present work is the construction, for the class of density and sound speed profiles defined by (1.3) and (1.4) of solutions of $A_\mu\,\phi = \lambda\phi$ that have prescribed asymptotic behaviors for $y \to \pm\infty$ and sufficient regularity in the parameters λ and μ to permit application of the methods of [5, 17].

The remainder of the introduction contains a description of the eigenfunctions and generalized eigenfunctions of A_μ, the corresponding normal mode functions for A and their physical interpretations.

The limiting form for $y \to \pm\infty$ of the equation $A_\mu\,\phi = \lambda\phi$ is the equation

$$\frac{d^2\phi}{dy^2} + (\lambda c^{-2}(\pm\infty) - \mu^2)\,\phi = 0 \tag{1.22}$$

whose general solution when $\lambda \neq c^2(\pm\infty)\mu^2$ can be written

$$\begin{cases} \phi = c_1\, e^{q_\pm'(\mu,\lambda)y} + c_2\, e^{-q_\pm'(\mu,\lambda)y}, \text{ or} \\ \\ \phi = c_1\, e^{iq_\pm(\mu,\lambda)y} + c_2\, e^{-iq_\pm(\mu,\lambda)y} \end{cases} \tag{1.23}$$

where

$$\left.\begin{cases} q_\pm'(\mu,\lambda) = (\mu^2 - \lambda c^{-2}(\pm\infty))^{1/2} > 0 \\ \\ q_\pm(\mu,\lambda) = -iq_\pm'(\mu,\lambda) \end{cases}\right\} \text{for } \lambda < c^2(\pm\infty)\mu^2, \tag{1.24}$$

and

$$\left.\begin{cases} q_\pm'(\mu,\lambda) = i\, q_\pm(\mu,\lambda) \\ \\ q_\pm(\mu,\lambda) = (\lambda\, c^{-2}(\pm\infty) - \mu^2)^{1/2} > 0 \end{cases}\right\} \text{for } \lambda > c^2(\pm\infty)\mu^2. \tag{1.25}$$

In particular, the solutions are oscillatory when $\lambda > c^2(\pm\infty)\mu^2$ and non-oscillatory when $\lambda < c^2(\pm\infty)\mu^2$. The first step in the analysis of A_μ will be to construct special solutions of $A_\mu\,\phi = \lambda\phi$ that have the asymptotic forms

$$\begin{cases} \left.\begin{aligned} \phi_1(y,\mu,\lambda) &= e^{q_+'(\mu,\lambda)y}\,[1 + o(1)], \quad y \to +\infty \\ \phi_2(y,\mu,\lambda) &= e^{-q_+'(\mu,\lambda)y}[1 + o(1)], \quad y \to +\infty \end{aligned}\right\} \lambda \neq c^2(\infty)\mu^2, \\ \left.\begin{aligned} \phi_3(y,\mu,\lambda) &= e^{q_-'(\mu,\lambda)y}\,[1 + o(1)], \quad y \to -\infty \\ \phi_4(y,\mu,\lambda) &= e^{-q_-'(\mu,\lambda)y}[1 + o(1)], \quad y \to -\infty \end{aligned}\right\} \lambda \neq c^2(-\infty)\mu^2. \end{cases} \tag{1.26}$$

It follows from an asymptotic calculation of the Wronskians that ϕ_1 and ϕ_2 are a solution basis for $A_\mu \phi = \lambda\phi$ when $\lambda \neq c^2(\infty)\mu^2$, while ϕ_3 and ϕ_4 are a basis when $\lambda \neq c^2(-\infty)\mu^2$.

The nature of the spectrum and eigenfunctions of A_μ can be inferred from (1.26). It will be assumed for definiteness that

$$(1.27) \qquad c(\infty) \leq c(-\infty).$$

It follows that if $\lambda < c^2(\infty)\mu^2$ then $A_\mu \phi = \lambda\phi$ has bounded solutions only if ϕ_2 and ϕ_3 are linearly dependent. Thus

$$(1.28) \qquad F(\mu,\lambda) = \rho^{-1} W(\phi_2,\phi_3) = 0$$

is an equation for the eigenvalues of A_μ, where W denotes the Wronskian and ρ^{-1} W is independent of y. The corresponding solutions

$$(1.29) \qquad \psi_k(y,\mu) = a_k(\mu)\ \phi_2(y,\mu,\lambda_k(\mu)) = a_k'(\mu)\ \phi_3(y,\mu,\lambda_k(\mu)),$$

where $\lambda = \lambda_k(\mu)$ is a root of (1.28), are square integrable on R and hence are eigenfunctions of A_μ. Moreover, A_μ can have no point eigenvalues $\lambda > c^2(\infty)\mu^2$, by (1.26). Thus $\sigma_0(A_\mu)$, the point spectrum of A_μ, lies in the interval $[c_m^2\mu^2, c^2(\infty)\mu^2]$. Criteria for $\sigma_0(A_\mu)$ to be empty, finite or countably infinite are given in §3.

It will be shown that the continuous spectrum of A_μ is $[c^2(\infty)\mu^2,\infty)$ and corresponding generalized eigenfunctions will be determined from (1.26). For $c^2(\infty)\mu^2 < \lambda < c^2(-\infty)\mu^2$ there is a single family of generalized eigenfunctions of the form

$$(1.30) \qquad \psi_0(y,\mu,\lambda) = a_0(\mu,\lambda)\ \phi_3(y,\mu,\lambda).$$

For $\lambda > c^2(-\infty)\mu^2$ there are two families defined by

$$(1.31)\qquad \begin{cases} \psi_+(y,\mu,\lambda) = a_+(\mu,\lambda)\ \phi_4(y,\mu,\lambda) \\ \psi_-(y,\mu,\lambda) = a_-(\mu,\lambda)\ \phi_1(y,\mu,\lambda) \end{cases}$$

It will be shown that these functions have the following asymptotic forms.

$$(1.32)\qquad \psi_k(y,\mu) \sim \begin{cases} c_k^+(\mu)\ e^{-q_+'(\mu,\lambda_k(\mu))y}, & y \to +\infty, \\ c_k^-(\mu)\ e^{q_-'(\mu,\lambda_k(\mu))y}, & y \to -\infty, \end{cases}$$

$$(1.33)\qquad \psi_0(y,\mu,\lambda) \sim c_0(\mu,\lambda) \begin{cases} e^{-iq_+(\mu,\lambda)y} + R_0(\mu,\lambda)\ e^{iq_+(\mu,\lambda)y}, & y \to +\infty, \\ T_0(\mu,\lambda)\ e^{q_-'(\mu,\lambda)y}, & y \to -\infty, \end{cases}$$

$$(1.34)\qquad \psi_+(y,\mu,\lambda) \sim c_+(\mu,\lambda) \begin{cases} e^{-iq_+(\mu,\lambda)y} + R_+(\mu,\lambda)\ e^{iq_+(\mu,\lambda)y}, & y \to +\infty, \\ T_+(\mu,\lambda)\ e^{-iq_-'(\mu,\lambda)y}, & y \to -\infty, \end{cases}$$

$$(1.35)\qquad \psi_-(y,p,\lambda) \sim c_-(\mu,\lambda) \begin{cases} T_-(\mu,\lambda)\ e^{iq_+(\mu,\lambda)y}, & y \to +\infty, \\ e^{iq_-(\mu,\lambda)y} + R_-(\mu,\lambda)\ e^{-iq_-(\mu,\lambda)y}, & y \to -\infty. \end{cases}$$

Here $a_k(\mu)$, $a_k'(\mu)$, $a_0(p,\lambda)$, $a_\pm(p,\lambda)$, $c_k^\pm(\mu)$, $c_0(\mu,\lambda)$, $c_\pm(\mu,\lambda)$, $R_0(\mu,\lambda)$, $R_\pm(\mu,\lambda)$, $T_0(\mu,\lambda)$ and $T_\pm(\mu,\lambda)$ are functions of μ and λ that will be calculated below.

Families of normal mode functions for A may be constructed from those for $A_{|p|}$ by the rule (1.14). The following notation will be used.

$$(1.36)\qquad \psi_{\pm}(x,y,p,\lambda) = (2\pi)^{-1} e^{ip\cdot x} \psi_{\pm}(y,|p|,\lambda),\ (p,\lambda) \in \Omega,$$

$$(1.37)\qquad \psi_0(x,y,p,\lambda) = (2\pi)^{-1} e^{ip\cdot x} \psi_0(y,|p|,\lambda),\ (p,\lambda) \in \Omega_0,$$

$$(1.38)\qquad \psi_k(x,y,p) = (2\pi)^{-1} e^{ip\cdot x} \psi_k(y,|p|), \quad p \in \Omega_k,\ k \geq 1,$$

The parameter domains Ω, Ω_0, Ω_k are defined by

$$(1.39)\qquad \Omega = \{(p,\lambda) \in R^3 \mid c^2(-\infty)\ |p|^2 < \lambda\},$$

$$(1.40)\qquad \Omega_0 = \{(p,\lambda) \in R^3 \mid c^2(\infty)\ |p|^2 < \lambda < c^2(-\infty)\ |p|^2\}$$

$$(1.41)\qquad \Omega_k = \{p \in R^2 \mid |p| \in \mathcal{O}_k\},\ k \geq 1,$$

where $\mathcal{O}_k$ is the set of $\mu > 0$ for which A_μ has a kth eigenvalue; see §6.

The three families have different wave-theoretic interpretations that are characterized by their asymptotic behaviors. Thus for $(p,\lambda) \in \Omega$ one has

$$(1.42)\qquad \psi_+(x,y,p,\lambda) \sim \frac{c_+(|p|,\lambda)}{2\pi} \begin{cases} e^{i(p\cdot x-q_+y)} + R_+\, e^{i(p\cdot x+q_+y)}, & y \to +\infty, \\ T_+\, e^{i(p\cdot x-q_-y)}, & y \to -\infty, \end{cases}$$

$$(1.43)\qquad \psi_-(x,y,p,\lambda) \sim \frac{c_-(|p|,\lambda)}{2\pi} \begin{cases} T_-\, e^{i(p\cdot x+q_+y)}, & y \to +\infty, \\ e^{i(p\cdot x+q_-y)} + R_-\, e^{i(p\cdot x-q_-y)}, & y \to -\infty, \end{cases}$$

where $q_\pm = q_\pm(|p|,\lambda)$, $R_+ = R_+(|p|,\lambda)$, etc. Hence $\psi_+(x,y,p,\lambda)$ behaves for $y \to +\infty$ like an incident plane wave with propagation vector

$\vec{k}_i = (p,-q_+)$ plus a specularly reflected wave with propagation vector $\vec{k}_r = (p,q_+)$, while for $y \to -\infty$ it behaves like a pure transmitted plane wave with propagation vector $\vec{k}_t = (p,-q_-)$. The incident and transmitted plane waves can be shown to satisfy Snell's law $n(\infty) \sin \theta(\infty) = n(-\infty) \sin \theta(-\infty)$ where $\theta(\infty)$ and $\theta(-\infty)$ are the angles between the y-axis and $\vec{k}_i$ and $\vec{k}_t$, respectively, and $n(\pm\infty) = c^{-1}(\pm\infty)$. $\psi_-(x,y,p,\lambda)$ has a similar interpretation.

For $(p,\lambda) \in \Omega_0$ one has

$$(1.44) \quad \psi_0(x,y,p,\lambda) \sim \frac{c_0(|p|,\lambda)}{2\pi} \begin{cases} e^{i(p\cdot x-q_+y)} + R_0\, e^{i(p\cdot x+q_+y)}, & y \to +\infty, \\ T_0\, e^{ip\cdot x}\, e^{q'_-y} & , y \to -\infty. \end{cases}$$

Hence for $y \to +\infty$ $\psi_0(x,y,p,\lambda)$ behaves like an incident plane wave plus a specularly reflected wave while for $y \to -\infty$ it is exponentially damped. This is analogous to the phenomenon of total reflection of a plane wave in a homogeneous medium of refractive index $n(\infty) = c^{-1}(\infty)$ at an interface with a medium of index $n(-\infty) = c^{-1}(-\infty) < n(\infty)$. Indeed, the condition $\lambda < c^2(-\infty)\ |p|^2$ is equivalent to the condition for total reflection: $n(\infty) \sin \theta(\infty) > n(-\infty)$.

For $p \in \Omega_k$, $k \geq 1$, one has

$$(1.45) \qquad \psi_k(x,y,p) \sim \begin{cases} \dfrac{c_k^+(|p|)}{2\pi}\, e^{ip\cdot x}\, e^{-q'_+y}, & y \to +\infty, \\[2ex] \dfrac{c_k^-(|p|)}{2\pi}\, e^{ip\cdot x}\, e^{q'_-y}, & y \to -\infty. \end{cases}$$

Hence the functions $\psi_k(x,y,p)$ can be interpreted as guided waves that are trapped by total reflection in the acoustic duct where $c(y) < c(\pm\infty)$.

They propagate in the direction $\vec{k} = (p,0)$ parallel to the duct and decrease exponentially with distance from it.

The coefficients $R_\pm$, R_0 and $T_\pm$, T_0 in (1.42), (1.43), (1.44) may be interpreted as reflection and transmission coefficients, respectively, for the scattering of plane waves by the stratified fluid. They will be shown to satisfy the conservation laws

$$q_\pm |R_\pm|^2 + q_\mp |T_\pm|^2 = q_\pm, \quad |R_0| = 1. \tag{1.46}$$

The completeness of the set $\{\psi_+, \psi_-, \psi_0, \psi_1, \psi_2, \ldots\}$ of normal mode functions is proved in §8 below.

The three families ψ_+, ψ_- and ψ_0 represent, collectively, the response of the stratified fluid to incident plane waves $\exp\{i(p \cdot x - qy)\}$, $(p,q) \in R^3$. To see this consider the mappings

$$\begin{cases} (p,q) = X_+(p,\lambda) = (p, q_+(|p|,\lambda)), & (p,\lambda) \in \Omega, \\ (p,q) = X_0(p,\lambda) = (p, q_+(|p|,\lambda)), & (p,\lambda) \in \Omega_0, \\ (p,q) = X_-(p,\lambda) = (p, -q_-(|p|,\lambda)), & (p,\lambda) \in \Omega. \end{cases} \tag{1.47}$$

X_+ is an analytic transformation of Ω onto the cone

$$C_+ = \{(p,q) \mid q > a\,|p|\} \tag{1.48}$$

where

$$a = ((c(-\infty)/c(\infty))^2 - 1)^{1/2} \geq 0. \tag{1.49}$$

Similarly, X_0 is an analytic transformation of Ω_0 onto the cone

$$C_0 = \{(p,q) \mid 0 < q < a\,|p|\} \tag{1.50}$$

and X_- is an analytic transformation of Ω onto the cone

$$C_- = \{(p,q) \mid q < 0\}. \tag{1.51}$$

Thus, the asymptotic forms of $\psi_\pm$ and ψ_0 for $y \to \pm\infty$ show that $\psi_+(x,y,p,\lambda)$ with $(p,\lambda) \in \Omega$ is the response of the fluid to a plane wave $\exp\{i(p\cdot x - qy)\}$ with $(p,q) \in C_+$, $\psi_0(x,y,p,\lambda)$ is the response to a plane wave with $(p,q) \in C_0$ and $\psi_-(x,y,p,\lambda)$ is the response to a plane wave with $(p,q) \in C_-$. Note that

$$R^3 = C_+ \cup C_0 \cup C_- \cup N \tag{1.52}$$

where N is a Lebesgue null set.

The interpretation of ψ_+, ψ_- and ψ_0 given above suggests the introduction of a composite eigenfunction

$$\phi_+(x,y,p,q) = (2\pi)^{-1} e^{ip\cdot x} \phi_+(y,p,q), \quad (p,q) \in C_+ \cup C_0 \cup C_-, \tag{1.53}$$

where

$$\phi_+(y,p,q) = \begin{cases} (2q)^{1/2} c(\infty) \, \psi_+(y,|p|,\lambda), \ (p,\lambda) = X_+^{-1}(p,q), \ (p,q) \in C_+, \\ (2q)^{1/2} c(\infty) \, \psi_0(y,|p|,\lambda), \ (p,\lambda) = X_0^{-1}(p,q), \ (p,q) \in C_0, \\ (2|q|)^{1/2} c(-\infty) \, \psi_-(y,|p|,\lambda), \ (p,\lambda) = X_-^{-1}(p,q), \\ \qquad\qquad (p,q) \in C_-. \end{cases} \tag{1.54}$$

The normalizing factors $(2q)^{1/2} c(\infty)$ and $(2|q|)^{1/2} c(-\infty)$ are the square roots of the Jacobians of X_+^{-1}, X_0^{-1} and X_-^{-1}. The function $\phi_+(x,y,p,q)$ is a solution of the differential equation

$$A \, \phi_+(\cdot,p,q) = \lambda(p,q) \, \phi_+(\cdot,p,q) \tag{1.55}$$

where

$$(1.56) \qquad \lambda(p,q) = \begin{cases} c^2(\infty)(|p|^2 + q^2), & (p,q) \in C_+ \cup C_0, \\ c^2(-\infty)(|p|^2 + q^2), & (p,q) \in C_-. \end{cases}$$

Its asymptotic behavior is described by

$$(1.57) \qquad \phi_+(x,y,p,q) \sim c(p,q) \begin{cases} e^{i(p\cdot x-qy)} + R_+ \, e^{i(p\cdot x+qy)}, & (p,q) \in C_+, \\ e^{i(p\cdot x-qy)} + R_0 \, e^{i(p\cdot x+qy)}, & (p,q) \in C_0, \\ T_- \, e^{i(p\cdot x+q_+(|p|,\lambda)y)} & , (p,q) \in C_-, \end{cases}$$

for $y \to +\infty$ and

$$(1.58) \qquad \phi_+(x,y,p,q) \sim c(p,q) \begin{cases} T_+ \, e^{i(p\cdot x-q_-(|p|,\lambda)y)}, & (p,q) \in C_+, \\ T_0 \, e^{ip\cdot x} \, e^{q_-'(|p|,\lambda)y}, & (p,q) \in C_0, \\ e^{i(p\cdot x-qy)} + R_- \, e^{i(p\cdot x+qy)}, & (p,q) \in C_- \end{cases}$$

for $y \to -\infty$. In §8 it is shown that one may take

$$(1.59) \qquad c(p,q) = \begin{cases} (2\pi)^{-3/2} \, c(\infty) \, \rho^{1/2}(\infty), & (p,q) \in C_+ \cup C_0, \\ (2\pi)^{-3/2} \, c(-\infty) \, \rho^{1/2}(-\infty), & (p,q) \in C_-, \end{cases}$$

and the completeness of the set $\{\phi_+,\psi_1,\psi_2,\ldots\}$ of normal mode functions is derived from that of $\{\psi_+,\psi_-,\psi_0,\psi_1,\psi_2,\ldots\}$.

Another family of normal mode functions for A is defined by

$$(1.60) \qquad \phi_-(x,y,p,q) = \overline{\phi_+(x,y,-p,q)}, \quad (p,q) \in C_+ \cup C_0 \cup C_-.$$

It is clear that $A\,\phi_- = \lambda(p,q)\,\phi_-$ and

$$(1.61) \qquad \phi_-(x,y,p,q) = (2\pi)^{-1}\, e^{ip\cdot x}\, \phi_-(y,p,q)$$

where

$$(1.62) \qquad \phi_-(y,p,q) = \overline{\phi_+(y,p,q)}.$$

The asymptotic behavior of ϕ_- for $y \to \pm\infty$ may be derived from (1.57), (1.58), (1.59). It is given by

$$(1.63) \qquad \phi_-(x,y,p,q) \sim c(p,q) \begin{cases} e^{i(p\cdot x+qy)} + \overline{R}_+\, e^{i(p\cdot x-qy)}, & (p,q) \in C_+, \\ e^{i(p\cdot x+qy)} + \overline{R}_0\, e^{i(p\cdot x-qy)}, & (p,q) \in C_0, \\ \overline{T}_-\, e^{i(p\cdot x-q_+(|p|,\lambda)y)}, & (p,q) \in C_-, \end{cases}$$

for $y \to +\infty$ and

$$(1.64) \qquad \phi_-(x,y,p,q) \sim c(p,q) \begin{cases} \overline{T}_+\, e^{i(p\cdot x+q_-(|p|,\lambda)y)}, & (p,q) \in C_+, \\ \overline{T}_0\, e^{ip\cdot x}\, e^{q_-'(|p|,\lambda)y}, & (p,q) \in C_0, \\ e^{i(p\cdot x+qy)} + \overline{R}_-\, e^{i(p\cdot x-qy)}, & (p,q) \in C_-, \end{cases}$$

for $y \to -\infty$. These relations clearly imply that $\phi_-(x,y,p,q)$ is not simply a multiple of $\phi_+(x,y,p,q)$. By contrast the guided mode functions have the symmetry property

$$(1.65) \qquad \psi_k(x,y,p) = \overline{\psi_k(x,y,-p)}, \quad k \geq 1,$$

because they are real-valued and depend on p only through $|p|$.

The completeness of the family $\{\phi_-,\psi_1,\psi_2,\ldots\}$ is derived from that of $\{\phi_+,\psi_1,\psi_2,\ldots\}$ in §8. The existence of the two families ϕ_+ and ϕ_- is a consequence of the invariance of the wave equation (1.1) under time reversal. The family ϕ_- is useful in the construction of asymptotic solutions for $t \to +\infty$ of (1.1); see [19].

The remainder of this paper is organized as follows. §2 contains a construction of the special solutions (1.26) of $A_\mu \phi = \lambda\phi$. In §3 the results of §2 are applied to characterize the point spectrum and continuous spectrum of A_μ. The eigenfunctions and generalized eigenfunctions of A_μ are constructed in §4. In §5 the Weyl-Kodaira theory is applied to construct an eigenfunction representation of the spectral family of A_μ. §6 contains an analysis of the dispersion relation (1.28) and the μ-dependence of the eigenvalues $\lambda_k(\mu)$ of A_μ. In §7 the results of §§5 and 6 are used to construct a normal mode representation of the spectral family of A. The normal mode expansions for A are derived in §8. The cases of semi-infinite and finite layers are discussed in §9. §10 contains concluding remarks concerning applications and extensions of the theory. A formulation of the Weyl-Kodaira theory appropriate for the analysis of A_μ is given in an Appendix.

The analytical work needed to derive and fully establish normal mode expansions for a large class of stratified fluids is necessarily intricate and lengthy. This is clear from examination of the simple case of the Pekeris model given in [17]. Therefore to make the work presented here as accessible as possible the concepts and results of each section are formulated in the first portions of the sections. Detailed proofs are placed at the ends of the sections and may be omitted without interrupting the exposition.

§2. Solutions of the Equation $A_\mu\ \phi = \zeta\phi$.

The special solutions $\phi_j(y,\mu,\lambda)$ $(j = 1,2,3,4)$ described in §1 are constructed in this section. Analytic continuations of these functions to complex values of λ are used in §§3 and 5 for the calculation of the spectral family of A_μ. Hence the more general case of solutions of $A_\mu\ \phi = \zeta\phi$ with $\zeta \in C$ will be treated.

The equation $A_\mu\ \phi = \zeta\phi$ cannot have solutions in the classical sense unless $c(y)$ and $\rho(y)$ are continuous and continuously differentiable, respectively. A suitable class of solutions is described by the following definition in which $AC(I)$ denotes the set of all functions that are absolutely continuous with respect to Lebesgue measure in the interval $I = (a,b) \subset R$.

Definition. A function $\phi : I = (a,b) \to C$ is said to be a solution of

$$A_\mu\ \phi(y) \equiv -c^2(y)\{\rho(y)(\rho^{-1}(y)\ \phi'(y))' - \mu^2\phi(y)\} = \zeta\phi(y) \tag{2.1}$$

in the interval I (where $\phi' = d\phi/dy$) if and only if

$$\phi \in AC(I),\ \rho^{-1}\phi' \in AC(I) \tag{2.2}$$

and (2.1) holds for almost all $y \in I$.

The following notation will be used in the definition and construction of the special solutions $\phi_j(y,\mu,\zeta)$. For each $\kappa \geq 0$

$$\begin{aligned} L(\kappa) &= \{\zeta \mid \operatorname{Re}\zeta < \kappa^{1/2}\}, \\ R(\kappa) &= \{\zeta \mid \operatorname{Re}\zeta > \kappa^{1/2}\}, \end{aligned} \tag{2.3}$$

(2.3 cont.) $$R^{\pm}(\kappa) = R(\kappa) \cap \{\zeta \mid \pm \operatorname{Im} \zeta \geq 0\}.$$

The definitions (1.24), (1.25) will be extended as follows.

$$\left.\begin{aligned} & q_{\pm}(\mu,\zeta) = (\zeta\, c^{-2}(\pm\infty) - \mu^2)^{1/2} \\ & -\pi/4 < \arg q_{\pm}(\mu,\zeta) < \pi/4 \\ & q_{\pm}'(\mu,\zeta) = i\, q_{\pm}(\mu,\zeta) \end{aligned}\right\} \zeta \in R(c(\pm\infty)\mu) \tag{2.4}$$

and

$$\left.\begin{aligned} & q_{\pm}'(\mu,\zeta) = (\mu^2 - \zeta\, c^{-2}(\pm\infty))^{1/2} \\ & -\pi/4 < \arg q_{\pm}'(\mu,\zeta) < \pi/4 \\ & q_{\pm}(\mu,\zeta) = -i\, q_{\pm}'(\mu,\zeta). \end{aligned}\right\} \zeta \in L(c(\pm\infty)\mu) \tag{2.5}$$

The results of this section will now be formulated.

Theorem 2.1. Under hypotheses (1.3), (1.4) on $\rho(y)$, $c(y)$ there exist functions

$$\phi_j : R \times R_+ \times (L(c(\infty)\mu) \cup R(c(\infty)\mu)) \to C,\ j = 1,2, \tag{2.6}$$

(where $R_+ = \{\mu \mid \mu \geq 0\}$) such that for every fixed $(\mu,\zeta) \in R_+ \times (L(c(\infty)\mu) \cup R(c(\infty)\mu))$, $\phi_j(y,\mu,\zeta)$ is a solution of (2.1) for $y \in R$ and $j = 1,2$ and

$$\left.\begin{aligned} & \phi_1(y,\mu,\zeta) = \exp\{q_+'(\mu,\zeta)y\}[1 + o(1)] \\ & \rho^{-1}(y)\, \phi_1'(y,\mu,\zeta) = \rho^{-1}(\infty)\, q_+'(\mu,\zeta) \exp\{q_+'(\mu,\zeta)y\}[1 + o(1)] \end{aligned}\right\} y \to +\infty, \tag{2.7}$$

and

$$
(2.8)\quad \left.\begin{aligned}
\phi_2(y,\mu,\zeta) &= \exp\{-q_+'(\mu,\zeta)y\}[1+o(1)] \\
\rho^{-1}(y)\,\phi_2'(y,\mu,\zeta) &= -\rho^{-1}(\infty)\; q_+'(\mu,\zeta)\,\exp\{-q_+'(\mu,\zeta)y\}[1+o(1)]
\end{aligned}\right\}\; y \to +\infty.
$$

Similarly, there exist functions

$$
(2.9)\qquad \phi_j : R \times R_+ \times (L(c(-\infty)\mu) \cup R(c(-\infty)\mu)) \to C,\; j = 3,4,
$$

such that for every fixed $(\mu,\zeta) \in R_+ \times (L(c(-\infty)\mu) \cup R(c(-\infty)\mu)$, $\phi_j(y,\mu,\zeta)$ is a solution of (2.1) for $y \in R$ and $j = 3, 4$ and

$$
(2.10)\quad \left.\begin{aligned}
\phi_3(y,\mu,\zeta) &= \exp\{q_-'(\mu,\zeta)y\}[1+o(1)] \\
\rho^{-1}(y)\,\phi_3'(y,\mu,\zeta) &= \rho^{-1}(-\infty)\; q_-'(\mu,\zeta)\,\exp\{q_-'(\mu,\zeta)y\}[1+o(1)]
\end{aligned}\right\}\; y \to -\infty,
$$

and

$$
(2.11)\quad \left.\begin{aligned}
\phi_4(y,\mu,\zeta) &= \exp\{-q_-'(\mu,\zeta)y\}[1+o(1)] \\
\rho^{-1}(y)\phi_4'(y,\mu,\zeta) &= -\rho^{-1}(-\infty)q_-'(\mu,\zeta)\,\exp\{-q_-'(\mu,\zeta)y\}[1+o(1)]
\end{aligned}\right\}\; y \to -\infty.
$$

The following three corollaries describe the dependence of the solutions $\phi_j(y,\mu,\zeta)$ on the parameters μ and ζ.

Corollary 2.2. The functions $\phi_j(y,\mu,\zeta)$ satisfy

$$
(2.12)\qquad \phi_j,\ \rho^{-1}\phi_j' \in C\left(R \times \bigcup_{\mu \geq 0} \{(\mu,\zeta) \mid \zeta \in L(c(\infty)\mu)\}\right)
$$

for $j = 1, 2$ and

$$(2.13) \qquad \phi_j,\ \rho^{-1}\phi_j' \in C\left(R \times \bigcup_{\mu\geq 0} \{(\mu,\zeta) \mid \zeta \in L(c(-\infty)\mu)\}\right)$$

for j = 3, 4. Moreover

$$(2.14) \qquad \begin{aligned} &\phi_1,\ \rho^{-1}\phi_1' \in C\left(R \times \bigcup_{\mu\geq 0} \{(\mu,\zeta) \mid \zeta \in R^+(c(\infty)\mu)\}\right), \\ &\phi_2,\ \rho^{-1}\phi_2' \in C\left(R \times \bigcup_{\mu\geq 0} \{(\mu,\zeta) \mid \zeta \in R^-(c(\infty)\mu)\}\right), \\ &\phi_3,\ \rho^{-1}\phi_3' \in C\left(R \times \bigcup_{\mu\geq 0} \{(\mu,\zeta) \mid \zeta \in R^-(c(-\infty)\mu)\}\right), \\ &\phi_4,\ \rho^{-1}\phi_4' \in C\left(R \times \bigcup_{\mu\geq 0} \{(\mu,\zeta) \mid \zeta \in R^+(c(-\infty)\mu)\}\right). \end{aligned}$$

Corollary 2.3. For each fixed $(y,\mu) \in R \times R_+$ the mappings

$$(2.15) \qquad \zeta \to \phi_j(y,\mu,\zeta),\ \zeta \to \rho^{-1}(y)\ \phi_j'(y,\mu,\zeta)$$

are analytic for

$$(2.16) \qquad \begin{aligned} &j = 1,\ \zeta \in L(c(\infty)\mu) \cup R^+(c(\infty)\mu)^{\text{int}}, \\ &j = 2,\ \zeta \in L(c(\infty)\mu) \cup R^-(c(\infty)\mu)^{\text{int}}, \\ &j = 3,\ \zeta \in L(c(-\infty)\mu) \cup R^-(c(-\infty)\mu)^{\text{int}}, \\ &j = 4,\ \zeta \in L(c(-\infty)\mu) \cup R^+(c(-\infty)\mu)^{\text{int}}, \end{aligned}$$

where $R^\pm(\kappa)^{\text{int}} = R(\kappa) \cap \{\zeta \mid \pm \text{Im}\ \zeta > 0\}$.

Corollary 2.4. The asymptotic estimates for ϕ_j and $\rho^{-1}\phi_j'$ of Theorem 2.1 hold uniformly for (μ,ζ) in any compact set Γ_j such that for

$$
(2.17)\quad
\begin{aligned}
&j = 1,\ \Gamma_1 \subset \bigcup_{\mu \geq 0} \{(\mu,\zeta) \mid \zeta \in L(c(\infty)\mu) \cup R^{+}(c(\infty)\mu)\},\\
&j = 2,\ \Gamma_2 \subset \bigcup_{\mu \geq 0} \{(\mu,\zeta) \mid \zeta \in L(c(\infty)\mu) \cup R^{-}(c(\infty)\mu)\},\\
&j = 3,\ \Gamma_3 \subset \bigcup_{\mu \geq 0} \{(\mu,\zeta) \mid \zeta \in L(c(-\infty)\mu) \cup R^{-}(c(-\infty)\mu)\},\\
&j = 4,\ \Gamma_4 \subset \bigcup_{\mu \geq 0} \{(\mu,\zeta) \mid \zeta \in L(c(-\infty)\mu) \cup R^{+}(c(-\infty)\mu)\}.
\end{aligned}
$$

The special solutions $\phi_j(y,\mu,\zeta)$ are not, in general, uniquely determined by the asymptotic conditions (2.7), (2.8), (2.10), (2.11). Indeed, if Re $q_+'(\mu,\lambda) > 0$ (resp., Re $q_+'(\mu,\lambda) < 0$) it is clear that any multiple of ϕ_2 (resp., ϕ_1) can be added to ϕ_1(resp., ϕ_2). A similar remark holds for ϕ_3 and ϕ_4. However, for each $\zeta \in C$ a sub-dominant solution (one with minimal growth at $y = \infty$ or $y = -\infty$) is unique. In particular, since

$$
(2.18)\quad
\begin{aligned}
&\text{Re } q_\pm'(\mu,\zeta) > 0 \text{ for } \zeta \in L(c(\pm\infty)\mu)\\
&\text{Re } q_\pm'(\mu,\zeta) \geq 0 \text{ for } \zeta \in R^{-}(c(\pm\infty)\mu)\\
&\text{Re } q_\pm'(\mu,\zeta) \leq 0 \text{ for } \zeta \in R^{+}(c(\pm\infty)\mu)
\end{aligned}
$$

one can prove

Corollary 2.5. The solution ϕ_2 is uniquely determined by (2.8) for all $\zeta \in L(c(\infty)\mu) \cup R^{-}(c(\infty)\mu)$. Similarly, ϕ_3 is uniquely determined by (2.10) for $\zeta \in L(c(-\infty)\mu) \cup R^{-}(c(-\infty)\mu)$, ϕ_1 is uniquely determined by (2.7) in $R^{+}(c(\infty)\mu)$ and ϕ_4 is uniquely determined by (2.11) in $R^{+}(c(-\infty)\mu)$.

When Re $\zeta = c^2(\pm\infty)\mu^2$ Theorem 2.1 provides no information about the asymptotic behavior for $y \to \pm\infty$ of solutions of $A_\mu\, \phi = \zeta\phi$. However, positive results can be obtained by strengthening hypothesis (1.4). The following extension of a known result [11, p. 209] will be used in §3.

<u>Theorem 2.6</u>. Assume that $\rho(y)$ and $c(y)$ satisfy hypothesis (1.3) and

$$(2.19) \qquad \int_0^\infty |\rho(y) - \rho(\infty)|\, dy < \infty, \int_0^\infty y^2\, |c(y) - c(\infty)|\, dy < \infty.$$

Then there exist functions

$$(2.20) \qquad \phi_j : R \times R_+ \to R,\ j = 1,2,$$

such that for every $\mu \in R_+$ the pair $\phi_1(y,\mu)$, $\phi_2(y,\mu)$ is a solution basis for $A_\mu\, \phi = c^2(\infty)\mu^2\phi$,

$$(2.21) \qquad \left.\begin{aligned} \phi_1(y,\mu) &= 1 + o(1) \\ \rho^{-1}(y)\ \phi_1'(y,\mu) &= o(1) \end{aligned}\right\}\ y \to +\infty,$$

and

$$(2.22) \qquad \left.\begin{aligned} \phi_2(y,\mu) &= \rho(\infty)y[1 + o(1)] \\ \rho^{-1}(y)\ \phi_2'(y,\mu) &= 1 + o(1) \end{aligned}\right\}\ y \to +\infty.$$

Lagrange's formula for A_μ may be written

$$(2.23) \qquad \int_{y_1}^{y_2} \{\psi\, A_\mu\, \phi - \phi\, A_\mu\, \psi\}\, c^{-2}(y)\, \rho^{-1}(y)\, dy = [\phi\psi](y_2) - [\phi\psi](y_1)$$

where

(2.24) $$[\phi\psi](y) = \phi(y)(\rho^{-1}(y)\ \psi'(y)) - \psi(y)(\rho^{-1}(y)\ \phi'(y)).$$

In particular, if ϕ and ψ are solutions of $A_\mu\ \phi = \zeta\phi$, $A_\mu\ \psi = \zeta\psi$ on an interval I then $[\phi\psi](y)$ = const. on I and $[\phi\psi](y) = 0$ on I if and only if ϕ and ψ are linearly dependent there. By combining these facts and Theorem 2.1 one can show that

(2.25) $$\begin{aligned} [\phi_1(\cdot,\mu,\zeta)\ \phi_2(\cdot,\mu,\zeta)] &= -2\ \rho^{-1}(\infty)\ q_+'(\mu,\zeta), \\ [\phi_3(\cdot,\mu,\zeta)\ \phi_4(\cdot,\mu,\zeta)] &= -2\ \rho^{-1}(-\infty) q_-'(\mu,\zeta), \end{aligned}$$

which imply

Corollary 2.7. The pair $\phi_1(y,\mu,\zeta)$, $\phi_2(y,\mu,\zeta)$ is a solution basis for $A_\mu\ \phi = \zeta\phi$ for all $(\mu,\zeta) \in R_+ \times (L(c(\infty)\mu) \cup R(c(\infty)\mu))$. Similarly, the pair $\phi_3(y,\mu,\zeta)$, $\phi_4(y,\mu,\zeta)$ is a solution basis for all $(\mu,\zeta) \in R_+ \times (L(c(-\infty)\mu) \cup R(c(-\infty)\mu))$.

This completes the formulation of the results of §2 and the proofs will now be given. The method of proof involves replacing $A_\mu\ \phi = \zeta\phi$ by an equivalent first order system. The latter can be regarded as a perturbation of the corresponding limit systems for $y \to \pm\infty$. In this way integral equations are established for solutions with prescribed asymptotic behavior for $y \to \infty$ or $y \to -\infty$ and these equations are solved by classical Banach space methods. This technique for constructing solutions with prescribed asymptotic behavior is well known - see for example [3, p. 1408] and [12, Ch. VII].

A first order system equivalent to $A_\mu\ \phi = \zeta\phi$. If $\phi(y)$ is any solution of (2.1) on an interval I and if

$$(2.26)\qquad \begin{cases} \psi_1(y) = \phi(y) \\ \psi_2(y) = \rho^{-1}(y)\ \phi'(y) \end{cases}$$

then ψ_1, $\psi_2 \in AC(I)$ (cf. (2.2)) and

$$(2.27)\qquad \begin{cases} \psi_1'(y) = \rho(y)\ \psi_2(y) \\ \psi_2'(y) = \rho^{-1}(y)[\mu^2 - \zeta\ c^{-2}(y)]\ \psi_1(y) \end{cases}$$

for almost every $y \in I$. Thus the column vector $\psi(y)$ with components $\psi_1(y)$, $\psi_2(y)$ is a solution of the first order linear system

$$(2.28)\qquad \psi'(y) = M(y,\mu,\zeta)\ \psi(y)$$

where

$$(2.29)\qquad M(y,\mu,\zeta) = \begin{pmatrix} 0 & \rho(y) \\ \rho^{-1}(y)[\mu^2 - \zeta\ c^{-2}(y)] & 0 \end{pmatrix}$$

Conversely, if $\psi \in AC(I)$ is a solution of (2.28), (2.29) and if $\phi(y) = \psi_1(y)$ then ϕ is a solution of (2.1). The solutions of Theorem 2.1 will be constructed by integrating (2.28), (2.29).

The limit system for $y \to +\infty$ and its solutions. By replacing $\rho(y)$, $c(y)$ in (2.28), (2.29) by $\rho(\infty)$, $c(\infty)$ one obtains the system

$$(2.30)\qquad \psi'(y) = M_0(\mu,\zeta)\ \psi(y)$$

where

$$(2.31) \qquad M_0(\mu,\zeta) = \begin{pmatrix} 0 & \rho(\infty) \\ \rho^{-1}(\infty)[\mu^2 - \zeta\, c^{-2}(\infty)] & 0 \end{pmatrix}.$$

$M_0(\mu,\zeta)$ has distinct eigenvalues $q_+'(\mu,\zeta)$, $-q_+'(\mu,\zeta)$ for $\zeta \in L(c(\infty)\mu) \cup R(c(\infty)\mu)$. The columns of

$$(2.32) \qquad B(\mu,\zeta) = \begin{pmatrix} 1 & 1 \\ \rho^{-1}(\infty)\; q_+'(\mu,\zeta) & -\rho^{-1}(\infty)\; q_+'(\mu,\zeta) \end{pmatrix}$$

are corresponding eigenvectors. Hence

$$(2.33) \qquad M_0(\mu,\zeta)\; B(\mu,\zeta) = B(\mu,\zeta)\; D(\mu,\zeta)$$

where

$$(2.34) \qquad D(\mu,\zeta) = \begin{pmatrix} q_+'(\mu,\zeta) & 0 \\ 0 & -q_+'(\mu,\zeta) \end{pmatrix}.$$

System (2.30), (2.31) may be integrated by the substitution

$$(2.35) \qquad \psi = B(\mu,\zeta)\; z.$$

It follows that $z'(y) = D(\mu,\zeta)\; z(y)$, whence

$$(2.36) \qquad \begin{cases} z_1(y) = c_1 \exp\{q_+'(\mu,\zeta)\; y\} \\ z_2(y) = c_2 \exp\{-q_+'(\mu,\zeta)\; y\} \end{cases}$$

and therefore

$$(2.37)\quad \begin{cases} \psi_1(y) = c_1 \exp\{q_+'(\mu,\zeta)y\} + c_2 \exp\{-q_+'(\mu,\zeta)y\} \\ \psi_2(y) = \rho^{-1}(\infty)\, q_+'(\mu,\zeta)(c_1 \exp\{q_+'(\mu,\zeta)y\} - c_2 \exp\{-q_+'(\mu,\zeta)y\}) \end{cases}$$

where c_1, c_2 are constants of integration.

Application of perturbation theory. System (2.28) may be regarded as a perturbation of the limit system (2.30). Thus if $N(y,\mu,\zeta)$ is defined by

$$(2.38)\qquad M(y,\mu,\zeta) = M_0(\mu,\zeta) + N(y,\mu,\zeta)$$

then

$$(2.39)\qquad N(y,\mu,\zeta) = \begin{pmatrix} 0 & a_1(y) \\ \mu^2\, a_2(y) + \zeta\, a_3(y) & 0 \end{pmatrix}$$

where

$$(2.40)\qquad \begin{cases} a_1(y) = \rho(y) - \rho(\infty) \\ a_2(y) = \rho^{-1}(y) - \rho^{-1}(\infty) \\ a_3(y) = -[\rho^{-1}(y)\, c^{-2}(y) - \rho^{-1}(\infty)\, c^{-2}(\infty)] \end{cases}$$

Note that each of these functions is in $L_1(y_0,\infty)$ for every $y_0 \in R$. For $a_1(y)$ this is part of hypothesis (1.4). For $a_2(y)$ and $a_3(y)$ it follows from (1.3) and (1.4). For example, one can write

$$(2.41)\qquad a_2(y) = (-\rho^{-1}(\infty)\, \rho^{-1}(y))(\rho(y) - \rho(\infty))$$

which exhibits a_2 as a product of a bounded measurable function and a function in $L_1(y_0,\infty)$.

On combining (2.28), (2.38) and making the substitution (2.35), one finds that (2.28) is equivalent to the system

$$(2.42) \qquad z'(y) = D(\mu,\zeta)\ z(y) + E(y,\mu,\zeta)\ z(y)$$

where

$$(2.43) \qquad E(y,\mu,\zeta) = B^{-1}(\mu,\zeta)\ N(y,\mu,\zeta)\ B(\mu,\zeta)$$

has components that are in $L_1(y_0,\infty)$. Solutions of (2.42) will be constructed which are asymptotically equal, for $y \to +\infty$, to the solutions (2.36) of $z' = D(\mu,\zeta)\ z$.

Proof of Theorem 2.1. The proof will be given for the function ϕ_1 only. The remaining cases can be proved by the same method. Solutions of (2.42) are related to the corresponding solutions of (2.1) by

$$(2.44) \qquad \begin{cases} \phi = z_1 + z_2 \\ \rho^{-1}\phi' = \rho^{-1}(\infty)\ q_+'(z_1 - z_2),\ q_+' = q_+'(\mu,\zeta). \end{cases}$$

Thus ϕ will be a solution of (2.1) that satisfies (2.7) if z is a solution of (2.42) that satisfies

$$(2.45) \qquad z_1 = \exp\{q_+'\ y\}\ \eta_1,\ z_2 = \exp\{q_+'\ y\}\ \eta_2$$

and

$$(2.46) \qquad \eta_1(y) = 1 + o(1),\ \eta_2(y) = o(1) \text{ for } y \to \infty.$$

Equations (2.42) and (2.45) imply

$$(2.47)\qquad \begin{cases} \eta_1' = \qquad\qquad E_{11}\,\eta_1 + E_{12}\,\eta_2, \\ \eta_2' = -2\,q_+'\,\eta_2 + E_{21}\,\eta_1 + E_{22}\,\eta_2, \end{cases}$$

and hence by integration

$$(2.48)\qquad \begin{aligned} \eta_1(y) &= c_1 + \int_{y_0}^{y} E_{1j}(y')\,\eta_j(y')\,dy' \\ \eta_2(y) &= \exp\{-2\,q_+'\,y\}\,c_2 + \int_{y_1}^{y} \exp\{-2\,q_+'(y-y')\}\,E_{2j}(y')\,\eta_j(y')\,dy' \end{aligned}$$

where c_1, c_2, y_0, y_1 are constants and the summation convention has been used (j is summed over $j = 1,2$).

<u>*Construction of ϕ_l for $\zeta \in L(c(\infty)\mu)$*</u>. By (2.18), Re $q_+'(\mu,\zeta) > 0$ for all $\zeta \in L(c(\infty)\mu)$. Thus to construct a solution of (2.47) that satisfies (2.46) it is natural to choose $c_1 = 1$, $c_2 = 0$, $y_0 = +\infty$ and y_1 finite in (2.48). This gives the system of integral equations

$$(2.49)\qquad \left.\begin{aligned} \eta_1(y) &= 1 - \int_{y}^{\infty} E_{1j}(y')\,\eta_j(y')\,dy' \\ \eta_2(y) &= \int_{y_1}^{y} \exp\{-2\,q_+'(y-y')\}\,E_{2j}(y')\,\eta_j(y')\,dy' \end{aligned}\right\}\quad y \geq y_1.$$

It is natural to study system (2.49) in the space

$$(2.50)\qquad X = CB([y_1,\infty),C^2)$$

of two-component vector functions of y whose components are continuous and bounded on $y_1 \leq y < \infty$. X is a Banach space with norm

$$\|\eta\| = \sup_{y \geq y_1} (|\eta_1(y)| + |\eta_2(y)|). \tag{2.51}$$

The system has the form

$$\eta(y) = \eta^0 + \int_{y_1}^{\infty} K(y,y')\, \eta(y')\, dy', \quad y \geq y_1, \tag{2.52}$$

where $\eta(y)$ and η^0 are column vectors with components $(\eta_1(y), \eta_2(y))$ and $(1,0)$, respectively, and the matrix kernel $K(y,y')$ is defined by

$$K_{1j}(y,y') = \begin{cases} 0 & , \ y_1 \leq y' < y, \\ -E_{1j}(y'), & y_1 \leq y \leq y' \end{cases} \tag{2.53}$$

$$K_{2j}(y,y') = \begin{cases} \exp\{-2\, q_+'(y - y')\}\, E_{2j}(y'), & y_1 \leq y' < y, \\ 0 & , \ y_1 \leq y \leq y', \end{cases} \tag{2.54}$$

and $j = 1,2$. The conditions $E_{jk} \in L_1(y_1,\infty)$ and $\mathrm{Re}\, q_+' > 0$ imply that the operator K defined by (2.52), (2.53) and (2.54) maps X into itself. To show that K is a bounded operator in X and estimate its norm note that

$$|(K\eta)_j(y)| \leq \|\eta\| \int_{y_1}^{\infty} (|E_{j1}(y)| + |E_{j2}(y)|)\, dy \tag{2.55}$$

for $j = 1,2$ and all $y \geq y_1$. It follows that

$$\|K\| \leq \int_{y_1}^{\infty} \sum_{j,k=1}^{2} |E_{jk}(y)|\, dy. \tag{2.56}$$

In particular, since $E_{jk} \in L_1(y_0,\infty)$ for every $y_0 \in R$, (2.56) implies that $\|K\| < 1$ for every sufficiently large y_1. For such a value of y_1

the equation

$$\eta = \eta^0 + K\eta \tag{2.57}$$

has a unique solution $\eta \in X$ given by

$$\eta = \sum_{m=0}^{\infty} K^m \eta^0. \tag{2.58}$$

Moreover, (2.57), or equivalently (2.49), implies that $\eta_1(y)$, $\eta_2(y)$ satisfy (2.47) for $y \geq y_1$. These functions then have unique continuations to solutions of (2.47) for all $y \in R$, by the classical existence and uniqueness theory for linear systems.

Of course, η_1 and η_2 are functions of μ and ζ as well as y because q_+' and the E_{jk} depend on these variables. The solution ϕ_1 of Theorem 2.1 will be defined by

$$\phi_1(y,\mu,\zeta) = \exp\{q_+'(\mu,\zeta)y\}(\eta_1(y,\mu,\zeta) + \eta_2(y,\mu,\zeta)) \tag{2.59}$$

To complete the proof that ϕ_1 is the desired function on $R \times R_+ \times L(c(\infty)\mu)$ it is only necessary to verify that (2.46) holds for each $(\mu,\zeta) \in R_+ \times L(c(\infty)\mu)$. It is clear from (2.49) that

$$|\eta_1(y) - 1| \leq \|\eta\| \int_y^{\infty} \sum_{j=1}^{2} |E_{1j}(y')|\, dy' = o(1),\ y \to +\infty. \tag{2.60}$$

For η_2, (2.49) implies that

$$|\eta_2(y)| \leq \|\eta\| \left(\int_{y_1}^{y_2} \exp\{-2q_+'(y-y')\} \sum_{j=1}^{2} |E_{2j}(y')|dy' + \int_{y_2}^{\infty} \sum_{j=1}^{2} |E_{2j}(y')|dy' \right) \tag{2.61}$$

for every $y_2 \geq y_1$ and every $y \geq y_2$. Hence for any fixed y_2 one has

$$(2.62) \qquad \limsup_{y\to\infty} |\eta_2(y)| \leq \|\eta\| \int_{y_2}^{\infty} \sum_{j=1}^{2} |E_{2j}(y')| \, dy'$$

because $\operatorname{Re} q_+' > 0$. Since y_2 in (2.62) is arbitrary it follows that $\eta_2(y) = o(1)$.

<u>Construction of ϕ_1 for $\zeta \in R(c(\infty)\mu)$</u>. By (2.3) $R(c(\infty)\mu)$ has the decomposition

$$(2.63) \qquad R(c(\infty)\mu) = R^+(c(\infty)\mu) \cup R^-(c(\infty)\mu)^{\text{int}}.$$

Moreover, for $\zeta \in R^-(c(\infty)\mu)^{\text{int}}$ one has $\operatorname{Re} q_+'(\mu,\zeta) > 0$ and hence the construction of the preceding case is valid. In the complementary case where $\zeta \in R^+(c(\infty)\mu)$ one has $\operatorname{Re} q_+'(\mu,\zeta) \leq 0$ and it is permissable to take $c_1 = 1$, $c_2 = 0$, $y_0 = y_1 = \infty$ in (2.48). The resulting system of integral equations

$$(2.64) \qquad \left.\begin{cases} \eta_1(y) = 1 - \displaystyle\int_y^{\infty} E_{1j}(y') \, \eta_j(y')dy' \\ \eta_2(y) = \quad - \displaystyle\int_y^{\infty} \exp\{-2q_+'(y-y')\} \, E_{2j}(y') \, \eta_j(y')dy' \end{cases}\right\} \; y \geq y_1$$

again defines an equation (2.57) in the Banach space X. Moreover, $|\exp\{-2q_+'(y-y')| \leq 1$ for $y \leq y' < \infty$ and (2.56) is again valid. It follows that for y_1 large enough (2.64) has a unique solution given by (2.58). The solution has a unique continuation to a solution of (2.47) on the interval $y \in R$. The validity of the asymptotic condition (2.46) is obvious from (2.64); cf. (2.60).

Proof of Corollary 2.2. Again the proof will be given for ϕ_1 only. Note that by (2.39)

$$N(y,\mu,\zeta) = N_1(y) + \mu^2 N_2(y) + \zeta N_3(y) \tag{2.65}$$

where the components of $N_j(y)$ are in $L_1(y_0,\infty)$ for $j = 1, 2, 3$ and every $y_0 \in R$. Thus by (2.43)

$$\begin{aligned} &E(y,\mu,\zeta) \\ &= B^{-1}(\mu,\zeta)N_1(y)B(\mu,\zeta) + \mu^2 B^{-1}(\mu,\zeta)N_2(y)B(\mu,\zeta) + \zeta B^{-1}(\mu,\zeta)N_3(y)B(\mu,\zeta) \end{aligned} \tag{2.66}$$

Proof of (2.12) for ϕ_1. Note that $q_+'(\mu,\zeta)$, and hence also $B(\mu,\zeta)$ and $B^{-1}(\mu,\zeta)$ are continuous functions on the set

$$\bigcup_{\mu>0} \{(\mu,\zeta) \mid \zeta \in L(c(\infty)\mu)\}. \tag{2.67}$$

Thus by using the estimate (2.56) for the operator $K = K(\mu,\zeta)$ in X one can show that for each compact subset Γ of the set (2.67) and each $\delta > 1$ there is a constant $y_1 = y_1(\Gamma,\delta)$ such that, taking $y_1 = y_1(\Gamma,\delta)$ in the definition of $K(\mu,\zeta)$, one has

$$\|K(\mu,\zeta)\| \leq \delta \text{ for all } (\mu,\zeta) \in \Gamma. \tag{2.68}$$

Hence the series (2.58) converges uniformly in X for $(\mu,\zeta) \in \Gamma$ which implies the continuity of ϕ_1 and $\rho^{-1}\phi_1'$ on the set $[y_1,\infty) \times \Gamma$. Their continuity on $R \times \Gamma$ then follows from the classical theorem on the continuous dependence of solutions of initial value problems on parameters. This implies the result (2.12) for ϕ_1 because Γ was an arbitrary compact subset of the set (2.67).

Proof of (2.14) for ϕ_1. The method used in the preceding case is applicable to the operator $K(\mu,\zeta)$ in X defined by (2.64).

Remark on Corollary 2.2. The argument given above can also be used to show that

$$(2.69) \qquad \phi_1,\ \rho^{-1}\phi_1' \in C\left(R \times \bigcup_{\mu>0} \{(\mu,\zeta) \mid \zeta \in R^-(c(\infty)\mu)^{\text{int}}\}\right).$$

However, the continuity of ϕ_1 and $\rho^{-1}\phi_1'$ on the set

$$(2.70) \qquad R \times \bigcup_{\mu>0} \{(\mu,\zeta) \mid \zeta \in R(c(\infty)\mu)\}$$

cannot be asserted since the constructions for $\zeta \in R^+(c(\infty)\mu)$ and $\zeta \in R^-(c(\infty)\mu)^{\text{int}}$ are different. Indeed, continuity of ϕ_1 on the set (2.70) is not to be expected since ϕ_1 is not uniquely determined when $\zeta \in R^-(c(\infty)\mu)^{\text{int}}$.

Proof of Corollary 2.3. The components of the matrix-valued function $E(y,\mu,\zeta)$ are analytic functions of $\zeta \in L(c(\infty)\mu) \cup R^+(c(\infty)\mu)^{\text{int}}$ for fixed values of y, μ. Hence the uniform convergence of the Neumann series (2.58) on compact subsets of this set, which follows from the proof of Corollary 2.2, implies the validity of Corollary 2.3 for ϕ_1. The remaining cases can be proved by the same method.

Proof of Corollary 2.4. The proof will be given for the case of ϕ_1 and (μ,ζ) in a compact subset Γ of the set (2.67). The remaining cases can be proved similarly. $\phi_1(y,\mu,\zeta)$ was defined by (2.59) and the functions $\eta_j(y,\mu,\zeta)$ satisfy

$$\eta_1(y,\mu,\zeta) - 1 = -\int_y^\infty E_{1j}(y',\mu,\zeta)\ \eta_j(y',\mu,\zeta)\ dy',$$

(2.71)

$$\eta_2(y,\mu,\zeta) = -\int_y^\infty \exp\ \{-2q_+'(\mu,\zeta)(y-y')\}\ E_{2j}(y',\mu,\zeta)\ \eta_j(y',\mu,\zeta)\ dy'.$$

It must be shown that these integrals tend to zero when $y \to \infty$, uniformly for $(\mu,\zeta) \in \Gamma$. Now (2.58) and the estimate (2.68) from the proof of Corollary 2.2 imply that $\|\eta(\cdot,\mu,\zeta)\| \le (1-\delta)^{-1}$ for all $(\mu,\zeta) \in \Gamma$. It follows that for fixed $y' \ge y \ge y_1(\Gamma,\delta)$ one has

$$|E_{kj}(y',\mu,\zeta)\ \eta_j(y',\mu,\zeta)| \le \|E(y',\mu,\zeta)\|\ \|\eta(y',\mu,\zeta)\|$$

(2.72)

$$\le (1-\delta)^{-1}\ \|E(y',\mu,\zeta)\|$$

for $(\mu,\zeta) \in \Gamma$. Now the continuity of $B(\mu,\zeta)$ implies that there is a $\gamma = \gamma(\Gamma)$ such that

(2.73) $$\|B(\mu,\zeta)\|\ \|B^{-1}(\mu,\delta)\|(1+\mu^2+|\zeta|) \le \gamma \text{ for } (\mu,\zeta) \in \Gamma$$

It follows from (2.66) that

(2.74) $$\|E(y',\mu,\zeta)\| \le \gamma \sum_{j=1}^{3} \|N_j(y')\|,\ (\mu,\zeta) \in \Gamma.$$

Combining (2.71), (2.72) and (2.74) gives

(2.75) $$|\eta_1(y,\mu,\zeta) - 1| \le \gamma(1-\delta)^{-1} \int_y^\infty \sum_{j=1}^{3} \|N_j(y')\|\ dy'$$

for all $y \ge y_1(\Gamma,\delta)$ and $(\mu,\zeta) \in \Gamma$. Since each $N_j \in L_2(y_0,\infty)$, (2.75) implies that $\eta_1(y,\mu,\zeta) - 1 = o(1)$ uniformly for $(\mu,\zeta) \in \Gamma$.

The case of $\eta_2(y,\mu,\zeta)$ is more complicated. Note that

$$|\eta_2(y,\mu,\zeta)| \leq \gamma(1-\delta)^{-1} \int_y^\infty \exp\{-2 \operatorname{Re} q_+'(\mu,\zeta)(y-y')\} \sum_{j=1}^{3} \|N_j(y')\| \, dy'. \tag{2.76}$$

Now the continuity of $q_+'(\mu,\zeta)$ and the definition of $L(c(\infty)\mu)$ imply that there is a $\kappa = \kappa(\Gamma) > 0$ such that

$$2 \operatorname{Re} q_+'(\mu,\zeta) \geq \kappa > 0 \text{ for all } (\mu,\zeta) \in \Gamma. \tag{2.77}$$

Combining (2.76), (2.77) one has, if $\gamma_1 = \gamma(1-\delta)^{-1}$,

$$\begin{aligned} &|\eta_2(y,\mu,\zeta)| \\ &\leq \gamma_1 \left\{\int_y^{y_2} \exp\{-\kappa(y-y')\} \sum_{j=1}^{3} \|N_j(y')\| dy' + \int_{y_2}^\infty \sum_{j=1}^{3} \|N_j(y')\| dy'\right\} \\ &\leq \gamma_1 \left\{\exp\{-\kappa(y-y_2)\} \int_{y_1}^\infty \sum_{j=1}^{3} \|N_j(y')\| dy' + \int_{y_2}^\infty \sum_{j=1}^{3} \|N_j(y')\| dy'\right\} \end{aligned} \tag{2.78}$$

for all $y_2 \geq y_1(\Gamma,\delta)$, $y \geq y_2$ and $(\mu,\zeta) \in \Gamma$.

Now let $\varepsilon > 0$ be given and choose $y_2 = y_2(\varepsilon,\Gamma,\delta) \geq y_1(\Gamma,\delta)$ such that

$$\gamma_1 \int_{y_2}^\infty \sum_{j=1}^{3} \|N_j(y')\| \, dy' < \varepsilon/2, \tag{2.79}$$

and hence

$$|\eta_2(y,\mu,\zeta)| \leq \gamma_1 \exp\{-\kappa(y-y_2)\} \int_{y_1}^\infty \sum_{j=1}^{3} \|N_j(y')\| \, dy' + \varepsilon/2 \tag{2.80}$$

for all $y \geq y_2(\varepsilon,\Gamma,\delta) \geq y_1(\Gamma,\delta)$ and $(\mu,\zeta) \in \Gamma$. Finally, choose a

$y_3(\varepsilon,\Gamma,\delta) \geq y_2(\varepsilon,\Gamma,\delta)$ such that

$$\text{(2.81)} \qquad \gamma_1 \exp\{-\kappa(y-y_2)\} \int_{y_1}^{\infty} \sum_{j=1}^{3} \|N_j(y')\| \, dy' < \varepsilon/2$$

for all $y \geq y_3(\varepsilon,\Gamma,\delta)$. It follows from (2.80) and (2.81) that $|\eta_2(y,\mu,\zeta)| < \varepsilon$ for all $y \geq y_3(\varepsilon,\Gamma,\delta)$ and $(\mu,\zeta) \in \Gamma$; i.e., $\eta_2(y,\mu,\zeta) = o(1)$ uniformly for $(\mu,\zeta) \in \Gamma$.

Proof of Corollary 2.5. It will be shown that $\phi_1(y,\mu,\zeta)$ is uniquely determined by (2.1) and (2.7) when $\zeta \in R^+(c(\infty)\mu)$. The other cases are proved similarly.

Assume that for some $\zeta \in R^+(c(\infty)\mu)$ there are two solutions of (2.1), (2.7). Then their difference $\phi(y)$ would satisfy (2.1) and $\phi(y) = o(1)$, $\rho^{-1}(y)\, \phi'(y) = o(1)$ because $\text{Re}\, q_+' \leq 0$ for $\zeta \in R^+(c(\infty)\mu)$. It follows that the corresponding pair $\eta_1(y)$, $\eta_2(y)$, defined by (2.44) and (2.45), would necessarily satisfy

$$\text{(2.82)} \qquad \begin{aligned} \eta_1(y) &= -\int_y^{\infty} E_{1j}(y')\, \eta_j(y')\, dy' \\ \eta_2(y) &= -\int_y^{\infty} \exp\{-2q_+'(y-y')\}\, E_{2j}(y')\, \eta_j(y')\, dy' \end{aligned}$$

since $|\exp\{-2q_+'(y-y')| \leq 1$ for $y \leq y'$. But (2.82) is equivalent to the equation $\eta = K\eta$ in X. If y_1 is chosen so large that $\|K\| < 1$ then $\eta = K\eta$ has the unique solution $\eta(y) \equiv 0$ for $y \geq y_1$. The unique continuation of this solution of (2.47) is then zero for all $y \in R$. Thus $\phi(y) \equiv 0$ for $y \in R$, which proves the uniqueness.

Proof of Theorem 2.6. The equation $A_\mu\, \phi = c^2(\infty)\, \mu^2 \phi$ is equivalent under the mapping (2.26) with the system (see (2.31), (2.38))

$$
(2.83) \qquad \begin{cases} \psi_1' = \rho(\infty)\psi_2 + B_1(y)\psi_2 \\ \psi_2' = B_2(y)\psi_1 \end{cases}
$$

where $B_1(y) = a_1(y) = \rho(\infty) - \rho(\infty)$ and

$$
(2.84) \qquad \begin{aligned} B_2(y) &= \mu^2\ \rho^{-1}(y)(1 - c^2(\infty)c^{-2}(y)) \\ &= \mu^2\ \rho^{-1}(y)\ c^{-2}(y)[c(y) + c(\infty)][c(y) - c(\infty)] \end{aligned}
$$

It follows from hypotheses (1.3) and (2.19) that

$$
(2.85) \qquad B_1(y),\ B_2(y),\ y\ B_2(y),\ y^2\ B_2(y) \in L_1(y_0,\infty)
$$

for every $y_0 \in R$ and every $\mu \geq 0$.

<u>Construction of ϕ_1</u>. *Application of the variation of constants* formula to the system (2.83) gives the integrated form

$$
(2.86) \qquad \begin{aligned} \psi_1(y) &= c_1 + \rho(\infty)\ c_2 y + \int_{y_0}^{y} \{\rho(\infty)(y-y')\ B_2(y')\ \psi_1(y') + B_1(y')\ \psi_2(y')\}dy' \\ \psi_2(y) &= c_2 + \int_{y_0}^{y} B_2(y')\ \psi_1(y')\ dy'. \end{aligned}
$$

Now ϕ_1 will satisfy $A_\mu\ \phi_1 = c^2(\infty)\mu^2\phi_1$ and the asymptotic condition (2.21) provided that $\psi_1 = \phi_1$, $\psi_2 = \rho^{-1}\phi_1'$ satisfies (2.86) and

$$
(2.87) \qquad \psi_1(y) = 1 + o(1),\ \psi_2(y) = o(1),\ y \to \infty.
$$

To construct such a solution take $c_1 = 1$, $c_2 = 0$ and $y_0 = \infty$ in (2.86). This gives the system

$$(2.88)\quad \begin{cases} \psi_1(y) = 1 - \displaystyle\int_y^\infty \{\rho(\infty)(y-y')\, B_2(y')\, \psi_1(y') + B_1(y')\, \psi_2(y')\}\, dy' \\ \psi_2(y) = \quad - \displaystyle\int_y^\infty B_2(y')\, \psi_1(y')\, dy' \end{cases}$$

or

$$(2.89)\qquad \psi(y) = \psi^0 + \int_{y_1}^\infty K(y,y')\, \psi(y')\, dy', \; y \geq y_1,$$

where $\psi(y)$ and ψ^0 have components $\psi_1(y)$, $\psi_2(y)$ and 1, 0 respectively, $K(y,y') \equiv 0$ for $y \geq y'$ and

$$(2.90)\qquad \begin{aligned} K_{11}(y,y') &= \rho(\infty)(y'-y)\, B_2(y') \\ K_{12}(y,y') &= -B_1(y') \\ K_{21}(y,y') &= -B_2(y') \\ K_{22}(y,y') &= 0 \end{aligned}$$

for $y \leq y'$. As in the proof of Theorem 2.1, one has

$$(2.91)\qquad |(K\psi)_j(y)| \leq \|\psi\| \int_{y_1}^\infty (|K_{j1}(y,y')| + |K_{j2}(y,y')|)\, dy'$$

and hence

$$(2.92)\qquad \begin{aligned} |(K\psi)_1(y)| &\leq \|\psi\| \left\{\rho(\infty) \int_y^\infty (y+y')|B_2(y')|dy' + \int_y^\infty |B_1(y')|dy'\right\} \\ &\leq \|\psi\| \left\{2\rho(\infty) \int_y^\infty y'\, |B_2(y')|dy' + \int_y^\infty |B_1(y')|dy'\right\} \end{aligned}$$

and

$$(2.93) \qquad |(K\psi)_2(y)| \leq \|\psi\| \int_y^\infty |B_2(y')|\, dy'$$

In particular,

$$(2.94) \qquad \|K\| \leq 2\rho(\infty) \int_{y_1}^\infty y'|B_2(y')|dy' + \int_{y_1}^\infty |B_1(y')|dy' + \int_{y_1}^\infty |B_2(y')|dy'$$

Thus (2.85) implies that K is contractive in X for y_1 large enough. Hence (2.88) has a unique solution on $[y_1,\infty)$ which can be continued as a solution of (2.83) to all $y \in R$. Moreover, (2.88), (2.92) and (2.93) imply that $|\psi_1(y) - 1| = |(K\psi)_1(y)| = o(1)$ and $|\psi_2(y)| = |(K\psi)_2(y)| = o(1)$. In fact (2.85) implies that $|\psi_2(y)| = o(y^{-2})$. Thus (2.87) is satisfied.

<u>Construction of ϕ_2</u>. ϕ_2 will satisfy $A_\mu\phi_2 = c^2(\infty)\mu^2\phi_2$ and the asymptotic condition (2.22) provided

$$(2.95) \qquad \begin{cases} \psi_1(y) = \phi_2(y) = \rho(\infty)y\ \eta_1(y) \\ \psi_2(y) = \rho^{-1}(y)\phi_2'(y) = \eta_2(y) \end{cases}$$

where

$$(2.96) \qquad \eta_1(y) = 1 + o(1),\ \eta_2(y) = 1 + o(1),\ y \to \infty.$$

Substituting in (2.86) with $c_1 = 0$, $c_2 = 1$ and $y_0 = \infty$ gives, after simplification,

$$\eta_1(y) = 1 - \int_y^\infty \{\rho(\infty)(y'-y^{-1}y'^2)B_2(y')\eta_1(y') + \rho^{-1}(\infty)y^{-1}B_1(y')\eta_2(y')\}dy'$$

(2.97)

$$\eta_2(y) = 1 - \int_y^\infty \rho(\infty)y'B_2(y')\eta_1(y')dy'.$$

or

(2.98)
$$\eta(y) = \eta^0 + \int_{y_1}^\infty K(y,y')\ \eta(y')\ dy',\ y \geq y_1$$

where $\eta(y)$ and η^0 have components $\eta_1(y)$, $\eta_2(y)$ and 1, 1 respectively, $K(y,y') \equiv 0$ for $y \geq y'$ and

(2.99)
$$K_{11}(y,y') = \rho(\infty)(-y' + y^{-1}y'^2)B_2(y')$$

$$K_{12}(y,y') = -\rho^{-1}(\infty)y^{-1}\ B_1(y')$$

$$K_{21}(y,y') = -\rho(\infty)\ y'\ B_2(y')$$

$$K_{22}(y,y') = 0$$

for $y \leq y'$. It follows from (2.91) that

$$|(K\eta)_1(y)| \leq \|\eta\| \left\{\rho(\infty)\int_y^\infty (y'+y^{-1}y'^2)|B_2(y')|dy' + \rho^{-1}(\infty)y^{-1}\int_y^\infty |B_1(y')|dy'\right\}$$

(2.100)

and

(2.101)
$$|(K\eta)_2(y)| \leq \|\eta\|\ \rho(\infty)\int_y^\infty y'\ |B_2(y')|\ dy'$$

In particular,

$$
\|K\| \le \rho(\infty) \int_{y_1}^{\infty} y' |B_2(y')| dy' + \rho(\infty) y_1^{-1} \int_{y_1}^{\infty} y'^2 |B_2(y')| dy'
$$

(2.102)

$$
+ \rho^{-1}(\infty) y_1^{-1} \int_{y_1}^{\infty} |B_1(y')| dy' + \rho(\infty) \int_{y_1}^{\infty} y' |B_2(y')| dy'
$$

Hence (2.85) implies that K is contractive in X for y_1 large enough and a unique solution is obtained as in the preceding case. Finally, (2.100) and (2.101) together with (2.85) imply that (2.96) is satisfied.

The existence of the special solutions ϕ_1, ϕ_2 has thus been proved. Their linear independence follows directly from (2.21), (2.22).

Proof of Corollary 2.7. This was verified by (2.25).

§3. Spectral Properties of A_μ.

The results of §2 are used in this section to derive precise results concerning the location and nature of the spectrum of A_μ. The notations $\sigma(A_\mu)$, $\sigma_0(A_\mu)$, $\sigma_c(A_\mu)$ and $\sigma_e(A_\mu)$ will be used to denote the spectrum, point spectrum, continuous spectrum and essential spectrum of A_μ, respectively. The definitions of [8, Ch. X] will be used. In particular, $\sigma_c(A_\mu)$ is a closed set and $\sigma_e(A_\mu)$ is the set of all non-isolated points of $\sigma(A_\mu)$. Note that the properties of A_μ described by (1.19) imply that $\sigma(A_\mu) \subset [c_m^2\mu^2, \infty)$.

The Point Spectrum of A_μ. Theorem 2.1 and its corollaries imply the following three lemmas concerning $\sigma_0(A_\mu)$.

Lemma 3.1. For all $\mu > 0$,

$$\sigma_0(A_\mu) \subset [c_m^2\mu^2, c^2(\infty)\mu^2]. \tag{3.1}$$

Lemma 3.2. For all $\mu > 0$,

$$\sigma(A_\mu) \cap [c_m^2\mu^2, c^2(\infty)\mu^2) \subset \sigma_0(A_\mu). \tag{3.2}$$

Moreover, $\sigma_0(A_\mu)$ is either a finite set (possibly empty) or a countable set with unique limit point $c^2(\infty)\mu^2$.

Lemma 3.3. The eigenvalues of A_μ that lie in the interval $[c_m^2\mu^2, c^2(\infty)\mu^2)$ are all simple.

The possibility that $c^2(\infty)\mu^2 \in \sigma_0(A_\mu)$ is not excluded by the hypotheses (1.3), (1.4) alone. Criteria for $c^2(\infty)\mu^2 \notin \sigma_0(A_\mu)$ are given below.

It will be convenient to use a notation that permits a unified discussion of the cases of finite and infinite point spectra $\sigma_0(A_\mu)$.

The number of eigenvalues in $[c_m^2\mu^2, c^2(\infty)\mu^2)$ will be denoted by $N(\mu) - 1$. Thus $N(\mu)$ is an extended integer-valued function of $\mu > 0$ $(1 \le N(\mu) \le +\infty)$. The eigenvalues of A_μ in $[c_m^2\mu^2, c^2(\infty)\mu^2)$, arranged in ascending order will be denoted by $\lambda_k(\mu)$, $1 \le j < N(\mu)$. Thus

$$c_m^2\mu^2 \le \lambda_1(\mu) < \lambda_2(\mu) < \cdots < c^2(\infty)\mu^2. \tag{3.3}$$

The corresponding eigenfunctions are

$$\psi_k(y,\mu) = a_k(\mu)\ \phi_2(y,\mu,\lambda_k(\mu)),\ k = 1,2,\ldots \tag{3.4}$$

where $a_k(\mu) > 0$ is chosen to make $\|\psi_k(\cdot,\mu)\| = 1$.

The Continuous and Essential Spectra of A_μ. Lemma 3.2 implies that $\sigma_e(A_\mu) \subset [c^2(\infty)\mu^2,\infty)$. Moreover, $\sigma_c(A_\mu)$ and $\sigma_e(A_\mu)$ are closed and $\sigma_c(A_\mu) \subset \sigma_e(A_\mu)$ [8, Ch. X]. The characterization of these sets will be completed in §5 by showing that $(c^2(\infty)\mu^2,\infty) \subset \sigma_c(A_\mu)$. These facts imply

Theorem 3.4. For all $\mu > 0$,

$$\sigma_c(A_\mu) = \sigma_e(A_\mu) = [c^2(\infty)\mu^2,\infty). \tag{3.5}$$

A direct proof of Theorem 3.4 can be given by using the special solutions of §2 and a criterion of Weyl; see [3, p. 1435].

It is known that the bottom point in the essential spectrum of a Sturm-Liouville operator A can be characterized by the oscillation properties of the solutions of $A\phi = \lambda\phi$ [3, p. 1469]. For the operator A_μ the characterization is described by

Corollary 3.5. The equation $A_\mu\ \phi = \lambda\phi$ is oscillatory (every real solution has infinitely many zeros) for every $\lambda > c^2(\infty)\mu^2$. The equation is non-oscillatory (every real solution has finitely many zeros) for every $\lambda < c^2(\infty)\mu^2$.

These results for A_μ follow directly from Theorem 2.1.

The Point Spectrum of A_μ (continued). The equation $A_\mu \phi = \lambda\phi$ may or may not be oscillatory for $\lambda = c^2(\infty)\mu^2$. This property is shown below to provide a criterion for $\sigma_0(A_\mu)$ to be finite. The basic tool in establishing such criteria is the classical oscillation theorem of Sturm. A version suitable for application to A_μ may be formulated as follows.

Let $I = (a,b)$ be an arbitrary interval $(-\infty \leq a < b \leq +\infty)$ and consider a pair of equations

$$L_j\phi \equiv (P_j^{-1}(y)\phi')' + Q_j(y)\phi = 0, \; j = 1,2, \tag{3.6}$$

where $P_j(y)$ and $Q_j(y)$ are defined and real valued for almost every $y \in I$, $P_j(y) > 0$ for almost every $y \in I$ and P_j, Q_j are Lebesgue integrable on compact subsets of I $(j = 1,2)$. A solution of (3.6) on I is a function $\phi \in AC(I)$ such that $P_j^{-1}\phi' \in AC(I)$ and (3.6) holds for almost all $y \in I$. Such solutions are uniquely determined by the values $\phi(y_0) = c_0$, $P_j^{-1}(y_0)\phi'(y_0) = c_1$ at any point $y_0 \in I$. Pairs of equations (3.6) such that

$$P_1(y) \leq P_2(y), \; Q_1(y) \leq Q_2(y) \text{ for almost all } y \in I \tag{3.7}$$

will be considered. When (3.7) holds the operator L_2 is said to be a Sturm majorant of operator L_1, and the operator L_1 is said to be a Sturm minorant of operator L_2, on I. Sturm's theorem may now be formulated as follows.

Theorem 3.6. Let $\phi_j(y) \not\equiv 0$ be solutions of $L_j\phi_j = 0$ on I $(j = 1,2)$ and assume that y_1 and y_2 are successive zeros of $\phi_1(y)$ in I, with $y_1 < y_2$. Moreover, let L_2 be a Sturm majorant of L_1 on (y_1,y_2). Then $\phi_2(y)$ has at least one zero in $[y_1,y_2)$. In addition, if either

$Q_1(y) < Q_2(y)$ or $P_1(y) < P_2(y)$ and $Q_2(y) \neq 0$ on a subset of (y_1,y_2) having positive measure then $\phi_2(y)$ has a zero in (y_1,y_2).

The special solution $\phi_3(y,\mu,\lambda)$ is real valued, tends to zero exponentially when $y \to -\infty$ and has finitely many zeros when $\lambda < c^2(\infty)\mu^2$. Theorem 3.6 will be shown to imply

Corollary 3.7. If $\lambda_1 < \lambda_2 < c^2(\infty)\mu^2$ then $\phi_3(y,\mu,\lambda_2)$ has at least as many zeros as $\phi_3(y,\mu,\lambda_1)$.

It will be convenient following [3, p. 1473] to introduce the sets

$$(3.8) \qquad I_k = I_k(\mu) = \{\lambda \mid \phi_3(y,\mu,\lambda) \text{ has exactly } k \text{ zeros}\},\ k = 0,1,2,\cdots.$$

Note that by Corollary 3.5 each $I_k \subset (-\infty, c^2(\infty)\mu^2]$. The point $c^2(\infty)\mu^2$ may or may not be in one of the sets I_k. Corollary 3.7 implies that each I_k is an interval and I_k lies to the left of I_{k+1} for $k = 0,1,2,\cdots$. It is important for the analysis of $\sigma_0(A_\mu)$ to know that the intervals $I_k \neq \phi$ for $k = 1,2,\cdots,N(\mu) - 1$. This is a corollary of the following fundamental oscillation theorem.

Theorem 3.8. If $\sigma_0(A_\mu) \neq \phi$ then for $k = 1,2,\cdots,N(\mu) - 1$ the eigenfunction $\psi_k(y,\mu)$ has precisely $k - 1$ zeros.

Corollary 3.9. If $\sigma_0(A_\mu) \neq \phi$ then

$$(3.9) \qquad I_k = (\lambda_k(\mu),\lambda_{k+1}(\mu)],\ k = 0,1,\cdots,N(\mu) - 2$$

(where $\lambda_0(\mu) \equiv -\infty$). Moreover, if $N(\mu) < \infty$ then $(\lambda_{N(\mu)-1}(\mu),c^2(\infty)\mu^2) \subset I_{N(\mu)-1}$.

Corollary 3.10. The number of eigenvalues that satisfy $\lambda_k(\mu) < \lambda < c^2(\infty)\mu^2$ is equal to the number of zeros of $\phi_3(y,\mu,\lambda)$.

<u>Criteria for the Finiteness of $\sigma_0(A_\mu)$</u>. The principal criterion for $\sigma_0(A_\mu)$ to be finite is described by

<u>Theorem 3.11</u>. $\sigma_0(A_\mu)$ is finite if and only if the equation $A_\mu\,\phi = c^2(\infty)\mu^2\phi$ is non-oscillatory on R. Hence $\sigma_0(A_\mu)$ is infinite if and only if $A_\mu\,\phi = c^2(\infty)\mu^2\phi$ is oscillatory on R.

It is shown below that Theorem 3.11 is a consequence of Theorem 3.8 and Sturm's comparison theorem.

<u>Corollary 3.12</u>. If $c(\infty) < c(-\infty)$ then $\sigma_0(A_\mu)$ is finite if and only if $\phi_3(y,\mu,c^2(\infty)\mu^2)$ has only a finite number of zeros.

Specific criteria for the finiteness of $\sigma_0(A_\mu)$ will now be obtained by deriving criteria for $A_\mu\,\phi = c^2(\infty)\mu^2\phi$ to be non-oscillatory and using Theorem 3.11. It will be assumed that $c(\infty) < c(-\infty)$ so that $A_\mu\,\phi = c^2(\infty)\mu^2\phi$ is non-oscillatory in neighborhoods of $y = -\infty$. Cases for which $c(\infty) = c(-\infty)$ may be treated by applying non-oscillation criteria at both $y = \infty$ and $y = -\infty$.

A criterion for $\sigma_0(A_\mu)$ to be finite is provided by Theorem 2.6. For under the conditions of the theorem $A_\mu\,\phi = c^2(\infty)\mu^2\phi$ has a solution basis ϕ_1, ϕ_2 satisfying (2.21), (2.22). It follows that the equation is non-oscillatory. This implies

<u>Theorem 3.13</u>. If $\rho(y)$, $c(y)$ satisfy (1.3), (1.4), $c(\infty) < c(-\infty)$ and

$$(3.10) \qquad \int_0^\infty y^2\,|c(y) - c(\infty)|\,dy < \infty$$

then $\sigma_0(A_\mu)$ is finite for every $\mu > 0$.

Alternative criteria for the finiteness of $\sigma_0(A_\mu)$ can be derived by constructing Sturm majorants of $A_\mu\,\phi = c^2(\infty)\mu^2\phi$ that are non-oscillatory and using Theorems 3.6 and 3.11. Similarly, criteria for $\sigma_0(A_\mu)$ to be infinite can be derived by constructing Sturm minorants that are oscillatory. Several criteria of this type will be given.

The equation $A_\mu \phi = \lambda\phi$ can be written

$$(\rho^{-1}(y)\phi')' + \rho^{-1}(y)(\lambda\, c^{-2}(y) - \mu^2)\phi = 0. \tag{3.11}$$

In particular, for $\lambda = c^2(\infty)\mu^2$ one has

$$(\rho^{-1}(y)\phi')' + \rho^{-1}(y)\mu^2(c^2(\infty)c^{-2}(y) - 1)\phi = 0. \tag{3.12}$$

The first factor $\rho^{-1}(y)$ in (3.12) is unimportant for the oscillation properties of the equation. Replacing it by ρ_M^{-1} gives the majorant

$$\phi'' + \rho_M\, \rho^{-1}(y)\mu^2\, (c^2(\infty)c^{-2}(y) - 1)\phi = 0. \tag{3.13}$$

Each non-oscillatory Sturm majorant of (3.13) gives a criterion for the finiteness of $\sigma_0(A_\mu)$. Since solutions of (3.13) are non-oscillatory on any interval $(-\infty, y_0)$ when $c(\infty) < c(-\infty)$, it is enough to construct majorants of (3.13) on intervals (y_0, ∞). An obvious non-oscillatory majorant for (3.13) is $\phi'' = 0$. Thus $\sigma_0(A_\mu)$ is finite for every $\mu > 0$ if there is a y_0 such that $c^2(\infty)c^{-2}(y) - 1 \le 0$ for all $y \ge y_0$; that is,

$$c(y) \ge c(\infty) \text{ for } y \ge y_0. \tag{3.14}$$

This means the graph of $c = c(y)$ lies above or on the limit line $c = c(\infty)$ in a neighborhood of $y = \infty$. Weaker hypotheses that include this case can be derived by comparing (3.13) with

$$\phi'' + \alpha\, y^{-2}\, \phi = 0 \tag{3.15}$$

which is oscillatory on (y_0, ∞) if $\alpha > 1/4$ and non-oscillatory if $\alpha \le 1/4$.

Oscillation theorems based on (3.15) were first given by A. Kneser [9]; see [3, p. 1463]. Comparison of (3.12) with (3.15) gives

Theorem 3.14. If $c(\infty) < c(-\infty)$ and

$$\limsup_{y\to\infty} y^2(c^{-2}(y) - c^{-2}(\infty)) \leq 0 \tag{3.16}$$

then $\sigma_0(A_\mu)$ is finite for all $\mu > 0$. Conversely, if

$$\liminf_{y\to\infty} y^2(c^{-2}(y) - c^{-2}(\infty)) > 0 \tag{3.17}$$

then there exists a $\mu_0 > 0$ such that $\sigma_0(A_\mu)$ is infinite for every $\mu \geq \mu_0$.

Note that the criterion (3.16) includes (3.14) as a special case. Note also that sufficient conditions for (3.16) or (3.17) to hold are the existence of constants y_0, K and $\varepsilon > 0$ such that

$$c(y) \geq c(\infty) - K\, y^{-2-\varepsilon} \quad \text{for } y \geq y_0 \tag{3.18}$$

or

$$c(y) \leq c(\infty) - K\, y^{-2} \quad \text{for } y \leq y_0, \tag{3.19}$$

respectively. In particular, $\sigma_0(A_\mu)$ is finite for all $\mu > 0$ if $c(y)$ approaches $c(\infty)$ from below sufficiently rapidly.

Criteria that Guarantee $\sigma_0(A_\mu) \neq \phi$. Such criteria may be derived by constructing Sturm minorants for $A_\mu\, \phi = c^2(\infty)\mu^2\phi$ whose solutions have zeros. If the minorant has solutions with infinitely many zeros then $\sigma_0(A_\mu)$ is infinite. If the minorant has a solution with finitely many zeros then it can be shown that $\phi_3(y,\mu,c^2(\infty)\mu^2)$ has at least as many zeros and one may use the following refinement of Theorem 3.11.

Theorem 3.15. If $A_\mu\, \phi = c^2(\infty)\mu^2\phi$ has a solution having a finite number k of zeros on R then the part of $\sigma(A_\mu)$ below $c^2(\infty)\mu^2$ is finite and has at least $k - 1$ and at most $k + 2$ points.

To apply the method in cases where $\sigma_0(A_\mu)$ is finite consider first the case $\rho(y) = \text{const.}$ so that (3.12) becomes

$$\phi'' + \mu^2(c^2(\infty)c^{-2}(y) - 1)\phi = 0 \tag{3.20}$$

Note that $c^2(\infty)c^{-2}(y) - 1 \geq c^2(\infty)c_0^{-2}(y) - 1$ for all $y \in R$ if and only if

$$c(y) \leq c_0(y) \text{ for all } y \in R. \tag{3.21}$$

If $c_0(y)$ can be chosen in such a way that

$$\phi'' + \mu^2(c^2(\infty)c_0^{-2}(y) - 1)\phi = 0 \tag{3.22}$$

has a solution on R with k zeros then $\sigma_0(A_\mu)$ will have at least k - 1 points by Theorem 3.15. In this way one can prove

<u>Theorem 3.16</u>. Let $\rho(y) = \text{const.}$ for all $y \in R$ and assume that there is a constant $c_0 \geq c_m$ and an interval $I = [a,b]$ with $b > a$ such that

$$c(y) \leq c_0 < c(\infty) \leq c(-\infty) \text{ for all } y \in I. \tag{3.23}$$

Then $\sigma_0(A_\mu) \neq \phi$ for all sufficiently large μ. In fact, $N(\mu) \to \infty$ when $\mu \to \infty$.

Theorem 3.16 can be proved by comparing c(y) with a suitable piece-wise constant function $c_0(y)$ that satisfies (3.21). An analogue of Theorem 3.16 can be proved in the general case where $\rho(y) \neq \text{const.}$ by making the change of variable $y \to \eta$ in (3.12), where

$$\eta = \int_0^y \rho(y')\, dy'. \tag{3.24}$$

The details, which are elementary but lengthy, are omitted.

This completes the formulation of the results of §3 and the proofs will now be given. Note that Lemma 3.1 is an immediate consequence of Theorem 2.1 which implies that for $\lambda > c^2(\infty)\mu^2$ the equation $A_\mu \phi = \lambda\phi$ has no solutions in $\mathcal{H}(R)$.

Proof of Lemma 3.2. The resolvent of A_μ is an integral operator in $\mathcal{H}(R)$ [3, XIII.3]

$$(3.25) \qquad (A_\mu - \zeta)^{-1} f(y) = \int_R G_\mu(y,y',\zeta)\, f(y')\, c^{-2}(y')\, \rho^{-1}(y')\, dy'.$$

$G_\mu(y,y',\zeta)$, the Green's function of A_μ, is known to have the form [3, p. 1329]

$$(3.26) \qquad G_\mu(y,y',\zeta) = [\phi_\infty \phi_{-\infty}]^{-1} \begin{cases} \phi_{-\infty}(y)\ \phi_\infty(y'), & y \le y', \\ \phi_\infty(y)\ \phi_{-\infty}(y'), & y \ge y', \end{cases}$$

where ϕ_∞ and $\phi_{-\infty}$ are non-trivial solutions of $A_\mu\, \phi = \zeta\phi$ that are in $L_2(0,\infty)$ and $L_2(-\infty,0)$, respectively. Thus for $\zeta \in L(c(\infty)\mu) \subset L(c(-\infty)\mu)$, $\phi_\infty = \phi_2$, $\phi_{-\infty} = \phi_3$ and one has

$$(3.27) \qquad G_\mu(y,y',\zeta) = [\phi_2\phi_3]^{-1} \begin{cases} \phi_3(y,\mu,\zeta)\ \phi_2(y',\mu,\zeta), & y \le y', \\ \phi_2(y,\mu,\zeta)\ \phi_3(y',\mu,\zeta), & y \ge y'. \end{cases}$$

It follows from Corollary 2.3 that $G_\mu(y,y',\zeta)$ is meromorphic in $L(c(\infty)\mu)$ with poles at the zeros of

$$(3.28) \qquad F(\mu,\zeta) = [\phi_2(\cdot,\mu,\zeta)\ \phi_3(\cdot,\mu,\zeta)].$$

As remarked in §1, these are precisely the eigenvalues of A_μ that are less than $c^2(\infty)\mu^2$. Their only possible limit point is $c^2(\infty)\mu^2$ since

$F(\mu,\zeta)$ is analytic in $L(c(\infty)\mu)$ by Corollary 2.3. These results imply the two statements of Lemma 3.2.

Proof of Lemma 3.3. This follows from Theorem 2.1 which implies that $A_\mu \phi = \lambda\phi$ always has at least one solution that is not in $\mathcal{H}(R)$.

Proof of Theorem 3.4. It was remarked above that $\sigma_c(A_\mu) \subset \sigma_e(A_\mu) \subset [c^2(\infty)\mu^2,\infty)$. Hence to prove (3.5) it is enough to show that $(c^2(\infty)\mu^2,\infty) \subset \sigma_c(A_\mu)$. This may be done by constructing a characteristic sequence for each $\lambda \in (c^2(\infty)\mu^2,\infty)$; i.e., a bounded sequence $\{\phi_n(y)\}$ in $\mathcal{H}(R)$ such that each $\phi_n \in D(A_\mu)$ and $(A_\mu - \lambda)\phi_n \to 0$ in $\mathcal{H}(R)$ but $\{\phi_n\}$ has no convergent subsequences. Indeed, a suitable sequence has the form $\phi_n(y) = \xi_n(y)\ \phi_3(y,\mu,\lambda)$ where $\xi_n \in D(A_0)$, $\xi_n(y) \equiv 1$ for $|y| \le n$, supp $\xi_n \subset [-n-1,n+1]$ and $\xi_n'(y)$ and $(\rho^{-1}(y)\xi_n'(y))'$ are bounded for all y and n. Such a sequence $\{\xi_n\}$ can be constructed but the details are lengthy. They will not be given here since the inclusion $(c^2(\infty)\mu^2,\infty) \subset \sigma_c(A_\mu)$ is proved in §5.

Proof of Corollary 3.5. For $\lambda \neq c^2(\infty)\mu^2$ every solution of $A_\mu \phi = \lambda\phi$ is a linear combination of $\phi_1(y,\mu,\lambda)$ and $\phi_2(y,\mu,\lambda)$ (Corollary 2.7). It follows from Theorem 2.1 that every real solution with $\lambda > c^2(\infty)\mu^2$ has infinitely many zeros in any interval (y_0,∞). On the other hand for $\lambda < c^2(\infty)\mu^2$ Theorem 2.1 implies that every real solution of $A_\mu \phi = \lambda\phi$ is either exponentially large or exponentially small for $y \to \pm\infty$. In every case $\phi(y)$ has constant sign outside of some interval $[-y_0,y_0]$ and hence can have only finitely many zeros.

Proof of Theorem 3.6. Results equivalent to Theorem 3.6 are proved in [6, Ch. XI] under the additional hypothesis that the P_j and Q_j are continuous. The same method will be shown to be applicable under the hypotheses of Theorem 3.6. The method is to study the phase plane curves

(3.29) $$(\xi,\eta) = (P_j^{-1}(y)\phi_j'(y),\phi_j(y)),\ y \in I,\ j = 1,2,$$

defined by the solutions $\phi_j(y)$ and to transform to polar coordinates (Prüfer transformation). Thus (3.29) can be written

(3.30) $$(\xi,\eta) = (r_j(y)\cos\theta_j(y),r_j(y)\sin\theta_j(y)),\ y \in I,\ j = 1,2.$$

Moreover, the curves (3.29) cannot pass through the origin because $\phi_j(y) \not\equiv 0$. Thus $r_j(y) > 0$ and $\theta_j(y)$ is uniquely defined by continuity and its value at the point $y_1 \in I$. Finally, $\theta_j \in AC(I)$ and (3.6) implies that θ_j is a solution of the first order equation

(3.31) $$\theta_j'(y) = P_j(y)\cos^2\theta_j(y) + Q_j(y)\sin^2\theta_j(y),\ y \in I.$$

To prove the first statement of Theorem 3.6 note that one can assume without loss of generality that $\phi_1(y) > 0$ for $y_1 < y < y_2$ and $\phi_2(y_1) \geq 0$. Thus $\theta_j(y)$ $(j = 1,2)$ may be defined as the unique solutions of (3.31) such that $\theta_1(y_1) = 0$ and $0 \leq \theta_2(y_1) < \pi$. It follows that $0 < \theta_1(y) < \pi$ for $y_1 < y < y_2$ and $\theta_1(y_2) = \pi$. It must be shown that $\phi_2(y)$ has a zero in $[y_1,y_2)$. If $\phi_2(y_1) = 0$ there is nothing to prove. If $\phi_2(y_1) > 0$ then $0 < \theta_2(y_1) < \pi$ and it follows from (3.31) and (3.7) that $\theta_2(y) > \theta_1(y)$ for all $y \geq y_1$ (see [6, p. 335]). In particular, $\theta_2(y_2) > \theta_1(y_2) = \pi$ whence by continuity $\theta_2(y_0) = \pi$ and therefore $\phi_2(y_0) = 0$ for some $y_0 \in (y_1,y_2)$.

To prove the second statement of Theorem 3.6 it is only necessary to remark that if $Q_1(y) < Q_2(y)$ or $P_1(y) < P_2(y)$ and $Q_2(y) \neq 0$ on a subset of (y_1,y_2) having positive measure then $\theta_2(y_2) > \pi$ even if $\theta_2(y_1) = 0$; see [6, p. 335]. This completes the proof.

Proof of Corollary 3.7. The function $\phi_3(y,\mu,\lambda)$ is a solution of equation (3.11). Thus if $\phi_j(y) = \phi_3(y,\mu,\lambda_j)$, $j = 1,2$, then

$$(\rho^{-1}(y)\phi_j')' + \rho^{-1}(y)(\lambda_j c^{-2}(y) - \mu^2)\phi_j = 0,\ j = 1,2. \tag{3.32}$$

These equations have the form (3.6) with $P_j(y) = \rho(y)$ and $Q_j(y) = \rho^{-1}(y)(\lambda_j c^{-2}(y) - \mu^2)$. Hence $P_1(y) = P_2(y)$ and $Q_1(y) < Q_2(y)$ for all $y \in R$, since $\rho(y)$ and $c(y)$ are always positive, and the second part of Theorem 3.6 is applicable. It follows that if $y_1 < y_2 < \cdots < y_k$ are the zeros of $\phi_1(y) = \phi_3(y,\mu,\lambda_1)$ then $\phi_2(y) = \phi_3(y,\mu,\lambda_2)$ has $k - 1$ zeros in the interval (y_1,y_k). Hence it will be enough to show that $\phi_3(y,\mu,\lambda_2)$ also has a zero in $(-\infty,y_1]$. To verify this apply Lagrange's formula (2.23) to $\phi_1(y)$ and $\phi_2(y)$ in $(-\infty,y_1]$. This is possible because $\phi_1(y)$ and $\phi_2(y)$ are exponentially small at $y = -\infty$. The result can be written

$$\begin{aligned}(\lambda_2 - \lambda_1) \int_{-\infty}^{y_1} \phi_1(y)\ \phi_2(y)\ c^{-2}(y)\ \rho^{-1}(y)\ dy \\ = \int_{-\infty}^{y_1} \{\phi_1\ A_\mu\ \phi_2 - \phi_2\ A_\mu\ \phi_1\}\ c^{-2}\ \rho^{-1}\ dy \\ = \phi_2(y_1)\ \{\rho^{-1}(y_1)\ \phi_1'(y_1)\}\end{aligned} \tag{3.33}$$

since $\phi_1(y_1) = 0$ and ϕ_1 and ϕ_2 vanish at $y = -\infty$. Now suppose that $\phi_2(y)$ has no zero in $(-\infty,y_1]$. Then $\phi_2(y_1) > 0$ because $\phi_2(y) > 0$ near $y = -\infty$ by Theorem 2.1. Moreover, $\phi_1(y) > 0$ for $-\infty < y < y_1$ and $\rho^{-1}(y_1)\ \phi_1'(y_1) < 0$ because y_1 is the first zero of ϕ_1. Thus the right hand side of (3.33) is negative. But the left hand side is clearly positive. This contradiction completes the proof.

<u>Proof of Theorem 3.8</u>. For regular Sturm-Liouville operators $A\phi = -\phi'' + q(y)\phi$ on finite intervals the oscillation theorem goes back to Sturm. For singular operators in $L_2(0,\infty)$ such that $q(y) \to +\infty$ when $y \to +\infty$ it was first proved by H. Weyl. More recently the result was proved by B. M. Levitan and I. S. Sargsjan [11, p. 201] under more general conditions on Q(y) that guarantee that the solutions of $A\phi = \lambda\phi$ are non-oscillatory on $0 \leq y < \infty$ for all $\lambda \in R$ (and hence $\sigma(A)$ is discrete). It will be shown here that the method of Levitan and Sargsjan is applicable to the case of Theorem 3.8.

The method of [11] is to regard the Sturm-Liouville problem for A on $0 \leq y < \infty$ as a limit of regular problems for A on $0 \leq y \leq b < \infty$ and to study the behavior of the eigenvalues and eigenfunctions as $b \to \infty$. Here the operator A_μ in $\mathcal{H}(R)$ will be regarded as a limit of the regular operator in $\mathcal{H}(a,b) = L_2(a,b;c^{-2}(y)\,\rho^{-1}(y)\,dy)$, $-\infty < a < b < \infty$, defined by A_μ and the boundary conditions $\phi(a) = \phi(b) = 0$. The corresponding operator in $\mathcal{H}(a,b)$ will be denoted by $A_{\mu,a,b}$. The limit $a \to -\infty$ will be studied first.

The operator A_μ is more general than the operator studied in [11]. However, examination of the proofs in [11] reveals that nothing is used but the Sturm comparison theorem, the convergence of the eigenvalues when $b \to \infty$, the continuity and asymptotic properties of the solution $\phi(y,\lambda)$ of $A\phi = \lambda\phi$ that satisfies the boundary condition at $y = 0$ and the non-oscillatory character of $A\phi = \lambda\phi$ in a λ-interval containing the point spectrum. All of these properties have been established for A_μ.

The solution of $A_\mu\,\phi = \lambda\phi$ that satisfies $\phi(b) = 0$, $\rho^{-1}(b)\,\phi'(b) = 1$ will be denoted by $\phi_b(y,\lambda)$. For $\lambda \neq c^2(-\infty)\mu^2$; $\phi_b(y,\lambda)$ is a linear combination of $\phi_3(y,\mu,\lambda)$ and $\phi_4(y,\mu,\lambda)$ and hence has the regularity properties of Corollaries 2.2 and 2.3. The eigenvalues of $A_{\mu,a,b}$ are the roots of the equation $\phi_b(a,\lambda) = 0$. They will be denoted by $\lambda_{k,a,b}$, $k = 1,2,\cdots$, with the convention that $\lambda_{k,a,b} < \lambda_{k+1,a,b}$. The corresponding eigenfunctions $\psi_{k,a,b}(y) = \phi_b(y,\lambda_{k,a,b})$ have precisely $k - 1$ zeros by the classical oscillation theorem. For the class of operators considered here this result can be proved by the method of [11, p. 17]. The zeros of $\psi_{k,a,b}(y)$ will be denoted by $y^{(k)}_{j,a,b}$, $1 \le j \le k - 1$.

The operator in $\mathcal{H}(-\infty,b)$ defined by A_μ and the boundary condition $\phi(b) = 0$ will be denoted by $A_{\mu,b}$. The methods used to study $\sigma(A_\mu)$ above can be used to show that $\sigma(A_{\mu,b}) \cap [c_m^2\mu^2, c^2(-\infty)\mu^2)$ $(\subset \sigma_0(A_{\mu,b}))$ is finite or countably infinite with unique limit point $c^2(-\infty)\mu^2$. The number of eigenvalues in $[c_m^2\mu^2, c^2(-\infty)\mu^2)$ will be denoted by $N(\mu,b) - 1$ $(\le +\infty)$ in analogy with the notation for A_μ, and the eigenvalues will be denoted by $\lambda_{k,b}$ $(\lambda_{k,b} < \lambda_{k+1,b})$. The eigenfunctions for $A_{\mu,b}$ are $\psi_{k,b}(y) = \phi_b(y,\lambda_{k,b})$.

The proof of the oscillation theorem for $A_{\mu,b}$ by the method of [11] will now be outlined. First,

$$\lim_{a\to-\infty} \lambda_{k,a,b} = \lambda_{k,b} \text{ for } 1 \le k < N(\mu,b). \tag{3.34}$$

This follows, for example, from the convergence of the Green's functions. It follows that

$$\lim_{a\to-\infty} \psi_{k,a,b}(y) = \psi_{k,b}(y) \text{ for } -\infty < y \le b, \tag{3.35}$$

uniformly on bounded subsets of $(-\infty,b]$. The proof of the oscillation theorem given in [11] is based on the following three lemmas.

Lemma 3.17. For each $k = 1,2,\cdots,N(\mu,b) - 1$ and each fixed $b \in R$ one has

$$(3.36) \qquad \sup_{a<b} y^{(k)}_{k-1,a,b} < \infty.$$

Lemma 3.18. For each $k = 1,2,\cdots,N(\mu,b) - 1$, each $j = 1,2,\cdots,k - 1$ and each fixed $b \in R$ one has

$$(3.37) \qquad \inf_{a<b} \left(y^{(k)}_{j+1,a,b} - y^{(k)}_{j,a,b}\right) > 0.$$

Lemma 3.19. For each $k = 1,2,\cdots,N(\mu,b) - 1$ and each fixed $b \in R$ one has

$$(3.38) \qquad \inf_{a<b} (\lambda_{k+1,a,b} - \lambda_{k,a,b}) > 0.$$

The proofs of these lemmas and the oscillation theorem for $A_{\mu,b}$ are the same as those given in [11, pp. 202-4] and will not be repeated here.

The proof of Theorem 3.8 may now be completed by regarding A_μ as a limit of $A_{\mu,b}$ for $b \to \infty$ and repeating the argument given above. The solution of $A_\mu \, \phi = \lambda\phi$ that satisfies the condition of square integrability at $y = -\infty$ is $\phi_3(y,\mu,\lambda)$ and is non-oscillatory for all $y \in R$ when $\lambda < c^2(\infty)\mu^2$ (Corollary 3.5). The remainder of the proof follows as before.

Proof of Corollary 3.9. Theorem 3.8 implies that $\lambda_{k+1}(\mu) \in I_k$ for $k = 0,1,2,\cdots,N(\mu)-2$. Moreover, a continuity argument based on Corollary 2.2 shows that the intervals I_k have the form $I_k = (a_k,a_{k+1}]$ where $a_0 < a_1 < \cdots < a_{N(\mu)-1}$ (see [3, p. 1475]). Thus to prove (3.9) it will suffice to prove that

$$F(\mu,a_k) = [\phi_2(\cdot,\mu,a_k)\ \phi_3(\cdot,\mu,a_k)] = 0 \tag{3.39}$$

for $k = 1,2,\cdots,N(\mu) - 1$. This proof will be based on the following two lemmas.

Lemma 3.20. Let $\lambda_0 < c^2(-\infty)\mu^2$ and let y_0 be a zero of $\phi_3(y,\mu,\lambda_0)$. Then to each sufficiently small $\varepsilon > 0$ there corresponds a $\delta > 0$ such that for $|\lambda - \lambda_0| < \delta$ the function $\phi_3(y,\mu,\lambda)$ has exactly one zero in the interval $|y - y_0| < \varepsilon$.

This result follows from Corollary 2.2 and the fact that $\rho^{-1}(y)\ \phi'(y)$ cannot vanish at a zero of a non-trivial solution of $A_\mu\ \phi = \lambda\phi$. For a proof see [11, p. 16].

For $\lambda \in I_k$ let $y_1(\lambda) < y_2(\lambda) < \cdots < y_k(\lambda)$ denote the zeros of $\phi_3(y,\mu,\lambda)$. Then each $y_j(\lambda)$ is uniquely defined for $\lambda \in \bigcup_{k\geq j} I_k$ and one has

Lemma 3.21. Each of the functions $y_j(\lambda)$ is continuous and strictly monotone increasing.

The continuity follows immediately from Lemma 3.20. The strict monotonicity follows from the proof of Corollary 3.7.

Proof of Corollary 3.9 (concluded). (3.39) will be proved by contradiction. Assume that $F(\mu,a_k) \neq 0$ and note that for $\lambda < c^2(\infty)\mu^2$ one has

$$\phi_3(y,\mu,\lambda) = c(\mu,\lambda)\ \phi_1(y,\mu,\lambda) + c'(\mu,\lambda)\ \phi_2(y,\mu,\lambda), \tag{3.40}$$

by Corollary 2.7. Moreover, Theorem 2.1 implies that

$$c(\mu,\lambda) = \rho(\infty)[\phi_2\phi_3]/2iq_+(\mu,\lambda) = \rho(\infty)\ F(\mu,\lambda)/2iq_+(\mu,\lambda). \tag{3.41}$$

Thus $c(\mu,a_k) \neq 0$ and by continuity (Corollary 2.2) there is an interval

$|\lambda - a_k| \leq \delta$ in which $c(\mu,\lambda) \neq 0$. It follows from (3.40) and the uniformity of the asymptotic estimates of Theorem 2.1 (Corollary 2.4) that there is an $M > 0$ such that

(3.42) $$|\phi_3(y,\mu,\lambda)| \geq 1 \text{ for all } y \geq M \text{ and } |\lambda - a_k| \leq \delta.$$

Note that Lemma 3.21 implies

(3.43) $$\lim_{\lambda \to a_k} y_j(\lambda) = y_j(a_k),\ j = 1,2,\cdots,k.$$

Now consider $y_{k+1}(\lambda)$ which is defined for $\lambda > a_k$. For $a_k < \lambda < a_k + \delta$ one has $y_{k+1}(\lambda) < M$ by (3.42). It follows that the limit

(3.44) $$\overline{y} = \lim_{\lambda \to a_k} y_{k+1}(\lambda)$$

exists and $\overline{y} \geq y_k(a_k)$. But $\phi_3(\overline{y},\mu,a_k) = 0$ by continuity and hence $\overline{y} = y_k(a_k)$. But this implies that for $\delta > 0$ small enough and $a_k < \lambda < a_k + \delta$ every neighborhood $|y - y_k(a_k)| < \varepsilon$ contains two zeros of $\phi_3(y,\mu,\lambda)$ in contradiction to Lemma 3.20. This completes the proof of (3.9).

The last statement of Corollary 3.9 follows from the proof of Theorem 3.8. Indeed, if $N(\mu) < \infty$ and $N(\mu) < N(\mu,b)$ then $N(\mu) \leq N(\mu,b) - 1$ and $\lambda_{N(\mu),b}$ is defined. In this case

(3.45) $$\liminf_{b \to \infty} \lambda_{N(\mu),b} \geq c^2(\infty)\mu^2$$

since otherwise $\lambda_{N(\mu),b(k)} \to \lambda_0 < c^2(\infty)\mu^2$ for some subsequence $\{b(k)\}$, which would imply that λ_0 was an additional eigenvalue of A_μ. If $N(\mu) = N(\mu,b) < \infty$ the same argument can be applied to the operator $A_{\mu,b}$.

<u>Proof of Corollary 3.10</u>. This follows immediately from Corollary 3.9.

Proof of Theorem 3.11. It will suffice to prove the second statement of the theorem. To this end let $\phi_b(y,\mu,\lambda)$ be the solution of $A_\mu\ \phi = \lambda\phi$ that satisfies $\phi_b(b,\mu,\lambda) = 0$, $\rho^{-1}(b)\ \phi_b'(b,\mu,\lambda) = 1$. Then $\phi_b(y,\mu,\lambda)$ and $\rho^{-1}(y)\ \phi_b'(y,\mu,\lambda)$ are continuous functions of $(y,\lambda) \in R^2$. Now assume that $A_\mu\ \phi = c^2(\infty)\mu^2\phi$ is oscillatory. Then $\phi_b(y,\mu,c^2(\infty)\mu^2)$ has infinitely many zeros. It follows by the method used to prove Lemma 3.20 that the number of zeros of $\phi_b(y,\mu,\lambda)$ tends to infinity as $\lambda \to c^2(\infty)\mu^2$. But then the same is true of $\phi_3(y,\mu,\lambda)$, by Theorem 3.6, and it follows from Corollary 3.10 that $\sigma_0(A_\mu)$ is infinite. To prove the converse note that if $\sigma_0(A_\mu)$ is infinite then Theorem 3.6, applied to the kth eigenfunction and any solution of $A_\mu\ \phi = c^2(\infty)\mu^2\phi$ implies that ϕ has $k - 2$ zeros. Since k is arbitrary it follows that $A_\mu\ \phi = c^2(\infty)\mu^2\phi$ is oscillatory.

Proof of Corollary 3.12. This follows immediately from Theorem 3.11. The hypothesis $c(\infty) < c(-\infty)$ is needed only to ensure that $\phi_3(y,\mu,c^2(\infty)\mu^2)$ is defined.

Proof of Theorem 3.13. This follows immediately from Corollary 3.12 and Theorem 2.6.

Proof of Theorem 3.14. To prove the first half of the theorem it will be shown that condition (3.16) implies the existence of a non-oscillatory majorant for equation (3.13) for every $\mu > 0$. This implies that (3.12), i.e., $A_\mu\ \phi = c^2(\infty)\mu^2\phi$, is non-oscillatory for every $\mu > 0$ and the finiteness of $\sigma_0(A_\mu)$ follows from Corollary 3.12.

To construct a majorant for (3.13) note that (3.16) implies that for every $\varepsilon > 0$ there is a $y_0 = y_0(\varepsilon)$ such that

$$y^2(c^2(\infty)c^{-2}(y) - 1)_+ \leq \varepsilon \text{ for all } y \geq y_0(\varepsilon), \tag{3.46}$$

where $\alpha_+ = \text{Max}(\alpha,0)$. It follows that for every $\mu > 0$ there is a $y_0 = y_0(\mu)$ such that

$$\rho_M \rho^{-1}(y)\, y^2 \,(c^2(\infty)c^{-2}(y) - 1) \le \rho_M \rho^{-1}(y)\, y^2\, (c^2(\infty)c^{-2}(y) - 1)_+$$

$$(3.47) \qquad \le \rho_M \rho_m^{-1}\, y^2(c^2(\infty)c^{-2}(y) - 1)_+$$

$$\le 1/4\mu^2 \text{ for all } y \ge y_0(\mu).$$

Hence for any $\mu > 0$ one has

$$(3.48) \qquad \rho_M \rho^{-1}(y)\, \mu^2(c^2(\infty)c^{-2}(y) - 1) \le 1/4y^2 \text{ for all } y \ge y_0(\mu).$$

It follows on comparing (3.13) with (3.15) with $\alpha = 1/4$ that (3.13) is non-oscillatory on $y_0(\mu) \le y < \infty$. It is non-oscillatory on $-\infty < y \le y_0(\mu)$ for any $\mu > 0$ because $c(\infty) < c(-\infty)$. This proves the first half of Theorem 3.14.

To prove the second half it will be shown that (3.17) implies the existence of a $\mu_0 > 0$ such that $A_\mu \phi = c^2(\infty)\mu^2\phi$ is oscillatory for every $\mu \ge \mu_0$. The result then follows from Theorem 3.11. To this end note that if ε satisfies

$$(3.49) \qquad 0 < \varepsilon < \liminf_{y\to\infty} y^2(c^2(\infty)c^{-2}(y) - 1)$$

then there is a $y_0 = y_0(\varepsilon)$ such that

$$(3.50) \qquad y^2(c^2(\infty)c^{-2}(y) - 1) \ge \varepsilon \text{ for all } y \ge y_0(\varepsilon).$$

In particular, given any $\alpha > 1/4$ there is a $\mu_0 > 0$ such that

$$(3.51) \qquad 0 < \rho_M \rho_m^{-1}\alpha/\mu_0^2 < \liminf_{y\to\infty} y^2(c^2(\infty)c^{-2}(y) - 1).$$

It follows that there is a $y_0 = y_0(\alpha)$ such that

(3.52) $$y^2(c^2(\infty)c^{-2}(y) - 1) \geq \rho_M \rho_m^{-1}\alpha/\mu_0^2 \text{ for all } y \geq y_0(\alpha).$$

This implies that

(3.53) $$\rho_m \rho_M^{-1} \mu_0^2(c^2(\infty)c^{-2}(y) - 1) \geq \alpha/y^2 \text{ for all } y \geq y_0(\alpha).$$

Hence, comparison of

(3.54) $$\phi'' + \rho_m \rho_M^{-1} \mu_0^2(c^2(\infty)c^{-2}(y) - 1)\phi = 0$$

and (3.15) with $\alpha > 1/4$ implies that (3.54) is oscillatory. But (3.54) is a Sturm minorant of (3.12); i.e., $A_\mu \phi = c^2(\infty)\mu^2\phi$, provided $\mu \geq \mu_0$. Hence the latter is oscillatory for all $\mu \geq \mu_0$.

Proof of Theorem 3.15. This result is proved in [3, p. 1481] for Sturm-Liouville operators with smooth coefficients. The proof is based on the oscillation theorem (Theorem 3.8), Sturm's comparison theorem and the continuous dependence of the zeros of solutions of $A_\mu \phi = \lambda\phi$ on λ (Lemma 3.20). Hence it extends immediately to the operator A_μ.

§4. Generalized Eigenfunctions of A_μ.

The eigenfunctions $\psi_k(y,\mu)$ corresponding to the point spectrum of A_μ were constructed in the preceding section. In this section the special solutions ϕ_j $(j = 1,2,3,4)$ of §2 are used to construct generalized eigenfunctions of A_μ corresponding to the points of the continuous spectrum. These functions will be used in §5 to construct the spectral family $\{\Pi_\mu(\lambda)\}$ of A_μ and to prove that $\sigma_c(A_\mu) = \sigma_e(A_\mu) = [c^2(\infty)\mu^2,\infty)$.

To construct the generalized eigenfunctions $\psi_0(y,\mu,\lambda)$, $\psi_\pm(y,\mu,\lambda)$ described in §1 recall that the special solutions $\phi_j(y,\mu,\lambda)$ are defined for all real $\lambda \neq c^2(\pm\infty)\mu^2$ and the pairs ϕ_1, ϕ_2 and ϕ_3, ϕ_4 are solution bases for $A_\mu\, \phi = \lambda\phi$ (Corollary 2.7). It follows that

$$(4.1)\qquad \begin{cases} \phi_j = c_{j3}\phi_3 + c_{j4}\phi_4, & j = 1,2, \\ \phi_j = c_{j1}\phi_1 + c_{j2}\phi_2, & j = 3,4. \end{cases}$$

The coefficients $c_{jk} = c_{jk}(\mu,\lambda)$ can be calculated by means of the bracket operation

$$(4.2)\qquad [\phi_j\phi_k](\mu,\lambda) = [\phi_j(\cdot,\mu,\lambda)\ \phi_k(\cdot,\mu,\lambda)]$$

of Lagrange's formula. Indeed, by forming the brackets of equations (4.1) with ϕ_4, ϕ_3, ϕ_2 and ϕ_1 in succession and using the asymptotic forms of Theorem 2.1 one finds

$$(4.3)\qquad \left.\begin{aligned} (-2iq_-)c_{j3} &= \rho(-\infty)[\phi_j\phi_4] \\ (2iq_-)c_{j4} &= \rho(-\infty)[\phi_j\phi_3] \end{aligned}\right\} \quad j = 1,2,$$

$$\text{(4.3 cont.)}\qquad \left.\begin{aligned} (-2iq_+)c_{j1} &= \rho(\infty)[\phi_j\phi_2] \\ (2iq_+)c_{j2} &= \rho(\infty)[\phi_j\phi_1] \end{aligned}\right\} \; j = 3,4.$$

In particular, Corollary 2.2 implies that each $c_{jk}(\mu,\lambda)$ is a continuous function for $\lambda \neq c^2(\pm\infty)\mu^2$. These relations will be used to determine the generalized eigenfunctions of A_μ. The notation

$$\text{(4.4)}\qquad \begin{aligned} \Lambda &= \Lambda(\mu) = \{\lambda \mid c^2(-\infty)\mu^2 < \lambda\}, \\ \Lambda_0 &= \Lambda_0(\mu) = \{\lambda \mid c^2(\infty)\mu^2 < \lambda < c^2(-\infty)\mu^2\} \end{aligned}$$

will be used. Note that $\Lambda_0 \neq \phi$ only if $c(\infty) < c(-\infty)$.

<u>The Spectral Interval Λ</u>. The generalized eigenfunctions of A_μ are the bounded solutions of the differential equation $A_\mu\,\phi = \lambda\phi$. For $\lambda \in \Lambda$, Theorem 2.1 and the relations (4.1) imply that all the solutions are bounded. It will be shown that the functions

$$\text{(4.5)}\qquad \left\{\begin{aligned} \psi_+(y,\mu,\lambda) &= a_+(\mu,\lambda)\;\phi_4(y,\mu,\lambda) \\ \psi_-(y,\mu,\lambda) &= a_-(\mu,\lambda)\;\phi_1(y,\mu,\lambda) \end{aligned}\right.$$

have the asymptotic forms described in §1. The completeness in $\Pi_\mu(\Lambda)\,\mathcal{H}(R)$ of these functions will be proved in §5. The pair (ϕ_2,ϕ_3), which provides an alternative basis, will not be treated explicitly here. It may be shown to correspond to the second family $\{\phi_-(y,p,q) \mid q > 0\}$ described at the end of §1.

It follows from (4.5), (4.1) and Theorem 2.1 that the asymptotic behavior of ψ_+ is given by

$$\psi_+(y,\mu,\lambda) = a_+ \begin{cases} c_{41}\, e^{iq_+y} + c_{42}\, e^{-iq_+y} + o(1), & y \to +\infty, \\ e^{-iq_-y} + o(1) & , y \to -\infty, \end{cases} \tag{4.6}$$

The equivalence of (4.6) and the asymptotic form (1.34) of §1 follows from

Lemma 4.1. For all $\mu > 0$ and $\lambda \in \Lambda$ the coefficients $c_{41}(\mu,\lambda)$, $c_{42}(\mu,\lambda)$ satisfy

$$\rho^{-1}(\infty)\, q_+\, |c_{42}|^2 = \rho^{-1}(\infty)\, q_+\, |c_{41}|^2 + \rho^{-1}(-\infty)\, q_- . \tag{4.7}$$

In particular, $c_{42}(\mu,\lambda) \neq 0$.

The proofs of Lemma 4.1 and subsequent lemmas are given at the end of the section.

The asymptotic forms (1.34) and (4.6) coincide if the coefficients satisfy

$$c_+ = a_+\, c_{42},\quad c_+\, T_+ = a_+,\quad c_+\, R_+ = a_+\, c_{41}. \tag{4.8}$$

In particular, the first relation and (4.3) imply that

$$c_+(\mu,\lambda) = a_+(\mu,\lambda)\, \frac{\rho(\infty)[\phi_4\phi_1](\mu,\lambda)}{2iq_+(\mu,\lambda)} . \tag{4.9}$$

The normalizing factor $a_+(\mu,\lambda)$ will be calculated in §5. The factors $R_+(\mu,\lambda)$, $T_+(\mu,\lambda)$ of (1.34) are independent of the normalization. Indeed, on combining (4.8), (4.9) and (4.3) one finds

$$T_+(\mu,\lambda) = \frac{2iq_+(\mu,\lambda)}{\rho(\infty)[\phi_4\phi_1](\mu,\lambda)} , \tag{4.10}$$

$$R_+(\mu,\lambda) = -\frac{[\phi_4\phi_2](\mu,\lambda)}{[\phi_4\phi_1](\mu,\lambda)}. \tag{4.11}$$

Note that the denominator $[\phi_4\phi_1]$ is not zero by Lemma 4.1 and relations (4.3).

The asymptotic behavior of ψ_- may be discussed similarly. Equations (4.5), (4.1) and Theorem 2.1 imply

$$\psi_-(y,\mu,\lambda) = a_- \begin{cases} e^{iq_+y} + o(1) & , \ y \to +\infty, \\ c_{13}\, e^{iq_-y} + c_{14}\, e^{-iq_-y} + o(1), & y \to -\infty, \end{cases} \tag{4.12}$$

and one has

<u>Lemma 4.2</u>. For all $\mu > 0$ and $\lambda \in \Lambda$ the coefficients $c_{13}(\mu,\lambda)$, $c_{14}(\mu,\lambda)$ satisfy

$$\rho^{-1}(-\infty)\; q_-\; |c_{13}|^2 = \rho^{-1}(-\infty)\; q_-\; |c_{14}|^2 + \rho^{-1}(\infty)\; q_+. \tag{4.13}$$

Comparison of (1.35) and (4.12) gives

$$c_- = a_-\, c_{13},\quad c_-\, T_- = a_-,\quad c_-\, R_- = a_-\, c_{14}. \tag{4.14}$$

Solving these equations for c_-, T_- and R_- and using (4.3) gives

$$c_-(\mu,\lambda) = a_-(\mu,\lambda)\,\frac{\rho(-\infty)[\phi_4\phi_1](\mu,\lambda)}{2iq_-(\mu,\lambda)} \tag{4.15}$$

$$T_-(\mu,\lambda) = \frac{2iq_-(\mu,\lambda)}{\rho(-\infty)[\phi_4\phi_1](\mu,\lambda)}, \tag{4.16}$$

$$R_-(\mu,\lambda) = -\frac{[\phi_1\phi_3](\mu,\lambda)}{[\phi_1\phi_4](\mu,\lambda)}. \tag{4.17}$$

The Spectral Interval Λ_0. For $\lambda \in \Lambda_0$, Theorem 2.1 and the relations (4.1) imply that the only bounded solutions of $A_\mu \phi = \lambda\phi$ are multiples of $\phi_3(y,\mu,\lambda)$. It will be shown that

$$\psi_0(y,\mu,\lambda) = a_0(\mu,\lambda)\ \phi_3(y,\mu,\lambda) \tag{4.18}$$

has the asymptotic form (1.33). Indeed, (4.18), (4.1) and Theorem 2.1 imply that

$$\psi_0(y,\mu,\lambda) = a_0 \begin{cases} c_{31}\ e^{iq_+y} + c_{32}\ e^{-iq_+y} + o(1), & y \to +\infty, \\ e^{q_-'y}[1 + o(1)] & , y \to -\infty. \end{cases} \tag{4.19}$$

The equivalence of (4.19) and (1.33) follows from

Lemma 4.3. For all $\mu > 0$ and $\lambda \in \Lambda_0$ one has

$$c_{32}(\mu,\lambda) = \overline{c_{31}(\mu,\lambda)} \neq 0. \tag{4.20}$$

Comparison of (1.33) and (4.19) gives

$$c_0 = a_0\ c_{32},\ c_0T_0 = a_0,\ c_0\ R_0 = a_0\ c_{31}. \tag{4.21}$$

Solving for c_0, T_0 and R_0 and using (4.3) gives

$$c_0(\mu,\lambda) = a_0(\mu,\lambda)\ \frac{\rho(\infty)[\phi_3\phi_1](\mu,\lambda)}{2iq_+(\mu,\lambda)}, \tag{4.22}$$

$$T_0(\mu,\lambda) = \frac{2iq_+(\mu,\lambda)}{\rho(\infty)[\phi_3\phi_1](\mu,\lambda)}, \tag{4.23}$$

$$R_0(\mu,\lambda) = -\frac{[\phi_3\phi_2](\mu,\lambda)}{[\phi_3\phi_1](\mu,\lambda)}. \tag{4.24}$$

The denominator $[\phi_3\phi_1] \neq 0$ by Lemma 4.3 and relations (4.3).

Finally, note that the conservation laws (1.46) hold; i.e.,

$$q_{\pm}\ |R_{\pm}|^2 + q_{\mp}\ |T_{\pm}|^2 = q_{\pm} \text{ for all } \lambda \in \Lambda, \tag{4.25}$$

$$|R_0| = 1 \text{ for all } \lambda \in \Lambda_0. \tag{4.26}$$

In fact, relations (4.25) are equivalent to relations (4.7) and (4.13) of Lemmas 4.1 and 4.2, as may be seen by combining (4.7), (4.8) and (4.13), (4.14). Similarly, relation (4.26) follows from Lemma 4.3 because (4.21) implies that $R_0 = c_{31}/c_{32}$.

<u>Proof of Lemma 4.1</u>. Relation (4.7) can be verified by calculating $[\phi_4\overline{\phi}_4]$ in two ways, using the asymptotic forms of ϕ_4 as $y \to \infty$ and $y \to -\infty$. Note that for $\lambda \in \Lambda$ one has $\overline{\phi}_4 = \phi_3$ by the uniqueness theorem (Corollary 2.5) and Theorem 2.1. Hence, calculating $[\phi_4\overline{\phi}_4]$ at $y = -\infty$ gives

$$[\phi_4\overline{\phi}_4] = [\phi_4\phi_3] = 2i\ \rho^{-1}(-\infty)q_-, \tag{4.27}$$

by (2.25). Next, relations (4.1) and Theorem 2.1 give (with the notation c.c. for complex conjugate)

$$\begin{aligned}
[\phi_4\overline{\phi}_4] &= \phi_4\{\rho^{-1}\overline{\phi}_4'\} - \text{c.c.} \\
&= (c_{41}\ \phi_1 + c_{42}\ \phi_2)\{\overline{c}_{41}\ \rho^{-1}\overline{\phi}_1' + \overline{c}_{42}\ \rho^{-1}\overline{\phi}_2'\} - \text{c.c.} \\
&= (c_{41}\ e^{iq_+y} + c_{42}\ e^{-iq_+y}) \\
&\quad \times \{\overline{c}_{41}\rho^{-1}(\infty)(-iq_+e^{-iq_+y}) + \overline{c}_{42}\rho^{-1}(\infty)(iq_+e^{iq_+y})\} - \text{c.c.} + o(1) \\
&= \rho^{-1}(\infty)(-iq_+)\{|c_{41}|^2 - c_{41}\overline{c}_{42}e^{2iq_+y} + \overline{c}_{41}c_{42}e^{-2iq_+y} - |c_{42}|^2\} - \text{c.c.} + o(1) \\
&= -2iq_+\ \rho^{-1}(\infty)\ \{|c_{41}|^2 - |c_{42}|^2\}.
\end{aligned} \tag{4.28}$$

Combining (4.27) and (4.28) gives (4.7).

Proof of Lemma 4.2. (4.13) can be verified by calculating $[\phi_1\bar{\phi}_1]$ in two ways, in analogy with the proof of Lemma 4.1. It can also be derived directly from (4.7) and the relations (4.3).

Proof of Lemma 4.3. Note that for $\lambda \in \Lambda_0$ the uniqueness theorem (Corollary 2.5) and Theorem 2.1 imply that ϕ_3 is real valued and $\bar{\phi}_1 = \phi_2$. Hence relations (4.3) imply

$$\bar{c}_{31} = \rho(\infty)\,\overline{[\phi_3\phi_2]}/2iq_+ = \rho(\infty)[\phi_3\phi_1]/2iq_+ = c_{32}. \tag{4.29}$$

Moreover, if $c_{32} = 0$ then $c_{31} = 0$ by (4.29) and hence by (4.3) one has

$$[\phi_3\phi_1] = [\phi_3\phi_2] = 0. \tag{4.30}$$

But this would imply that ϕ_1 and ϕ_2 are linearly dependent which contradicts (2.25). Hence $c_{32} \neq 0$.

§5. The Spectral Family of A_μ.

The eigenfunctions and generalized eigenfunctions of §4 are used in this section to construct the spectral family $\{\Pi_\mu(\lambda)\}$ of A_μ. The construction if based on the Weyl-Kodaira theory as presented in the Appendix. Note that the operator A_μ has the form (A.1) with $I = R$, $p(y) = \rho(y)$, $q(y) = \mu^2 \rho^{-1}(y)$ and $w(y) = c^{-2}(y)\,\rho^{-1}(y)$. It is clear that p, q and w satisfy (A.2), (A.3), (A.4) when $\rho(y)$ and $c(y)$ are Lebesgue measurable and satisfy (1.3).

It will be convenient to decompose R into the disjoint union

$$R = \Lambda_d \cup \{c^2(\infty)\mu^2\} \cup \Lambda_0 \cup \{c^2(-\infty)\mu^2\} \cup \Lambda \tag{5.1}$$

if $c(\infty) < c(-\infty)$ and

$$R = \Lambda_d \cup \{c^2(\infty)\mu^2\} \cup \Lambda \tag{5.2}$$

if $c(\infty) = c(-\infty)$ where

$$\Lambda_d = \Lambda_d(\mu) = (-\infty, c^2(\infty)\mu^2), \tag{5.3}$$

and Λ_0 and Λ are defined by (4.4). The spectral measures of the components of (5.1) and (5.2) will be studied separately.

<u>The Spectral Family in Λ</u>. The spectral measure $\Pi_\mu(\Delta)$ of intervals $\Delta = (a,b) \subset \Lambda$ will be calculated by applying the Weyl-Kodaira theorem to A_μ in Λ. The solution pair

$$\begin{cases} \psi_1(y,\lambda) = \phi_4(y,\mu,\lambda) \\ \psi_2(y,\lambda) = \phi_1(y,\mu,\lambda) \end{cases} \tag{5.4}$$

will be used to obtain a spectral representation in terms of the generalized eigenfunctions $\psi_{\pm}(y,\mu,\lambda)$ defined by (4.5). The normalizing factors $a_{\pm}(\mu,\lambda)$ will be chosen after the matrix measure for (ψ_1,ψ_2) has been determined. Note that the pair (ψ_1,ψ_2) satisfies the hypotheses of the Weyl-Kodaira theorem. Indeed, (A.11) follows from Corollary 2.2 and (A.12) from Lemma 4.1.

The Weyl-Kodaira theorem implies that

$$\|\Pi_\mu(\Delta)f\|^2 = \int_\Delta \overline{\hat{f}_j(\lambda)}\, \hat{f}_k(\lambda)\, m_{jk}(d\lambda),\ \Delta \subset \Lambda, \tag{5.5}$$

for all $f \in \mathcal{H}(R)$ where $(m_{jk}(\Delta))$ is the spectral measure on Λ associated with the basis (5.4) and

$$\hat{f}_j(\lambda) = \lim_{M\to\infty} \int_{-M}^{M} \overline{\psi_j(y,\lambda)}\, f(y)\, w(y)\, dy, \tag{5.6}$$

the integrals converging in $L_2(\Lambda,m)$. Thus to complete the determination of $\Pi_\mu(\Delta)$ for $\Delta \subset \Lambda$ it is only necessary to calculate $\{m_{jk}(\Delta)\}$. Now $\Pi_\mu(\Delta)$ can be calculated from the resolvent

$$R_\mu(\zeta) = (A_\mu - \zeta)^{-1} \tag{5.7}$$

by means of Stone's theorem (see, e.g., [16, p. 79]). For $\Delta \subset \Lambda$ the theorem takes the form

$$\|\Pi_\mu(\Delta)f\|^2 = \lim_{\varepsilon\to 0+} \frac{1}{2\pi i} \int_\Delta (f,[R_\mu(\lambda+i\varepsilon) - R_\mu(\lambda-i\varepsilon)]f)d\lambda \tag{5.8}$$

because $\sigma_0(A_\mu) \cap \Lambda = \phi$ by Lemma 3.1. Moreover, $R_\mu(\zeta)$ is an integral operator in $\mathcal{H}(R)$ whose kernel, the Green's function of A_μ, can be represented by the analytic continuation of the basis (5.4) into the ζ-plane. This procedure, whose details are presented in the proofs at the end of the section, leads to

Theorem 5.1. For all $f \in \mathcal{H}(R)$ and all $\Delta \subset \Lambda$ the spectral measure $\Pi_\mu(\Delta)$ satisfies

$$(5.9) \qquad \|\Pi_\mu(\Delta)f\|^2 = \int_\Delta \{A_+^2(\mu,\lambda)\ |\hat{f}_1(\lambda)|^2 + A_-^2(\mu,\lambda)\ |\hat{f}_2(\lambda)|^2\}\, d\lambda$$

where

$$(5.10) \qquad A_\pm^2(\mu,\lambda) = \frac{q_\pm(\mu,\lambda)}{\pi\ \rho(\pm\infty)\,|[\phi_1\phi_4]|^2}\ .$$

Corollary 5.2. The matrix measure $(m_{jk}(\Delta))$ for the basis (5.4) is given by

$$(5.11) \qquad m_{11}(\Delta) = \int_\Delta A_+^2(\mu,\lambda)d\lambda,\ m_{22}(\Delta) = \int_\Delta A_-^2(\mu,\lambda)d\lambda$$

and $m_{12}(\Delta) = m_{21}(\Delta) = 0$ for all $\Delta \subset \Lambda$.

These results suggest an appropriate choice of the normalization factors $a_\pm(\mu,\lambda)$ of (4.5). Note that if instead of the basis (5.4) one takes (4.5) then (5.9) becomes

$$(5.12) \qquad \|\Pi_\mu(\Delta)f\|^2 = \int_\Delta \{A_+^2\ |a_+|^{-2}\ |\hat{f}_+|^2 + A_-^2\ |a_-|^{-2}\ |\hat{f}_-|^2\}\, d\lambda$$

where

$$(5.13) \qquad \hat{f}_{\pm}(\mu,\lambda) = \lim_{M\to\infty} \int_{-M}^{M} \overline{\psi_{\pm}(y,\mu,\lambda)}\, f(y)\, w(y)\, dy$$

converge in the space $L_2(\Lambda,m')$ corresponding to (ψ_+,ψ_-). This suggests the choice $|a_{\pm}|^2 = A_{\pm}^2$ or

$$(5.14) \qquad a_{\pm}(\mu,\lambda) = e^{i\theta_{\pm}(\mu,\lambda)}\, A_{\pm}(\mu,\lambda)$$

where $\theta_{\pm}(\mu,\lambda)$ is an arbitrary real valued continuous function. The matrix measure $\{m'_{jk}\}$ for (ψ_+,ψ_-) is independent of the choice of the phase factors $\exp\{i\,\theta_{\pm}(\mu,\lambda)\}$ and one could take $\theta_{\pm}(\mu,\lambda) \equiv 0$. However, it will be more convenient to choose $\theta_{\pm}(\mu,\lambda)$ in §8 in a way that simplifies the asymptotic form of the normal mode functions $\phi_{\pm}(y,p,q)$. Theorem 5.1 and the above remarks imply

<u>Corollary 5.3</u>. If the basis $(\psi_+(y,\mu,\lambda),\ \psi_-(y,\mu,\lambda))$ is normalized by (5.14) then for all $\Delta \subset \Lambda$ one has

$$(5.15) \qquad \|\Pi_{\mu}(\Delta)f\|^2 = \int_{\Delta} (|\hat{f}_+(\mu,\lambda)|^2 + |\hat{f}_-(\mu,\lambda)|^2)\, d\lambda$$

and the matrix measure $(m'_{jk}(\Delta))$ for (ψ_+,ψ_-) is given by

$$(5.16) \qquad m'_{11}(\Delta) = m'_{22}(\Delta) = |\Delta|$$

and $m'_{12}(\Delta) = m'_{21}(\Delta) = 0$ where $|\Delta|$ is the Lebesgue measure of Δ.

<u>The Spectral Family in Λ_0</u>. The spectral measure of intervals $\Delta \subset \Lambda_0$ will be calculated by applying the Weyl-Kodaira theorem to A_{μ}, Λ_0 and the solution pair

$$(5.17) \qquad \begin{cases} \psi_1(y,\lambda) = \phi_3(y,\mu,\lambda), \\ \psi_2(y,\lambda) = \phi_1(y,\mu,\lambda). \end{cases}$$

The function ϕ_3 is chosen to obtain a representation in terms of the generalized eigenfunction ψ_0 defined by (4.18). The second function could be replaced by any independent solution of $A_\mu\,\phi = \lambda\phi$. The pair (5.17) satisfies (A.11) by Corollary 2.2 and (A.13) by Lemma 4.3.

Calculation of the spectral measure in Λ_0 by the method described above leads to

Theorem 5.4. For all $f \in \mathcal{H}(R)$ and all $\Delta \subset \Lambda_0$ one has

$$(5.18) \qquad \|\Pi_\mu(\Delta)f\|^2 = \int_\Delta A_0^2(\mu,\lambda)\ |\hat{f}_1(\lambda)|^2\, d\lambda$$

where

$$(5.19) \qquad A_0^2(\mu,\lambda) = \frac{q_+(\mu,\lambda)}{\pi\,\rho(\infty)\,|[\phi_1\phi_3]|^2}\ .$$

Hence the matrix measure $(m_{jk}(\Delta))$ associated with the basis (5.17) is given by

$$(5.20) \qquad m_{11}(\Delta) = \int_\Delta A_0^2(\mu,\lambda)d\lambda$$

and $m_{12}(\Delta) = m_{21}(\Delta) = m_{22}(\Delta) = 0$ for all $\Delta \subset \Lambda_0$.

On replacing (5.17) by the basis $(\psi_0(y,\mu,\lambda),\ \phi_1(y,\mu,\lambda))$ and defining the normalizing factor by

$$(5.21)\qquad a_0(\mu,\lambda) = e^{i\,\theta_0(\mu,\lambda)}\,A_0(\mu,\lambda)$$

where $\theta_0(\mu,\lambda)$ is an arbitrary real valued continuous function one obtains

Corollary 5.5. If $a_0(\mu,\lambda)$ is defined by (5.21) then for all $\Delta \subset \Lambda_0$ one has

$$(5.22)\qquad \|\Pi_\mu(\Delta)f\|^2 = \int_\Delta |\hat{f}_0(\mu,\lambda)|^2\,d\lambda$$

where

$$(5.23)\qquad \hat{f}_0(\mu,\lambda) = \lim_{M\to\infty}\int_{-M}^{M} \overline{\psi_0(y,\mu,\lambda)}\,f(y)\,w(y)\,dy.$$

In particular, the matrix measure $(m'_{jk}(\Delta))$ for the pair (ψ_0,ϕ_1) is given by

$$(5.24)\qquad m'_{11}(\Delta) = |\Delta|$$

and $m'_{12}(\Delta) = m'_{21}(\Delta) = m'_{22}(\Delta) = 0$ for all $\Delta \subset \Lambda_0$, and the integral in (5.23) converges in $L_2(\Lambda_0)$.

The Spectral Family in Λ_d. The portion of $\sigma(A_\mu)$ in Λ_d was shown in §3 to be the set of eigenvalues $\{\lambda_k(\mu) \mid 1 \le k < N(\mu)\}$. Moreover, each $\lambda_k(\mu)$ is a simple eigenvalue with normalized eigenfunction $\psi_k(y,\mu)$ defined by (3.4), and corresponding orthogonal projection $P_{\mu k}$ defined by

$$(5.25)\qquad P_{\mu k}\,f(y) = (\psi_k(\cdot,\mu),f)\,\psi_k(y,\mu).$$

Hence, recalling that by convention $\Pi_\mu(\lambda) = \Pi_\mu(\lambda + 0)$, one has

$$\Pi_\mu(\lambda)\ f(y) = \sum_{\lambda_k(\mu)\leq\lambda} (\psi_k(\cdot,\mu),f)\ \psi_k(y,\mu),\ \lambda \in \Lambda_d. \tag{5.26}$$

The notation in (5.26) denotes a summation over all indices k such that $\lambda_k(\mu) \leq \lambda$. The sum in (5.26) is finite for all $\lambda \in \Lambda_d$.

The Spectral Measure of the Points $c^2(\infty)\mu^2$ and $c^2(-\infty)\mu^2$. When $c(\infty) < c(-\infty)$ one always has $\Pi_\mu(\{c^2(-\infty)\mu^2\}) = 0$ because in this case the special solutions $\phi_1(y,\mu,\lambda)$, $\phi_2(y,\mu,\lambda)$ are defined for $\lambda = c^2(-\infty)\mu^2$ and Theorem 2.1 implies that $A_\mu\ \phi = c^2(-\infty)\mu^2\phi$ has no solutions in $\mathcal{H}(R)$. The point $\lambda = c^2(\infty)\mu^2$ may be an eigenvalue of A_μ. In each case this question must be decided by determining the behavior of solutions of $A_\mu\ \phi = c^2(\infty)\mu^2\phi$ for $y \to \pm\infty$. Theorem 2.6 gives simple sufficient conditions for $\Pi_\mu(\{c^2(\infty)\mu^2\}) = 0$. For simplicity, it will be assumed in the remainder of the report that $c^2(\infty)\mu^2 \notin \sigma_0(A_\mu)$. In cases where $c^2(\infty)\mu^2$ is an eigenvalue a corresponding term must be added to the eigenfunction expansion.

The Eigenfunction Expansion for A_μ. Combining the representations of Π_μ obtained above, one finds the representation

$$\begin{aligned}\|\Pi_\mu(\lambda)f\|^2 &= \sum_{k=1}^{N(\mu)-1} H(\lambda - \lambda_k(\mu))\ |\hat{f}_k(\mu)|^2 \\ &+ \int_{\Lambda_0} H(\lambda - \lambda')\ |\hat{f}_0(\mu,\lambda')|^2\ d\lambda' \\ &+ \int_{\Lambda_0} H(\lambda - \lambda')(|\hat{f}_+(\mu,\lambda')|^2 + |\hat{f}_-(\mu,\lambda')|^2)\ d\lambda'\end{aligned} \tag{5.27}$$

where $H(\lambda) = 1$ for $\lambda \geq 0$, $H(\lambda) = 0$ for $\lambda < 0$, (5.28) $$\hat{f}_k(\mu) = \int_R \overline{\psi_k(y,\mu)}\, f(y)\, c^{-2}(y)\, \rho^{-1}(y)\, dy, \quad 1 \le k < N(\mu),$$

(5.29) $$\hat{f}_0(\mu,\lambda) = L_2(\Lambda_0)\text{-}\lim_{M\to\infty} \int_{-M}^{M} \overline{\psi_0(y,\mu,\lambda)}\, f(y)\, c^{-2}(y)\, \rho^{-1}(y)\, dy,$$

and

(5.30) $$\hat{f}_\pm(\mu,\lambda) = L_2(\Lambda)\text{-}\lim_{M\to\infty} \int_{-M}^{M} \overline{\psi_\pm(y,\mu,\lambda)}\, f(y)\, c^{-2}(y)\, \rho^{-1}(y)\, dy.$$

In particular, on making $\lambda \to \infty$ one obtains the Parseval relation for A_μ:

(5.31) $$\|f\|^2 = \sum_{k=1}^{N(\mu)-1} |\hat{f}_k(\mu)|^2 + \int_{\Lambda_0} |\hat{f}_0(\mu,\lambda)|^2 d\lambda + \int_{\Lambda} (|\hat{f}_+(\mu,\lambda)|^2 + |\hat{f}_-(\mu,\lambda)|^2) d\lambda.$$

Thus the correspondence

(5.32) $$f \to \Psi_\mu f = (\hat{f}_+(\mu,\cdot), \hat{f}_-(\mu,\cdot), \hat{f}_0(\mu,\cdot), \hat{f}_1(\mu), \hat{f}_2(\mu), \cdots)$$

defines an isometric mapping Ψ_μ of $\mathcal{H}(R)$ into the direct sum space

(5.33) $$L_2(\Lambda) + L_2(\Lambda) + L_2(\Lambda_0) + C^{N(\mu)-1}$$

and one has

Theorem 5.6. Ψ_μ is a unitary operator from $\mathcal{H}(R)$ to the space (5.33).

This result will be shown to follow from the Weyl-Kodaira theorem and the corresponding properties of the partially isometric operators

$$\text{(5.34)}\qquad \begin{aligned} \Psi_{\mu\pm} &: \mathcal{H}(R) \to L_2(\Lambda), \\ \Psi_{\mu 0} &: \mathcal{H}(R) \to L_2(\Lambda_0), \\ \Psi_{\mu k} &: \mathcal{H}(R) \to C,\ 1 \le k < N(\mu), \end{aligned}$$

defined by

$$\text{(5.35)}\qquad \begin{aligned} \Psi_{\mu\pm} f &= \hat{f}_{\pm}(\mu,\cdot), \\ \Psi_{\mu 0} f &= \hat{f}_0(\mu,\cdot), \\ \Psi_{\mu k} f &= \hat{f}_k(\mu),\ 1 \le k < N(\mu). \end{aligned}$$

In fact, the Weyl-Kodaira theorem implies

<u>Theorem 5.7</u>. The operators (5.34) are partially isometric and if orthogonal projections in $\mathcal{H}(R)$ are defined by

$$\text{(5.36)}\qquad P_{\mu\pm} = \Psi^*_{\mu\pm}\ \Psi_{\mu\pm},\ P_{\mu k} = \Psi^*_{\mu k}\ \Psi_{\mu k},\ 0 \le k < N(\mu),$$

then

$$\text{(5.37)}\qquad \begin{aligned} P_{\mu+} + P_{\mu-} &= \Pi_\mu(\Lambda), \\ P_{\mu 0} &= \Pi_\mu(\Lambda_0), \\ \sum_{k=1}^{N(\mu)-1} P_{\mu k} &= \Pi_\mu(\Lambda_d). \end{aligned}$$

Moreover,

$$\text{(5.38)}\qquad \begin{aligned} \Psi_{\mu\pm}\ \Psi^*_{\mu\pm} &= 1 \text{ in } L_2(\Lambda(\mu)), \\ \Psi_{\mu 0}\ \Psi^*_{\mu 0} &= 1 \text{ in } L_2(\Lambda_0(\mu)), \\ \Psi_{\mu k}\ \Psi^*_{\mu k} &= 1 \text{ in } C,\ 1 \le k < N(\mu). \end{aligned}$$

<u>Corollary 5.8</u>. The inverse isometries $\Psi^*_{\mu\pm}$, $\Psi^*_{\mu k}$, $0 \le k < N(\mu)$, are given by

$$(5.39)\qquad \begin{aligned} (\Psi^*_{\mu\pm} f_\pm)(y) &= \mathcal{H}(R)\text{-}\lim_{\delta\to 0, M\to\infty} \int_{c^2(-\infty)\mu^2+\delta}^{M} \psi_\pm(y,\mu,\lambda)\; f_\pm(\lambda)d\lambda, \\ (\Psi^*_{\mu_0} f_0)(y) &= \mathcal{H}(R)\text{-}\lim_{\delta\to 0} \int_{c^2(\infty)\mu^2+\delta}^{c^2(-\infty)\mu^2-\delta} \psi_0(y,\mu,\lambda)\; f_0(\lambda)d\lambda, \\ (\Psi^*_{\mu k} f_k)(y) &= f_k\, \psi_k(y,\mu),\; 1 \le k < N(\mu). \end{aligned}$$

The spectral property of the unitary operator Ψ_μ is described by

<u>Corollary 5.9</u>. For every $f \in D(A_\mu)$ one has

$$(5.40)\qquad \Psi_\mu\, A_\mu f = (\lambda\hat{f}_+(\mu,\lambda), \lambda\hat{f}_-(\mu,\lambda), \lambda\hat{f}_0(\mu,\lambda), \lambda_1(\mu)\hat{f}_1(\mu), \cdots).$$

This completes the formulation of the results of §5 and the proofs will now be given.

<u>Proof of Theorem 5.1</u>. The integral representation of the resolvent will be used. Thus, as in the proof of Theorem 3.2,

$$(5.41)\qquad R_\mu(\zeta)\; f(y) = \int_R G_\mu(y,y',\zeta)\; f(y')\; w(y')\; dy',\quad \zeta \notin \sigma(A_\mu),$$

where $w(y) = c^{-2}(y)\ \rho^{-1}(y)$, and the Green's function G_μ has the form

$$(5.42)\qquad G_\mu(y,y',\zeta) = [\phi_\infty(\cdot,\zeta)\phi_{-\infty}(\cdot,\zeta)]^{-1}\phi_{-\infty}(y_<,\zeta)\;\phi_\infty(y_>,\zeta)$$

where $y_< = y_<(y,y') = \text{Min}\,(y,y')$, $y_> = y_>(y,y') = \text{Max}\,(y,y')$ and $\phi_{\pm\infty}(y,\zeta)$ are non-trivial solutions of $A_\mu\, \phi = \zeta\phi$ such that $\phi_{-\infty}(\cdot,\zeta) \in L_2(-\infty,0)$ and $\phi_\infty(\cdot,\zeta) \in L_2(0,\infty)$. To identify these solutions note that by (2.4)

$$(5.43)\qquad \begin{aligned} &\operatorname{Im} q_{\pm}(\mu,\zeta) > 0 \text{ for } \zeta \in R^{+}(c(-\infty)\mu)^{\text{int}} \\ &\operatorname{Im} q_{\pm}(\mu,\zeta) < 0 \text{ for } \zeta \in R^{-}(c(-\infty)\mu)^{\text{int}} \end{aligned}$$

and

$$(5.44)\qquad e^{\pm y\, q_{\pm}'} = e^{\pm i\, y\, q_{\pm}} = e^{\mp y \operatorname{Im} q_{\pm} \pm i\, y \operatorname{Re} q_{\pm}}.$$

It follows from Theorem 2.1 that

$$(5.45)\qquad \left.\begin{aligned} \phi_{\infty}(y,\zeta) &= \phi_1(y,\mu,\zeta) \\ \phi_{-\infty}(y,\zeta) &= \phi_4(y,\mu,\zeta) \end{aligned}\right\} \text{ for } \zeta \in R^{+}(c(-\infty)\mu)^{\text{int}}$$

and

$$(5.46)\qquad \left.\begin{aligned} \phi_{\infty}(y,\zeta) &= \phi_2(y,\mu,\zeta) \\ \phi_{-\infty}(y,\zeta) &= \phi_3(y,\mu,\zeta) \end{aligned}\right\} \text{ for } \zeta \in R^{-}(c(-\infty)\mu)^{\text{int}}.$$

Now the functions appearing in (5.45) and (5.46) have continuous extensions to $R^{+}(c(-\infty)\mu)$ and $R^{-}(c(-\infty)\mu)$, respectively, by Corollary 2.2. Indeed, (5.42), (5.45), (5.46) and Corollary 2.2 imply

<u>Lemma 5.10</u>. For all $\lambda \in \Lambda$ one has

$$(5.47)\qquad G_{\mu}(y,y',\lambda+i0) = [\phi_1\phi_4]^{-1}_{(\lambda)}\ \phi_4(y_<,\mu,\lambda)\ \phi_1(y_>,\mu,\lambda),$$

$$(5.48)\qquad G_{\mu}(y,y',\lambda-i0) = [\phi_2\phi_3]^{-1}_{(\lambda)}\ \phi_3(y_<,\mu,\lambda)\ \phi_2(y_>,\mu,\lambda),$$

and the limits are uniform on compact subsets of $R \times R \times \Lambda$.

To prove Theorem 5.1 it is clearly sufficient to verify (5.9) for the functions f of a dense subset of $\mathcal{H}(R)$. It will be convenient to use the subset

$$\mathcal{H}^{com}(R) = \mathcal{H}(R) \cap \{f \mid \text{supp } f \text{ is compact}\}. \tag{5.49}$$

Note that for $f \in \mathcal{H}^{com}(R)$ the integrand in Stone's formula (5.8) can be written

$$\frac{1}{2\pi i}(f,[R_\mu(\lambda+i\varepsilon) - R_\mu(\lambda-i\varepsilon)]f)$$

(5.50)

$$= \frac{1}{2\pi i}\int_R \int_R \{G_\mu(y,y',\lambda+i\varepsilon) - G_\mu(y,y',\lambda-i\varepsilon)\}\,\overline{f(y)}\,f(y')\,w(y)\,w(y')\,dydy'$$

Lemma 5.10 implies that for all $f \in \mathcal{H}^{com}(R)$ this expression tends to a limit

$$(f,H_\mu(\lambda)f) = \int_R \int_R H_\mu(y,y',\lambda)\,\overline{f(y)}\,f(y')\,w(y)\,w(y')\,dydy', \tag{5.51}$$

uniformly on compact subsets $\Delta \subset \Lambda$, where

$$H_\mu(y,y',\lambda) = \frac{1}{2\pi i}\{G_\mu(y,y',\lambda+i0) - G_\mu(y,y',\lambda-i0)\}. \tag{5.52}$$

It follows that

$$\|\Pi_\mu(\Delta)f\|^2 = \int_\Delta (f,H_\mu(\lambda)f)\,d\lambda \tag{5.53}$$

for all $f \in \mathcal{H}^{com}(R)$. The proof of Theorem 5.1 will be completed by calculating $H_\mu(y,y',\lambda)$. Note that the well-known property $R_\mu(\zeta)^* = R_\mu(\bar{\zeta})$ implies that $\overline{G_\mu(y',y,\zeta)} = G_\mu(y,y',\bar{\zeta})$ and hence

$$G_\mu(y,y',\lambda-i0) = \overline{G_\mu(y',y,\lambda+i0)}. \tag{5.54}$$

Equations (5.52) and (5.54) imply that $H_\mu(y,y',\lambda)$ is Hermitian symmetric:

$$\overline{H_\mu(y',y,\lambda)} = H_\mu(y,y',\lambda). \tag{5.55}$$

Hence, it will be enough to calculate $H_\mu(y,y',\lambda)$ for $y \le y'$.

Calculation of $H_\mu(y,y',\lambda)$. Definition (2.4) of $q_\pm(\mu,\zeta)$ implies that

$$\overline{q_\pm'(\mu,\overline{\zeta})} = -q_\pm'(\mu,\zeta) \text{ for } \zeta \in R(c(-\infty)\mu). \tag{5.56}$$

It follows from Theorem 2.1 and Corollary 2.5 that

$$\left.\begin{aligned} \phi_1(y,\mu,\zeta) &= \overline{\phi_2(y,\mu,\overline{\zeta})} \\ \phi_4(y,\mu,\zeta) &= \overline{\phi_3(y,\mu,\overline{\zeta})} \end{aligned}\right\} \text{ for } \zeta \in R^+(c(-\infty)\mu) \tag{5.57}$$

and

$$\left.\begin{aligned} \phi_2(y,\mu,\zeta) &= \overline{\phi_1(y,\mu,\overline{\zeta})} \\ \phi_3(y,\mu,\zeta) &= \overline{\phi_4(y,\mu,\overline{\zeta})} \end{aligned}\right\} \text{ for } \zeta \in R^-(c(-\infty)\mu). \tag{5.58}$$

It follows that

$$\begin{aligned} \phi_1(y',\mu,\lambda) &= \overline{\phi_2(y',\mu,\lambda)} \\ &= \overline{c_{21}(\mu,\lambda)}\ \overline{\phi_1(y',\mu,\lambda)} + \overline{c_{24}(\mu,\lambda)}\ \overline{\phi_4(y',\mu,\lambda)} \end{aligned} \tag{5.59}$$

where

$$c_{21} = [\phi_2\phi_4]/[\phi_1\phi_4],\quad c_{24} = [\phi_2\phi_1]/[\phi_4\phi_1]. \tag{5.60}$$

Similarly

$$\phi_2(y',\mu,\lambda) = \overline{\phi_1(y',\mu,\lambda)} \tag{5.61}$$

and

$$\phi_3(y,\mu,\lambda) = c_{31}(\mu,\lambda)\ \phi_1(y,\mu,\lambda) + c_{34}(\mu,\lambda)\ \phi_4(y,\mu,\lambda) \tag{5.62}$$

where

$$c_{31} = [\phi_3\phi_4]/[\phi_1\phi_4],\quad c_{34} = [\phi_3\phi_1]/[\phi_4\phi_1]. \tag{5.63}$$

Combining these relations and Lemma 5.10, one finds for $\lambda \in \Lambda$, $y \leq y'$

$$\begin{aligned} G_\mu(y,y',\lambda+i0) &= [\phi_1\phi_4]^{-1}\ \phi_4(y,\mu,\lambda)\ \phi_1(y',\mu,\lambda) \\ &= [\phi_1\phi_4]^{-1}\{\overline{c}_{21}\ \phi_4(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)} + \overline{c}_{24}\ \phi_4(y,\mu,\lambda)\ \overline{\phi_4(y',\mu,\lambda)}\} \end{aligned} \tag{5.64}$$

and

$$\begin{aligned} G_\mu(y,y',\lambda-i0) &= [\phi_2\phi_3]^{-1}\ \phi_3(y,\mu,\lambda)\ \phi_2(y',\mu,\lambda) \\ &= [\phi_2\phi_3]^{-1}\{c_{31}\ \phi_1(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)} + c_{34}\ \phi_4(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)}\}. \end{aligned} \tag{5.65}$$

Combining (5.52), (5.64) and (5.65) gives

$$\begin{aligned} H_\mu(y,y',\lambda) = \frac{1}{2\pi i}\Bigg(&\frac{\overline{c}_{24}}{[\phi_1\phi_4]}\ \phi_4(y,\mu,\lambda)\ \overline{\phi_4(y',\mu,\lambda)} - \frac{c_{31}}{[\phi_2\phi_3]}\ \phi_1(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)} \\ &+ \left\{\frac{\overline{c}_{21}}{[\phi_1\phi_4]} - \frac{c_{34}}{[\phi_2\phi_3]}\right\}\ \phi_4(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)}\Bigg) \end{aligned} \tag{5.66}$$

To calculate the coefficients in (5.66) recall that $\overline{\phi_1(y,\mu,\lambda)} = \phi_2(y,\mu,\lambda)$, $\overline{\phi_3(y,\mu,\lambda)} = \phi_4(y,\mu,\lambda)$ for $\lambda \in \Lambda$. It follows from (5.60), (5.63) and

(2.25) that

$$(5.67)\qquad \frac{\overline{c_{24}}}{[\phi_1\phi_4]} = \frac{-[\phi_1\phi_2]}{|[\phi_1\phi_4]|^2} = \frac{2i\rho^{-1}(\infty)q_+}{|[\phi_1\phi_4]|^2} = 2\pi i\ A_+^2,$$

$$(5.68)\qquad \frac{-c_{31}}{[\phi_2\phi_3]} = \frac{-[\phi_3\phi_4]}{|[\phi_1\phi_4]|^2} = \frac{2i\rho^{-1}(-\infty)q_-}{|[\phi_1\phi_4]|^2} = 2\pi i\ A_-^2,$$

$$(5.69)\qquad \frac{\overline{c_{21}}}{[\phi_1\phi_4]} - \frac{c_{34}}{[\phi_2\phi_3]} = \frac{[\phi_1\phi_3]}{[\phi_1\phi_4][\phi_2\phi_3]} - \frac{[\phi_3\phi_1]}{[\phi_4\phi_1][\phi_2\phi_3]} = 0.$$

Thus (5.66) can be written

$$(5.70)\qquad \begin{aligned} H_\mu(y,y',\lambda) &= A_+^2\ \phi_4(y,\mu,\lambda)\ \overline{\phi_4(y',\mu,\lambda)} + A_-^2\ \phi_1(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)} \\ &= A_+^2\ \psi_1(y,\lambda)\ \overline{\psi_1(y',\lambda)} + A_-^2\ \psi_2(y,\lambda)\ \overline{\psi_2(y',\lambda)}. \end{aligned}$$

This was proved for $y \le y'$. However, both sides of (5.70) are Hermitian symmetric and it therefore holds for all $(y,y') \in R^2$. Multiplying (5.70) by $\overline{f(y)}\ f(y')\ w(y)\ w(y')$ and integrating over R^2 gives

$$(5.71)\qquad (f,H_\mu(\lambda)f) = A_+^2\ |\hat{f}_1(\lambda)|^2 + A_-^2\ |\hat{f}_2(\lambda)|^2$$

and combining (5.53) and (5.71) gives (5.9). This completes the proof of Theorem 5.1.

Proof of Corollary 5.2. It follows from (5.5) and (5.9) that

$$(5.72)\qquad \begin{aligned} (f,\Pi_\mu(\Delta)g) &= \int_\Delta \overline{\hat{f}_j(\lambda)}\ \hat{g}_j(\lambda)\ m_{jk}(d\lambda) \\ &= \int_\Delta \{A_+^2\ \overline{\hat{f}_1(\lambda)}\ \hat{g}_1(\lambda) + A_-^2\ \overline{\hat{f}_2}(\lambda)\ \hat{g}_2(\lambda)\}\ d\lambda \end{aligned}$$

for all $f,g \in \mathcal{H}(R)$. Moreover, the Weyl-Kodaira theorem implies that

$\hat{f} = (\hat{f}_1,\hat{f}_2)$ and $\hat{g} = (\hat{g}_1,\hat{g}_2)$ can be arbitrary vectors in $L_2(\Lambda,m)$. Now all vectors $\hat{f}$ with bounded Borel functions as components are in $L_2(\Lambda,m)$. Thus the first of equations (5.11) can be obtained from (5.72) by taking $\hat{f}_1 = \hat{g}_1 = \chi_\Delta$, the characteristic function of Δ, and $\hat{f}_2 = \hat{g}_2 = 0$. The remaining equations of Corollary 5.2 are obtained *similarly*.

<u>Proof of Theorem 5.4</u>. The proof follows that of Theorem 5.1. Equations (5.41), (5.42) for the resolvent are still valid. However, instead of (5.43) one has

$$\begin{aligned} &\operatorname{Im} q_+(\mu,\zeta) > 0 \text{ for } \zeta \in R^+(c(\infty)\mu)^{\text{int}}, \\ &\operatorname{Im} q_+(\mu,\zeta) < 0 \text{ for } \zeta \in R^-(c(\infty)\mu)^{\text{int}}, \\ &\operatorname{Im} q_-(\mu,\zeta) < 0 \text{ for } \zeta \in R(c(\infty)\mu) \cap L(c(-\infty)\mu). \end{aligned} \tag{5.73}$$

Thus by Theorem 2.1

$$\left.\begin{aligned} \phi_\infty(y,\zeta) &= \phi_1(y,\mu,\zeta) \\ \phi_{-\infty}(y,\zeta) &= \phi_3(y,\mu,\zeta) \end{aligned}\right\} \text{ for } \zeta \in R^+(c(\infty)\mu)^{\text{int}} \cap L(c(-\infty)\mu), \tag{5.74}$$

and

$$\left.\begin{aligned} \phi_\infty(y,\zeta) &= \phi_2(y,\mu,\zeta) \\ \phi_{-\infty}(y,\zeta) &= \phi_3(y,\mu,\zeta) \end{aligned}\right\} \text{ for } \zeta \in R^-(c(\infty)\mu)^{\text{int}} \cap L(c(-\infty)\mu). \tag{5.75}$$

Hence Corollary 2.2 implies

<u>Lemma 5.11</u>. For all $\lambda \in \Lambda_0$ and $y \le y'$ one has

$$G_\mu(y,y',\lambda+i0) = [\phi_1\phi_3]^{-1}\, \phi_3(y,\mu,\lambda)\, \phi_1(y',\mu,\lambda), \tag{5.76}$$

$$G_\mu(y,y',\lambda-i0) = [\phi_2\phi_3]^{-1}\, \phi_3(y,\mu,\lambda)\, \phi_2(y',\mu,\lambda), \tag{5.77}$$

and the limits are uniform on compact subsets of $R \times R \times \Lambda_0$.

Proceeding to the calculation of $H_\mu(y,y',\lambda)$ one has

$$(5.78)\qquad \left.\begin{aligned} \overline{q_+'(\mu,\overline{\zeta})} &= -q_-'(\mu,\zeta) \\ \overline{q_-'(\mu,\overline{\zeta})} &= \ \ q_-'(\mu,\zeta) \end{aligned}\right\} \text{ for } \zeta \in R(c(\infty)\mu) \cap L(c(-\infty)\mu)$$

whence

$$(5.79)\qquad \phi_1(y,\mu,\zeta) = \overline{\phi_2(y,\mu,\overline{\zeta})} \text{ for } \zeta \in \ddot{R}^+(c(\infty)\mu) \cap L(c(-\infty)\mu)$$

and

$$(5.80)\qquad \phi_3(y,\mu,\zeta) = \overline{\phi_3(y,\mu,\overline{\zeta})} \text{ for } \zeta \in R(c(\infty)\mu) \cap L(c(-\infty)\mu).$$

It follows that

$$(5.81)\qquad \begin{aligned} \phi_1(y',\mu,\lambda) &= \overline{\phi_2(y',\mu,\lambda)} \\ &= \overline{c_{21}(\mu,\lambda)}\ \overline{\phi_1(y',\mu,\lambda)} + \overline{c_{23}(\mu,\lambda)}\ \overline{\phi_3(y,\mu,\lambda)} \end{aligned}$$

where

$$(5.82)\qquad c_{21} = [\phi_2\phi_3]/[\phi_1\phi_3] \text{ and } c_{23} = [\phi_2\phi_1]/[\phi_3\phi_1].$$

These relations and Lemma 5.11 imply that for $\lambda \in \Lambda_0$ and $y \le y'$,

$$(5.83)\qquad \begin{aligned} &G_\mu(y,y',\lambda+i0) \\ &= [\phi_1\phi_3]^{-1}\{\overline{c}_{21}\ \phi_3(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)} + \overline{c}_{23}\phi_3(y,\mu,\lambda)\ \overline{\phi_3(y',\mu,\lambda)}\} \end{aligned}$$

and

$$(5.84)\qquad G_\mu(y,y',\lambda-i0) = [\phi_2\phi_3]^{-1}\ \phi_3(y,\mu,\lambda)\ \overline{\phi_1(y',\mu,\lambda)}.$$

Hence

$$H_\mu(y,y',\lambda)$$

(5.85)

$$= \frac{1}{2\pi i}\left\{\frac{\bar{c}_{23}}{[\phi_1\phi_3]}\phi_3(y,\mu,\lambda)\overline{\phi_3(y',\mu,\lambda)} + \left\{\frac{\bar{c}_{21}}{[\phi_1\phi_3]} - \frac{1}{[\phi_2\phi_3]}\right\}\phi_3(y,\mu,\lambda)\overline{\phi_1(y',\mu,\lambda)}\right\}.$$

From the relations $\overline{\phi_1(y,\mu,\lambda)} = \phi_2(y,\mu,\lambda)$, $\overline{\phi_3(y,\mu,\lambda)} = \phi_3(y,\mu,\lambda)$, together with (5.82) and (2.25) it follows that

$$\frac{\bar{c}_{23}}{[\phi_1\phi_3]} = \frac{-[\phi_1\phi_2]}{|[\phi_1\phi_3]|^2} = \frac{2i\rho^{-1}(\infty)q_+}{|[\phi_1\phi_3]|^2} = 2\pi i\ A_0^2, \tag{5.86}$$

and

$$\frac{\bar{c}_{21}}{[\phi_1\phi_3]} - \frac{1}{[\phi_2\phi_3]} = \frac{[\phi_1\phi_3]}{[\phi_1\phi_3][\phi_2\phi_3]} - \frac{1}{[\phi_2\phi_3]} = 0. \tag{5.87}$$

Thus (5.85) can be written

$$H_\mu(y,y',\lambda) = A_0^2\ \phi_3(y,\mu,\lambda)\ \overline{\phi_3(y',\mu,\lambda)} = A_0^2\ \psi_1(y,\lambda)\ \overline{\psi_1(y',\lambda)} \tag{5.88}$$

for $y \le y'$ and hence for all $(y,y') \in R^2$. It follows by integration that for all $f \in \mathcal{H}^{com}(R)$ one has

$$(f,H_\mu(\lambda)f) = A_0^2\ |\hat{f}_0(\lambda)|^2,\ \lambda \in \Lambda_0. \tag{5.89}$$

Combining (5.89) and (5.53) gives (5.18). Finally, (5.18) implies (5.20) by the argument used to prove Corollary 5.2. This completes the proof of Theorem 5.4.

Proof of Theorem 5.6. Ψ_μ is isometric by (5.31), (5.32). Hence to prove that Ψ_μ is unitary it is only necessary to prove that it is surjective. But this is an immediate consequence of Theorem 5.7, equations (5.38). The latter are implied by the Weyl-Kodaira theorem.

Proof of Theorem 5.7 and Corollaries 5.8 and 5.9. These results are direct consequences of the Weyl-Kodaira theorem.

§6. The Dispersion Relations for the Guided Modes.

The eigenvalues of A_μ determine the relation between the wave number $|p|$ and frequency $\omega = \sqrt{\lambda_k(|p|)}$, or dispersion relation, for the guided mode functions $\psi_k(x,y,p)$. The functions $\lambda_k(\mu)$ also appear in the definition (3.4) of $\psi_k(y,\mu)$. The purpose of this section is to provide the information concerning the μ-dependence of $\lambda_k(\mu)$ and $\psi_k(y,\mu)$ that is needed for the spectral analysis of A in §§7-8.

The domain of definition of the function $\lambda_k(\mu)$ is the set

$$\mathcal{O}_k = \{\mu \mid N(\mu) \geq k + 1\}, \ k = 1,2,\cdots. \tag{6.1}$$

Note that $\mathcal{O}_k$ is not empty if and only if $1 \leq k < N_0$ where

$$N_0 = \sup_{\mu>0} N(\mu) \leq +\infty. \tag{6.2}$$

Clearly, if $N_0 < +\infty$ then $N_0 - 1$ is the maximum number of eigenvalues of A_μ for $\mu > 0$. If $N_0 = +\infty$ then either $\sigma_0(A_\mu)$ is infinite for some $\mu > 0$ or $\sigma_0(A_\mu)$ is finite for all $\mu > 0$ and $N(\mu) \to \infty$ when $\mu \to \infty$. Theorem 3.14 implies that both cases occur. The principal result of this section is

Theorem 6.1. For $1 \leq k < N_0$ the set $\mathcal{O}_k$ is open and $\lambda_k : \mathcal{O}_k \to R$ is an analytic function.

The proof of this result given below is based on analytic perturbation theory as developed in [8].

The curves $\lambda = \lambda_k(\mu)$, $\mu \in \mathcal{O}_k$, can never meet or cross because each eigenvalue is simple and the corresponding eigenfunction $\psi_k(y,\mu)$ has exactly k zeros (Theorem 3.8). Thus for $1 \leq k < N_0 - 1$ one has

$$(6.3) \qquad c_m^2\mu^2 \le \lambda_k(\mu) < \lambda_{k+1}(\mu) < c^2(\infty)\mu^2, \ \mu \in \mathcal{O}_{k+1}.$$

Moreover, if $\mathcal{O}_k$ is unbounded then (6.3) implies

$$(6.4) \qquad c_m^2 \le \liminf_{\mu\to\infty} \mu^{-2}\lambda_k(\mu) \le \limsup_{\mu\to\infty} \mu^{-2}\lambda_k(\mu) \le c^2(\infty).$$

In particular, if $\mathcal{O}_k$ is unbounded then

$$(6.5) \qquad \lim_{\mu\to\infty} \lambda_k(\mu) = +\infty.$$

A related property is given by

<u>Theorem 6.2</u>. For $1 \le k < N_0$ the function $\lambda_k(\mu)$ is strictly monotone increasing; i.e., for all $\mu_1, \mu_2 \in \mathcal{O}_k$ one has

$$(6.6) \qquad \lambda_k(\mu_1) < \lambda_k(\mu_2) \text{ when } \mu_1 < \mu_2.$$

The proof of (6.6) given below is based on a variational characterization of $\lambda_k(\mu)$.

By Theorem 6.1, $\mathcal{O}_k$ is open and is therefore a union of disjoint open intervals. Hence the curve $\lambda = \lambda_k(\mu)$ consists of one or more disjoint analytic arcs. It is interesting that these arcs can terminate only on the curve $\lambda = c^2(\infty)\mu^2$. More precisely, one has

<u>Corollary 6.3</u>. Let μ_0 be a boundary point of $\mathcal{O}_k$. Then

$$(6.7) \qquad \lim_{\mu\to\mu_0} \lambda_k(\mu) = c^2(\infty)\mu_0^2.$$

It is clear from Theorem 6.2 and (6.3) that the limit in (6.7) exists and does not exceed $c^2(\infty)\mu_0^2$. The equality (6.7) is proved below.

The result (6.4) can be improved by strengthening the hypotheses concerning $c(y)$. A result of this type is

<u>Theorem 6.4</u>. Let $c_m < c(\infty)$ and assume that for each $\varepsilon > 0$ there is an interval $I(\varepsilon) \subset R$ such that

(6.8) $$c(y) \le c_m + \varepsilon \text{ for all } y \in I(\varepsilon).$$

Then $N_0 = +\infty$, $\mathcal{O}_k$ is unbounded for each $k \ge 1$ and $\lambda_k(\mu) \sim c_m^2\mu^2$ when $\mu \to \infty$ in $\mathcal{O}_k$; i.e.,

(6.9) $$\lim_{\mu\to\infty} \mu^{-2} \lambda_k(\mu) = c_m^2.$$

The analyticity of $\lambda_k(\mu)$ and Corollary 2.2 imply the continuity of the eigenfunctions $\psi_k(y,\mu)$. More precisely, one has

<u>Corollary 6.5</u>. For $1 \le k < N_0$ the function $\psi_k(y,\mu)$ satisfies

(6.10) $$\psi_k,\ \rho^{-1}\ \psi_k' \in C(R \times \mathcal{O}_k).$$

This completes the formulation of the results of §6.

<u>Proof of Theorem 6.1</u>. The analytic perturbation theory of [8, Ch. VII] will be used. Note that the operator A_μ may be defined for all $\mu \in C$ by (1.16), (1.18) and is a closed operator in $\mathcal{H}(R)$. Moreover, the domain $D(A_\mu)$ is independent of μ and is a Hilbert space with respect to the norm defined by

(6.11) $$\|\phi\|^2_{D(A_\mu)} = \|\phi\|^2_{\mathcal{H}(R)} + \|\phi'\|^2_{\mathcal{H}(R)} + \|(\rho^{-1}\phi')'\|^2_{\mathcal{H}(R)}.$$

It follows that $\mu \to A_\mu$ is holomorphic in the generalized sense. Indeed, in the definition of [8, p. 366] one may take $Z = D(A_\mu)$ (independent of μ) and define $U(\mu) : Z \to \mathcal{H}(R)$ to be the identification map. Then $U(\mu)$ is bounded holomorphic (in fact, constant) and

(6.12) $$V(\mu)\phi = A_\mu\ U(\mu)\phi = -c^2\rho\ [(\rho^{-1}\phi')' - \rho^{-1}\mu^2\phi]$$

is holomorphic for all $\phi \in Z$. Thus A_μ is holomorphic. It follows by [8, p. 370] that each $\lambda_k(\mu)$ has a Puiseux expansion at each point $\mu_0 \in \mathcal{O}_k$. But each $\lambda_k(\mu_0)$ is simple (Lemma 3.3) and hence the Puiseux series can contain no fractional powers of $\mu - \mu_0$ [8, p. 71]. Thus λ_k is in fact analytic at each $\mu_0 \in \mathcal{O}_k$. This proves both statements of Theorem 6.1.

Proof of Theorem 6.2 and Corollary 6.3. The eigenvalue $\lambda_k(\mu)$ can be characterized by the variational principle [3, pp. 1543-4]

$$\lambda_k(\mu) = \inf_{M \in S_k} \sup_{\substack{\phi \in M \cap D(A_\mu) \\ \|\phi\|=1}} (A_\mu \phi, \phi) \tag{6.13}$$

where S_k denotes the set of all k-dimensional subspaces of $\mathcal{H}(R)$. Moreover, $D(A_\mu)$ is independent of μ and

$$(A_\mu \phi, \phi) = \int_R (|\phi'(y)|^2 + \mu^2 \, |\phi(y)|^2) \, \rho^{-1}(y) \, dy \tag{6.14}$$

for all $\phi \in D(A_\mu)$ [8, p. 322]. Hence if $\mu_1 < \mu_2$ then

$$(A_{\mu_1} \phi, \phi) < (A_{\mu_2} \phi, \phi) \tag{6.15}$$

for all $\phi \in D(A_{\mu_1}) = D(A_{\mu_2})$. In particular, (6.13) and (6.15) imply that if $\mu_1, \mu_2 \in \mathcal{O}_k$ then

$$\lambda_k(\mu_1) \leq \lambda_k(\mu_2) \tag{6.16}$$

which proves the weak monotonicity of λ_k. It will be convenient to use (6.16) to prove Corollary 6.3 before proving the strong monotonicity.

To prove Corollary 6.3 note that (6.16) and (6.3) imply that the limit in (6.7) exists and does not exceed $c^2(\infty)\mu_0^2$. But if

$\lim_{\mu \to \mu_0} \lambda_k(\mu) = \lambda_k^0 < c^2(\infty)\mu_0^2$ then Corollary 2.2 implies that μ_0, λ_k^0 satisfy

$$F(\mu_0, \lambda_k^0) = 0; \tag{6.17}$$

see (3.28). Moreover, $F(\mu,\lambda)$ is analytic at μ_0, λ_k^0 by Corollary 2.3. It follows that λ_k^0 is an eigenvalue of A_{μ_0} and hence $\mu_0 \in \mathcal{O}_k$ by Theorem 6.1. This contradicts the assumption that μ_0 is a boundary point of $\mathcal{O}_k$.

To prove that each $\lambda_k(\mu)$ is strictly monotonic in $\mathcal{O}_k$ two cases will be considered. First, if λ_1, λ_2 are in the same component of $\mathcal{O}_k$, say $(a,b) \subset \mathcal{O}_k$, then $\lambda_k(\mu_1) = \lambda_k(\mu_2)$ would imply that $\lambda_k(\mu) = \text{const.}$ in $[\mu_1, \mu_2]$ and hence in (a,b), since $\lambda_k(\mu)$ by Theorem 6.1. But this contradicts Corollary 6.3 since

$$c^2(\infty)a^2 = \lim_{\mu \to a} \lambda_k(\mu) < \lim_{\mu \to b} \lambda_k(\mu) = c^2(\infty)b^2 \tag{6.18}$$

In the second case μ_1 and μ_2 lie in different components of $\mathcal{O}_k$, say $\mu_1 \in (a_1, b_1) \subset \mathcal{O}_k$ and $\mu_2 \in (a_2, b_2) \subset \mathcal{O}_k$ with $b_1 \leq a_2$. In this case, by the preceding argument one has

$$\lambda_k(\mu_1) < c^2(\infty)b_1^2 \leq c^2(\infty)a_2^2 < \lambda_k(\mu_2) \tag{6.19}$$

which completes the proof.

<u>Proof of Theorem 6.4 (sketch)</u>. The proof is based on the method proposed for the proof of Theorem 3.16 and the variational principle (6.13). Note that the hypothesis (6.8) and Theorem 3.16, generalized to non-constant $\rho(y)$, imply that $N(\mu) \to \infty$ for $\mu \to \infty$. Hence $N_0 = +\infty$ and each $\mathcal{O}_k$ is unbounded.

To prove (6.9) choose piece-wise constant functions $c_0(y)$ and $c^0(y)$ such that

(6.20) $$c_0(y) \le c(y) \le c^0(y) \text{ for all } y \in R,$$

(6.21) $$c_0(y) = c_m < c_0(\infty) \text{ on an interval } I_0, \text{ and}$$

(6.22) $$c_m + \varepsilon = c^0(y) < c^0(\infty) \text{ on an interval } I(\varepsilon).$$

The notation

(6.23) $$\begin{aligned} \mathcal{H}_0(R) &= L_2(R, c_0^{-2}(y)\, \rho^{-1}(y)\, dy) \\ \mathcal{H}^0(R) &= L_2(R, c^{0\,-2}(y)\, \rho^{-1}(y)\, dy) \end{aligned}$$

will be used. The three spaces $\mathcal{H}(R)$, $\mathcal{H}_0(R)$ and $\mathcal{H}^0(R)$ have equivalent norms. In particular, if $\|\phi\|_0$ and $\|\phi\|^0$ denote the norms in $\mathcal{H}_0(R)$ and $\mathcal{H}^0(R)$, respectively, then by (6.20)

(6.24) $$\|\phi\|^0 \le \|\phi\| \le \|\phi\|_0 .$$

Now note that the variational principle can be formulated in the homogeneous form

(6.25) $$\lambda_k(\mu) = \inf_{\mathcal{M}\in S_k} \sup_{\substack{\phi\in\mathcal{M}\cap D(A_\mu) \\ \phi\ne 0}} \frac{(A_\mu\phi,\phi)}{\|\phi\|^2}$$

where

(6.26) $$\frac{(A_\mu\phi,\phi)}{\|\phi\|^2} = \frac{\int_R (|\phi'|^2+\mu^2|\phi|^2)\rho^{-1}(y)\,dy}{\|\phi\|^2} .$$

Both the numerator in (6.26) and $D(A_\mu)$ are independent of $c(y)$. Hence, if $A_{0\mu}$ and A_μ^0 denote the operators corresponding to $\rho(y)$, $c_0(y)$ and $\rho(y)$, $c^0(y)$ respectively, then (6.24) and (6.26) imply

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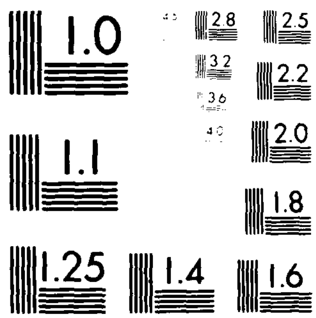

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(6.27) $$\frac{(A_{0\mu}\phi,\phi)}{\|\phi\|_0^2} \le \frac{(A_\mu\phi,\phi)}{\|\phi\|^2} \le \frac{(A_\mu^0\phi,\phi)}{\|\phi\|^{02}}$$

for all $\phi \in D(A_{0\mu}) = D(A_\mu) = D(A_\mu^0)$ such that $\phi \neq 0$. The variational principle (6.25) and (6.27) imply that

(6.28) $$\lambda_{0k}(\mu) \le \lambda_k(\mu) \le \lambda_k^0(\mu)$$

for all sufficiently large μ, where $\{\lambda_{0k}(\mu)\}$ and $\{\lambda_k^0(\mu)\}$ are the eigenvalues of $A_{0\mu}$ and A_μ^0, respectively.

The proof of (6.9) can now be completed by showing by direct calculation that (see [17])

(6.29) $$\lim_{\mu\to\infty} \mu^{-2}\,\lambda_{0k}(\mu) = c_m^2$$
$$\lim_{\mu\to\infty} \mu^{-2}\,\lambda_k^0(\mu) = (c_m + \varepsilon)^2 .$$

It follows from (6.28), (6.29) that

(6.30) $$c_m^2 \le \liminf_{\mu\to\infty} \mu^{-2}\,\lambda_k(\mu) \le \limsup_{\mu\to\infty} \mu^{-2}\,\lambda_k(\mu) \le (c_m + \varepsilon)^2 .$$

Equation (6.9) follows because, by hypothesis, $\varepsilon > 0$ is arbitrary.

Proof of Corollary 6.5. The result (6.10) is immediate from Corollary 2.2, the relation

(6.31) $$\psi_k(y,\mu) = \phi_3(y,\mu,\lambda_k(\mu))/\|\phi_3(\cdot,\mu,\lambda_k(\mu))\|$$

and Theorem 6.1.

§7. The Spectral Family of A.

The acoustic propagator A was defined in §1 and shown to be a selfadjoint non-negative operator in the Hilbert space $\mathcal{H}$. In this section the spectral family $\{\Pi(\lambda) \mid \lambda \geq 0\}$ of A is constructed by means of the normal mode functions $\{\psi_+, \psi_-, \psi_0, \psi_1, \cdots\}$. The method of construction is to use Fourier analysis in the variables $x \in R^2$ to reduce A to the operator $A_{|p|}$ and then to use the spectral representation of $\Pi_{|p|}(\lambda)$ developed in §5. The construction is given in Theorems 7.1-7.4. In the remainder of the section the proofs of Theorems 7.1-7.4 are developed in a series of lemmas and auxiliary theorems.

The formal definitions of the normal mode functions $\psi_{\pm}(x,y,p,\lambda)$, $\psi_0(x,y,p,\lambda)$ and $\psi_k(x,y,p)$ were given in §1, equations (1.29)-(1.31) and (1.36)-(1.41). The definitions were completed by the construction of the special solutions $\phi_j(y,\mu,\lambda)$ in §2 and of the normalizing factors $a_{\pm}(\mu,\lambda)$, $a_0(\mu,\lambda)$, $a_k(\mu)$ in §5. The construction of $\Pi(\lambda)$ will be based on these normal mode functions and the corresponding *generalized Fourier transforms*. Formally the latter are the scalar products of functions $f \in \mathcal{H}$ with the normal mode functions. The following notation will be used.

$$\tilde{f}_{\pm}(p,\lambda) = \int_{R^3} \overline{\psi_{\pm}(x,y,p,\lambda)}\, f(x,y)\, c^{-2}(y)\, \rho^{-1}(y)\, dxdy, \tag{7.1}$$

$$\tilde{f}_0(p,\lambda) = \int_{R^3} \overline{\psi_0(x,y,p,\lambda)}\, f(x,y)\, c^{-2}(y)\, \rho^{-1}(y)\, dxdy, \tag{7.2}$$

$$\tilde{f}_k(p) = \int_{R^3} \overline{\psi_k(x,y,p)}\, f(x,y)\, c^{-2}(y)\, \rho^{-1}(y)\, dxdy, \quad k \geq 1. \tag{7.3}$$

Of course these integrals need not converge since the normal mode functions are not in $\mathcal{H}$. Instead, they will be interpreted as Hilbert space limits as in the Plancherel theory of the Fourier transform. This interpretation will be based on the following three theorems.

Theorem 7.1. If $f \in L_1(R^3)$ then the integrals in (7.1), (7.2), (7.3) are absolutely convergent for $(p,\lambda) \in \Omega$, $(p,\lambda) \in \Omega_0$ and $p \in \Omega_k$, respectively, and

$$\tilde{f}_{\pm} \in C(\Omega),\ \tilde{f}_0 \in C(\Omega_0),\ \tilde{f}_k \in C(\Omega_k),\ k \geq 1. \tag{7.4}$$

For each $f \in \mathcal{H}$ and $M > 0$ define

$$f_M(x,y) = \begin{cases} f(x,y) \text{ if } |x| \leq M \text{ and } |y| \leq M, \\ 0 \quad \text{if } |x| > M \text{ or } |y| > M. \end{cases} \tag{7.5}$$

It is clear that $f_M \to f$ in $\mathcal{H}$ when $M \to \infty$. Moreover, $f_M \in \mathcal{H} \cap L_1(R^3)$ and one has

Theorem 7.2. For every $f \in \mathcal{H}$ and $M > 0$,

$$\tilde{f}_{M\pm} \in L_2(\Omega),\ \tilde{f}_{M0} \in L_2(\Omega_0),\ \tilde{f}_{Mk} \in L_2(\Omega_k),\ k \geq 1, \tag{7.6}$$

and the Parseval relation holds:

$$\|f_M\|^2_{\mathcal{H}} = \|f_{M+}\|^2_{L_2(\Omega)} + \|\tilde{f}_{M-}\|^2_{L_2(\Omega)} + \sum_{k=0}^{N_0-1} \|\tilde{f}_{Mk}\|^2_{L_2(\Omega_k)} \tag{7.7}$$

The relation (7.7) suggests the introduction of the direct sum space

$$\tilde{\mathcal{H}} = L_2(\Omega) + L_2(\Omega) + \sum_{k=0}^{N_0-1} L_2(\Omega_k). \tag{7.8}$$

$\tilde{\mathcal{H}}$ is a Hilbert space with norm defined by

$$(7.9) \qquad \|h\|^2_{\tilde{\mathcal{H}}} = \|h_+\|^2_{L_2(\Omega)} + \|h_-\|^2_{L_2(\Omega)} + \sum_{k=0}^{N_0-1} \|h_k\|^2_{L_2(\Omega_k)};$$

see [3, p. 1783]. Theorem 7.2 implies that for each $f \in \mathcal{H}$ and $M > 0$, the sequence $\tilde{f}_M = (\tilde{f}_{M+}, \tilde{f}_{M-}, \tilde{f}_{M0}, \tilde{f}_{M1}, \cdots) \in \tilde{\mathcal{H}}$ and

$$(7.10) \qquad \|f_M\|_{\mathcal{H}} = \|\tilde{f}_M\|_{\tilde{\mathcal{H}}}.$$

For arbitrary $f \in \mathcal{H}$ the generalized Fourier transforms associated with A are defined by

Theorem 7.3. For all $f \in \mathcal{H}$, $\{\tilde{f}_M\}$ is a Cauchy sequence in $\tilde{\mathcal{H}}$, for $M \to \infty$, and hence

$$(7.11) \qquad \lim_{M\to\infty} \tilde{f}_M = \tilde{f} = (\tilde{f}_+, \tilde{f}_-, \tilde{f}_0, \tilde{f}_1, \cdots)$$

exists in $\tilde{\mathcal{H}}$. In particular, each of the limits

$$(7.12) \qquad \begin{aligned} \tilde{f}_\pm &= L_2(\Omega)\text{-}\lim_{M\to\infty} \tilde{f}_{M\pm} \\ \tilde{f}_0 &= L_2(\Omega_0)\text{-}\lim_{M\to\infty} \tilde{f}_{M0} \\ \tilde{f}_k &= L_2(\Omega_k)\text{-}\lim_{M\to\infty} \tilde{f}_{Mk}, \quad k \geq 1, \end{aligned}$$

exists and the Parseval relation

$$(7.13) \qquad \|f\|^2_{\mathcal{H}} = \|\tilde{f}\|^2_{\tilde{\mathcal{H}}} = \|\tilde{f}_+\|^2_{L_2(\Omega)} + \|\tilde{f}_-\|^2_{L_2(\Omega)} + \sum_{k=0}^{N_0-1} \|\tilde{f}_k\|^2_{L_2(\Omega_k)}$$

holds for every $f \in \mathcal{H}$.

Theorem 7.3 associates with each $f \in \mathcal{H}$ a family of generalized Fourier transforms $\tilde{f} = (\tilde{f}_+, \tilde{f}_-, \tilde{f}_0, \tilde{f}_1, \cdots) \in \tilde{\mathcal{H}}$ such that

$$(7.14)\qquad \tilde{f}_\pm(p,\lambda) = L_2(\Omega)\text{-}\lim_{M\to\infty} \int_{-M}^{M} \int_{|x|\le M} \overline{\psi_\pm(x,y,p,\lambda)}\, f(x,y) c^{-2}(y) \rho^{-1}(y) dx dy,$$

$$(7.15)\qquad \tilde{f}_0(p,\lambda) = L_2(\Omega_0)\text{-}\lim_{M\to\infty} \int_{-M}^{M} \int_{|x|\le M} \overline{\psi_0(x,y,p,\lambda)}\, f(x,y) c^{-2}(y) \rho^{-1}(y) dx dy,$$

$$(7.16)\qquad \tilde{f}_k(p) = L_2(\Omega_k)\text{-}\lim_{M\to\infty} \int_{-M}^{M} \int_{|x|\le M} \overline{\psi_k(x,y,p)}\, f(x,y) c^{-2}(y) \rho^{-1}(y) dx dy, \qquad k \ge 1.$$

It is easy to verify that if $f \in \mathcal{H} \cap L_1(R^3)$ then the functions $(\tilde{f}_+, \tilde{f}_-, \tilde{f}_0, \tilde{f}_1, \cdots)$ defined by Theorems 7.1 and 7.3 are equivalent and hence the notation is unambiguous. A construction of the spectral family $\{\Pi(\mu)\}$ based on these functions is described by

<u>Theorem 7.4</u>. For all $f, g \in \mathcal{H}$ and all real $\mu \ge 0$, $\Pi(\mu)$ satisfies the relation

$$(7.17)\qquad \begin{aligned} (f, \Pi(\mu) g) = &\int_\Omega H(\mu-\lambda)\left(\overline{\tilde{f}_+(p,\lambda)}\, \tilde{g}_+(p,\lambda) + \overline{\tilde{f}_-(p,\lambda)}\, \tilde{g}_-(p,\lambda)\right) dp d\lambda \\ &+ \int_{\Omega_0} H(\mu-\lambda)\, \overline{\tilde{f}_0(p,\lambda)}\, \tilde{g}_0(p,\lambda)\, dp d\lambda \\ &+ \sum_{k=1}^{N_0-1} \int_{\Omega_k} H(\mu-\lambda_k(|p|))\, \overline{\tilde{f}_k(p)}\, \tilde{g}_k(p)\, dp \end{aligned}$$

where $H(\mu) = 1$ for $\mu \ge 0$ and $H(\mu) = 0$ for $\mu < 0$.

The remainder of §7 presents the proofs of these theorems. The proof of Theorem 7.1 will be based on

<u>Lemma 7.5</u>. The normal mode functions satisfy

$$(7.18)\qquad \begin{aligned} &\psi_\pm(x,y,p,\lambda) \in C(R^3 \times \Omega), \\ &\psi_0(x,y,p,\lambda) \in C(R^3 \times \Omega_0), \\ &\psi_k(x,y,p) \in C(R^3 \times \Omega_k),\ k \geq 1. \end{aligned}$$

Moreover, for each compact set $K \subset \Omega$ there exists a constant M_K such that

$$(7.19)\qquad |\psi_\pm(x,y,p,\lambda)| \leq M_K \text{ for all } (x,y) \in R^3 \text{ and } (p,\lambda) \in K.$$

Similarly, for each compact $K \subset \Omega_0$ there exists a constant M_K such that

$$(7.20)\qquad |\psi_0(x,y,p,\lambda)| \leq M_K \text{ for all } (x,y) \in R^3 \text{ and } (p,\lambda) \in K,$$

and for each $k \geq 1$ and compact $K \subset \Omega_k$ there exists a constant M_K such that

$$(7.21)\qquad |\psi_k(x,y,p)| \leq M_K \text{ for all } (x,y) \in R^3 \text{ and } p \in K.$$

<u>Proof of Lemma 7.5</u>. To prove (7.18) note that, by (1.31), (1.36)

$$(7.22)\qquad \psi_+(x,y,p,\lambda) = (2\pi)^{-1}\, e^{ip\cdot x}\, a_+(|p|,\lambda)\, \phi_4(y,|p|,\lambda)$$

where $a_+(\mu,\lambda)$ is defined by (5.14). The continuity of $\phi_4(y,|p|,\lambda)$ on $R \times \Omega$ follows from Corollary 2.2. The continuity of $a_+(|p|,\lambda)$ on Ω follows from (5.10), (5.14) and the assumed continuity of the phase function $\theta_+(\mu,\lambda)$. Thus the continuity of ψ_+ on $R^3 \times \Omega$ follows from (7.22). The proofs for ψ_- and ψ_0 are similar and will not be given. The continuity of ψ_k, $k \geq 1$, follows from Corollary 6.5.

To prove (7.19) for ψ_+ note that (7.22) and the continuity of $a_+(|p|,\lambda)$ imply that it is enough to prove the existence of a constant M_K such that

$$(7.23) \qquad |\phi_4(y,|p|,\lambda)| \leq M_K \text{ for all } y \in R \text{ and } (p,\lambda) \in K.$$

Now the uniformity of the asymptotic estimates (2.11) on the compact sets Γ_4 of Corollary 2.4 implies that

$$(7.24) \qquad \phi_4(y,|p|,\lambda) = \exp\{-i\,y\,q_-(|p|,\lambda)\}[1 + o(1)], \; y \to -\infty,$$

uniformly for $(p,\lambda) \in K$. Hence, there exists a constant y_K such that

$$(7.25) \qquad |\phi_4(y,|p|,\lambda)| \leq 2 \text{ for all } y \leq -y_K \text{ and } (p,\lambda) \in K.$$

Similarly, using the relation

$$(7.26) \qquad \phi_4(y,\mu,\lambda) = c_{41}(\mu,\lambda)\,\phi_1(y,\mu,\lambda) + c_{42}(\mu,\lambda)\,\phi_2(y,\mu,\lambda)$$

from (4.1), (4.2), (4.3), the continuity of $c_{41}(\mu,\lambda)$ and $c_{42}(\mu,\lambda)$ and the uniformity of the asymptotic estimates for ϕ_1, ϕ_2 when $y \to +\infty$, one finds that there exist constants y_K', M_K' such that

$$(7.27) \qquad |\phi_4(y,|p|,\lambda)| \leq M_K' \text{ for all } y \geq y_K' \text{ and } (p,\lambda) \in K.$$

Finally, the continuity of $\phi_4(y,|p|,\lambda)$ on $R \times \Omega$, which follows from Corollary 2.2, implies the existence of a constant M_K'' such that

$$(7.28) \qquad |\phi_4(y,|p|,\lambda)| \leq M_K'' \text{ for } -y_K \leq y \leq y_K' \text{ and } (p,\lambda) \in K.$$

Combining (7.25), (7.27) and (7.28) gives the estimate (7.23) with $M_K = \text{Max}\,(2,M_K',M_K'')$.

The proofs of (7.19) for ψ_- and of (7.20) and (7.21) can be given by the same method. This completes the discussion of Lemma 7.5.

Proof of Theorem 7.1. Consider the function $\tilde{f}_+(p,\lambda)$. The absolute convergence of the integral in (7.1) for each $(p,\lambda) \in \Omega$ follows from (7.19). To prove that $\tilde{f}_+ \in C(\Omega)$ let $(p_0,\lambda_0) \in \Omega$ and let $K \subset \Omega$ be compact and contain (p_0,λ_0) in its interior. Then by Lemma 7.5

$$(7.29) \qquad |\overline{\psi_+(x,y,p,\lambda)}\, f(x,y)| \le M_K\, |f(x,y)| \text{ for } (x,y) \in R^3,\ (p,\lambda) \in K.$$

Hence, the continuity of $\tilde{f}_+$ at (p_0,λ_0) follows from (7.18) and (7.29) by Lebesgue's dominated convergence theorem. The continuity of $\tilde{f}_-$, $\tilde{f}_0$ and $\tilde{f}_k$ follows by the same argument. This completes the proof of Theorem 7.1.

Relationship of A to A_μ. As a preparation for the proofs of Theorems 7.2, 7.3 and 7.4 the operator A will be related to $A_{|p|}$ by Fourier analysis in the variables $x \in R^2$. To this end note that if $u \in \mathcal{H}$ then Fubini's theorem implies that $u(\cdot,y) \in L_2(R^2)$ for almost every $y \in R$. Thus if $F : L_2(R^2) \to L_2(R^2)$ denotes the Fourier transform in $L_2(R^2)$ then the Plancherel theory implies that

$$(7.30) \qquad \hat{u}(p,y) = (Fu)(p,y) = L_2(R^2)\text{-}\lim_{M\to\infty} (2\pi)^{-1} \int_{|x|\le M} e^{-ip\cdot x}\, u(x,y)dx$$

exists for almost every $y \in R$ and

$$(7.31) \qquad \int_{R^2} |\hat{u}(p,y)|^2\, dp = \int_{R^2} |u(x,y)|^2\, dx \text{ for a.e. } y \in R.$$

Another application of Fubini's theorem gives

Lemma 7.6. $u \in \mathcal{H}$ if and only if $\hat{u} = Fu \in \mathcal{H}$ and the mapping $F : \mathcal{H} \to \mathcal{H}$ is unitary. In particular

$$\|\hat{u}\|_{\mathcal{H}} = \|u\|_{\mathcal{H}} \text{ for all } u \in \mathcal{H}. \tag{7.32}$$

The Fourier transform of the acoustic propagator A will be denoted by $\hat{A}$. Thus

$$\hat{A} = F A F^{-1}, \; D(\hat{A}) = F\, D(A). \tag{7.33}$$

A more detailed characterization of $D(\hat{A})$ is needed to relate $\hat{A}$ to $A_{|p|}$. It will be based on

Lemma 7.7. Let $u \in \mathcal{H}$. Then $D_j u \in \mathcal{H}$ ($j = 1,2$) if and only if $p_j \hat{u}(p,y) \in \mathcal{H}$ and

$$F D_j u = p_j F u, \; j = 1,2. \tag{7.34}$$

Similarly, $D_y u \in \mathcal{H}$ if and only if $D_y \hat{u} \in \mathcal{H}$ and

$$F D_y u = D_y F u. \tag{7.35}$$

Proof of Lemma 7.7. The distributional derivatives $D_j u$, $D_y u$ may be characterized as temperate distributions on the Schwartz space $S(R^3)$ of rapidly decreasing testing functions [7]. $S(R^3)$ is mapped onto itself by F. The proof of (7.34) is essentially the same as in the standard Plancherel theory. To verify (7.35) note that the distribution-theoretic definition of $D_y u \in \mathcal{H}$ is

$$\int_{R^3} D_y u(x,y)\, \phi(x,y) dxdy = -\int_{R^3} u(x,y)\, D_y \phi(x,y) dxdy \text{ for all } \phi \in S(R^3). \tag{7.36}$$

Application of Parseval's relation gives

$$(7.37) \qquad \int_{R^3} (F\, D_y u)\hat{\phi}\ dpdy = -\int_{R^3} (Fu)(F\, D_y \phi) dpdy \text{ for all } \phi \in S(R^3).$$

Now for $\phi \in S(R^3)$ it is easy to verify that

$$(7.38) \qquad D_y \hat{\phi} = D_y F \phi = F\, D_y \phi.$$

Thus (7.37) is equivalent to

$$(7.39) \qquad \int_{R^3} (F\, D_y u)\hat{\phi}\ dpdy = -\int_{R^3} (Fu)\, D_y \hat{\phi}\ dpdy \text{ for all } \hat{\phi} \in S(R^3)$$

which in turn is equivalent to (7.35).

Application of Lemma 7.7 to $\hat{A}$ gives

Lemma 7.8. The operator $\hat{A}$ is characterized by the relations

$$(7.40) \qquad F\, L_2^1(R^3) = \{\hat{u} \mid p_1\hat{u},\ p_2\hat{u} \text{ and } D_y\hat{u} \text{ are in } \mathcal{H}\},$$

$$(7.41) \qquad D(\hat{A}) = F\, L_2^1(R^3) \cap \{\hat{u} \mid D_y\ (\rho^{-1} D_y \hat{u}) - |p|^2\ \hat{u} \in \mathcal{H}\}, \text{ and}$$

$$(7.42) \qquad \hat{A}\ \hat{u} = -c^2\ \{\rho D_y(\rho^{-1} D_y \hat{u}) - |p|^2\ \hat{u}\},\ \hat{u} \in D(\hat{A}).$$

Proof of Lemma 7.8. These results follow from application of Lemma 7.7 to the definition of $L_2^1(R^3)$, $D(A)$ and A - equations (1.2), (1.8) and (1.9).

Corollary 7.9. For all $u \in D(A)$ one has

$$(7.43) \qquad \hat{u}(p,\cdot) \in D(A_{|p|}) \text{ and}$$

$$(\hat{A}\,\hat{u})(p,\cdot) = A_{|p|}\,\hat{u}(p,\cdot) \tag{7.44}$$

for almost every $p \in R^2$.

Corollary 7.9 is an immediate consequence of Lemma 7.6, Lemma 7.8 and Fubini's theorem.

The Sets $\mathcal{H}^{com}$, $\mathcal{H}'$, $\mathcal{H}''$ and $\mathcal{H}'''$. The following subsets of $\mathcal{H}$ will be used in the proofs of Theorems 7.2, 7.3 and 7.4.

$$\mathcal{H}^{com} = \mathcal{H} \cap \{f \mid \text{supp } f \text{ is compact}\}, \tag{7.45}$$

$$\mathcal{H}' = F^{-1}\mathcal{D}(R^3) = \{f \mid \hat{f} = Ff \in \mathcal{D}(R^3)\}, \tag{7.46}$$

$$\mathcal{H}'' = \{f(x,y) = f_1(x)f_2(y) \mid f_1 \in F^{-1}\mathcal{D}(R^2), f_2 \in \mathcal{H}(R), \text{supp } f_2 \text{ compact}\}, \tag{7.47}$$

$$\mathcal{H}''' = \text{span } \mathcal{H}'' = \{f = \sum_{\alpha=1}^{m} a_\alpha\, f_\alpha \mid a_\alpha \in C, f_\alpha \in \mathcal{H}''\}. \tag{7.48}$$

In (7.46) and (7.47), $\mathcal{D}(R^n)$ denotes the Schwartz space of testing functions with compact support [7]. The sets $\mathcal{H}^{com}$, $\mathcal{H}'$ and $\mathcal{H}'''$ are linear submanifolds of $\mathcal{H}$ which are dense in $\mathcal{H}$. Indeed, it is well known that $\mathcal{D}(R^3)$ is dense in $L_2(R^3)$. This fact implies that $\mathcal{H}^{com}$ is dense in $\mathcal{H}$. The denseness of $\mathcal{H}'$ in $\mathcal{H}$ follows from that of $\mathcal{D}(R^3)$ and the unitarity of F. The denseness of $\mathcal{H}'''$ follows from the fact that $\mathcal{H}$ is the tensor product of $L_2(R^2)$ and $\mathcal{H}(R)$.

It is clear that each of the sets $\mathcal{H}^{com}$, $\mathcal{H}'$ and $\mathcal{H}'''$ is a subset of $\mathcal{H} \cap L_1(R^3)$. Hence, for f in one of these sets, the transforms $\tilde{f}_\pm$, $\tilde{f}_0$ and $\tilde{f}_k$ defined by (7.1)-(7.3) are continuous functions by Theorem 7.1. An alternative characterization is given by

Lemma 7.10. If $f \in \mathcal{H}^{com} \cup \mathcal{H}' \cup \mathcal{H}''$ then

$$\tilde{f}_{\pm}(p,\lambda) = \int_R \overline{\psi_{\pm}(y,|p|,\lambda)}\, \hat{f}(p,y)\, c^{-2}(y)\, \rho^{-1}(y)\, dy, \tag{7.49}$$

$$\tilde{f}_0(p,\lambda) = \int_R \overline{\psi_0(y,|p|,\lambda)}\, \hat{f}(p,y)\, c^{-2}(y)\, \rho^{-1}(y)\, dy, \tag{7.50}$$

$$\tilde{f}_k(p) = \int_R \overline{\psi_k(y,|p|)}\, \hat{f}(p,y)\, c^{-2}(y)\, \rho^{-1}(y)\, dy,\ k \geq 1. \tag{7.51}$$

Proof of Lemma 7.10. Equations (7.49)-(7.51) follow from (7.1)-(7.3) on substituting the definitions (1.36)-(1.38) of the normal mode functions and carrying out the x-integration. These operations are justified for $f \in L_1(R^3)$ by Lemma 7.5 and Fubini's theorem.

Corollary 7.11. If $f \in \mathcal{H}'$ and $f(x,y) = f_1(x)\, f_2(y)$ then

$$\tilde{f}_{\pm}(p,\lambda) = \hat{f}_1(p) f_{2\pm}(|p|,\lambda) = [F\, \Phi_{|p|\pm} f](p,\lambda) = [\Phi_{|p|\pm} Ff](p,\lambda), \tag{7.52}$$

$$\tilde{f}_0(p,\lambda) = \hat{f}_1(p) f_{20}(|p|,\lambda) = [F\, \Phi_{|p|0} f](p,\lambda) = [\Phi_{|p|0}\, Ff](p,\lambda), \tag{7.53}$$

$$\tilde{f}_k(p) = \hat{f}_1(p) \tilde{f}_{2k}(|p|) = [F\, \Phi_{|p|k} f](p) = [\Phi_{|p|k} Ff](p). \tag{7.54}$$

These results follow immediately from Lemma 7.10 and the results of §5.

The notation

$$R(T,\zeta) = (T - \zeta)^{-1} \tag{7.55}$$

will be used for the resolvent of T. The proofs of Theorems 7.2, 7.3 and 7.4 will be based on Stone's theorem relating $R(A,\zeta)$ and the spectral family of A, together with the following three lemmas relating A and $A_{|p|}$.

Lemma 7.12. Let $f \in \mathcal{H}'''$ and let $u_\zeta = R(A,\zeta)f$ or, equivalently, $\hat{u}_\zeta = R(\hat{A},\zeta)\hat{f}$. Then

$$\hat{u}_\zeta(p,y) = \int_R G_{|p|}(y,y',\zeta)\ \hat{f}(p,y')\ c^{-2}(y')\ \rho^{-1}(y')\ dy' \tag{7.56}$$

for almost every $(p,y) \in R^3$ where $G_\mu(y,y',\zeta)$ is the Green's function for A_μ.

Proof of Lemma 7.12. It is enough to verify (7.56) for functions $f \in \mathcal{H}''$. Let $f(x,y) = f_1(x)f_2(y)$ be such a function so that $\hat{f}(p,y) = \hat{f}_1(p)f_2(y)$. Now $\hat{u}_\zeta \in D(\hat{A})$ and hence by Corollary 7.9

$$((\hat{A} - \zeta)\hat{u}_\zeta)(p,y) = ((A_{|p|} - \zeta)\hat{u}_\zeta)(p,y) = \hat{f}_1(p)f_2(y) \tag{7.57}$$

for almost every $(p,y) \in R^3$. It follows that

$$\hat{u}_\zeta(p,y) = [R(A_{|p|},\zeta)\hat{f}_1(p)f_2](y) = \hat{f}_1(p)[R(A_{|p|},\zeta)f_2](y) \tag{7.58}$$

which is equivalent to (7.56) because

$$[R(A_{|p|},\zeta)f_2](y) = \int_R G_{|p|}(y,y',\zeta)\ f_2(y')\ c^{-2}(y')\ \rho^{-1}(y')\ dy'. \tag{7.59}$$

Lemma 7.13. Let $f(x,y) = f_1(x)\ f_2(y)$ and $g(x,y) = g_1(x)g_2(y)$ be elements of $\mathcal{H}''$ and let $\zeta = \lambda + i\varepsilon$ with $\lambda \in R$ and $\varepsilon > 0$. Then for all $\mu \in R$ one has

$$\begin{aligned} &\int_{-1}^{\mu} (f,[R(A,\zeta) - R(A,\overline{\zeta})]g)d\lambda \\ &= \int_{R^2} \overline{\hat{f}_1(p)}\ \hat{g}_1(p) \int_{-1}^{\mu} (f_2,[R(A_{|p|},\zeta) - R(A_{|p|},\overline{\zeta})]g_2)d\lambda dp. \end{aligned} \tag{7.60}$$

Proof of Lemma 7.13. The Plancherel theory implies that

$$(7.61)\qquad (f,[R(A,\zeta) - R(A,\bar{\zeta})]g) = (\hat{f},[R(\hat{A},\zeta) - R(\hat{A},\bar{\zeta})]\hat{g}).$$

Combining this and (7.58) gives, by Fubini's theorem

$$(7.62)\qquad (f,[R(A,\zeta) - R(A,\bar{\zeta})]g) = \int_{R^2} \overline{\hat{f}_1(p)}\ \hat{g}_1(p)(f_2,[R(A_{|p|},\zeta) - R(A_{|p|},\bar{\zeta})]g_2)dp.$$

Now a standard estimate for the resolvent of a selfadjoint operator [8, p. 272] implies that

$$(7.63)\qquad |\overline{\hat{f}_1(p)}\ \hat{g}_1(p)(f_2,[R(A_{|p|},\zeta) - R(A_{|p|},\bar{\zeta})]g_2)| \le \frac{2}{\varepsilon}\ |\overline{\hat{f}_1(p)}\hat{g}_1(p)|\|f_2\|\|g_2\|$$

Thus integrating (7.62) over $-1 \le \lambda \le \mu$ gives (7.60) by Fubini's theorem.

Lemma 7.14. *The spectral family* $\{\Pi(\mu)\}$ *satisfies relation (7.17) of Theorem 7.4 for all* $f,g \in \mathcal{H}'''$.

Proof of Lemma 7.14. Stone's theorem in its general form is [16, p. 79]

$$(7.64)\qquad \lim_{\varepsilon\to 0+} \frac{1}{\pi i}\int_a^b (f,[R(A,\lambda+i\varepsilon) - R(A,\lambda-i\varepsilon)]g)d\lambda = (f,[\Pi(b) + \Pi(b-) - \Pi(a) - \Pi(a-)]g).$$

In the present case $\sigma(A) \subset [0,\infty)$ and (7.64) implies

$$(7.65)\qquad \frac{1}{2}(f,[\Pi(\mu) + \Pi(\mu-)]f) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0+}\int_{-1}^{\mu} (f,[R(A,\lambda+i\varepsilon) - R(A,\lambda-i\varepsilon)]f)d\lambda.$$

Moreover, since $\Pi(\mu-) = \lim_{\delta\to 0+} \Pi(\mu - \delta)$ and $\lim_{\delta\to 0+} \Pi((\mu - \delta)-) = \Pi(\mu-)$, (7.65) implies

$$(7.66) \qquad (f,\Pi(\mu-)f) = \frac{1}{2\pi i} \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \int_{-1}^{\mu} (f,[R(A,\lambda+i\varepsilon) - R(A,\lambda-i\varepsilon)]f)d\lambda.$$

If $f(x,y) = f_1(x)f_2(y) \in \mathcal{H}'$ then combining (7.60) with $g = f$ and (7.66) gives

$$(7.67) \qquad \begin{aligned} &(f,\Pi(\mu-)f) \\ &= \frac{1}{2\pi i} \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \int_{R^2} |\hat{f}_1(p)|^2 \int_{-1}^{\mu-\delta} (f_2,[R(A_{|p|},\lambda+i\varepsilon) - R(A_{|p|},\lambda-i\varepsilon)]f_2)d\lambda dp. \end{aligned}$$

Now application of the spectral theorem to $A_{|p|}$ gives

$$(7.68) \qquad (f_2,[R(A_{|p|},\lambda+i\varepsilon) - R(A_{|p|},\lambda-i\varepsilon)]f_2) = \int_R \frac{2i\varepsilon}{(\lambda-\lambda')^2+\varepsilon^2} (f_2,\Pi_{|p|}(d\lambda')f_2).$$

It follows that

$$(7.69) \qquad \begin{aligned} &\left|\int_{-1}^{\mu-\delta} (f_2,[R(A_{|p|},\lambda+i\varepsilon) - R(A_{|p|},\lambda-i\varepsilon)]f_2)d\lambda\right| \\ &\qquad \le \int_R \left(\int_{-1}^{\mu-\delta} \frac{2\varepsilon}{(\lambda-\lambda')^2+\varepsilon^2} d\lambda\right)(f_2,\Pi_{|p|}(d\lambda')f_2). \end{aligned}$$

Moreover,

$$(7.70) \qquad \int_{-1}^{\mu-\delta} \frac{2\varepsilon}{(\lambda-\lambda')^2+\varepsilon^2} d\lambda \le \int_R \frac{2\varepsilon}{(\lambda-\lambda')^2+\varepsilon^2} d\lambda = 2\pi$$

for all $\lambda' \in R$ and all $\varepsilon > 0$. Combining (7.69) and (7.70) gives

$$(7.71)\qquad \left|\int_{-1}^{\mu-\delta} (f_2,[R(A_{|p|},\lambda+i\varepsilon) - R(A_{|p|},\lambda-i\varepsilon)]f_2)d\lambda\right| \leq 2\pi \, \| f_2 \|$$

for all $p \in R^2$ and all $\mu \geq 0$, $\delta > 0$ and $\varepsilon > 0$. In addition Stone's theorem applied to $A_{|p|}$ gives

$$(7.72)\qquad \frac{1}{2\pi i} \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \int_{-1}^{\mu-\delta} (f_2,[R(A_{|p|},\lambda+i\varepsilon) - R(A_{|p|},\lambda-i\varepsilon)]f_2)d\lambda = (f_2,\Pi_{|p|}(\mu-)f_2).$$

Equations (7.67), (7.72) and the estimate (7.71) imply, by Lebesgue's dominated convergence theorem,

$$(7.73)\qquad (f,\Pi(\mu-)f) = \int_{R^2} |\hat{f}_1(p)|^2 \, (f_2,\Pi_{|p|}(\mu-)f_2) \, dp.$$

It follows by polarization that

$$(7.74)\qquad (f,\Pi(\mu-)g) = \int_{R^2} \overline{\hat{f}_1(p)} \, \hat{g}_1(p) \, (f_2,\Pi_{|p|}(\mu-)g_2) \, dp$$

for all $f,g \in \mathcal{H}''$. The same argument applied to (7.65) gives the relation

$$(7.75)\qquad (f,[\Pi(\mu)+\Pi(\mu-)]g) = \int_{R^2} \overline{\hat{f}_1(p)} \, \hat{g}_1(p)(f_2,[\Pi_{|p|}(\mu)+\Pi_{|p|}(\mu-)]g_2)dp$$

Subtracting (7.74) from (7.75) gives

$$(7.76)\qquad (f,\Pi(\mu)g) = \int_{R^2} \overline{\hat{f}_1(p)} \, \hat{g}_1(p)(f_2,\Pi_{|p|}(\mu)g_2)dp$$

for all $f,g \in \mathcal{H}''$. To prove the relation (7.17) for $f,g \in \mathcal{H}''$ note that the construction of Π_μ given by (5.27) implies, by polarization

$$(f_2,\Pi_{|p|}(\mu)g_2)$$

(7.77)

$$= \int_{\Lambda(|p|)} H(\mu - \lambda)[\overline{\hat{f}_{2+}(|p|,\lambda)}\ \hat{g}_{2+}(|p|,\lambda) + \overline{\hat{f}_{2-}(|p|,\lambda)}\ \hat{g}_{2-}(|p|,\lambda)]d\lambda$$

$$+ \int_{\Lambda_0(|p|)} H(\mu - \lambda)\ \overline{\hat{f}_{20}(|p|,\lambda)}\ \hat{g}_{20}(|p|,\lambda)\ d\lambda$$

$$+ \sum_{k=1}^{N(|p|)-1} H(\mu - \lambda_k(|p|))\ \overline{\hat{f}_{2k}(|p|)}\ g_{2k}(|p|)$$

for all $p \in R^2$ such that $|p| > 0$, where

$$\Lambda(|p|) = \{\lambda \mid c^2(-\infty)|p|^2 < \lambda\}, \text{ and}$$

(7.78)

$$\Lambda_0(|p|) = \{\lambda \mid c^2(\infty)|p|^2 < \lambda < c^2(-\infty)|p|^2\}.$$

Substituting this into (7.76) and recalling the definitions of Ω, Ω_0 and Ω_k $(k \geq 1)$ gives (7.19) for $f,g \in \mathcal{H}''$. The relation extends immediately to all $f,g \in \mathcal{H}'''$ by linearity. This completes the proof of Lemma 7.14.

<u>Proof of Theorem 7.2</u>. Let $f \in \mathcal{H}$ and $M > 0$ be given. Then since $\mathcal{H}'''$ is dense in $\mathcal{H}$ there exists a sequence $\{g_n\}$ in $\mathcal{H}'''$ such that $g_n \to f_M$ in $\mathcal{H}$. Note that since

$$\|f_M - g_n\|^2_{\mathcal{H}} = \int_{-M}^{M} \int_{R^2} |f(x,y) - g_n(x,y)|^2\ c^{-2}(y)\ \rho^{-1}(y)dxdy$$

(7.79)

$$+ \int_{|y|\geq M} \int_{R^2} |g_n(x,y)|^2\ c^{-2}(y)\ \rho^{-1}(y)\ dxdy$$

it may be assumed that $g_n(x,y) = 0$ for $|y| > M$. Now Lemma 7.14 implies that Parseval's relation

$$(7.80) \qquad \|g\|^2_{\mathcal{H}} = \|\tilde{g}\|^2_{\tilde{\mathcal{H}}} = \|\tilde{g}_+\|^2 + \|\tilde{g}_-\|^2 + \sum_{k=0}^{N_0-1} \|\tilde{g}_k\|^2$$

holds for all $g \in \mathcal{H}'''$, where $\tilde{g} = (\tilde{g}_+, \tilde{g}_-, \tilde{g}_0, \tilde{g}_1, \cdots)$. On applying (7.80) to the differences $g_n - g_m$ it is found that $\tilde{g}_n = (\tilde{g}_{n+}, \tilde{g}_{n-}, \tilde{g}_{n0}, \tilde{g}_{n1}, \cdots)$ is a Cauchy sequence in $\tilde{\mathcal{H}}$. Hence there exists a limit

$$(7.81) \qquad \lim_{n\to\infty} \tilde{g}_n = h = (h_+, h_-, h_0, h_1, \cdots) \in \tilde{\mathcal{H}},$$

since $\tilde{\mathcal{H}}$ is a Hilbert space. To complete the proof of Theorem 7.2 it will be enough to show that

$$(7.82) \qquad \begin{aligned} \tilde{f}_{M\pm}(p,\lambda) &= h_+(p,\lambda) \text{ for a.e. } (p,\lambda) \in \Omega, \\ \tilde{f}_{M0}(p,\lambda) &= h_0(p,\lambda) \text{ for a.e. } (p,\lambda) \in \Omega_0, \\ \tilde{f}_{Mk}(p) &= h_k(p) \text{ for a.e. } p \in \Omega_k,\ k \geq 1. \end{aligned}$$

This clearly implies (7.6) since $h \in \tilde{\mathcal{H}}$. Moreover, since Hilbert space convergence implies convergence of the norms, the relation (7.80) for $g_n \in \mathcal{H}'''$ implies

$$(7.83) \qquad \|f_M\|^2_{\mathcal{H}} = \lim_{n\to\infty} \|g_n\|^2_{\mathcal{H}} = \lim_{n\to\infty} \|\tilde{g}_n\|^2_{\tilde{\mathcal{H}}} = \|h\|^2_{\tilde{\mathcal{H}}} = \|h_+\|^2 + \|h_-\|^2 + \sum_{k=0}^{N_0-1} \|h_k\|^2$$

which is equivalent to (7.7) when (7.82) holds.

Relation (7.82) will be proved for $\tilde{f}_{M+}$. The proofs for the remaining cases are entirely similar. To prove that $\tilde{f}_{M+}(p,\lambda) = h_+(p,\lambda)$ for a.e. (p,λ) in Ω note that if K is any compact subset of Ω then $\tilde{f}_{M+} \in C(K) \subset L_2(K)$ by Theorem 7.1 and

$$(7.84) \qquad \|\tilde{f}_{M+} - h_+\|_{L_2(K)} = \lim_{n\to\infty} \|\tilde{f}_{M+} - \tilde{g}_{n+}\|_{L_2(K)}$$

Now by Lemma 7.10

$$(7.85) \quad \tilde{f}_{M+}(p,\lambda) - \tilde{g}_{n+}(p,\lambda) = \int_{-M}^{M} \overline{\psi_+(y,|p|,\lambda)}\, [\hat{f}_M(p,y) - \hat{g}_n(p,y)] c^{-2}(y)\rho^{-1}(y)dy.$$

Hence by Lemma 7.5 and Schwarz's inequality

$$(7.86) \quad \begin{aligned} |\tilde{f}_{M+}(p,\lambda) - \tilde{g}_{n+}(p,\lambda)| &\le M_K \int_{-M}^{M} |\hat{f}_M(p,y) - \hat{g}_n(p,y)|\; c^{-2}(y)\rho^{-1}(y)\,dy \\ &\le M_K \left(\int_{-M}^{M} c^{-2}(y)\rho^{-1}(y)dy\right)^{1/2} \left(\int_{-M}^{M} |\hat{f}_M(p,y) - \hat{g}_n(p,y)|^2 c^{-2}(y)\rho^{-1}(y)dy\right)^{1/2} \end{aligned}$$

for all $(p,\lambda) \in K$. It follows that there is a constant $C = C(K,M)$ such that

$$(7.87) \qquad \|\tilde{f}_{M+} - \tilde{g}_{n+}\|_{L_2(K)} \le C\, \|\hat{f}_M - \hat{g}_n\|_{\mathcal{H}} = C\, \|f_M - g_n\|_{\mathcal{H}} .$$

Since $g_n \to f_M$ in $\mathcal{H}$, (7.87) implies that the limit in (7.84) is zero and hence $\tilde{f}_{M+}(p,\lambda) = h_+(p,\lambda)$ for a.e. $(p,\lambda) \in K$. This completes the proof since $K \subset \Omega$ was an arbitrary compact set.

<u>Proof of Theorem 7.3</u>. To prove that $\{\tilde{f}_M\}$ is a Cauchy sequence in $\tilde{\mathcal{H}}$ for $M \to \infty$, let $M > 0$ and $N > 0$ be arbitrary numbers and let $\{g_n^M\}$, $\{g_n^N\}$ be sequences in $\mathcal{H}''$ such that $g_n^M \to f_M$, $g_n^N \to f_N$ in $\mathcal{H}$. Then, as proved above, $\tilde{g}_n^M \to \tilde{f}_M$ and $\tilde{g}_n^N \to \tilde{f}_N$ in $\tilde{\mathcal{H}}$ and Parseval's relation (7.80) holds for $\tilde{g}_n^M - \tilde{g}_n^N$. Passage to the limit $n \to \infty$ gives

$$(7.88) \qquad \|f_M - f_N\|^2_{\mathcal{H}} = \|\tilde{f}_M - \tilde{f}_N\|^2_{\tilde{\mathcal{H}}}$$

which implies that $\{\tilde{f}_M\}$ is a Cauchy sequence because $f_M \to f$ in $\mathcal{H}$ when $M \to \infty$. Finally, one gets (7.13) for arbitrary $f \in \mathcal{H}$ by passage to the limit in (7.9).

Proof of Theorem 7.4. It will be enough to prove (7.17) for $f = g \in \mathcal{H}$ since the general case then follows by polarization. Now by Lemma 7.14

$$
\begin{aligned}
(g,\Pi(\mu)g) = &\int_{\Omega} H(\mu-\lambda)(|\tilde{g}_+(p,\lambda)|^2 + |\tilde{g}_-(p,\lambda)|^2)dpd\lambda \\
&+ \int_{\Omega_0} H(\mu-\lambda)|\tilde{g}_0(p,\lambda)|^2\, dpd\lambda \\
&+ \sum_{k=1}^{N_0-1} \int_{\Omega_k} H(\mu - \lambda_k(|p|))\ |\tilde{g}_k(p)|^2\, dp
\end{aligned}
\tag{7.89}
$$

for all $g \in \mathcal{H}''$. Let $f \in \mathcal{H}$, $M > 0$ and let $\{g_n\}$ be a sequence in $\mathcal{H}''$ such that $g_n \to f_M$ in $\mathcal{H}$. Then it follows from the proof of Theorem 7.2 that $\tilde{g}_n \to \tilde{f}_M$ in $\tilde{\mathcal{H}}$. Replacing g by g_n in (7.89) and making $n \to \infty$ gives (7.89) with $g = f_M$. If $N_0 = +\infty$ then passage to the limit is justified because the right-hand side of (7.89) is majorized by

$$
\|\tilde{g}_+\|^2 + \|\tilde{g}_-\|^2 + \sum_{k=0}^{N_0-1} \|\tilde{g}_k\|^2 < \infty. \tag{7.90}
$$

Thus (7.89) is valid with $g = f_M$ where $f \in \mathcal{H}$ and $M > 0$ are arbitrary. Making $M \to \infty$ and repeating the above argument gives (7.89) with $g = f \in \mathcal{H}$, by Theorem 7.3.

Another proof may be obtained by noting that the left-hand side of (7.89) is a bounded quadratic form on $\mathcal{H}$, while the right-hand side is a bounded quadratic function of $\tilde{g} = \Psi g$ because of the majorization by (7.90). Thus (7.17) follows from the boundedness of Ψ and the fact that (7.89) holds for g in the dense set $\mathcal{H}''$.

§8. Normal Mode Expansions for A.

The normal mode expansions for the acoustic propagator A that are the main results of this report are formulated and proved in this section. The starting point is the representation of the spectral family of A given by Theorem 7.4. The main result, Theorem 8.8, shows that the family $\{\psi_+, \psi_-, \psi_0, \psi_1, \cdots\}$ is a complete orthogonal family of normal modes for A. Theorem 8.9 shows that it provides a spectral representation of A. These results are shown to imply that the families $\{\phi_+, \psi_1, \psi_2, \cdots\}$ and $\{\phi_-, \psi_1, \psi_2, \cdots\}$, defined in §1, are also complete orthogonal families of normal modes for A and provide alternative spectral representations.

The basic representation space for A associated with the family $\{\psi_+, \psi_-, \psi_0, \psi_1, \cdots\}$ is the direct sum space

(8.1) $$\tilde{\mathcal{H}} = L_2(\Omega) + L_2(\Omega) + \sum_{k=0}^{N_0-1} L_2(\Omega_k)$$

introduced in §7. Theorem 7.3 associates with each $f \in \mathcal{H}$ an element $\tilde{f} \in \tilde{\mathcal{H}}$. The Parseval relation (7.13) implies that the linear operator

(8.2) $$\Psi : \mathcal{H} \to \tilde{\mathcal{H}}$$

defined by

(8.3) $$\Psi\, f = \tilde{f} \text{ for all } f \in \mathcal{H}$$

is an isometry; i.e.,

(8.4) $$\|\Psi f\|_{\tilde{\mathcal{H}}} = \|f\|_{\mathcal{H}} \text{ for all } f \in \mathcal{H}.$$

The principal result of this section is

Theorem 8.1. The operator Ψ is unitary; i.e.,

(8.5) $$\Psi^* \Psi = 1 \text{ in } \mathcal{H}, \text{ and}$$

(8.6) $$\Psi \Psi^* = 1 \text{ in } \tilde{\mathcal{H}}.$$

Relations (8.5) and (8.6) generalize the completeness and orthogonality properties, respectively, of the eigenfunction expansions for operators with discrete spectra. Relation (8.5) is equivalent to (8.4) and thus follows from Theorem 7.3. Relation (8.6) is shown below to follow from the unitarity of the operator Ψ_μ associated with A_μ (Theorem 5.6).

The completeness relation (8.5) implies that every $f \in \mathcal{H}$ has a normal mode expansion based on the family $\{\psi_+, \psi_-, \psi_0, \psi_1, \cdots\}$. The orthogonality relation (8.6) implies that the space $\tilde{\mathcal{H}}$ is isomorphic to $\mathcal{H}$ and thus provides a parameterization of the set of all states $f \in \mathcal{H}$ of the acoustic field. These implications of Theorem 8.1 will be developed in a series of corollaries.

The normal mode expansion will be based on the linear operators

(8.7) $$\Psi_\pm : \mathcal{H} \to L_2(\Omega)$$

(8.8) $$\Psi_k : \mathcal{H} \to L_2(\Omega_k), \; 0 \leq k < N_0,$$

defined for all $f \in \mathcal{H}$ by

(8.9) $$\Psi_\pm f = \tilde{f}_\pm,$$

(8.10) $$\Psi_k f = \tilde{f}_k, \quad 0 \le k < N_0,$$

where $\tilde{f}_\pm$, $\tilde{f}_k$ are defined as in Theorem 7.3. It is clear that

(8.11) $$\Psi f = (\Psi_+ f, \Psi_- f, \Psi_0 f, \Psi_1 f, \cdots)$$

for all $f \in \mathcal{H}$ and, by (8.4),

(8.12) $$\|\Psi_+ f\|^2 + \|\Psi_- f\|^2 + \sum_{k=0}^{N_0-1} \|\Psi_k f\|^2 = \|f\|^2.$$

In particular, each of the operators $\Psi_\pm$, Ψ_k is bounded with norm not exceeding 1. The normal mode expansion for A, in abstract form, is given by

<u>Corollary 8.2</u>. The family $\{\Psi_+, \Psi_-, \Psi_0, \Psi_1, \cdots\}$ satisfies

(8.13) $$1 = \Psi_+^* \Psi_+ + \Psi_-^* \Psi_- + \sum_{k=0}^{N_0-1} \Psi_k^* \Psi_k$$

where 1 is the identity operator in $\mathcal{H}$ and the series in (8.13) converges strongly.

It will be shown that (8.13) is equivalent to the completeness relation (8.5). The orthogonality relation (8.6) will be shown to be equivalent to the relations described by

<u>Corollary 8.3</u>. The family $\{\Psi_+, \Psi_-, \Psi_0, \Psi_1, \cdots\}$ satisfies the relations

(8.14) $$\Psi_\pm \Psi_\pm^* = 1 \text{ in } L_2(\Omega),$$

(8.15) $$\Psi_k \Psi_k^* = 1 \text{ in } L_2(\Omega_k), \quad 0 \le k < N_0.$$

In addition, $\Psi_+ \Psi_-^* = \Psi_- \Psi_+^* = 0$, $\Psi_\pm \Psi_k^* = 0$, $\Psi_k \Psi_\pm^* = 0$ and $\Psi_k \Psi_\ell^* = 0$ for all k and $\ell \neq k$ such that $0 \leq k, \ell < N_0$.

Relations (8.14), (8.15) imply that each of the operators Ψ_+, Ψ_-, Ψ_k $(0 \leq k < N_0)$ is partially isometric [8, p. 258]. It follows that the operators in $\mathcal{H}$ defined by

$$(8.16) \qquad \begin{cases} P_\pm = \Psi_\pm^* \Psi_\pm \\ P_k = \Psi_k^* \Psi_k, \; 0 \leq k < N_0, \end{cases}$$

are orthogonal projections in $\mathcal{H}$ onto subspaces

$$(8.17) \qquad \begin{cases} \mathcal{H}_\pm = P_\pm \mathcal{H} \\ \mathcal{H}_k = P_k \mathcal{H}, \; 0 \leq k < N_0. \end{cases}$$

Combining this with Corollaries 8.2 and 8.3 gives

Corollary 8.4. $\{P_+, P_-, P_0, P_1, \cdots\}$ is a complete family of orthogonal projections in $\mathcal{H}$; i.e., the range spaces defined by (8.17) are mutually orthogonal and

$$(8.18) \qquad 1 = P_+ + P_- + \sum_{k=0}^{N_0-1} P_k.$$

The spaces (8.17) are subspaces of $\mathcal{H}$ and hence the direct sum space

$$(8.19) \qquad \mathcal{H}_+ + \mathcal{H}_- + \sum_{k=0}^{N_0-1} \mathcal{H}_k$$

may be identified with the set of all

(8.20)
$$f = f_+ + f_- + \sum_{k=0}^{N_0-1} f_k$$

in $\mathcal{H}$ such that

(8.21)
$$f_\pm \in \mathcal{H}_\pm, \; f_k \in \mathcal{H}_k \text{ for } 0 \le k < N_0$$

and

(8.22)
$$\|f_+\|^2 + \|f_-\|^2 + \sum_{k=0}^{N_0-1} \|f_k\|^2 < \infty.$$

With this convention, Corollary 8.4 implies

<u>Corollary 8.5</u>. $\mathcal{H}$ has the decomposition

(8.23)
$$\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_- + \sum_{k=0}^{N_0-1} \mathcal{H}_k.$$

The definitions of the operators $\Psi_\pm$, Ψ_k and equations (7.14)-(7.16) imply

<u>Corollary 8.6</u>. The operators $\Psi_\pm$, Ψ_k have the representations

(8.24)
$$(\Psi_\pm f)(p,\lambda) = L_2(\Omega)\text{-}\lim_{M\to\infty} \int_{R_M^3} \overline{\psi_\pm(x,y,p,\lambda)}\, f(x,y)c^{-2}(y)\rho^{-1}(y)dxdy,$$

(8.25)
$$(\Psi_0 f)(p,\lambda) = L_2(\Omega_0)\text{-}\lim_{M\to\infty} \int_{R_M^3} \overline{\psi_0(x,y,p,\lambda)}\, f(x,y)c^{-2}(y)\rho^{-1}(y)dxdy,$$

(8.26)
$$(\Psi_k f)(p) = L_2(\Omega_k)\text{-}\lim_{M\to\infty} \int_{R_M^3} \overline{\psi_k(x,y,p)}\, f(x,y)c^{-2}(y)\rho^{-1}(y)dxdy,$$

for $1 \le k < N_0$ where $R_M^3 = R^3 \cap \{(x,y) \mid |x| \le M, |y| \le M\}$.

Of course, the family of sets $\{R^3_M \mid M > 0\}$ can be replaced by any family $\{K_M \mid M > 0\}$ of compact sets whose characteristic functions tend to 1 almost everywhere in R^3 when $M \to \infty$. Corollary 8.6 implies a similar representation for the adjoint operators $\Psi^*_\pm$, Ψ^*_k. It is formulated as

Corollary 8.7. The operators $\Psi^*_\pm$, Ψ^*_k have the representations

$$(8.27)\qquad (\Psi^*_\pm g_\pm)(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega^M} \psi_\pm(x,y,p,\lambda)\; g_\pm(p,\lambda)\,dp\,d\lambda,$$

$$(8.28)\qquad (\Psi^*_0 g_0)(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega^M_0} \psi_0(x,y,p,\lambda)\; g_0(p,\lambda)\,dp\,d\lambda,$$

$$(8.29)\qquad (\Psi^*_k g_k)(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega^M_k} \psi_k(x,y,p)\; g_k(p)\,dp,$$

for $1 \le k < N_0$ where Ω^M and Ω^M_k $(0 \le k < N_0)$ are families of compact subsets of Ω and Ω_k whose characteristic functions tend to 1 almost everywhere in Ω and Ω_k, respectively, when $M \to \infty$.

By combining Theorem 8.1 and Corollaries 8.2, 8.6 and 8.7 the following explicit formulation of the normal mode expansion if obtained.

Theorem 8.8. Every $f \in \mathcal{H}$ has a representation

$$(8.30)\qquad f(x,y) = f_+(x,y) + f_-(x,y) + \sum_{k=0}^{N_0-1} f_k(x,y),$$

convergent in $\mathcal{H}$, where $f_\pm \in \mathcal{H}_\pm$ and f_k, $0 \le k < N_0$, are given by

$$(8.31)\qquad f_\pm(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega^M} \psi_\pm(x,y,p,\lambda)\; \tilde{f}_\pm(p,\lambda)\,dp\,d\lambda,$$

$$f_0(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega_0^M} \psi_0(x,y,p,\lambda)\, \tilde{f}_0(p,\lambda)\,dp\,d\lambda, \tag{8.32}$$

$$f_k(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{\Omega_k^M} \psi_k(x,y,p)\, \tilde{f}_k(p)\,dp, \tag{8.33}$$

for $1 \leq k < N_0$ and $\tilde{f}_\pm$, $\tilde{f}_k$ are defined by (7.14)-(7.16). Conversely, if $\tilde{f} = (\tilde{f}_+,\tilde{f}_-,\tilde{f}_0,\tilde{f}_1,\cdots)$ is any vector in $\tilde{\mathcal{H}}$ then (8.30)-(8.33) define a vector $f \in \mathcal{H}$ such that $\tilde{f}_\pm$, $\tilde{f}_k$ are related to f by (7.14)-(7.16).

Theorem 8.9. The unitary operator Ψ defines a spectral representation for A in $\tilde{\mathcal{H}}$ in the sense that for all $f \in D(A)$ one has

$$(\Psi_\pm A\, f)(p,\lambda) = \lambda(\Psi_\pm f)(p,\lambda) = \lambda\, \tilde{f}_\pm(p,\lambda), \tag{8.34}$$

$$(\Psi_0 A\, f)(p,\lambda) = \lambda(\Psi_0 f)(p,\lambda) = \lambda\, \tilde{f}_0(p,\lambda), \tag{8.35}$$

$$(\Psi_k A\, f)(p) = \lambda_k(|p|)(\Psi_k f)(p) = \lambda_k(|p|)\, \tilde{f}_k(p), \tag{8.36}$$

for $1 \leq k < N_0$.

Corollary 8.10. The complete family of orthogonal projections $\{P_+,P_-,P_0,P_1,\cdots\}$ reduces A; i.e.,

$$P_\pm A \subset A\, P_\pm,\ P_k A \subset A\, P_k \tag{8.37}$$

for $0 \leq k < N_0$ and if

$$\begin{aligned} A_\pm &= A\, P_\pm = P_\pm\, A\, P_\pm \\ A_k &= A\, P_k = P_k\, A\, P_k,\ 0 \leq k < N_0, \end{aligned} \tag{8.38}$$

denote the parts of A in $\mathcal{H}_\pm$ and $\mathcal{H}_k$ then

$$A = A_+ + A_- + \sum_{k=0}^{N_0-1} A_k. \tag{8.39}$$

The Family $\{\phi_+(x,y,p,q) \mid (p,q) \in R^3 - N\}$. It will now be shown how the normal mode expansion of Theorem 8.8 can be reformulated in terms of the family $\{\phi_+,\psi_1,\psi_2,\cdots\}$. To begin equation (1.59) for the normalizing factor c(p,q) will be verified.

First, recall that the normalizing factors $a_\pm(\mu,\lambda)$, $a_0(\mu,\lambda)$ for $\psi_\pm$, ψ_0 are related to the factors $c_\pm(\mu,\lambda)$, $c_0(\mu,\lambda)$ in their asymptotic forms, equations (1.33)-(1.35), by

$$c_\pm = T_\pm^{-1}\, a_\pm,\; c_0 = T_0^{-1}\, a_0; \tag{8.40}$$

see (4.8), (4.14), (4.21). Combining (8.40) with equations (4.10), (4.16), (4.23) for $T_\pm$, T_0 and equations (5.10), (5.14), (5.19), (5.21) defining $a_\pm$, a_0 gives

$$c_\pm(\mu,\lambda) = \left(\frac{\rho(\pm\infty)}{4\pi\; q_\pm(\mu,\lambda)}\right)^{1/2},\; c_0(\mu,\lambda) = \left(\frac{\rho(\infty)}{4\pi\; q_+(\mu,\lambda)}\right)^{1/2}, \tag{8.41}$$

provided that the phase factors $e^{i\theta_\pm}$, $e^{i\theta_0}$ are defined by

$$\begin{aligned} e^{i\theta_\pm(\mu,\lambda)} &= i[\phi_3\phi_2]/|[\phi_3\phi_2]|,\; \lambda \in \Lambda(\mu), \\ e^{i\theta_0(\mu,\lambda)} &= i[\phi_3\phi_2]/|[\phi_3\phi_2]|,\; \lambda \in \Lambda_0(\mu). \end{aligned} \tag{8.42}$$

On combining (8.41) and the definition of $\phi_+(x,y,p,q)$, equations (1.53) and (1.54), and the asymptotic forms (1.33)-(1.35) for $\psi_\pm$, ψ_0 one obtains the asymptotic forms (1.57), (1.58) for ϕ_+ with the normalizing factor c(p,q) defined by (1.59).

To derive the normal mode expansion for the family $\{\phi_+, \psi_1, \psi_2, \cdots\}$ from that for $\{\psi_+, \psi_-, \psi_0, \psi_1, \psi_2, \cdots\}$ it is clearly sufficient to restrict attention to the subspace

$$\mathcal{H}_f = \mathcal{H}_+ + \mathcal{H}_- + \mathcal{H}_0 \tag{8.43}$$

and the corresponding orthogonal projection

$$P_f = P_+ + P_- + P_0. \tag{8.44}$$

Thus if $h \in \mathcal{H}$ and $h_f = P_f h = h_+ + h_- + h_0$ then Theorem 8.8 implies that

$$\begin{aligned} h_f(x,y) = & \int_\Omega \psi_+(x,y,p,\lambda)\ \tilde{h}_+(p,\lambda)dpd\lambda + \int_{\Omega_0} \psi_0(x,y,p,\lambda)\ \tilde{h}_0(p,\lambda)dpd\lambda \\ & + \int_\Omega \psi_-(x,y,p,\lambda)\ \tilde{h}_-(p,\lambda)dpd\lambda \end{aligned} \tag{8.45}$$

where the integrals converge in $\mathcal{H}$. Changing the variables of integration in the three integrals by means of the mappings X_+, X_0 and X_-, respectively, and using the definition (1.53), (1.54) of ϕ_+ gives

$$\begin{aligned} h_f(x,y) = & \int_{C_+} \psi_+(x,y,p,q)\ \tilde{h}_+(p,\lambda)c(\infty)(2q)^{1/2}\ dpdq \\ & + \int_{C_0} \phi_+(x,y,p,q)\ \tilde{h}_0(p,\lambda)c(\infty)(2q)^{1/2}\ dpdq \\ & + \int_{C_-} \phi_+(x,y,p,q)\ \tilde{h}_-(p,\lambda)c(-\infty)(2|q|)^{1/2}\ dpdq \\ = & \int_{R^3-N} \phi_+(x,y,p,q)\ \hat{h}_+(p,q)dpdq \end{aligned} \tag{8.46}$$

where $\lambda = \lambda(p,q)$ is defined by (1.56) and

$$(8.47)\qquad \hat{h}_+(p,q) = \begin{cases} (2q)^{1/2}\, c(\infty)\, \tilde{h}_+(p,\lambda(p,q)), & (p,q) \in C_+, \\ (2q)^{1/2}\, c(\infty)\, \tilde{h}_0(p,\lambda(p,q)), & (p,q) \in C_0, \\ (2|q|)^{1/2}\, c(-\infty)\, \tilde{h}_-(p,\lambda(p,q)), & (p,q) \in C_-. \end{cases}$$

It is easy to verify by considering the three cones C_+, C_0 and C_- separately that

$$(8.48)\qquad \hat{h}_+(p,q) = \int_{R^3} \overline{\phi_+(x,y,p,q)}\, h(x,y)\, c^{-2}(y)\, \rho^{-1}(y)dxdy$$

where the integral converges in $L_2(R^3)$. Moreover, it can be verified by direct calculation, using the Parseval relation of Theorem 7.3, that

$$(8.49)\qquad \|h_f\|_{\mathcal{H}} = \|\hat{h}_+\|_{L_2(R^3)}$$

for all $h \in \mathcal{H}$. These considerations suggest

<u>Theorem 8.11</u>. For all $h \in \mathcal{H}$ the limit

$$(8.50)\qquad \hat{h}_+(p,q) = L_2(R^3)\text{-}\lim_{M\to\infty} \int_{R^3_M} \overline{\phi_+(x,y,p,q)}\, h(x,y)c^{-2}(y)\rho^{-1}(y)dxdy$$

exists. Moreover, the mapping $\Phi_+ : \mathcal{H} \to L_2(R^3)$ defined by $\Phi_+ h = \hat{h}_+$ is a partial isometry such that

$$(8.51)\qquad \Phi_+ \Phi_+^* = 1 \text{ and } \Phi_+^* \Phi_+ = P_f,$$

and the adjoint mapping $h_f = \Phi_+^* \hat{h}_+$ is given by

$$(8.52)\qquad h_f(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{(R^3-N)_M} \phi_+(x,y,p,q)\, \hat{h}_+(p,q)dpdq.$$

Finally, Φ_+ is a spectral mapping for A in the sense that for all

$h \in D(A)$ one has

$$(8.53) \qquad (\Phi_+ A\, h)(p,q) = \lambda(p,q)\ \Phi_+ h(p,q)$$

where $\lambda(p,q)$ is defined by (1.56).

Note that Theorem 8.11 is simply a reformulation of Theorem 8.8 and not a new theorem.

The Family $\{\phi_-(x,y,p,q) \mid (p,q) \in R^3 - N\}$. The analogue of Theorem 8.11 for the family $\{\phi_-\}$ will be formulated as

Corollary 8.12. For all $h \in \mathcal{H}$ the limit

$$(8.54) \qquad \hat{h}_-(p,q) = L_2(R^3)\text{-}\lim_{M\to\infty} \int_{R_M^3} \overline{\phi_-(x,y,p,q)}\ h(x,y)c^{-2}(y)\rho^{-1}(y)dxdy$$

exists and the mapping $\Phi_- : \mathcal{H} \to L_2(R^3)$ defined by $\Phi_- h = \hat{h}_-$ is a partial isometry such that

$$(8.55) \qquad \Phi_-\ \Phi_-^* = 1 \text{ and } \Phi_-^*\ \Phi_- = P_f.$$

Moreover,

$$(8.56) \qquad h_f(x,y) = \mathcal{H}\text{-}\lim_{M\to\infty} \int_{R_M^3} \phi_-(x,y,p,q)\ \hat{h}_-(p,q)dpdq.$$

Finally, for all $h \in D(A)$, one has

$$(8.57) \qquad (\Phi_- A\, h)(p,q) = \lambda(p,q)\ \Phi_- h(p,q).$$

These results are direct corollaries of Theorem 8.11. This follows from the observations that $f(p,q) \to f(-p,q)$ defines a unitary transformation in $L_2(R^3)$ while $f \to \overline{f}$ defines a unitary transformation in both $\mathcal{H}$ and $L_2(R^3)$.

This completes the formulation of the results of §8. The proofs will now be presented.

The proofs of Theorem 8.1 and Corollaries 8.2 and 8.3 will be based on a lemma concerning bounded linear operators from a Hilbert space into a direct sum space. To formulate it let $\mathcal{H}$ and $\mathcal{H}_k$ $(k \in N)$ denote Hilbert spaces, where N is a finite or denumerable set, and define

$$(8.58) \qquad \tilde{\mathcal{H}} = \sum_{k \in N} \mathcal{H}_k .$$

Elements of $\tilde{\mathcal{H}}$ will be written $g = \{g_k\}$ where $g_k \in \mathcal{H}_k$ and

$$(8.59) \qquad \|g\|^2 = \sum_{k \in N} \|g_k\|_k^2 < \infty$$

if $\|\cdot\|_k$ is the norm in $\mathcal{H}_k$. If

$$(8.60) \qquad B : \mathcal{H} \to \tilde{\mathcal{H}}$$

is a bounded linear operator then

$$(8.61) \qquad Bf = \{(Bf)_k\} = \{B_k f\}$$

where

$$(8.62) \qquad B_k : \mathcal{H} \to \mathcal{H}_k$$

is a bounded linear operator. With this notation one has

<u>Lemma 8.13</u>. The adjoint operator B^*: $\tilde{\mathcal{H}} \to \mathcal{H}$ is given by

$$(8.63) \qquad B^* g = \sum_{k \in N} B_k^* g_k , \; g = \{g_k\} \in \tilde{\mathcal{H}},$$

where the series converges in $\mathcal{H}$. Hence

$$B^*B = \sum_{k\in N} B_k^* B_k, \tag{8.64}$$

where the series converges strongly, and

$$B\,B^*g = \left\{\sum_{j\in N} B_k B_j^* g_j\right\},\ g = \{g_k\} \in \tilde{\mathcal{H}}, \tag{8.65}$$

where the series converge in $\mathcal{H}_k$.

<u>Proof</u>. If N is finite then (8.63) follows directly from the definition of B^*. If N is denumerably infinite then (8.63) holds for all $g \in \tilde{\mathcal{H}}$ with finitely many non-zero components g_k. But any $g \in \tilde{\mathcal{H}}$ can be approximated in $\tilde{\mathcal{H}}$ by such vectors and B^* is bounded. The convergence in $\mathcal{H}$ of the series in (8.63) follows. Equation (8.64) follows on applying (8.63) to the vector (8.61). (8.65) follows on applying B to the vector (8.63).

<u>Corollary 8.14</u>. $B^*B = 1$ in $\mathcal{H}$ if and only if

$$\sum_{k\in N} B_k^* B_k = 1 \text{ in } \mathcal{H}, \tag{8.66}$$

the series converging strongly.

<u>Corollary 8.15</u>. $B\,B^* = 1$ in $\tilde{\mathcal{H}}$ if and only if

$$B_k B_j^* = \delta_{jk} \text{ for all } k,j \in N \tag{8.67}$$

where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$.

<u>Proof of Corollaries 8.14 and 8.15</u>. (8.66) is an immediate consequence of (8.64) and $B^*B = 1$. (8.67) follows from $B\,B^* = 1$ and (8.65) on taking $g_{(j)} = \{\delta_{jk}\, g_k\}$, j fixed.

On applying Corollary 8.14 to the direct sum space (8.1) and the operator Ψ it is seen that (8.13) of Corollary 8.2 is equivalent to (8.5) of Theorem 8.1. Similarly, Corollary 8.15 implies that the relations of Corollary 8.3 are equivalent to (8.6) of Theorem 8.1.

Proof of Corollary 8.2. It was shown that (8.5) follows from Theorem 7.3. Thus (8.13) is valid by Corollary 8.14.

Proof of Corollary 8.3. The spectral mapping $\Psi_\mu = (\Psi_{\mu+}, \Psi_{\mu-}, \Psi_{\mu 0}, \Psi_{\mu 1}, \cdots)$ for A_μ is unitary for all $\mu > 0$ by Theorem 5.6. It follows from Corollary 8.15 that the analogue of Corollary 8.3 holds for Ψ_μ; see (5.38). It will be shown that Corollary 8.3 follows from these relations. For brevity only the relation $\Psi_0\, \Psi_0^* = 1$ will be proved. The remaining relations can be proved by the same method.

For the proof of $\Psi_0\, \Psi_0^* = 1$ define

$$\hat{\Psi}_0 = \Psi_0\, F^{-1} : \mathcal{H} \to L_2(\Omega_0). \tag{8.68}$$

Then

$$\Psi_0\, \Psi_0^* = \hat{\Psi}_0\, \hat{\Psi}_0^* \tag{8.69}$$

and it will suffice to prove that $\hat{\Psi}_0\, \hat{\Psi}_0^* = 1$. The following lemma will be used.

Lemma 8.16. For all $g \in \mathcal{D}(\Omega_0)$ one has

$$(\hat{\Psi}_0^*\, g)(p,y) = \int_{\Lambda_0(|p|)} \psi_0(y,|p|,\lambda)\, g(p,\lambda)d\lambda = [\Psi^*_{|p|_0}\, g(p,\cdot)](y). \tag{8.70}$$

Proof of Lemma 8.16. Let $f \in \mathcal{H}' = F^{-1}\, \mathcal{D}(R^3)$. Then $\hat{f} \in \mathcal{D}(R^3)$ and by Lemma 7.10 one has

$$(\hat{\Psi}_0^* g, \hat{f}) = (g, \hat{\Psi}_0 \hat{f}) = (g, \Psi_0 f) = (g, \tilde{f}_0)$$

(8.71)

$$= \int_{\Omega_0} \overline{g(p,\lambda)} \int_R \overline{\psi_0(y,|p|,\lambda)}\, \hat{f}(p,y) c^{-2}(y) \rho^{-1}(y)\, dp\, d\lambda .$$

Now $\psi_0(y,|p|,\lambda)$ is defined for $(y,p,\lambda) \in R \times \Omega_0$. Extend the definition to all $(y,p,\lambda) \in R \times R^3$ by

$$\psi_0(y,|p|,\lambda) = 0 \text{ for } y \in R,\ (p,\lambda) \in R^3 - \Omega_0, \tag{8.72}$$

and apply Fubini's theorem to the integral in (8.71). This gives

$$(\hat{\Psi}_0^* g, \hat{f}) = \int_{R^3} \left(\int_R \overline{\psi_0(y,|p|,\lambda) g(p,\lambda)}\, d\lambda \right) \hat{f}(p,y) c^{-2}(y) \rho^{-1}(y)\, dp\, dy, \tag{8.73}$$

which implies (8.70) because $\mathcal{D}(R^3)$ is dense in $\mathcal{H}$ and supp $\psi_0(y,|p|,\cdot) \subset \Lambda_0(|p|)$, by (8.72).

<u>Proof of Corollary 8.3 (completed)</u>. The relation $\hat{\Psi}_0\, \hat{\Psi}_0^* = 1$ is equivalent to

$$(\hat{\Psi}_0\, \hat{\Psi}_0^* f, g) = (\hat{\Psi}_0^* f, \hat{\Psi}_0^* g) = (f,g) \tag{8.74}$$

for all $f,g \in L_2(\Omega_0)$. Moreover, since $\hat{\Psi}_0^*$ is bounded it will suffice to verify (8.74) for f and g in the dense set $\mathcal{D}(\Omega_0)$. Now Lemma 8.16 implies that for all $f,g \in \mathcal{D}(\Omega_0)$ one has

$$(\hat{\Psi}_0^* f, \hat{\Psi}_0^* g) = \int_{R^3} \overline{[\Psi_{|p|0}^* f(p,\cdot)]}\,(y)\ [\Psi_{|p|0}^* g(p,\cdot)](y)\, dp\, dy$$

(8.75)

$$= \int_{R^2} (\Psi_{|p|0}^* f(p,\cdot), \Psi_{|p|0}^* g(p,\cdot))\, dp$$

But by (5.38), $\Psi_{|p|_0}\Psi^*_{|p|_0} = 1$ for all $|p| > 0$ and hence the last integral equals

$$(8.76)\qquad \int_{R^2} (f(p,\cdot),g(p,\cdot))_{\Lambda_0(|p|)}dp = (f,g)_{L_2(\Omega_0)}$$

which proves (8.74). This completes the proof.

Proof of Theorem 8.1. It was shown that (8.5) follows from Theorem 7.3. Relation (8.6) follows from Corollary 8.3, by Corollary 8.15.

Proof of Corollaries 8.4 and 8.5. Corollary 8.4 follows immediately from the definitions (8.16) and Corollaries 8.2 and 8.3. Corollary 8.5 follows from Corollary 8.4.

Proof of Corollary 8.6. This is just a restatement of Theorem 7.3, equation (7.12).

Proof of Corollary 8.7. Equation (8.27) for Ψ_+^* will be verified. The proofs of the remaining equation are similar. It is clearly sufficient to verify (8.27) for functions $g_+ \in L_2(\Omega)$ with compact supp $g_+ \subset \Omega^M$. If g_+ is such a function and $f \in \mathcal{H}^{com}$ then

$$(8.77)\qquad \begin{aligned} (f,\Psi_+^* g) &= (\Psi_+ f, g_+) = (\tilde{f}_+, g_+) \\ &= \int_{\Omega^M} \left[\int_{R^3} \psi_+(x,y,p,\lambda)\,\overline{f(x,y)}\,c^{-2}(y)\rho^{-1}(y)dxdy\right] g_+(p,\lambda)dpd\lambda \\ &= \int_{R^3} \overline{f(x,y)} \left[\int_{\Omega^M} \psi_+(x,y,p,\lambda)g_+(p,\lambda)dpd\lambda\right] c^{-2}(y)\rho^{-1}(y)dxdy. \end{aligned}$$

This relation implies (8.27) for g_+ because $\mathcal{H}^{com}$ is dense in $\mathcal{H}$.

Proof of Theorem 8.8. The representation (8.30), (8.31) follows from Corollaries 8.2 and 8.7. The converse follows from the unitarity of Ψ, Theorem 8.1.

Proof of Theorem 8.9. Only equation (8.35) will be proved. The other equations can be proved by the same method. To prove (8.35) let $f \in D(A)$, $g \in \mathcal{D}(\Omega_0)$ and note that

$$(\Psi_0 A\, f, g) = (A\, f, \Psi_0^* g) = (\hat{A}\, \hat{f}, \hat{\Psi}_0^* g). \tag{8.78}$$

Now if $h = \hat{\Psi}_0^* g$ then $h \in \mathcal{H}$, by Corollary 8.3 and, by Lemma 8.16,

$$h(p,y) = \int_{\Lambda_0(|p|)} \psi_0(y,|p|,\lambda)\, g(p,\lambda) d\lambda, \quad (p,y) \in R^3. \tag{8.79}$$

A distribution-theoretic calculation of $\hat{A}\, h$ gives

$$\hat{A}\, h(p,y) = \int_{\Lambda_0(|p|)} \psi_0(y,|p|,\lambda)\, \lambda\, g(p,\lambda) d\lambda. \tag{8.80}$$

In particular, since $\lambda\, g(p,\lambda) \in L_2(\Omega_0)$, one has $\hat{A}\, h \in \mathcal{H}$ by Corollary 8.3 and hence $h \in D(\hat{A})$. Combining this with (8.78) gives

$$\begin{aligned} (\Psi_0 A\, f, g) &= (\hat{A}\, \hat{f}, h) = (\hat{f}, \hat{A}\, h) \\ &= \int_{R^3} \overline{\hat{f}(p,y)} \left[\int_{\Lambda_0(|p|)} \psi_0(y,|p|,\lambda)\, \lambda\, g(p,\lambda) d\lambda \right] c^{-2}(y) \rho^{-1}(y) dp dy. \end{aligned} \tag{8.81}$$

Writing $k(p,\lambda) = \lambda\, g(p,\lambda) \in \mathcal{D}(\Omega_0)$ and applying Lemma 8.16 again gives

$$(8.82)\quad \begin{aligned}(\Psi_0 A f, g) &= (\hat{f}, \hat{\Psi}_0^* k) = (f, \Psi_0^* k) \\ &= (\Psi_0 f, k) = \int_\Omega \overline{\tilde{f}_0(p,\lambda)}\, k(p,\lambda)dpd\lambda \\ &= \int_\Omega \overline{\lambda\, \tilde{f}_0(p,\lambda)}\, g(p,\lambda)dpd\lambda .\end{aligned}$$

This implies (8.35) because the functions $g \in \mathcal{D}(\Omega_0)$ are dense in $L_2(\Omega_0)$.

Proof of Corollary 8.10. It is only necessary to verify relations (8.37). By [8, p. 530] these relations are equivalent to

$$(8.83)\qquad P_\pm \Pi(\mu) = \Pi(\mu) P_\pm \;, \quad P_k \Pi(\mu) = \Pi(\mu) P_k$$

for $0 \leq k < N_0$ and all $\mu \in R$. These equations are immediate consequences of Theorem 7.4. For example (7.17) and the definition of P_+ implies that

$$(8.84)\qquad (f, P_+ \Pi(\mu) g) = (f, \Pi(\mu) P_+ g) = \int_\Omega H(\mu-\lambda)\, \overline{\tilde{f}_+(p,\lambda)}\, \tilde{g}_+(p,\lambda)dpd\lambda$$

for all $f,g \in \mathcal{H}$, which implies $P_+\Pi(\mu) = \Pi(\mu)P_+$. The other relations are proved similarly.

Proof of Theorem 8.11. Equation (8.50) can be verified by applying the definition (8.47) to h_M and using the convergence statement of Theorem 7.3. Relations (8.51) follow from (8.47) and Corollary 8.3 by direct calculation. Equation (8.52) can be verified by reversing the steps in the calculation (8.45), (8.46). Relation (8.53) follows from Theorem 8.9 and equation (8.47).

Proof of Corollary 8.12. This was indicated immediately after the statement of the Corollary.

§9. Semi-Infinite and Finite Layers.

The purpose of this section is to present extensions of the preceding analysis to the cases of semi-inifinite and finite layers of stratified fluid. The methods and results are entirely analogous to those developed above. Therefore the presentation emphasizes the modifications required in these cases. Proofs are indicated briefly or omitted.

Semi-Infinite Layers. With a suitable choice of coordinates the region occupied by the fluid is described by the domain

$$R_+^3 = \{(x,y) \mid y > 0\}. \tag{9.1}$$

The acoustic field is assumed to satisfy either the Dirichlet or the Neumann boundary condition on the boundary of R_+^3. Physically, these conditions correspond to the cases where the boundary plane is free and rigid, respectively. The functions $\rho(y)$ and $c(y)$ are assumed to be Lebesgue measurable and satisfy

$$0 < \rho_m \le \rho(y) \le \rho_M < \infty,\ 0 < c_m \le c(y) \le c_M < \infty \tag{9.2}$$

for all $y > 0$ and

$$\int_0^\infty |\rho(y) - \rho(\infty)|dy < \infty,\ \int_0^\infty |c(y) - c(\infty)|dy < \infty. \tag{9.3}$$

The acoustic propagator A defined by (1.2) determines selfadjoint operators A^0 and A^1 in

$$\mathcal{H}_+ = L_2(R_+^3, c^{-2}(y)\rho^{-1}(y)dxdy) \tag{9.4}$$

corresponding to the two boundary conditions. The domains of A^0 and A^1 are subsets of

$$L_2^1(A,R_+^3) = L_2^1(R_+^3) \cap \{u \mid \nabla\cdot(\rho^{-1}\nabla u) \in L_2(R_+^3)\}. \tag{9.5}$$

The operator A^0 corresponding to the Dirichlet condition is defined by

$$D(A^0) = L_2^1(A,R_+^3) \cap \{u \mid u(x,0+) = 0 \text{ in } L_2(R^2)\}. \tag{9.6}$$

Sobolev's embedding theorem for $L_2^1(R_+^3)$ [1] implies that the boundary values $u(x,0+)$ are defined in $L_2(R^2)$.

The Neumann condition will be interpreted in the generalized sense employed in [15]; i.e.,

$$\int_{R_+^3} \{\nabla\cdot(\rho^{-1}\nabla u)v + \rho^{-1}\nabla u \cdot \nabla v\}dxdy = 0 \tag{9.7}$$

for all $v \in L_2^1(R_+^3)$. Thus the operator A^1 corresponding to the Neumann condition is defined by

$$D(A^1) = L_2^1(A,R_+^3) \cap \{u \mid (9.7) \text{ holds for all } v \in L_2^1(R_+^3)\}. \tag{9.8}$$

The operators are defined by $A^j u = Au$ for all $u \in D(A^j)$ and one has

$$A^j = A^{j*} \geq 0,\ j = 0,\ 1. \tag{9.9}$$

This is most easily proved by introducing the corresponding sesquilinear forms, as in §1, and using Kato's representation theorem [8, p. 322].

The spectral analysis of A^0 and A^1 may be based on the corresponding reduced propagators A^0 and A^1 in

$$\mathcal{H}(R_+) = L_2(R_+,c^{-2}(y)\rho^{-1}(y)dy),\ R_+ = \{y \mid y > 0\}. \tag{9.10}$$

They are defined by

$$D(A_\mu^0) = L_2^1(R_+) \cap \{\psi \mid (\rho^{-1}\psi')' \in \mathcal{H}(R_+) \text{ and } \psi(0+) = 0\}, \tag{9.11}$$

$$D(A_\mu^1) = L_2^1(R_+) \cap \{\psi \mid (\rho^{-1}\psi')' \in \mathcal{H}(R_+) \text{ and } (\rho\psi')(0+) = 0\}, \tag{9.12}$$

$$A_\mu^j\psi = A_\mu\psi \text{ for } \psi \in D(A_\mu^j),\ j = 0,1, \tag{9.13}$$

and one has

$$A_\mu^j = A_\mu^{j*} \geq c_m^2\ \mu^2,\ j = 0,1. \tag{9.14}$$

The results of §3 can be extended to these operators. Thus

$$\sigma_c(A_\mu^j) = \sigma_e(A_\mu^j) = [c^2(\infty)\mu^2,\infty), \tag{9.15}$$

$$\sigma(A_\mu^j) \cap (-\infty,c^2(\infty)\mu^2) \subset \sigma_0(A_\mu^j), \tag{9.16}$$

and the eigenvalues in this interval are all simple. They will be denoted by $\lambda_k^j(\mu)$, $1 \leq k < N^j(\mu) \leq +\infty$.

<u>Eigenfunctions of A_μ^j.</u> These functions will be denoted by $\psi_k^j(y,\mu)$, $1 \leq k < N^j(\mu)$. They are uniquely defined by the conditions $\psi_k^j(\cdot,\mu) \in D(A_\mu^j)$, $\|\psi_k^j(\cdot,\mu)\|_{\mathcal{H}(R_+)} = 1$,

$$(A_\mu - \lambda_k^j(\mu))\ \psi_k^j(y,\mu) = 0 \text{ for } y \in R_+, \text{ and} \tag{9.17}$$

$$\psi_k^0(0+,\mu) = 0,\ (\rho^{-1}\psi_k^{1\,\prime})(0+,y) = 0. \tag{9.18}$$

The asymptotic behavior of $\psi_k^j(y,\mu)$ for $y \to +\infty$ is given by

$$\psi_k^j(y,\mu) \sim c_k^j(\mu) \exp\{-y\ q'(\mu,\lambda_k^j(\mu))\},\ y \to \infty, \tag{9.19}$$

where

$$q'(\mu,\lambda) = (\mu^2 - \lambda\, c^{-2}(\infty))^{1/2} > 0. \tag{9.20}$$

The function $\psi_k^j(y,\mu)$ has precisely $k - 1$ zeros.

<u>Generalized Eigenfunctions of A_μ^j</u>. For $\lambda > c^2(\infty)\mu^2$, A_μ^j has a single family of generalized eigenfunctions $\{\psi^j(\cdot,\mu,\lambda) \mid \lambda > c^2(\infty)\mu^2\}$. They are determined up to normalization by the conditions

$$(A_\mu - \lambda)\psi^j(y,\mu,\lambda) = 0 \text{ for } y \in R_+, \text{ and} \tag{9.21}$$

$$\psi^0(0+,\mu,\lambda) = 0, \ (\rho^{-1}\psi^{1\,\prime})(0+,\mu,\lambda) = 0. \tag{9.22}$$

Their asymptotic behavior for $y \to \infty$ is given by

$$\psi^j(y,\mu,\lambda) \sim c^j(\mu,\lambda)[e^{-iyq(\mu,\lambda)} + R^j(\mu,\lambda)\, e^{iyq(\mu,\lambda)}], \ y \to \infty, \tag{9.23}$$

where

$$q(\mu,\lambda) = (\lambda\, c^{-2}(\infty) - \mu^2)^{1/2} > 0. \tag{9.24}$$

The normalizing factors $c^j(\mu,\lambda)$ are calculated below.

<u>Generalized Eigenfunctions of A^j</u>. These are defined by

$$\psi^j(x,y,p,\lambda) = (2\pi)^{-1}\, e^{ip\cdot x}\, \psi^j(y,|p|,\lambda), \ (p,\lambda) \in \Omega, \tag{9.25}$$

$$\psi_k^j(x,y,p) = (2\pi)^{-1}\, e^{ip\cdot x}\, \psi_k^j(y,|p|), \ p \in \Omega_k, \ k \geq 1 \tag{9.26}$$

where

$$\Omega = \{(p,\lambda) \mid \lambda > c^2(\infty)|p|^2\} \tag{9.27}$$

$$\Omega_k = \{p \mid |p| \in \mathcal{O}_k\}, \ k \geq 1 \tag{9.28}$$

and $O_k = \{\mu \mid N^j(\mu) \geq k + 1\}$, as before. The wave-theoretic interpretations of the eigenfunctions (9.25), (9.26) may be derived from the asymptotic forms (9.19), (9.23) as in §1. With these definitions the following analogue of Theorem 7.4 holds.

Theorem 9.1. The spectral families $\{\Pi^j(\mu)\}$ of A^j satisfy

$$(9.29)\qquad \begin{aligned}(f,\Pi^j(\mu)f) = \int_\Omega H(\mu-\lambda)\ |\tilde{f}^j(p,\lambda)|^2\ dpd\lambda \\ + \sum_{k=1}^{N_0^j-1} \int_{\Omega_k} H(\mu - \lambda_k^j(|p|))\ |\tilde{f}_k^j(p)|^2 dp\end{aligned}$$

for all $f \in \mathcal{H}_+$ where $N_0^j = \sup_{\mu>0} N^j(\mu)$ and

$$(9.30)\qquad \tilde{f}^j(p,\lambda) = L_2(\Omega)\text{-}\lim_{M\to\infty} \int_0^M \int_{|x|\leq M} \overline{\psi^j(x,y,p,\lambda)}\ f(x,y)c^{-2}(y)\rho^{-1}(y)dxdy,$$

$$(9.31)\qquad \tilde{f}_k^j(p) = L_2(\Omega_k)\text{-}\lim_{M\to\infty} \int_0^M \int_{|x|\leq M} \overline{\psi_k^j(x,y,p)}\ f(x,y)c^{-2}(y)\rho^{-1}(y)dxdy.$$

Relation to the Infinite Layer Problem. Theorem 9.1 can be derived by the method employed for the infinite layer problem in the preceding sections. However, it can also be deduced directly from Theorem 7.4. To this end one extends $\rho(y)$, $c(y)$ to all $y \in R$ as even functions:

$$(9.32)\qquad \rho(-y) = \rho(y),\ c(-y) = c(y),\ y > 0.$$

Then it follows from (9.2), (9.3) that the extended functions satisfy (1.3), (1.4) with

(9.33) $$\rho(-\infty) = \rho(\infty), \quad c(-\infty) = c(\infty).$$

The corresponding operator in $\mathcal{H}$ will be denoted by A, as before. Property (9.33) implies that

(9.34) $$q_{\pm}(\mu,\lambda) = q(\mu,\lambda), \quad q'_{\pm}(\mu,\lambda) = q'(\mu,\lambda)$$

where the latter are defined by (9.20), (9.24). Moreover, the special solutions $\phi_j(y,\mu,\lambda)$ of §3 satisfy

(9.35) $$\begin{cases} \phi_1(-y,\mu,\lambda) = \phi_4(y,\mu,\lambda) \\ \phi_2(-y,\mu,\lambda) = \phi_3(y,\mu,\lambda) \end{cases}$$

and it is easy to verify that the eigenfunction $\psi_k(y,\mu)$ of A_μ is even (resp., odd) when k is odd (resp., even). It follows that the eigenfunctions of A_μ^j can be calculated from those of A_μ by the rule

(9.36) $$\begin{cases} \psi_k^0(y,\mu) = \sqrt{2}\ \psi_{2k}(y,\mu), & y \geq 0,\ k = 1,2,\cdots \\ \psi_k^1(y,\mu) = \sqrt{2}\ \psi_{2k-1}(y,\mu), & y \geq 0,\ k = 1,2,\cdots \end{cases}$$

The factor $\sqrt{2}$ is to renormalize ψ_k from R to R_+.

Concerning the generalized eigenfunctions, note that there is no $\psi_0(y,\mu,\lambda)$ for A_μ because $c(-\infty) = c(\infty)$ and (9.35) implies that

(9.37) $$\psi_{\pm}(-y,\mu,\lambda) = \psi_{\mp}(y,\mu,\lambda).$$

It follows that the coefficients $R_\pm$, $T_\pm$ in (1.33), (1.34) satisfy

(9.38) $$R_+(\mu,\lambda) = R_-(\mu,\lambda), \quad T_+(\mu,\lambda) = T_-(\mu,\lambda)$$

and the generalized eigenfunctions of A_μ^j can be calculated from those of A_μ by the rule

$$
(9.39)\qquad
\begin{aligned}
\psi^0(y,\mu,\lambda) &= \psi_+(y,\mu,\lambda) - \psi_+(-y,\mu,\lambda),\ y \geq 0,\\
\psi^1(y,\mu,\lambda) &= \psi_+(y,\mu,\lambda) + \psi_+(-y,\mu,\lambda),\ y \geq 0.
\end{aligned}
$$

The factors $c^j(\mu,\lambda)$ and $R^j(\mu,\lambda)$ of (9.23) are given by

$$
(9.40)\qquad c^0(\mu,\lambda) = c^1(\mu,\lambda) = (\rho(\infty)/4\pi q(\mu,\lambda))^{1/2}\ ,
$$

$$
(9.41)\qquad R^0 = R_\pm - T_\pm,\ R^1 = R_\pm + T_\pm.
$$

Theorem 9.1 can be deduced from Theorem 7.4 by introducing the operators

$$
(9.42)\qquad J_j : \mathcal{H}(R_+^3) \to \mathcal{H},\ j = 0,1,
$$

defined by

$$
(9.43)\qquad J_0\ u(x,y) = \begin{cases} u(x,y), & (x,y) \in R_+^3,\\ -u(x,-y), & (x,-y) \in R_+^3, \end{cases}
$$

and

$$
(9.44)\qquad J_1\ u(x,y) = \begin{cases} u(x,y), & (x,y) \in R_+^3,\\ u(x,-y), & (x,-y) \in R_+^3. \end{cases}
$$

J_0 and J_1 are bounded linear operators and using the fact that A has coefficients satisfying (9.32) one can show that the resolvents of A^j and A are related by

(9.45) $$R(A^j,\zeta) = \frac{1}{2} J_j^* R(A,\zeta) J_j, \ j = 0,1.$$

From this and Stone's theorem (7.64) it follows that

(9.46) $$\Pi^j(\mu) = \frac{1}{2} J_j^* \Pi(\mu) J_j, \ j = 0,1.$$

Theorem 9.1 follows directly from these relations and Theorem 7.4.

Finite Layers. In this case, with a suitable choice of coordinates the region occupied by the fluid is described by the domain

(9.47) $$R_h^3 = R^3 \cap \{(x,y) \mid 0 < y < h\}$$

where $h > 0$. The case of a fluid layer with a free surface at $y = 0$ and a rigid bottom at $y = h$ will be discussed.

The acoustic propagator A and boundary conditions determine a selfadjoint operator A^h in

(9.48) $$\mathcal{H}_h = L_2(R_h^3, c^{-2}(y)\rho^{-1}(y)dxdy).$$

To define the domain of A^h let

(9.49) $$L_2^{1,0}(R_h^3) = L_2^1(R_h^3) \cap \{u \mid u(x,0+) = 0 \text{ in } L_2(R_h^3)\}.$$

The Dirichlet condition at $y = 0$ will be enforced by requiring $D(A^h) \subset L_2^{1,0}(R_h^3)$. The Neumann condition at $y = h$ will be interpreted in the generalized sense that

(9.50) $$\int_{R_h^3} \{\nabla\cdot(\rho^{-1}\nabla u)v + \rho^{-1}\nabla u\cdot\nabla v\}dxdy = 0 \text{ for all } v \in L_2^{1,0}(R_h^3).$$

Thus

(9.51) $$D(A^h) = L_2^{1,0}(R_h^3) \cap L_2^1(A,R_h^3) \cap \{u \mid (9.50) \text{ holds}\}$$

and $A^h u = Au$ for all $u \in D(A^h)$. As before

(9.52) $$A^h = A^{h*} \geq 0.$$

The corresponding reduced propagator A_μ^h in $\mathcal{H}(R_h)$ $= L_2(R_h, c^{-2}(y)\rho^{-1}(y)dy)$ ($R_h = \{y \mid 0 < y < h\}$) is defined by

(9.53) $$D(A_\mu^h) = L_2^1(R_h) \cap \{\psi \mid (\rho^{-1}\psi')' \in \mathcal{H}(R_+), \psi(0+) = 0, (\rho^{-1}\psi')(h+) = 0\},$$

and $A_\mu^h \psi = A_\mu \psi$ for all $\psi \in D(A_\mu^h)$. One has

(9.54) $$A_\mu^h = A_\mu^{h*} \geq c_m^2 \mu^2$$

as before. In the present case A_μ^h is a regular Sturm-Liouville operator. Hence

(9.55) $$\sigma(A_\mu^h) = \sigma_0(A_\mu^h)$$

and if $\lambda_k^h(\mu)$, $1 \leq k < \infty$, denotes the eigenvalues then $\lambda_k^h(\mu) \to \infty$ when $k \to \infty$. Note that in this case $\mathcal{O}_k = R_+$ and $\Omega_k = R^2$. If $\psi_k^h(y,\mu)$ denotes the corresponding eigenfunctions and $\psi_k^h(x,y,p) = (2\pi)^{-1} e^{ip\cdot x} \psi_k^h(y,|p|)$ then the spectral family $\{\Pi^h(\mu)\}$ for A^h satisfies

(9.56) $$(f, \Pi^h(\mu) f) = \sum_{k=1}^{\infty} \int_{R^2} H(\mu - \lambda_k^h(|p|)) \, |\tilde{f}_k^h(p)|^2 \, dp$$

where

(9.57) $$\tilde{f}_k^h(p) = L_2(R^2)\text{-}\lim_{M\to\infty} \int_0^h \int_{|x|\leq M} \overline{\psi_k^h(x,y,p)} \, f(x,y) c^{-2}(y) \rho^{-1}(y) dx dy.$$

These results can be proved by the method developed in §§2-7.

§10. Concluding Remarks.

The spectral analysis of the acoustic propagator A has been developed to provide a foundation for the study of acoustic wave propagation in stratified fluids. The result developed in this report can be used to analyze transient sound fields in stratified fluids following the method developed for the Pekeris model in [18]. It can also be used to establish a limiting absorption theorem and corresponding theory of steady-state sound fields in stratified fluids. A third application is to the analysis of the scattering and trapping of acoustic waves by the acoustic ducts produced by minima of c(y). These applications will be presented in separate reports.

A number of extensions of the theory are possible. Thus, other classes of density and sound speed profiles could be studied. Examples include cases where $\rho(y)$ and/or $c(y)$ tend to zero or infinity at $y = \pm\infty$ or at finite boundary points or interior points. The Weyl-Kodaira theory is applicable to all such operators that lead to selfadjoint realizations of A_μ. Some of these cases will be of interest for applications.

Another extension of the theory that holds great interest for applications is to the analysis of the scattering of sound by obstacles immersed in stratified fluids. Mathematically, this problem leads to the spectral analysis of the acoustic propagators for stratified fluids in domains exterior to bounded sets. The analysis presented above is a necessary preparation for such a study.

Appendix. The Weyl-Kodaira Theory.

The general Sturm-Liouville operator may be written

$$L\,\phi(y) = w^{-1}(y)\{-(p^{-1}(y)\phi')' + q(y)\phi\}. \tag{A.1}$$

The basic spectral theory of such operators was established by H. Weyl [14] and K. Kodaira [10]. The purpose of this Appendix is to present a version of the Weyl-Kodaira theory that is applicable to the operator A_μ of this report.

It is true that expositions of the Weyl-Kodaira theory are available in [2, 3, 11] and a number of other textbooks and monographs. However, in these and most of the book and periodical literature, hypotheses are made concerning the form of the operator, or the continuity or differentiability of the coefficients, that limit the applicability of the theory. Thus most authors assume that $w(y) \equiv 1$ and many take $p(y) \equiv 1$ as well. Moreover, it is usual to assume that the coefficients are smooth functions or at least continuous. It is known that if the coefficients are sufficiently regular then L can be reduced to the Schrodinger form $L\,\phi = -\phi'' + q(y)\phi$ by changes of the independent and dependent variables [11, p. 2]. However, this technique is not applicable to operators with singular coefficients. Here a version of the Weyl-Kodaira theory is presented that is applicable to operators (A.1) with locally integrable coefficients. The concepts needed for this extension of the theory are available in the classic book of Coddington and Levinson [2].

The operator (A.1) will be studied on an arbitrary interval $I = \{y \mid -\infty \le a < y < b \le +\infty\}$. The coefficients will be assumed to have the properties

(A.2) $p(y)$, $q(y)$, $w(y)$ are defined and real valued for almost all $y \in I$,

(A.3) $p(y) > 0$ and $w(y) > 0$ for almost all $y \in I$,

(A.4) $p(y)$, $q(y)$, $w(y)$ are in $L_1^{\ell oc}(I)$,

where $L_1^{\ell oc}(I) = \{f(y) \mid f \in L_1(K)$ for every compact $K \subset I\}$. It is natural to study L in the Hilbert space $\mathcal{H}(I,w)$ with scalar product

$$(u,v) = \int_I \overline{u(y)}\, v(y)\, w(y)dy. \tag{A.5}$$

In the general theory of singular Sturm-Liouville operators two operators in $\mathcal{H}(I,w)$ are associated with L. The first is the maximal operator L_1 defined by

$$\text{(A.6)} \quad \begin{cases} D(L_1) = \mathcal{H}(I,w) \cap AC(I) \cap \{u \mid p^{-1}u' \in AC(I), Lu \in \mathcal{H}(I,w)\}, \\ \\ L_1 u = Lu \text{ for all } u \in D(L_1). \end{cases}$$

The second is the minimal operator L_0 defined by

$$\text{(A.7)} \quad \begin{cases} D(L_0) = D(L_1) \cap \{u \mid (L_1u,v) = (u,L_1v) \text{ for all } v \in D(L_1)\}, \\ \\ L_0 u = Lu \text{ for all } u \in D(L_0). \end{cases}$$

It can be shown that L_0 is densely defined and closed and satisfies

$$L_0 \subset L_0^* = L_1. \tag{A.8}$$

It follows that every selfadjoint realization of L in $\mathcal{H}(I,w)$ must satisfy

$$L_0 \subset L \subset L_1. \tag{A.9}$$

If $L_0 = L_0^* = L_1$ then L is said to be essentially selfadjoint. The classification of the selfadjoint realizations of L by means of boundary conditions at a and b will not be reviewed here. For essentially selfadjoint operators no boundary conditions are needed (Weyl's limit point case). The operator A_μ of §1 is essentially selfadjoint since its maximal operator is selfadjoint (cf. (1.18), (1.19)).

The Weyl-Kodaira theory provides spectral representations of the selfadjoint realizations of singular Sturm-Liouville operators. Each representation is derived from a basis of solutions of $L\psi = \lambda\psi$ and a corresponding 2×2 positive matrix measure $m(\lambda) = (m_{jk}(\lambda))$ [3, p. 1337ff]. The representation spaces are the Lebesgue spaces $L_2(\Lambda,m)$ associated with m, with norm defined by

$$\|F\|^2_{\Lambda,m} = \int_\Lambda \sum_{j,k=1}^{2} \overline{F_j(\lambda)}\, F_k(\lambda)\, m_{jk}(d\lambda). \tag{A.10}$$

The following version of the Weyl-Kodaira theory is adapted from [3, pp. 1351-6].

<u>Theorem (Weyl-Kodaira)</u>. Let L be a selfadjoint realization of L in $\mathcal{H}(I,w)$ with spectral family $\{\Pi_L(\lambda)\}$. Let $\Lambda = (\lambda_1,\lambda_2) \subset R$ and let $\psi_j(y,\lambda)$ $(j = 1,2)$ be a pair of functions with the properties

(A.11) $\psi_j(y,\lambda) \in C(I \times \Lambda)$, $j = 1,2$,

(A.12) The pair $\psi_j(y,\lambda)$ $(j = 1,2)$ is a solution basis for $L\psi = \lambda\psi$ on I for each $\lambda \in \Lambda$.

Then there exists a unique 2×2 positive matrix measure $m = (m_{jk})$ on Λ with the following properties.

(A.13) For all $f \in \mathcal{H}(I,w)$ there exists the limit

$$\hat{f}(\lambda) = (\hat{f}_1(\lambda),\hat{f}_2(\lambda)) = \underset{a'\to a,\, b'\to b}{L_2(\Lambda,m)\text{-lim}} \int_{a'}^{b'} f(y)\ \overline{(\psi_1(y,\lambda)},\overline{\psi_2(y,\lambda))}w(y)dy.$$

(A.14) The mapping $U : \mathcal{H}(I,w) \to L_2(\Lambda,m)$ defined by $Uf = \hat{f}$ is a partial isometry with initial set $\Pi_L(\Lambda)\ \mathcal{H}(I,w)$ and final set $L_2(\Lambda,m)$.

(A.15) The inverse isomorphism of $L_2(\Lambda,m)$ onto $\Pi_L(\Lambda)\ \mathcal{H}(I,w)$ is given by

$$(U^*F)(y) = \underset{\mu_1\to\lambda_1,\, \mu_2\to\lambda_2}{\mathcal{H}(I,w)\text{-lim}} \int_{\mu_1}^{\mu_2} \sum_{j,k=1}^{2} \psi_j(y,\lambda)F_k(\lambda)m_{jk}(d\lambda).$$

(A.16) For all Borel functions $\Psi(\lambda)$ on R with supp $\Psi \subset \Lambda$, one has

$$U\ D(\Psi(L)) = L_2(\Lambda,m) \cap \{\hat{f} \mid \Psi(\lambda)\hat{f}(\lambda) \in L_2(\Lambda,m)\},\ \text{and}$$

$$(U\ \Psi(L)f)(\lambda) = \Psi(\lambda)\hat{f}(\lambda).$$

<u>Discussion of the Proof</u>. The theorem is proved in [3] under the hypotheses $w(y) \equiv 1$, $p(y)$, $q(y) \in C^\infty(I)$ and $p(y) > 0$. To prove it under hypotheses (A.2), (A.3), (A.4) one may first prove it for the special case of the basis $\phi_j(y,\lambda)$ that satisfies $\phi_j^{(k-1)}(c,\lambda) = \delta_j^k$ where $a < c < b$. The functions $\phi_j(y,\lambda)$ are entire functions of λ and the theorem can be proved by the classical limit-point, limit-circle method of Weyl as

presented in [2]. The general case can then be obtained by a change of basis from $\phi_j(y,\lambda)$ to $\psi_j(y,\lambda)$. In fact, this was the procedure used by Kodaira in his original paper [10]. The first uniqueness results for m are due to E. A. Coddington and V. A. Marčenko (see [3]). The uniqueness proof given in [3] can be extended to the case treated here.

As emphasized by Dunford and Schwartz, the utility of the Weyl-Kodaira theorem is due to the possibility of using different bases ψ_j for different portions of the spectrum of L. When a basis has been chosen one need only calculate the measure m. A general procedure for doing this, due to E. C. Titchmarsh [3, p. 1364] is known for cases in which the $\psi_j(y,\lambda)$ have *analytic continuations to a neighborhood of* Λ in the complex plane. However, such continuations are not always available. A procedure that is applicable when the $\psi_j(y,\lambda)$ have a one-sided continuation into the complex plane is illustrated in §5 above.

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13. ABSTRACT

Abstract.

A spectral analysis and normal mode expansions are developed for the acoustic propagator

$$Au = -c^2(y)\,\rho(y)\,\nabla\cdot(\rho^{-1}(y)\,\nabla u)$$

of a stratified fluid with sound speed $c(y)$ and density $\rho(y)$ at depth y. For an infinite fluid it is assumed that the (in general discontinuous) functions $c(y)$, $\rho(y)$ are uniformly positive and bounded and satisfy

$$\pm\int_0^{\pm\infty} |c(y) - c(\pm\infty)|\, dy < \infty, \quad \pm\int_0^{\pm\infty} |\rho(y) - \rho(\pm\infty)|\, dy < \infty.$$

Semi-infinite and finite fluid layers are also treated. The work provides a basis for the analysis of transient and steady-state sound fields in such fluids.

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