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STATISTICAL CHARACTERIZATION OF ROUGH TERRAIN

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A technique is presented to statistically characterize rough terrain surfaces. The approach is described in terms of a specific example but in principle is quite general and can be adapted to fit any number of situations. The starting point is the use of observed surface height data to generate statistical parameters for probability density functions (PDF) that potentially characterize the data. This involves the use of parameter estimation techniques. The estimated parameters are then used in an hypothesis test to determine the best PDF for the given data. The accuracy of the analysis depends on the available

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data, the estimators employed, and the determination of appropriate PDF's. The example consists of a large number of terrain regions in an area of eastern Massachusetts. Experimental data in the form of electromagnetic scattering from the surface are available for this site. Because of the number of cases involved, the complexity of the multivariate height distributions, and the type of measurements available, only a single observation of the multivariate data is used in the present analysis. Techniques to improve the parameter estimation are being pursued. The results of the statistical analysis in terms of mean height, variance, correlation, PDF, and a geologic feature characterizing each subregion are presented. These will be used in an electromagnetic scattering formulation to allow comparison with the experimental data. Once agreement is reached, other areas of interest can then be analyzed in a similar fashion.

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1. INTRODUCTION

In this report, a general technique for characterizing data will be developed. To illustrate the procedure, a particular application to terrain features will be used as an example.

The general problem we are formulating is the following. We consider a set of data and whatever external requirements, results, experience, and limitations are available, and propose statistical distributions which might characterize the data. Next, we mathematically derive a multivariate joint probability density function, PDF, for each case. We then use statistical estimation theory to evaluate the parameters of the PDF's, and finally, we apply an hypothesis test to decide which is the more appropriate PDF for the data set or subsets. This general procedure involves a number of topics and some further discussion may clarify these aspects.

Consider some arbitrary, randomly distributed physical quantities: $Z_1, Z_2, \ldots, Z_N$—either discrete or continuous. One set of measurements of these physical quantities, consisting of one value of each of N variables: $z_1, z_2, \ldots, z_N$ is considered as a single representation of a random process governed by some multivariate PDF. The PDF can have different forms, Gaussian, exponential,

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uniform, binomial, and so on. Properties of the multivariate PDF are discussed in numerous texts, such as Papoulis[1] or Mood and Graybill.[2] In general, the PDF is a function of several parameters such as the mean values of each of the variables μ_i, the variances σ_i^2, and the covariances γ_{ij}, where i = 1, 2, ..., N, and j = 1, 2, ..., N.

When starting from observed values, the situation in its most general state would be that neither the form of the PDF is known, nor the parameters which enter into the expression for the PDF. In order to obtain some information on these parameters, estimation theory is used.[3] The next question is that of the form of the PDF in which the parameters are incorporated. This is not known a priori and must be determined. An hypothesis testing procedure is developed to accomplish this.

The particular hypothesis testing procedure selected in the present case allows only a binary decision process to be considered. Hence, the discrimination is restricted to two forms of the PDF. The test is based upon the maximum a posteriori probability criterion.[4] This is equivalent to the minimum error probability criterion. The explicit mathematical expressions involved in the hypothesis test are worked out for the case of choosing whether an N - variate Gaussian or exponential PDF would be more likely to produce the occurrence of the data set $[z_1]$. However, the procedure for hypothesis testing may be applied quite generally, Indeed, where the number of cases allow, a better test would be to see which density would be more likely to result in several realizations rather than the single one used here.

The specific problem used as an example in this report is that of characterizing a large terrain region that is considered to be made up of smaller subareas ($\sim$4 km^2). The main feature of interest is the distribution of heights within these subregions. The nature of this specific problem places some constraints on the form of the particular statistical analysis carried out for the report.

The goal of the particular case is the development of mathematical descriptions of each of the subareas for use in calculation of the scattering of electromagnetic waves from the uneven terrain surface. Each region is characterized by a geologic code and several statistical parameters. In particular, we are concerned with being able to associate a PDF with the range of heights in the subregions and to determine parameters that make the general PDF explicit.

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1. Papoulis, A. (1965) Probability, Random Variables and Stochastic Processes, McGraw-Hill.
2. Mood, A.M., and Graybill, F.A. (1963) Introduction to the Theory of Statistics, McGraw Hill.
3. Jenkins, G.M., and Watts, D.G. (1968) Spectral Analysis and Its Applications, Holden-Day.
4. Whalen, A.D. (1971) Detection of Signals in Noise, Academic Press.

The data set $[z_i]$ which is analyzed in detail in this report is such that each value of the random variable corresponds to a point in the x - y plane, so that $z_i = z_i(x_k, y_\ell)$, where i = 1, 2, 3, ... N, and N is the total number of grid points in the x - y plane. Here, x_k denotes the k^{th} equally spaced x-value along the x-axis and y_ℓ denotes the ℓ^{th} y-value along the y-axis, where k = 1, 2, ..., $\sqrt{N}$ and ℓ = 1, 2, ..., $\sqrt{N}$. Thus, the N points are distributed in the x - y plane so as to form a rectangular grid. This restricts the analysis to the class of data sets where the covariance matrix

$$R_{mn} = \langle (z_m - \bar{z})(z_n - \bar{z}) \rangle \tag{1}$$

can be assumed to have the form:

$$R_{mn} = \sigma^2 \exp(-\tau_{mn}^2/T^2) \tag{2}$$

where

σ^2 = variance $\langle (z_p - \bar{z})(z_p - \bar{z}) \rangle$

$\bar{z}$ = mean value

$\langle\rangle$ = denotes expectation value

T = correlation length

and

$\tau_{mn}^2 = (x_m - x_n)^2 + (y_m - y_n)^2$ that is,

= square of distance between grid point (x_m, y_m) and point (x_n, y_n).

The motivation for assuming a covariance matrix in the form given by Eq. (2) arises because of the fact that it leads to a tractable mathematical expression for the incoherent power that is scattered when an electromagnetic wave is reflected from a rough surface.[5,6] Then, the data set $\{z_i\}$ corresponds to the heights at particular points on a rough surface.

Also, in the hypothesis testing procedure the binary decision process is worked out in detail here for the case where the PDF is either an N - variable Gaussian or exponential. This specialization is also motivated by the theory of electromagnetic wave scattering from rough surfaces.[5,6]

The performance of a radar system in detecting and tracking targets depends upon the electromagnetic scattering characteristics of the terrain surrounding the radar. Rough terrain will contribute to the radar clutter return and to the multipath return[5,6] which involves wave scattering from both the target and terrain surrounding the radar. These two aspects can be described by the theory of scattering from rough surfaces if the properties of the terrain surrounding the radar are known. The properties of the terrain pertinent to rough surface scattering include the form of the PDF for the surface height distribution, the correlation length when R_{mn} is assumed to have the form given by Eq. (2), the mean surface height, the variance in surface height, and the complex dielectric constant characterizing the surface.

In this introduction, we have proposed a general technique for statistical characterization and discussed how it has been applied to a specific case. The particular external considerations used in defining appropriate forms and constraints for the formulation have been included. In the main body of this report we will develop the successive steps used in the analysis as exemplified by the terrain height data characterization. The first aspect is the parameter estimation required for the eventual selection of an appropriate PDF.

2. PARAMETER ESTIMATION

The starting point in these procedures is a data set or sets which are assumed to represent samples characterized by appropriate PDF's. For the example with which we are concerned our potential PDF's are Gaussian or exponential. The first step in discriminating between these is to construct estimators of the parameters required for the two different densities. The estimators can be considered as functions defined on the N-dimensional sample space. The particular estimate obtained from the given data is then regarded as a realization of the random variables

5. Beckmann, P., and Spizzichino, A. (1963) The Scattering of Electromagnetic Waves from Rough Surfaces, Macmillan Co.

6. Ruck, G.T., Barrick, D.E., Stuart, W.D., and Krichbaum, C.K. (1970) Radar Cross Section Handbook, Vol. 2, Plenum Press.

representing the estimator. The parameters which will be discussed are the means μ_i, the variances σ_i^2, and the covariance function γ_{ij}, for the two possible PDF's.

For a completely rigorous approach to estimation, we could use a technique such as maximum likelihood estimation. That estimate of the parameter maximizes the probability of having obtained the observed values. However, when the data represents a multivariate correlated distribution, the procedures for maximizing the likelihood functions become complex[7] and require multiple observations of the N-variate samples. As an alternative, we would consider the accuracy of the estimator by minimizing the bias and/or variance of the estimator, which are defined in terms of the lower order moments of the estimator PDF.

The bias of an estimator is defined as the difference between the true value of the parameter and the expected value of the estimator. When the bias is zero, the PDF of the estimator is centered exactly at the true value of the parameter, and the estimator is termed unbiased. This implies that as the number of observations increases, the estimate of a parameter approaches the true value of the parameter. Also, if the variance of an estimator is small, the PDF of the estimator has a narrower width centered around its mean value. If the bias and variance of an estimator both tend to zero as the number of observations increases, then the estimator is termed consistent. However, not all estimators are consistent, since reducing the variance may sometimes increase the bias. A compromise between minimum variance and minimum bias is often achieved by minimizing the mean square error of the estimator. The problem with this, however, is that assignment of values to the bias and variance of the estimator depends on a knowledge of the PDF of the estimator which in turn is related to the PDF of the variates. In general, unless the variates are independent with a Gaussian PDF, it is not easy to establish the appropriate PDF's for the parameter estimators and hence the bias and variance cannot readily be evaluated.

For a specific problem, that is, a single terrain region which we describe by a set of N height values, it would be possible to carry out p-repeated measurements $[z_i]_1, \ldots, [z_i]_p$ of the N-variates at appropriate positions and then do a maximum likelihood estimate (mle) of the parameters along the lines suggested by Morrison[7] for a multinormal case. If his analysis is extended to our multivariate exponential case, though, we see that the mle for the mean and the covariance (and variance) estimators are not as straightforward. Both forms would have to be calculated for consistency. In our actual case, there are a multitude of regions each with different density functions; then the full mle analysis would have to be done for each case and again would become computationally unappealing.

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7. Morrison, D.F. (1976) <u>Multivariate Statistical Methods</u>, McGraw-Hill.

This discussion has pointed out why we did not conduct an assessment of the accuracy of the parameter estimators in any strict sense. This may not be the case for other situations and based on the particular circumstances, some assessment of the confidence limits and error bounds may cause the user to select some more appropriate forms for his estimators.

For our case of terrain heights, the number of subareas to be evaluated and the difficulty of obtaining successive measurements of our N-variates that would essentially be independent observations (given the digitized terrain maps available for analysis) led us to the use of a single N-variate sample. We thus had to make a number of assumptions. We have assumed that the mean height for each variate is the same and the variances of each of the heights about their means also are equal. With these assumptions it becomes reasonable to use the sample mean as the estimator for the mean height:

$$\hat{\bar{Z}} = (1/N) \sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}} Z(x_i, y_j)$$

and as the estimator for the variance, the sample variance:

$$\hat{\sigma}^2 = (1/N) \sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}} (Z_{ij} - \hat{\bar{Z}})^2$$

where the mean is given by its estimated value.

In constructing an estimator for the covariance matrix parameter γ_{zz}, the process is slightly more complicated. We assume that the height distribution is stationary over the individual subregions, although the relations for distinct areas may differ. This is a spatial equivalent to a time series that is stationary for some interval although not over a very long duration. This again requires the means and variances of the PDF's to be independent of location in the subregion and the covariances to be dependent only on a discrete separation factor for all spatial positions. This concept of position being introduced into the covariance relations has been discussed in the introduction but there are a number of aspects to this dependence which affect the form of the estimator, and hence should be given further attention.

The objective of our covariance analysis is to generate a functional form for the covariance between variates at two points of the grid structure that is based solely on their separation (that is, the process is stationary). In particular we define a correlation length T, where T represents the distance at which a normalized covariance function, c_{zz} has decreased to the value e^{-1} where $c_{zz} = \frac{\gamma_{zz}}{\sigma^2}$. Based on the grid, c_{zz} takes on discrete values and $c_{zz}(0) = 1$. For the calculation of T, we assume a parabolic fall-off with distance and carry out a least squares fit to the function. Separate x and y direction calculations were made and averaged for uniformity. In addition, the fall-off is rapid enough so the least squares fits were based on averaging the first $\sqrt{N}$ x and y separations respectively.

One implication of this procedure is that direct estimation of the entire covariance matrix was not carried out. As indicated, the short range covariances were used to determine a value for T and subsequently the complete covariance matrix was estimated by appropriate functional dependence $\hat{\gamma}_{zz}(m,n) = \hat{\sigma}^2 e^{-\tau_{mn}^2/\hat{T}^2}$. If we had not been interested in writing the covariance matrix in functional form we could have chosen the alternative approach of specifying $\hat{\gamma}'_{zz}(\tau)$ where as before $\tau_{mn}^2 = (x_m - x_n)^2 + (y_m - y_n)^2$ for every pair of variates in the terrain region.

The actual forms used are:

$$\hat{c}_{1zz}(k) = (1/N\hat{\sigma}^2)\left[\sum_{j=1}^{\sqrt{N}} \sum_{i=1}^{\sqrt{N}-k} Z_{ij} Z_{i+k\,j} - \hat{Z}\left[\sum_{j=1}^{\sqrt{N}} \sum_{i=k+1}^{\sqrt{N}-k} Z_{ij}\right] - \frac{k\hat{Z}^2}{\sqrt{N}}\right]$$

and

$$\hat{c}_{2zz}(k) = (1/N\hat{\sigma}^2)\left[\sum_{i=1}^{\sqrt{N}} \sum_{j=1}^{\sqrt{N}-k} Z_{ij} Z_{i\,j+k} - \hat{Z}\left[\sum_{i=1}^{\sqrt{N}} \sum_{j=k+1}^{\sqrt{N}-k} Z_{ij}\right] - \frac{k\hat{Z}^2}{\sqrt{N}}\right].$$

Under the assumptions that have been made, including stationarity over the region, the above forms for the estimators are similar to the form used in Jenkins and Watts.[3]

The least squares results for these values give us correlation lengths for the x and y directions and their average is used as the value of T that applies to the entire region. This then generates the complete covariance matrix.

The whole question of the estimation theory used in the particular case discussed in the report is complex. Care should be exercised by other users of this approach to apply estimators which are appropriate to their knowledge of the data they are analyzing. In the present case we are in the process of considering the use of multiple observations to establish better estimators for the parameters of the probability density. A further question of interest is the relation of the estimators to the outcome of the hypothesis testing procedure in which they are used. This aspect is also being addressed.

In addition to the estimators, the hypothesis testing also requires the forms of the PDF's being evaluated. The derivation of the two PDF's used in the present example constitutes the next topic.

3. MULTIVARIATE EXPONENTIAL PROBABILITY DENSITY

In our analysis of the terrain characteristics, we want to establish whether the heights within the selected regions are more closely distributed according to Gaussian or exponential probabilities. As the first step in this determination we selected estimators for the mean and variance of the distribution to be verified. Here we establish the general N - variate forms for the two PDF's in which we are interested. We are assuming that the N - variates are jointly distributed and all have the same mean and variance.

The form for the Gaussian multivariate PDF is well known.[2] Let $\underset{\sim}{Z}' = (z_1', z_2' \ldots z_N')$ be an N - dimensional random variable in vector form. Then

$$P_G(z_1', z_2', \cdots z_N') = \left(|\underset{\sim}{R}|^{1/2}(2\pi)^{N/2}\right)^{-1} e^{-(1/2)(\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1}(\underset{\sim}{z}' - \underset{\sim}{\mu})} \tag{6}$$

where $\underset{\sim}{R}$ is a positive definite symmetric matrix whose elements are constants and $\underset{\sim}{\mu}$ is a vector $\underset{\sim}{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$ whose elements are constants.

For our case $\mu_i = \mu$, $\forall$ i and the matrix $\underset{\sim}{R}$ is the covariance matrix for the variates where $R_{ij} = \sigma_{ij}$, $\sigma_{ii} = \sigma^2$, $\forall$ i and $\rho_{ij} = \frac{\sigma_{ij}}{\sigma^2}$. Thus

$$\underset{\sim}{R} = \sigma^2 \begin{pmatrix} 1 & \rho_{12} & \cdots\cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots\cdots & \rho_{2N} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho_{N1} & \rho_{N2} & \cdots\cdots & 1 \end{pmatrix} \tag{7}$$

Note that the quadratic form for this case is given by

$$Q^2 = (\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu}) .$$

To complete the analysis it is still necessary to derive an equivalent form for a multivariate exponential PDF. In that case the appropriate coefficients will be required. The general form would be

$$P_E(z_1', z_2', \ldots, z_N') = C_1 e^{-C_2[(\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1}(\underset{\sim}{z}' - \underset{\sim}{\mu})]^{1/2}}$$

where the quadratic form is defined as in the case for the Gaussian PDF. The question to be resolved before this can be used is what values must be assigned to C_1 and C_2.

The values can be determined by evaluating the zeroth and second moment integrals. For a probability density, the zeroth case must equal unity and the second moment equals the variance. The two integrals turn out to be similar in many aspects of their evaluation. We first discuss the zeroth moment case and then relate the second moment evaluation to that discussion.

The original integral has the form:

$$I_o = C_1 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dz_1' \, dz_2' \cdots dz_N' \, e^{-C_2[(\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu})]^{1/2}} .$$

We substitute $\underset{\sim}{Z} = (\underset{\sim}{z}' - \underset{\sim}{\mu})$ and obtain:

$$I_o = C_1 \int_{-\infty}^{\infty} \cdots \int_{\infty}^{\infty} dz_1 \, dz_2 \cdots dz_N \, e^{-C_2[\underset{\sim}{Z}^T \underset{\sim}{R}^{-1} \underset{\sim}{Z}]^{1/2}} .$$

The next simplification uses the fact that the quadratic form is unchanged by a coordinate transformation that diagonalizes the matrix $\underset{\sim}{R}^{-1}$. This transformation can be achieved by a relation involving the eigenvalues, λ_i $(i = 1, N)$ and corresponding eigenvectors of $\underset{\sim}{R}^{-1}$. Let $\underset{\sim}{A}$ be a matrix of eigenvectors, then $\underset{\sim}{Y} = \underset{\sim}{A}^T \underset{\sim}{Z}$ reduces the quadratic form and the integral to:[8]

$$I_o = C_1 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dy_1 \, dy_2 \cdots dy_N \, e^{-C_2[\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_N y_N^2]^{1/2}} .$$

For convenience we transform again to remove the eigenvalues from the exponent:

$$I_o = C_1 (\lambda_1 \lambda_2 \cdots \lambda_N)^{-1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dW_1 \, dW_2 \cdots dW_N \, e^{-C_2[W_1^2 + W_2^2 + \cdots W_N^2]^{1/2}}$$

Up to this point the relation between successive sets of variables has been clear. The actual evaluation of this integral requires an additional coordinate transformation. For that case, the Jacobian will have to be considered in detail as a result of the complexity of the relations between the two sets of variables. This is discussed in Appendix A and only the result will be shown here.

8. Hildebrand, F. B. (1965) Methods of Applied Mathematics, Prentice-Hall.

$$I_o = C_1(\lambda_1\lambda_2 \cdots \lambda_N)^{-1/2} \int_o^{\infty} r^{N-1} e^{-C_2 r} dr \int_o^{2\pi} d\theta_1 \int_o^{\pi} \sin^{N-2} \theta_{N-1} d\theta_{N-1}$$

$$\cdots \int_o^{\pi} \sin \theta_2 \, d\theta_2 \quad \text{or}$$

$$I_o = (\lambda_1\lambda_2 \cdots \lambda_N)^{-1/2} 2\pi C_1 C_2^{-N} (N-1)! \int_o^{\pi} \sin^{N-2} \theta_{N-1} \, d\theta_{N-1} \int_o^{\pi} \sin^{N-3} \theta_{N-2}$$

$$d\theta_{N-2} \cdots \int_o^{\pi} \sin \theta_2 \, d\theta_2 .$$

After considerable mathematical manipulation, a closed form solution is obtainable. The details of its evaluation are presented in Appendix B. The end result is

$$I_o = 2(\pi)^{N/2} (\lambda_1\lambda_2 \cdots \lambda_N)^{-1/2} \Gamma(N) C_1 C_2^{-N} \Big/ \Gamma(N/2) = 1 . \tag{8}$$

This gives us the first of the two relations for the two unknown quantities C_1 and C_2.

For the other relation we consider the second moment integral for each of the individual variates:

$$I_2(z_i') = \sigma^2 = C_1 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (z_i' - \mu_i)^2 e^{-C_2[(\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu})]^{1/2}} dz_1' \, dz_2' \cdots dz_N' .$$

Note that $N\sigma^2 = \sum_{i=1}^{N} I_2(z_i')$. After a change of variables we have:

$$N\sigma^2 = C_1 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (Z_1^2 + Z_2^2 + \cdots + Z_N^2) e^{-C_2[\underset{\sim}{Z}^T \underset{\sim}{R}^{-1} \underset{\sim}{Z}]^{1/2}} dZ_1 \cdots dZ_N .$$

As in the previous case we next decorrelate the variables using the eigenvector matrix transformation. Here though, there is the additional multiplying term which has to be considered. Note that $Z_1^2 + Z_2^2 + \cdots + Z_N^2 = \underset{\sim}{Z}^T \underset{\sim}{Z}$ and $\underset{\sim}{Z} = \underset{\sim}{A}\, \underset{\sim}{y}$.

Then

$$\underset{\sim}{Z}^T \underset{\sim}{Z} = (\underset{\sim}{y}^T \underset{\sim}{A}^T)(\underset{\sim}{A}\,\underset{\sim}{y}) = \underset{\sim}{y}^T \underset{\sim}{y} = y_1^2 + y_2^2 + \cdots + y_N^2 .$$

If we again use the relation $W_i^2 = \lambda_i y_i^2$ we have:

$$N\sigma^2 = C_1(\lambda_1 \lambda_2 \cdots \lambda_N)^{-1/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{W_1^2}{\lambda_1} + \frac{W_2^2}{\lambda_2} + \cdots + \frac{W_N^2}{\lambda_N} \right)$$

$$e^{-C_2[W_1^2 + W_2^2 + \cdots W_N^2]^{1/2}} dW_1 \cdots dW_N .$$

As before, the relations discussed in Appendix A are used to carry out a final transformation. The corresponding detailed evaluation of the integral is covered in Appendix B and the end result is

$$N^2\sigma^2 = 2(\pi)^{N/2}(\lambda_1\lambda_2 \cdots \lambda_N)^{-1/2}\Gamma(N+2)\sum_{i=1}^{N}\lambda_i^{-1}\,C_1 C_2^{-(N+2)}\Big/\Gamma(N/2). \tag{9}$$

This is the required second relation for the two unknowns.

To simplify we note that $\sum_{i=1}^{N}\lambda_i^{-1}$ is the trace of the inverse of $\underset{\sim}{R}^{-1}$ or $\sum_{i=1}^{N}\lambda_i^{-1} = \mathrm{Tr}\,(\underset{\sim}{R}^{-1})^{-1} = \mathrm{Tr}\,(\underset{\sim}{R}) = N\sigma^2$.

Substitution and rearrangement leads to the result:

$$C_1 = \left[|R|^{1/2}\,(2)^{\frac{N+1}{2}}(2\pi)^{\frac{N-1}{2}}\,\Gamma\left(\frac{N+1}{2}\right)\right]^{-1}(N+1)^{N/2},$$

$C_2^2 = (N+1)$; and finally

$$P_E(z_1', z_2', \cdots z_N') = \left[|R|^{1/2}\,(2)^{\frac{N+1}{2}}(2\pi)^{\frac{N-1}{2}}\,\Gamma\left(\frac{N+1}{2}\right)\right]^{-1}$$

$$(N+1)^{N/2}\,e^{-[(N+1)(\underset{\sim}{z}'-\underset{\sim}{\mu})^T\underset{\sim}{R}^{-1}(\underset{\sim}{z}'-\underset{\sim}{\mu})]^{1/2}}\,. \tag{10}$$

We have now arrived at the form of our N-variate exponential PDF. Some final comment as to $P_E(z_1', z_2', \ldots, z_N')$ satisfying the requirements of a probability density function may be appropriate. First, the zeroth moment relation was chosen to satisfy the requirement that the cumulative distribution over all space be unity. Secondly, since the quadratic form is non-negative and R is positive definite, this implies $0 \le P_E(z_1', z_2', \ldots, z_N') \le 1$. Thus, the criteria for a probability density are indeed satisfied by this form.

The determination of the appropriate expressions for the two probability densities in which we are interested allows us to proceed to the actual incorporation of these results into a test for their degree-of-fit to the data. This will be the subject of the next section.

4. HYPOTHESIS TEST

As has been discussed in the introduction, the concern for our particular case is to determine whether the various terrain subregions are better described by a Gaussian or exponential PDF. This binary decision could be generalized depending on the data involved and indeed, could even be made a more complex decision process if the appropriate criteria are available.

The form of the hypothesis test used here is based on the maximum a posteriori probability criterion.[4] This is equivalent to a minimum error probability criterion. We assign hypothesis H_1 to the Gaussian case and hypothesis H_o to the exponential. Then the likelihood ratio parameter,

$$\lambda \triangleq \frac{P_1 (Z_1, Z_2, \cdots Z_N)}{P_o (Z_1, Z_2, \cdots Z_N)} .$$

Let $P(H_o)$ be the probability that hypothesis H_o is true. Then the decision rule may be written as: Choose H_1 if

$$\lambda \geq \frac{P(H_o)}{1 - P(H_o)} .$$

For our case we assume that it is equally likely that hypothesis H_1 or H_o is true and the decision rule reduces to whether or not $\lambda \geq 1$. Note that it may be possible to alter the probability that H_o is true based on external evidence (such as the type of terrain).

This formulation represents the decision as a quotient of the two PDF's of interest. The next step is to derive the specific test ratio by introducing the two expressions for the N-variate joint PDF's from Eq. (6) and Eq. (10):

$$\frac{P_1}{P_o} = \left(\frac{|R|^{1/2} (2)^{\frac{N+1}{2}} (2\pi)^{\frac{N-1}{2}} \Gamma\left(\frac{N+1}{2}\right)}{(2\pi)^{N/2} |R|^{1/2} (N+1)^{N/2}} \right) \frac{e^{-\frac{Q^2}{2}}}{e^{-\sqrt{N+1}Q}}$$

where again $Q^2 = (\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu})$. After some manipulation this reduces to:

$$\lambda = \frac{P_1}{P_o} = \left(\frac{\Gamma\left(\frac{N+1}{2}\right) e^{(N+1)/2}}{\sqrt{\pi} \left(\frac{N+1}{2}\right)^{N/2}} \right) e^{-\frac{1}{2}(Q - \sqrt{N+1})^2} .$$

For convenience, we rewrite the test in logarithmic form and assert that H_1 is true if $\ell n\ \lambda \geq 0$. We then have the result: H_1 is true if

$$-\frac{1}{2}(Q - \sqrt{N+1})^2 \geq \frac{1}{2} \ell n\ \pi - \frac{1}{2}(N+1) + \frac{N}{2} \ell n \left(\frac{N+1}{2}\right) - \ell n \left[\Gamma\left(\frac{N+1}{2}\right)\right].$$

For the data sets used in the example of this report $N = 100$ so we can translate the decision parameter to a specific value for this case. We then say that the terrain heights in a given subregion are distributed according to a Gaussian PDF provided

$$9.22 \leq Q \leq 10.88$$

or

$$85.01 \leq (\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu}) \leq 118.37$$

and correspondingly, the heights are better described by the exponential PDF when $Q^2 > 118.37$ or $Q^2 < 85.01$

Since the hypothesis test has been formulated in terms of the quadratic form, Q^2, the characterization of the various subregions depends on the determination of that quantity for the respective data subsets. The next section addresses this question by pointing out some of the computational complications associated with these large, jointly distributed data sets.

5. POSITIVE DEFINITENESS OF COVARIANCE MATRIX AND PROBLEMS OF INVERTING LARGE MATRICES

It is important in these studies to explicitly demonstrate the positive definiteness of the quadratic form

$$Q^2 = (\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu}).$$

This is necessary for two reasons: (1) The positive definiteness of Q^2 insures that the functional forms for both the Gaussian and exponential N-variate PDF's are such that they satisfy the requirements of a true mathematical probability density function, (2) the quadratic form Q^2 contains the inverse of the covariance matrix $\underset{\sim}{R}^{-1}$. If the matrix $\underset{\sim}{R}$ is large there can be difficulties inverting it on a digital computer. If, however, $\underset{\sim}{R}$ is both positive definite and symmetric then special, particularly efficient algorithms exist for the inversion process.

We will first address this latter aspect. For the purposes of the proof we consider a more general form which involves the matrix $\underset{\sim}{R}$ rather than $\underset{\sim}{R}^{-1}$:

$$S^2 = \underset{\sim}{u}^T \underset{\sim}{R} \underset{\sim}{u}^*$$

where $\underset{\sim}{u}$ is any N-dimensional vector and (*) denotes complex conjugate. Furthermore, by definition, S^2 is positive definite if $\underset{\sim}{u}^T \underset{\sim}{R} \underset{\sim}{u}^* \geq 0$, where the equality holds if and only if $\underset{\sim}{u}$ is identically equal to the zero vector.

Now, as the first step in showing that S^2 is positive definite, we use the well known formula for the Fourier transform of a Gaussian function

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{i\omega t} \, dt = \sqrt{(\pi/\alpha)} \, e^{-\omega^2/4\alpha} \tag{11}$$

where Re $(\alpha) > 0$.

This is introduced as a result of our assumption that $R_{mn} = \sigma^2 e^{-\tau_{mn}^2/T^2}$

Relating the Gaussian form to our correlation case we note that the distance between the ordinates is equally spaced in the x - y plane so

$$(y_m - y_n)^2 = (\Delta y)^2 (m - n)^2$$

and likewise for the distance between abscissae

$$(x_m - x_n)^2 = (\Delta x)^2 (m - n)^2 .$$

Let $\alpha_1 = (1/4)(T/\Delta x)^2$ and $\alpha_2 = (1/4)(T/\Delta y)^2$ so that

$$e^{-(x_m - x_n)^2/T^2} = \sqrt{(\alpha_1/\pi)} \int_{-\infty}^{\infty} e^{-\alpha_1 t_1^2} e^{i(m-n)t_1} \, dt_1 \tag{12a}$$

and

$$e^{-(y_m - y_n)^2/T^2} = \sqrt{(\alpha_2/\pi)} \int_{-\infty}^{\infty} e^{-\alpha_2 t_2^2} e^{i(m-n)t_2} \, dt_2 . \tag{12b}$$

Thus

$$\begin{aligned}
S^2 &= \underset{\sim}{u}^T \underset{\sim}{R} \underset{\sim}{u}^* \\
&= \sum_{m=1}^{N} \sum_{n=1}^{N} u_m u_n^* \, e^{-(x_m - x_n)^2/T^2} e^{-(y_m - y_n)^2/T^2} \\
&= \sum_{m=1}^{N} \sum_{n=1}^{N} u_m u_n^* \left(\frac{\sqrt{\alpha_1 \alpha_2}}{\pi}\right) \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \, e^{i(m-n)t_1} e^{i(m-n)t_2} e^{-\alpha_1 t_1^2 - \alpha_2 t_2^2} \\
&= \left(\frac{\sqrt{\alpha_1 \alpha_2}}{\pi}\right) \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \, e^{-\alpha_1 t_1^2 - \alpha_2 t_2^2} \sum_{m=1}^{N} u_m e^{im(t_1+t_2)} \sum_{n=1}^{N} u_n^* e^{-in(t_1+t_2)}
\end{aligned} \tag{13}$$

This leads to the result

$$S^2 = \sqrt{\frac{\alpha_1 \alpha_2}{\pi^2}} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \, e^{-\alpha_1 t_1^2 - \alpha_2 t_2^2} \left| \sum_{\ell=1}^{N} u_1 \, e^{i\ell(t_1+t_2)} \right|^2 , \quad (14)$$

and thus $S^2 \geq 0$. It remains to be shown that $S^2 = 0$ iff $u = 0$. Trivially, if $\underset{\sim}{u} = 0$, $S^2 = 0$. Also, if $S^2 = \underset{\sim}{u}^T \underset{\sim}{R} \underset{\sim}{u} = 0$ then $\sum_{\ell=1}^{N} u_\ell e^{i\ell t} = 0$ follows from above where $t = t_1 + t_2$. Next, introducing successive complex conjugates of the exponential and integrating yields $\int_0^{2\pi} e^{i\ell t} e^{-ikt} dt = 2\pi \, \delta_{\ell k}$. Thus $\sum_{\ell=1}^{N} u_\ell e^{i\ell t} = 0$ implies $u_k = 0$ for $k = 1, 2, \cdots, N$ so that $S^2 = 0$ iff $\underset{\sim}{u}$ is equal to the zero vector and we have completed the proof that the quadratic form S^2 is positive definite. (Part of this proof is based upon the work of Fante.[9])

S^2 being positive definite implies that the eigenvalues λ_i of the matrix $\underset{\sim}{R}$ are all positive. Also, if λ_i is an eigenvalue of $\underset{\sim}{R}$, $i = 1, \ldots, N$, then the eigenvalues of $\underset{\sim}{R}^{-1}$ are λ_i^{-1} and these are all positive. Thus a quadratic form associated with $\underset{\sim}{R}^{-1}$ will be positive definite.[8] Since $Q^2 = (\underset{\sim}{z}' - \underset{\sim}{\mu})^T \underset{\sim}{R}^{-1} (\underset{\sim}{z}' - \underset{\sim}{\mu})$ is an example where the vectors are real, then Q^2 is positive definite.

The next topic is that of matrix inversion. The data sets analyzed in this report were such that $N = 100$, so that the matrix $\underset{\sim}{R}$ to be inverted was of order 100 by 100 elements having an extended range of magnitude. This combination can introduce complications to the inversion process.

The first technique used for inversion was a Cholesky reduction or square root method (Forsythe and Moler,[10] pp 14 et seq). The specific algorithm that was used required that $\underset{\sim}{R}$ be symmetric and positive definite. Both of these criteria are satisfied. Because the matrix $\underset{\sim}{R}$ was particularly large, round-off error accumulated during the Cholesky reduction was so large that the verification of the inverse based on $\underset{\sim}{R} \cdot \underset{\sim}{R}^{-1} = \underset{\sim}{I}$, the unit matrix, resulted in a matrix with off diagonal elements sometimes greater than one, instead of 10^{-10} or less which would be expected for the computer used.

To circumvent this loss in accuracy, the more conventional Gauss elimination method with pivoting was used (Forsythe and Moler[10]). Also, double-precision accuracy was retained in the computational procedure. When this also had unreliable results in some cases, a threshold value of 10^{-3} was set so that any element in the $\underset{\sim}{R}$ matrix less than or equal to the threshold was automatically set equal to zero. When these procedures were applied to the inversion problem, $\underset{\sim}{R}$ times $\underset{\sim}{R}^{-1}$

9. Fante, R. L., Private communication.

10. Forsythe, G., and Moler, C. B. (1967) Computer Solution of Linear Algebraic Systems, Prentice-Hall, pp 114 et seq.

yielded a matrix whose diagonal elements were equal to $1 \pm \epsilon$, where $\epsilon \sim 10^{-12}$ and the off-diagonal elements were $\sim\epsilon$.

There are numerous techniques for improving the accuracy when obtaining the inverse of a matrix. References Forsythe and Moler[10] and Berezin and Zhidkov[11] discuss at least a dozen methods. The choice of a particular method depends upon the peculiar properties of the original matrix (that is, the order of the matrix, positive definiteness, symmetry, size of matrix elements, and so on). Another factor in the choice of an inversion method depends upon trade-offs in computer storage requirements vs computation time.

In applying the procedure outlined in this report to other data sets and PDF's, the appropriate forms of the correlation of the variates have to be considered and individual decisions made as to inversion procedures where necessary.

6. STATISTICAL CHARACTERIZATION OF A SPECIFIC SITE

The specific example used in this report is a site in Eastern Massachusetts selected because electromagnetic scattering data from the terrain is available.[12] A rectangular area around the Discrete Address Beacon System (DABS) site was designated. The rectangular area was 43.3 km long and 42.8 km wide. The rectangular area was then subdivided into smaller rectangular cells, each with sides of 2050 m by 1852 m. Each cell is then further subdivided into a 10 by 10 grid of points. A data base of topographic elevations for this area is available at the Electromagnetic Compatibility Analysis Center (ECAC). This was prepared from Defense Mapping Agency (DMA) supplied digitized terrain maps at 1:250,000 scale size.

The statistical analysis techniques discussed in Sections 2 and 4 of this report were then applied to each cell, where now $[z_i']$ represents the set of topographic elevations at the grid points in each cell ($i = 1, 2, 3, \ldots 100$). In addition to parameter estimates, each cell has a geologic feature code associated with it. Table 1 indicates the particular geologic codes used.

Table 2 indicates the results of applying the statistical procedures indicated in this report to the DABS radar site. The rectangular region was subdivided into 528 cells, with twenty-two cells extending along the x-axis of the grid and twenty-four cells extending along the y-axis. The (x, y) coordinates of a particular cell are referenced with respect to the center of that cell. The origin of the (x, y) coordinate system is taken at the center of the cell in the extreme southwest corner

11. Berezin, I.S., and Zhidkov, N.P. (1965) Computing Methods, Vol. II, Pergamon Press.

12. McGarty, T.P. (1974) Models of Multipath Propagation Effects in a Ground-to-Air Surveillance System, AD 777241.

of the rectangular region (see Figure 1). In Table 2, the columns labelled X and Y refer to the (x, y) coordinates of a particular cell. The indexing of the cells is such that as one moves down the X and Y columns, increments equal to one cell length in the x-direction are taken with the y-coordinate fixed until the extreme eastern edge of the rectangular region is reached (see Figure 1). Then, the sequencing returns to the extreme western edge with the y-coordinate increased by one cell width and increments again are taken in the x-direction. This procedure is repeated until the extreme northeast corner is reached.

The column labelled IC contains the geologic codes. The column labelled SIG lists the mean heights. VAR refers to the variance in surface height, TA to the correlation length and TEST to the results of the hypothesis testing procedure. The units of length are in meters.

It may be recalled from Section 4 that (since TEST = Q^2 - 118.37) the heights in a given region are Gaussian distributed when $-33.36 \leq \text{TEST} \leq 0.0$ and exponentially distributed otherwise. When TEST = 0.0 and VAR is very small, the region is essentially a smooth surface (no roughness).

One observation that can be made is that when the magnitude of TEST is very large, so is the correlation length; TA for those cases is comparable to one-half the cell size or even larger. When that occurs, the determinant of the covariance matrix $\underset{\sim}{R}$ becomes very small. As a result it becomes more difficult to obtain an accurate inverse of $\underset{\sim}{R}$ due to rapid build-up of round-off error. A related problem with the results is that for some cases TEST < -118.37. This could only occur if the quadratic form, Q^2 were not positive definite. This contradiction of that theoretically imposed condition suggests the presence of additional machine-induced errors. Those results can not be valid. Further investigations of these aspects are being conducted, along with assessment of the parameter estimation techniques.

In order to use these results in the rough surface electromagnetic calculations, one additional aspect should be noted. For the types of geological features that describe the respective regions, data exists on the associated complex dielectric constants at microwave frequencies.[13,14]

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13. Lytle, R.J. (1974) Measurement of earth medium electrical characteristics: Techniques, results and applications, IEEE Trans. on Geoscience Electronics, GE-12:81.

14. Long, M.W. (1975) Radar Reflectivity of Land and Sea, Lexington Books.

Table 1. Geologic Code for the Site

0	-	Sea Water	4	-	Cleared Area
1	-	Woods	5	-	Town (City)
2	-	Swamp	6	-	Road
3	-	Lake	7	-	Intermittent Lake*

*The Massachusetts area has few true intermittent lakes. The code was used when many small lakes or ponds appear in a cell (rather than a large, all water cell).

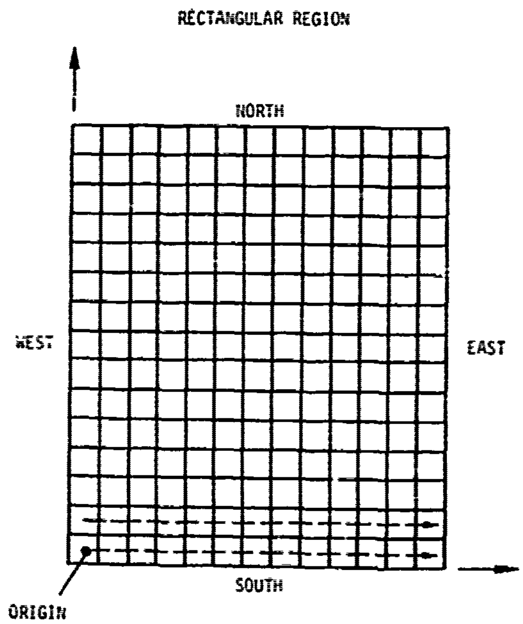

Figure 1. Subdivision of Site Into Sample Regions

Table 2. Details of the Statistical Analysis

X	Y	IC	SIG	VAR	TA	TEST
0.0	0.0	1	.141E+03	.786E+02	.537E+03	.630E+04
2051.1	0.0	1	.140E+03	.814E+02	.461E+03	.130E+04
4102.2	0.0	1	.141E+03	.183E+03	.422E+03	.259E+04
6153.3	0.0	4	.119E+03	.110E+03	.402E+03	.904E+03
8204.4	0.0	1	.116E+03	.266E+03	.431E+03	-.392E+04
10255.5	0.0	5	.111E+03	.177E+03	.436E+03	.114E+04
12306.6	0.0	4	.178E+03	.322E+03	.469E+03	-.326E+04
14357.7	0.0	1	.111E+03	.530E+02	.468E+03	.345E+04
16408.8	0.0	1	.109E+03	.240E+03	.460E+03	.310E+04
18459.9	0.0	1	.825E+02	.163E+03	.629E+03	.569E+05
20511.0	0.0	1	.647E+02	.537E+02	.412E+03	.479E+03
22562.1	0.0	1	.657E+02	.866E+02	.689E+03	-.377E+05
24613.2	0.0	2	.610E+02	.279E-09	.524E+03	0.
26664.3	0.0	5	.595E+02	.678E+01	.672E+03	.101E+06
28715.4	0.0	1	.597E+02	.617E+01	.560E+03	-.274E+05
30766.5	0.0	5	.520E+02	.940E+02	.701E+03	.373E+07
32817.6	0.0	1	.418E+02	.474E+01	.463E+03	.437E+04
34868.7	0.0	5	.411E+02	.715E+01	.417E+03	.350E+04
36919.8	0.0	1	.500E+02	.282E+02	.701E+03	.140E+00
38970.9	0.0	1	.571E+02	.287E+02	.525E+03	.109E+05
41022.0	0.0	5	.610E+02	.314E+02	.443E+03	.934E+03
43073.1	0.0	5	.595E+02	.231E+03	.461E+03	.190E+04
0.0	1851.6	1	.142E+03	.195E+03	.583E+03	.780E+04
2051.1	1851.6	1	.148E+03	.220E+03	.435E+03	-.249E+03
4102.2	1851.6	1	.169E+03	.549E+03	.560E+03	-.167E+05
6153.3	1851.6	1	.128E+03	.293E+03	.783E+03	.167E+00
8204.4	1851.6	1	.957E+02	.112E+03	.570E+03	-.609E+04
10255.5	1851.6	7	.945E+02	.119E+03	.453E+03	.686E+04
12306.6	1851.6	4	.142E+03	.300E+03	.667E+03	.309E+06
14357.7	1851.6	1	.126E+03	.443E+03	.657E+03	-.130E+06
16408.8	1851.6	1	.106E+03	.445E+02	.469E+03	.339E+04
18459.9	1851.6	1	.117E+03	.194E+03	.572E+03	.194E+06
20511.0	1851.6	1	.736E+02	.998E+02	.471E+03	.277E+04
22562.1	1851.6	2	.716E+02	.896E+02	.631E+03	.879E+05
24613.2	1851.6	2	.622E+02	.617E+01	.760E+03	.610E+03
26664.3	1851.6	2	.645E+02	.819E+02	.511E+03	.115E+05
28715.4	1851.6	5	.607E+02	.174E+02	.445E+03	.559E+04
30766.5	1851.6	5	.612E+02	.587E+02	.409E+03	-.122E+05
32817.6	1851.6	5	.485E+02	.530E+02	.470E+03	.130E+05
34868.7	1851.6	5	.486E+02	.944E+02	.548E+03	-.307E+06
36919.8	1851.6	4	.490E+02	.412E+02	.669E+03	.326E+06
38970.9	1851.6	4	.695E+02	.312E+02	.804E+03	-.691E+06
41022.0	1851.6	4	.673E+02	.587E+02	.373E+03	.307E+03
43073.1	1851.6	5	.663E+02	.145E+03	.371E+03	.107E+04
0.0	3703.2	7	.176E+03	.611E+03	.730E+03	.593E+06
2051.1	3703.2	7	.185E+03	.420E+03	.639E+03	.549E+05
4102.2	3703.2	7	.166E+03	.350E+03	.462E+03	.101E+04
6153.3	3703.2	4	.132E+03	.946E+02	.370E+03	.577E+03
8204.4	3703.2	1	.966E+02	.326E+03	.759E+03	.469E+07
10255.5	3703.2	1	.792E+02	.120E+03	.741E+03	.111E+07
12306.6	3703.2	1	.113E+03	.416E+03	.562E+03	-.605E+04
14357.7	3703.2	4	.147E+03	.272E+03	.366E+03	.412E+03
16408.8	3703.2	7	.119E+03	.339E+03	.587E+03	.111E+05
18459.9	3703.2	1	.101E+03	.425E+03	.443E+03	.703E+03
20511.0	3703.2	1	.745E+02	.814E+02	.432E+03	-.970E+03
22562.1	3703.2	1	.942E+02	.241E+03	.532E+03	-.135E+03
24613.2	3703.2	5	.657E+02	.264E+02	.552E+03	.215E+04
26664.3	3703.2	5	.739E+02	.619E+02	.443E+03	.627E+04
28715.4	3703.2	1	.565E+02	.534E+02	.454E+03	.702E+04
30766.5	3703.2	1	.500E+02	.456E+02	.360E+03	.071E+03
32817.6	3703.2	4	.478E+02	.474E+02	.460E+03	.171E+04
34868.7	3703.2	4	.450E+02	.407E+02	.470E+03	.451E+04
36919.8	3703.2	7	.519E+02	.197E+02	.516E+03	.122E+05
38970.9	3703.2	4	.695E+02	.713E+02	.611E+03	.270E+05
41022.0	3703.2	1	.680E+02	.887E+02	.449E+03	.611E+04
43073.1	3703.2	5	.647E+02	.330E+02	.357E+03	.452E+03

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	5554.8	7	.135E+03	.442E+03	.629E+03	.511E+06
2051.1	5554.8	1	.208E+03	.360E+03	.438E+03	.108E+04
4102.2	5554.8	7	.135E+03	.226E+03	.476E+03	.396E+04
6153.3	5554.8	4	.141E+03	.124E+03	.531E+03	-.344E+04
8204.4	5554.8	1	.101E+03	.459E+03	.708E+03	-.486E+08
10255.5	5554.8	1	.744E+02	.411E+02	.447E+03	-.237E+03
12306.6	5554.8	1	.337E+02	.277E+03	.557E+03	-.105E+05
14357.7	5554.8	4	.136E+03	.641E+03	.541E+03	.507E+04
16408.8	5554.8	7	.137E+03	.321E+03	.644E+03	.194E+06
18459.9	5554.8	1	.959E+02	.140E+03	.421E+03	.162E+04
20511.0	5554.8	1	[illegible]	.509E+02	.773E+03	.423E+07
22562.1	5554.8	1	[illegible]	.374E+02	.569E+03	.935E+05
24613.2	5554.8	1	.763E+02	.259E+03	.635E+03	.173E+07
26664.3	5554.8	5	.622E+02	.731E+02	.547E+03	-.111E+06
28715.4	5554.8	1	.549E+02	.149E+03	.428E+03	.129E+04
30766.5	5554.8	1	.438E+02	.172E+02	.375E+03	.709E+03
32817.6	5554.8	1	.383E+02	.856E+01	.533E+03	.258E+04
34868.7	5554.8	7	.394E+02	.926E+01	.476E+03	-.119E+03
36919.6	5554.8	1	.434E+02	.151E+02	.370E+03	.389E+03
38970.9	5554.8	1	.463E+02	.580E+02	.551E+03	-.367E+05
41022.0	5554.8	5	.487E+02	.121E+03	.652E+03	.225E+06
43073.1	5554.8	5	.532E+02	.147E+03	.452E+03	.768E+04
0.0	7406.4	1	.219E+03	.578E+03	.548E+03	-.920E+05
2051.1	7406.4	1	.171E+03	.273E+03	.363E+03	-.369E+03
4102.2	7406.4	1	.141E+03	.369E+03	.552E+03	-.183E+04
6153.3	7406.4	4	.112E+03	.893E+02	.643E+03	.528E+04
8204.4	7406.4	1	.101E+03	.298E+03	.867E+03	.935E+07
10255.5	7406.4	1	.822E+02	.940E+02	.538E+03	-.344E+04
12306.6	7406.4	1	.717E+02	.179E+02	.453E+03	.483E+04
14357.7	7406.4	1	.183E+03	.328E+03	.686E+03	.497E+06
16408.8	7406.4	1	.141E+03	.376E+03	.460E+03	-.189E+04
18459.9	7406.4	1	.124E+03	.310E+03	.621E+03	.531E+06
20511.0	7406.4	1	.876E+02	.637E+02	.453E+03	.515E+04
22562.1	7406.4	1	.907E+02	.508E+02	.366E+03	.789E+02
24613.2	7406.4	1	.824E+02	.220E+03	.644E+03	.122E+06
26664.3	7406.4	7	.559E+02	.142E+02	.431E+03	-.831E+03
28715.4	7406.4	1	.628E+02	.137E+03	.426E+03	.874E+03
30766.5	7406.4	1	.472E+02	.272E+02	.384E+03	.594E+04
32817.6	7406.4	1	.462E+02	.867E+02	.523E+03	.327E+05
34868.7	7406.4	1	.434E+02	.441E+02	.522E+03	.113E+05
36919.8	7406.4	2	.487E+02	.215E+02	.423E+03	.162E+04
38970.9	7406.4	2	.418E+02	.447E+01	.586E+03	.305E+05
41022.0	7406.4	5	.404E+02	.876E+01	.681E+03	.486E+06
43073.1	7406.4	5	.404E+02	.876E+01	.681E+03	.378E+06
0.0	9258.0	1	.237E+03	.883E+03	.729E+03	.775E+06
2051.1	9258.0	7	.179E+03	.660E+03	.746E+03	.259E+07
4102.2	9258.0	5	.179E+03	.428E+03	.545E+03	.176E+05
6153.3	9258.0	5	.967E+02	.876E+02	.376E+03	.181E+04
8204.4	9258.0	7	.894E+02	.164E+03	.424E+03	-.997E+04
10255.5	9258.0	7	.969E+02	.128E+03	.522E+03	.866E+04
12306.6	9258.0	1	.617E+02	.308E+03	.517E+03	.167E+05
14357.7	9258.0	1	.107E+03	.317E+03	.568E+03	-.166E+05
16408.8	9258.0	4	.115E+03	.364E+03	.689E+03	.133E+06
18459.9	9258.0	1	.136E+03	.151E+03	.623E+03	.395E+06
20511.0	9258.0	1	.869E+02	.116E+03	.358E+03	.469E+03
22562.1	9258.0	1	.101E+03	.155E+03	.457E+03	.159E+05
24613.2	9258.0	1	.722E+02	.233E+02	.653E+03	.181E+07
26664.3	9258.0	1	.721E+02	.546E+01	.368E+03	.288E+03
28715.4	9258.0	7	.609E+02	.172E+03	.437E+03	.118E+03
30766.5	9258.0	1	.511E+02	.675E+02	.678E+03	.138E+06
32817.6	9258.0	1	.610E+02	.118E+03	.475E+03	-.138E+04
34868.7	9258.0	1	.558E+02	.116E+03	.683E+03	.345E+06
36919.8	9258.0	2	.570E+02	.444E+02	.631E+03	.198E+06
38970.9	9258.0	2	.769E+02	.148E+02	.474E+03	.186E+03
41022.0	9258.0	1	.414E+02	.617E+01	.568E+03	-.274E+05
43073.1	9258.0	1	.488E+02	.916E+01	.358E+03	.386E+03

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	11109.6	7	.246E+03	.855E+03	.528E+03	-.759E+05
2051.1	11109.6	1	.153E+03	.234E+03	.648E+03	.115E+06
4102.2	11109.6	5	.109E+03	.152E+03	.550E+03	.919E+03
6153.3	11109.6	5	.106E+03	.290E+03	.640E+03	.220E+06
8204.4	11109.6	1	.106E+03	.187E+03	.518E+03	.319E+04
10255.5	11109.6	1	.112E+03	.928E+02	.531E+03	-.272E+04
12306.6	11109.6	4	.801E+02	.188E+03	.560E+03	-.230E+05
14357.7	11109.6	5	.871E+02	.549E+03	.688E+03	.160E+06
16408.8	11109.6	5	.111E+03	.115E+03	.325E+03	.244E+03
18459.9	11109.6	4	.127E+03	.9[illegible]1E+03	.558E+03	.6[illegible]E+04
20511.0	11109.6	1	.104E+03	.871E+03	.739E+03	.233E+07
22562.1	11109.6	4	.974E+02	.198E+03	.460E+03	.637E+03
24613.2	11109.6	7	.768E+02	.312E+02	.435E+03	.108E+04
26664.3	11109.6	5	.768E+02	.575E+02	.352E+03	.349E+03
28715.4	11109.6	1	.745E+02	.547E+02	.467E+03	.705E+04
30766.5	11109.6	1	.604E+02	.146E+03	.427E+03	-.438E+04
32817.6	11109.6	1	.646E+02	.922E+02	.483E+03	.154E+04
34868.7	11109.6	1	.572E+02	.116E+03	.673E+03	.179E+06
36919.8	11109.6	1	.611E+02	.554E+01	.362E+03	-.786E+03
38970.9	11109.6	1	.382E+02	.473E+02	.687E+03	-.635E+06
41022.0	11109.6	5	.427E+02	.111E+02	.545E+03	.337E+04
43073.1	11109.6	5	.464E+02	.393E+02	.520E+03	.150E+04
0.0	12961.2	7	.231E+03	.225E+04	.636E+03	-.685E+07
2051.1	12961.2	1	.135E+03	.138E+03	.485E+03	.529E+04
4102.2	12961.2	5	.100E+03	.565E+02	.372E+03	.443E+03
6153.3	12961.2	5	.127E+03	.208E+03	.441E+03	.786E+03
8204.4	12961.2	7	.119E+03	.812E+02	.776E+03	.207E+07
10255.5	12961.2	7	.114E+03	.508E+02	.448E+03	.552E+04
12306.6	12961.2	1	.766E+02	.492E+02	.584E+03	.154E+05
14357.7	12961.2	5	.911E+02	.157E+03	.642E+03	.226E+06
16408.8	12961.2	5	.993E+02	.226E+03	.454E+03	.455E+04
18459.9	12961.2	5	.896E+02	.119E+03	.468E+03	.593E+04
20511.0	12961.2	1	.106E+03	.611E+03	.420E+03	.472E+04
22562.1	12961.2	1	.771E+02	.177E+03	.359E+03	.351E+03
24613.2	12961.2	1	.858E+02	.5[illegible]4E+02	.368E+03	.412E+03
26664.3	12961.2	4	.841E+02	.136E+03	.334E+03	.603E+03
28715.4	12961.2	1	.695E+02	.219E+02	.467E+03	.341E+04
30766.5	12961.2	1	.627E+02	.112E+03	.635E+03	.361E+07
32817.6	12961.2	7	.637E+02	.561E+02	.521E+03	.313E+05
34868.7	12961.2	2	.656E+02	.685E+02	.534E+03	.663E+04
36919.8	12961.2	1	.514E+02	.463E+02	.567E+03	-.205E+05
38970.9	12961.2	7	.403E+02	.222E+02	.434E+03	-.508E+03
41022.0	12961.2	5	.432E+02	.164E+02	.375E+03	.634E+03
43073.1	12961.2	5	.544E+02	.507E+02	.569E+03	-.562E+04
0.0	14812.8	5	.206E+03	.193E+04	.725E+03	.289E+07
2051.1	14812.8	5	.179E+03	.319E+03	.467E+03	.347E+04
4102.2	14812.8	5	.114E+03	.278E+03	.552E+03	.351E+04
6153.3	14812.8	5	.140E+03	.395E+03	.664E+03	.309E+06
8204.4	14812.8	7	.105E+03	.897E+02	.570E+03	.248E+05
10255.5	14812.8	7	.955E+02	.574E+02	.467E+03	.833E+04
12306.6	14812.8	1	.752E+02	.923E+02	.429E+03	.791E+03
14357.7	14812.8	1	.869E+02	.110E+03	.554E+03	.135E+03
16408.8	14812.8	1	.764E+02	.293E+02	.452E+03	.835E+04
18459.9	14812.8	5	.860E+02	.349E+02	.680E+03	.657E+06
20511.0	14812.8	7	.900E+02	.913E+02	.435E+03	.226E+03
22562.1	14812.8	7	.788E+02	.208E+03	.452E+03	.888E+03
24613.2	14812.8	1	.728E+02	.553E+02	.444E+03	.768E+04
26664.3	14812.8	2	.839E+02	.826E+02	.356E+03	.517E+03
28715.4	14812.8	1	.649E+02	.597E+02	.565E+03	-.338E+04
30766.5	14812.8	2	.625E+02	.451E+02	.540E+03	.137E+05
32817.6	14812.8	7	.674E+02	.605E+02	.512E+03	-.481E+05
34868.7	14812.8	4	.607E+02	.616E+02	.544E+03	.292E+05
36919.8	14812.8	5	.531E+02	.389E+02	.527E+03	-.839E+05
38970.9	14812.8	5	.542E+02	.624E+02	.725E+03	.307E+07
41022.0	14812.8	5	.472E+02	.118E+03	.422E+03	-.152E+04
43073.1	14812.8	1	.604E+02	.479E+02	.434E+03	-.103E+04

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	16664.4	5	.171E+03	.427E+03	.873E+03	-.762E+08
2051.1	16664.4	5	.117E+03	.292E+03	.461E+03	.349E+04
4102.2	16664.4	5	.133E+03	.269E+03	.376E+03	.934E+03
6153.3	16664.4	5	.121E+03	.161E+03	.521E+03	.341E+03
8204.4	16664.4	1	.960E+02	.279E+02	.447E+03	-.428E+04
10255.5	16664.4	1	.101E+03	.397E+02	.541E+03	.234E+05
12306.6	16664.4	1	.949E+02	.239E+03	.806E+03	.116E+08
14357.7	16664.4	1	.724E+02	.838E+01	.341E+03	.413E+03
16408.8	16664.4	1	.748E+02	.207E+02	.341E+03	.348E+03
18459.9	16664.4	1	.775E+02	.677E+02	.397E+03	.131E+04
20511.0	16664.4	7	.770E+02	.510E+02	.418E+03	.250E+04
22562.1	16664.4	1	.732E+02	.148E+02	.432E+03	-.808E+03
24613.2	16664.4	7	.727E+02	.139E+02	.361E+03	-.954E+02
26664.3	16664.4	2	.726E+02	.285E+02	.523E+03	.173E+05
28715.4	16654.4	1	.792E+02	.229E+03	.525E+03	.342E+05
30766.5	16664.4	1	.719E+02	.617E+01	.560E+03	-.273E+05
32817.6	16664.4	2	.649E+02	.413E+02	.384E+03	.757E+04
34868.7	16664.4	5	.604E+02	.227E+02	.382E+03	.347E+04
36919.8	16664.4	5	.548E+02	.398E+02	.433E+03	.872E+03
38970.9	16664.4	5	.627E+02	.447E+02	.365E+03	.588E+03
41022.0	16664.4	5	.440E+02	.820E+02	.787E+03	.343E+07
43073.1	16664.4	1	.619E+02	.845E+02	.538E+03	.155E+05
0.0	18516.0	7	.154E+03	.543E+03	.647E+03	.107E+06
2051.1	18516.0	5	.125E+03	.165E+03	.654E+03	-.165E+07
4102.2	18516.0	1	.157E+03	.703E+02	.466E+03	.172E+04
6153.3	18516.0	1	.118E+03	.274E+03	.532E+03	-.233E+02
8204.4	18516.0	1	.110E+03	.166E+03	.544E+03	.661E+04
10255.5	18516.0	1	.107E+03	.826E+02	.343E+03	.328E+03
12306.6	18516.0	1	.987E+02	.143E+03	.591E+03	.404E+05
14357.7	18516.0	1	.738E+02	.831E+02	.560E+03	-.319E+04
16408.8	18516.0	1	.748E+02	.452E+02	.360E+03	.474E+03
18459.9	18516.0	4	.955E+02	.295E+03	.438E+03	.550E+03
20511.0	18516.0	1	.909E+02	.181E+03	.424E+03	-.381E+05
22562.1	18516.0	7	.765E+02	.506E+02	.568E+03	-.134E+06
24613.2	18516.0	1	.682E+02	.382E+02	.432E+03	-.112E+04
26664.3	18516.0	1	.655E+02	.178E+02	.462E+03	.114E+04
28715.4	18516.0	1	.977E+02	.984E+02	.457E+03	.986E+03
30766.5	18516.0	5	.736E+02	.161E+02	.416E+03	.199E+04
32817.6	18516.0	1	.583E+02	.151E+02	.434E+03	.224E+03
34868.7	18516.0	5	.624E+02	.601E+02	.322E+03	-.354E+03
36919.8	18516.0	5	.442E+02	.196E+02	.490E+03	.477E+04
38970.9	18516.0	5	.565E+02	.623E+02	.437E+03	-.186E+03
41022.0	18516.0	1	.776E+02	.533E+02	.643E+03	.522E+05
43073.1	18516.0	1	.444E+02	.803E+02	.635E+03	.537E+06
0.0	20367.6	7	.158E+03	.700E+02	.543E+03	.128E+05
2051.1	20367.6	1	.138E+03	.234E+03	.370E+03	.507E+02
4102.2	20367.6	1	.166E+03	.104E+03	.443E+03	.723E+03
6153.3	20367.6	1	.148E+03	.243E+03	.675E+03	.225E+06
8204.4	20367.6	7	.122E+03	.151E+03	.480E+03	.101E+04
10255.5	20367.6	1	.105E+03	.315E+03	.770E+03	-.614E+07
12306.6	20367.6	1	.103E+03	.661E+02	.441E+03	.245E+04
14357.7	20367.6	1	.713E+02	.366E+02	.350E+03	.426E+03
16408.8	20367.6	1	.785E+02	.206E+03	.706E+03	.111E+07
18459.9	20367.6	5	.101E+03	.201E+03	.453E+03	.138E+05
20511.0	20367.6	1	.977E+02	.282E+03	.657E+03	-.201E+06
22562.1	20367.6	1	.754E+02	.592E+02	.481E+03	-.201E+04
24613.2	20367.6	1	.782E+02	.521E+02	.610E+03	.245E+05
26664.3	20367.6	1	.636E+02	.206E+03	.421E+03	.147E+04
28715.4	20367.6	5	.739E+02	.296E+03	.647E+03	.274E+06
30766.5	20367.6	5	.660E+02	.850E+02	.353E+03	.344E+03
32817.6	20367.6	5	.724E+02	.164E+03	.453E+03	.280E+03
34868.7	20367.6	5	.606E+02	.157E+03	.397E+03	.773E+03
36919.8	20367.6	1	.790E+02	.884E+01	.656E+03	.462E+05
38970.9	20367.6	1	.431E+02	.128E+02	.615E+03	.161E+06
41022.0	20367.6	1	.322E+02	.134E+02	.569E+03	-.748E+04
43073.1	20367.6	7	.396E+02	.494E+02	.599E+03	.102E+06

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	22219.2	7	.186E+03	.126E+04	.528E+03	.262E+05
2051.1	22219.2	1	.166E+03	.277E+03	.652E+03	.107E+06
4102.2	22219.2	1	.159E+03	.915E+02	.451E+03	.106E+04
6153.3	22219.2	1	.131E+03	.368E+03	.652E+03	.409E+06
8204.4	22219.2	7	.117E+03	.921E+02	.534E+03	.217E+04
10255.5	22219.2	1	.119E+03	.225E+03	.550E+03	-.246E+05
12306.6	22219.2	1	.955E+02	.275E+03	.633E+03	-.623E+06
14357.7	22219.2	1	.774E+02	.740E+02	.439E+03	.977E+03
16408.8	22219.2	1	.70[illegible]E+02	.111E+03	.774E+03	.307E+07
18459.9	22219.2	5	.103E+03	.101E+03	.675E+03	.333E+06
20511.0	22219.2	1	.904E+02	.355E+03	.761E+03	.742E+07
22562.1	22219.2	1	.703E+02	.122E+03	.547E+03	-.627E+05
24613.2	22219.2	1	.814E+02	.220E+03	.346E+03	.362E+03
26664.3	22219.2	7	.685E+02	.251E+03	.438E+03	.784E+03
28715.4	22219.2	7	.595E+02	.678E+01	.672E+03	.103E+06
30766.5	22219.2	5	.576E+02	.132E+03	.367E+03	.533E+03
32817.6	22219.2	5	.707E+02	.387E+03	.518E+03	.727E+04
34868.7	22219.2	5	.661E+02	.196E+03	.576E+03	.155E+05
36919.8	22219.2	5	.303E+02	.916E+01	.638E+03	-.291E+06
38970.9	22219.2	5	.341E+02	.156E+02	.525E+03	.138E+05
41022.0	22219.2	1	.34[illegible]E+02	.292E+02	.557E+03	-.209E+04
43073.1	22219.2	1	.463E+02	.357E+02	.445E+03	.955E+04
0.0	24070.8	1	.240E+03	.894E+03	.644E+03	.521E+05
2051.1	24070.8	1	.183E+03	.540E+03	.629E+03	.275E+06
4102.2	24070.8	1	.157E+03	.296E+03	.739E+03	.116E+07
6153.3	24070.8	1	.132E+03	.181E+03	.594E+03	.229E+06
8204.4	24070.8	1	.119E+03	.153E+03	.451E+03	-.182E+04
10255.5	24070.8	1	.107E+03	.286E+03	.572E+03	.499E+05
12306.6	24070.8	1	.869E+02	.390E+03	.615E+03	.159E+06
14357.7	24070.8	1	.966E+02	.326E+03	.770E+03	.458E+07
16408.8	24070.8	7	.636E+02	.149E+02	.583E+03	.251E+04
18459.9	24070.8	7	.947E+02	.151E+03	.647E+03	.799E+05
20511.0	24070.8	1	.925E+02	.255E+03	.362E+03	.529E+03
22562.1	24070.8	1	.754E+02	.815E+02	.475E+03	.617E+03
24613.2	24070.8	7	.838E+02	.396E+03	.664E+03	.418E+06
26664.3	24070.8	1	.731E+02	.102E+03	.471E+03	.136E+05
28715.4	24070.8	1	.667E+02	.370E+02	.429E+03	-.721E+03
30766.5	24070.8	5	.559E+02	.253E+02	.375E+03	.721E+03
32817.6	24070.8	5	.419E+02	.878E+01	.341E+03	.411E+03
34868.7	24070.8	5	.419E+02	.592E+02	.450E+03	.196E+04
36919.8	24070.8	5	.415E+02	.104E+02	.524E+03	.121E+05
38970.9	24070.8	5	.329E+02	.393E+02	.642E+03	.171E+06
41022.0	24070.8	5	.372E+02	.839E+02	.432E+03	.115E+04
43073.1	24070.8	7	.422E+02	.180E+02	.529E+03	-.900E+04
0.0	25922.4	7	.224E+03	.636E+03	.587E+03	.675E+05
2051.1	25922.4	1	.174E+03	.650E+03	.454E+03	.141E+04
4102.2	25922.4	1	.179E+03	.283E+03	.779E+03	.357E+07
6153.3	25922.4	1	.125E+03	.316E+03	.527E+03	.128E+05
8204.4	25922.4	1	.103E+03	.310E+03	.635E+03	.161E+07
10255.5	25922.4	7	.822E+02	.561E+02	.525E+03	.204E+05
12306.6	25922.4	1	.104E+03	.160E+03	.561E+03	.574E+04
14357.7	25922.4	1	.934E+02	.278E+03	.620E+03	.339E+06
16408.8	25922.4	1	.683E+02	.153E+03	.576E+03	.153E+05
18459.9	25922.4	7	.763E+02	.119E+03	.635E+03	.169E+07
20511.0	25922.4	7	.934E+02	.259E+03	.473E+03	-.229E+04
22562.1	25922.4	1	.905E+02	.812E+02	.754E+03	.195E+07
24613.2	25922.4	7	.77[illegible]E+02	.255E+03	.524E+03	.969E+04
26664.3	25922.4	1	.797E+02	.116E+03	.752E+03	.593E+06
28715.4	25922.4	1	.560E+02	.516E+02	.480E+03	.848E+03
30766.5	25922.4	5	.586E+02	.176E+03	.628E+03	.600E+05
32817.6	25922.4	7	.732E+02	.627E+02	.548E+03	-.114E+05
34868.7	25922.4	5	.294E+02	.493E+02	.613E+03	.655E+05
36919.8	25922.4	5	.307E+02	.779E+02	.538E+03	.252E+05
38970.9	25922.4	5	.268E+02	.528E+02	.718E+03	.292E+07
41022.0	25922.4	5	.709E+02	.147E+03	.451E+03	-.109E+04
43073.1	25922.4	7	.428E+02	.123E+02	.338E+03	.428E+03

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	27774.0	7	.22[illegible]	.34[illegible]+03	.371E+03	.285E+03
2051.1	27774.0	1	.1[illegible]	.[illegible]E+03	.737E+03	.628E+07
4102.2	27774.0	1	.156E+03	.64[illegible]E+03	.556E+03	-.197E+03
6153.3	27774.0	1	.111E+03	.61[illegible]E+03	.742E+03	.289E+07
8204.4	27774.0	1	[illegible]	.845E+02	.512E+03	.230E+05
10255.5	27774.0	1	.[illegible]28E+02	.206E+03	.518E+03	.123E+03
12306.6	27774.0	1	.121E+03	.20[illegible]E+03	.673E+03	-.211E+07
14357.7	27774.0	1	.113E+03	.15[illegible]E+03	.427E+03	-.451E+03
16408.8	27774.0	1	.[illegible]11[illegible]E+02	.177E+03	.613E+03	.164E+06
18459.9	27774.0	5	.562E+02	.357E+02	.456E+03	-.474E+04
20511.0	27774.0	5	[illegible]	.238E+02	.775E+03	.281E+07
22562.1	27774.0	1	[illegible]	.753E+02	.477E+03	.557E+03
24613.2	27774.0	7	.50[illegible]E+02	.20[illegible]E+02	.354E+03	.355E+03
26664.3	27774.0	1	.744E+02	.818E+02	.415E+03	.176E+04
28715.4	27774.0	1	.49[illegible]E+02	.471E+02	.611E+03	.133E+06
30766.5	27774.0	7	.47[illegible]E+02	.116E+03	.428E+03	.366E+03
32817.6	27774.0	1	.32[illegible]E+02	.156E+03	.635E+03	.647E+06
34868.7	27774.0	5	.45[illegible]E+02	.673E+02	.458E+03	-.578E+04
36919.8	27774.0	5	[illegible]	.286E+02	.347E+03	.322E+03
38970.9	27774.0	5	[illegible]	.752E+02	.436E+03	-.354E+03
41022.0	27774.0	7	.43[illegible]E+02	.155E+03	.458E+03	.387E+04
43073.1	27774.0	7	.287[illegible]E+02	.174E+03	.433E+03	.199E+04
0.0	29625.6	1	[illegible]	.470E+03	.772E+03	-.222E+07
2051.1	29625.6	1	.135E+03	.690E+03	.648E+03	.239E+06
4102.2	29625.6	1	.1[illegible]	.155E+03	.637E+03	.932E+06
6153.3	29625.6	1	.927[illegible]	.749E+02	.435E+03	.579E+03
8204.4	29625.6	1	.106E+03	.291E+03	.566E+03	-.472E+05
10255.5	29625.6	1	.140E+03	.587E+03	.631E+03	.686E+05
12306.6	29625.6	1	.116E+03	.262E+03	.550E+03	-.173E+05
14357.7	29625.6	1	.394[illegible]	.271E+03	.756E+03	.534E+07
16408.8	29625.6	1	.76[illegible]E+02	.107E+03	.727E+03	.196E+06
18459.9	29625.6	1	.61[illegible]	.728E+02	.520E+03	.208E+05
20511.0	29625.6	1	.57[illegible]E+02	.137E+02	.415E+03	.337E+04
22562.1	29625.6	1	.633E+02	.199E+02	.445E+03	.469E+04
24613.2	29625.6	1	.555E+02	.490E+02	.383E+03	.814E+03
26664.3	29625.6	1	.64[illegible]E+02	.591E+02	.410E+03	-.306E+04
28715.4	29625.6	7	.5[illegible]	.59[illegible]E+02	.746E+03	-.713E+07
30766.5	29625.6	7	.371E+02	.815E+02	.356E+03	.470E+03
32817.6	29625.6	7	.51[illegible]	.61[illegible]E+02	.696E+03	.279E+06
34868.7	29625.6	4	.542E+02	.111E+03	547E+03	-.462E+06
36919.8	29625.6	5	.360E+02	.479E+02	.425E+03	-.253E+04
38970.9	29625.6	5	.395E+02	.442E+02	.651E+03	.336E+06
41022.0	29625.6	1	.477E+02	.114E+03	.580E+03	.256E+06
43073.1	29625.6	7	.414E+ [illegible]	.842E+02	.374E+03	.758E+03
0.0	31477.2	1	.213E+03	.420E+03	.476E+03	.154E+04
2051.1	31477.2	1	.13[illegible]E+03	.971E+03	.769E+03	.897E+07
4102.2	31477.2	1	.125E+03	.902E+03	.558E+03	.609E+04
6153.3	31477.2	1	.130E+03	.770E+03	.560E+03	.915E+03
8204.4	31477.2	1	.122E+03	.286E+03	.437E+03	.132E+04
10255.5	31477.2	1	.145E+0[illegible]	.237E+03	.604E+03	.799E+05
12306.6	31477.2	1	.90[illegible]	.278E+03	.675E+03	.206E+06
14357.7	31477.2	1	.701E+02	.116E+03	.551E+03	-.263E+05
16408.8	31477.2	1	.766E+02	.178E+03	.446E+03	.183E+04
18459.9	31477.2	1	.707E+02	.155E+03	.657E+03	-.674E+05
20511.0	31477.2	1	.61[illegible]	.386E+02	.451E+03	.884E+04
22562.1	31477.2	1	.629E+02	.125E+02	.415E+03	.245E+04
24613.2	31477.2	1	.625E+02	.776E+02	.434E+03	.627E+03
26664.3	31477.2	1	.671E+0[illegible]	.78[illegible]E+02	.477E+03	.138E+04
28715.4	31477.2	1	.499E+02	.716E+02	.366E+03	.198E+03
30766.5	31477.2	7	.324E+02	.971E+02	.448E+03	.824E+04
32817.6	31477.2	7	.712E+02	.231E+03	.552E+03	-.475E+05
34868.7	31477.2	5	.56[illegible]	.1[illegible]1E+03	.363E+03	.354E+03
36919.8	31477.2	1	.412E+02	.328E+02	.374E+03	.778E+03
38970.9	31477.2	1	.49[illegible]	.706E+02	.463E+03	.774E+04
41022.0	31477.2	1	.657E+02	.114E+03	.511E+03	.303E+04
43073.1	31477.2	7	.541E+02	.113E+03	.525E+03	.794E+04

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	33328.8	1	.236E+03	.579E+03	.435E+03	.888E+03
2051.1	33328.8	1	.155E+03	.844E+03	.571E+03	.114E+05
4102.2	33328.8	1	.162E+03	.818E+03	.762E+03	-.194E+08
6153.3	33328.8	1	.168E+03	.457E+03	.666E+03	.595E+06
8204.4	33328.8	1	.126E+03	.638E+03	.728E+03	.401E+07
10255.5	33328.8	1	.103E+03	.497E+03	.615E+03	.165E+06
12306.6	33328.8	1	.796E+02	.314E+03	.437E+03	.457E+03
14357.7	33328.8	1	.719E+02	.773E+02	.407E+03	.336E+04
16408.8	33328.8	1	.946E+02	.856E+02	.441E+03	.107E+04
18459.9	33328.8	1	.743E+02	.174E+03	.861E+03	.135E+08
20511.0	33328.8	1	.579E+02	.137E+02	.332E+03	.453E+03
22562.1	33328.8	7	.653E+02	.306E+02	.437E+03	-.414E+03
24613.2	33328.8	1	.632E+02	.755E+02	.512E+03	.203E+04
26664.3	33328.8	1	.814E+02	.125E+03	.429E+03	-.112E+04
28715.4	33328.8	7	.504E+02	.186E+03	.776E+03	.312E+07
30766.5	33328.8	1	.397E+02	.151E+02	.425E+03	-.232E+04
32817.6	33328.8	7	.649E+02	.116E+03	.509E+03	.326E+04
34868.7	33328.8	2	.744E+02	.489E+03	.513E+03	.154E+04
36919.8	33328.8	2	.409E+02	.524E+02	.430E+03	.169E+04
38970.9	33328.8	2	.429E+02	.434E+02	.355E+03	.321E+03
41022.0	33328.8	1	.541E+02	.113E+03	.441E+03	.189E+04
43073.1	33328.8	1	.493E+02	.904E+02	.689E+03	-.460E+06
0.0	35180.4	1	.241E+03	.783E+03	.522E+03	.104E+05
2051.1	35180.4	1	.165E+03	.861E+03	.682E+03	.358E+06
4102.2	35180.4	1	.181E+03	.188E+03	.542E+03	.428E+04
6153.3	35180.4	1	.151E+03	.619E+03	.788E+03	.166E+08
8204.4	35180.4	1	.126E+03	.375E+03	.541E+03	-.751E+04
10255.5	35180.4	1	.961E+02	.448E+03	.713E+03	.124E+07
12306.6	35180.4	1	.839E+02	.178E+03	.527E+03	-.110E+04
14357.7	35180.4	1	.963E+02	.181E+03	.449E+03	.279E+04
16408.8	35180.4	1	.112E+03	.156E+03	.640E+03	.173E+06
18459.9	35180.4	1	.748E+02	.179E+03	.791E+03	.804E+07
20511.0	35180.4	1	.577E+02	.204E+02	.443E+03	.152E+04
22562.1	35180.4	1	.611E+02	.572E+02	.372E+03	.954E+03
24613.2	35180.4	5	.671E+02	.483E+02	.631E+03	.883E+06
26664.3	35180.4	5	.777E+02	.606E+03	.638E+03	-.521E+06
28715.4	35180.4	7	.509E+02	.243E+03	.675E+03	.137E+06
30766.5	35180.4	1	.485E+02	.453E+02	.552E+03	.849E+03
32817.6	35180.4	1	.722E+02	.533E+03	.596E+03	.875E+05
34868.7	35180.4	2	.805E+02	.181E+03	.468E+03	.204E+04
36919.8	35180.4	2	.490E+02	.130E+03	.344E+03	.605E+03
38970.9	35180.4	7	.443E+02	.541E+02	.377E+03	.663E+03
41022.0	35180.4	7	.529E+02	.147E+03	.614E+03	.161E+06
43073.1	35180.4	1	.533E+02	.282E+03	.423E+03	-.336E+04
0.0	37032.0	1	.266E+03	.356E+03	.443E+03	.882E+04
2051.1	37032.0	4	.197E+03	.967E+03	.558E+03	-.951E+04
4102.2	37032.0	1	.202E+03	.210E+03	.415E+03	.564E+03
6153.3	37032.0	1	.142E+03	.820E+03	.779E+03	.649E+07
8204.4	37032.0	1	.106E+03	.464E+03	.636E+03	.172E+07
10255.5	37032.0	7	.867E+02	.913E+02	.411E+03	.106E+04
12306.6	37032.0	1	.132E+03	.450E+03	.718E+03	.418E+07
14357.7	37032.0	1	.109E+03	.164E+03	.455E+03	.266E+05
16408.8	37032.0	1	.113E+03	.152E+03	.445E+03	.119E+05
18459.9	37032.0	1	.853E+02	.214E+03	.611E+03	.122E+08
20511.0	37032.0	1	.595E+02	.175E+02	.366E+03	.506E+03
22562.1	37032.0	5	.545E+02	.571E+02	.416E+03	.213E+04
24613.2	37032.0	5	.547E+02	.278E+02	.450E+03	.697E+04
26664.3	37032.0	5	.471E+02	.432E+02	.526E+03	-.720E+06
28715.4	37032.0	7	.388E+02	.165E+03	.545E+03	.792E+04
30766.5	37032.0	1	.529E+02	.234E+03	.556E+03	-.176E+05
32817.6	37032.0	7	.924E+02	.394E+03	.435E+03	.156E+04
34868.7	37032.0	7	.102E+03	.632E+03	.460E+03	-.103E+04
36919.8	37032.0	1	.596E+02	.437E+03	.750E+03	.262E+07
38970.9	37032.0	7	.419E+02	.987E+01	.413E+03	-.916E+03
41022.0	37032.0	1	.498E+02	.104E+03	.762E+03	.466E+07
43073.1	37032.0	7	.777E+02	.420E+03	.375E+03	.118E+04

Table 2. Details of the Statistical Analysis (Cont)

X	Y	IC	SIG	VAR	TA	TEST
0.0	38883.6	5	.274E+03	.344E+03	.446E+03	-.110E+05
2051.1	38883.6	1	.234E+03	.759E+03	.620E+03	.165E+06
4102.2	38883.6	1	.205E+03	.347E+03	.445E+03	.477E+04
6153.3	38883.6	1	.152E+03	.366E+03	.381E+03	.790E+03
8204.4	38883.6	1	.114E+03	.524E+03	.675E+03	.272E+06
10255.5	38883.6	1	.117E+03	.659E+03	.736E+03	.190E+07
12306.6	38883.6	1	.166E+03	.359E+03	.454E+03	.383E+04
14357.7	38883.6	1	.124E+03	[illegible]	.584E+03	.304E+04
16408.8	38883.6	7	.100E+03	.196E+03	.416E+03	.185E+04
18459.9	38883.6	1	.936E+02	.201E+03	.739E+03	.372E+08
20511.0	38883.6	7	.267E+02	.279E+03	.729E+03	.990E+06
22562.1	38883.6	5	[illegible]	.306E+03	.620E+03	.269E+06
24613.2	38883.6	5	.441E+02	.256E+02	.461E+03	.640E+04
26664.3	38883.6	5	.416E+02	.994E+01	.347E+03	.558E+03
28715.4	38883.6	5	.342E+02	.795E+02	.646E+03	.226E+05
30766.5	38883.6	7	.577E+02	.112E+03	.580E+03	.902E+04
32817.6	38883.6	1	.955E+02	.376E+03	.771E+03	-.384E+06
34868.7	38883.6	1	.117E+03	.471E+03	.453E+03	.182E+04
36919.8	38883.6	1	.572E+02	[illegible]	.457E+03	-.110E+05
38970.9	38883.6	1	.519E+02	.271E+02	.460E+03	.428E+04
41022.0	38883.6	7	.514E+02	.247E+02	.596E+03	.187E+06
43073.1	38883.6	1	.567E+02	.387E+02	.550E+03	.501E+05
0.0	40735.2	1	.281E+03	.542E+03	.439E+03	.112E+04
2051.1	40735.2	1	.232E+03	.964E+03	.792E+03	.294E+07
4102.2	40735.2	1	.196E+03	.961E+02	.410E+03	.684E+03
6153.3	40735.2	1	.142E+03	.683E+03	.423E+03	.521E+03
8204.4	40735.2	1	.112E+03	.296E+03	.474E+03	-.122E+03
10255.5	40735.2	1	.127E+03	.461E+03	.490E+03	.460E+04
12306.6	40735.2	1	.136E+03	.567E+03	.538E+03	.214E+05
14357.7	40735.2	7	[illegible]	.547E+03	.817E+03	.102E+08
16408.8	40735.2	1	.107E+03	.413E+03	.409E+03	.924E+03
18459.9	40735.2	1	.760E+02	.233E+03	.469E+03	.386E+04
20511.0	40735.2	1	[illegible]	.361E+03	.731E+03	.361E+07
22562.1	40735.2	5	.681E+02	.130E+03	.744E+03	.220E+07
24613.2	40735.2	1	.505E+02	.335E+02	.526E+03	.542E+05
26664.3	40735.2	5	.490E+02	.815E+02	.699E+03	.549E+06
28715.4	40735.2	5	.799E+02	.114E+03	.742E+03	.186E+07
30766.5	40735.2	1	.616E+02	.349E+02	.414E+03	.177E+03
32817.6	40735.2	1	.974E+02	.257E+03	.411E+03	-.730E+0[illegible]
34868.7	40735.2	1	.827E+02	.221E+03	.690E+03	.619E+0[illegible]
36919.8	40735.2	1	.733E+02	.764E+03	.541E+03	-.209E+05
38970.9	40735.2	1	.702E+02	.590E+03	.578E+03	.679E+04
41022.0	40735.2	1	.612E+02	.306E+02	.693E+03	.335E+06
43073.1	40735.2	1	.524E+02	.758E+02	.611E+03	.329E+05
0.0	42586.8	1	.227E+03	.120E+04	.449E+03	.108E+05
2051.1	42586.8	1	.207E+03	.399E+03	.366E+03	.210E+03
4102.2	42586.8	1	.217E+03	.475E+03	.456E+03	-.199E+04
6153.3	42586.8	1	.170E+03	.339E+03	.345E+03	.642E+03
8204.4	42586.8	1	.133E+03	.542E+03	.631E+03	.966E+06
10255.5	42586.8	1	.125E+03	.583E+02	.582E+03	.168E+04
12306.6	42586.8	1	.152E+03	.413E+03	.645E+03	.100E+06
14357.7	42586.8	1	.140E+03	.531E+03	.776E+03	-.171E+06
16408.8	42586.8	1	.956E+02	.706E+03	.618E+03	.480E+06
18459.9	42586.8	1	.659E+02	.126E+03	.593E+03	.559E+05
20511.0	42586.8	1	.571E+02	.856E+01	.428E+03	-.124E+04
22562.1	42586.8	5	.633E+02	.107E+03	.497E+03	-.864E+04
24613.2	42586.8	5	.569E+02	.194E+02	.533E+03	.485E+04
26664.3	42586.8	5	.392E+02	.114E+03	.573E+03	.256E+05
28715.4	42586.8	7	.484E+02	.801E+02	.771E+03	.240E+07
30766.5	42586.8	1	.813E+02	.164E+03	.612E+03	.213E+06
32817.6	42586.8	1	.669E+02	.739E+03	.513E+03	.158E+04
34868.7	42586.8	1	.649E+02	[illegible]	.431E+03	-.994E+03
36919.8	42586.8	1	.791E+02	.232E+03	.714E+03	.793E+06
38970.9	42586.8	7	.799E+02	.535E+03	.635E+03	.279E+07
41022.0	42586.8	1	.636E+02	.886E+02	.618E+03	.271E+06
43073.1	42586.8	4	.779E+02	.793E+02	.457E+03	.257E+03

7. CONCLUSIONS

In Section 6 we pointed out that the site chosen for characterization is one for which electromagnetic scattering data is available. We have also discussed how the procedures of this report were driven by the intention to use the results in rough surface scattering calculations. Briefly, the complex dielectric properties of a geological area, together with knowledge of the mean height, variance in height, correlation length, and PDF for the surface heights is sufficient for the calculation of the specular and diffuse reflection of a radar wave from the surface. The results of the calculations using the statistical properties will then be compared to the data of McGarty.[12]

The comparison involves several aspects. Computer programs are being developed which incorporate a number of existing models of rough surface scattering cross sections,[5,6,15,16] into a more general model for calculating the coherent and incoherent power detected by a monopulse receiving antenna. The scattering geometry that is used in the model is shown in Figure 2. The terrain statistics of Table 2 will be used as one input to this program and the results compared with the experimental values.

Once the characterization approach to terrain scattering analysis yields reasonable agreement with data, the scattering at additional sites can be characterized and these results used to evaluate site terrain effects in detection and tracking of low flying targets.

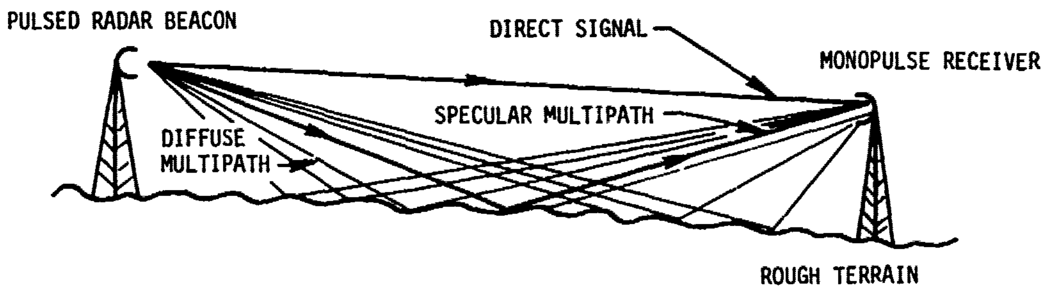

Figure 2. Reflection of Radar Waves From Rough Terrain

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Appendix A

The Form of the Jacobian for Some Sets of Related Variables

In the evaluation of multiple integrals it is often helpful to carry out a change of variables. This process requires the determination of the Jacobian expressing the relation between two sets of variables.

In particular we are concerned with the sets of variables $(w_1, w_2, \ldots, w_N)$ and $(r, \theta_1, \theta_2, \ldots, \theta_{N-1})$ and the set of appropriate connective relations. If we can express these relations such that:

(1) w_1 is a function of $r, w_2, \ldots, w_N$ with r alone varying;

(2) w_2 is a function of $r, \theta_1, w_3, \ldots, w_N$ with θ_1 alone varying;

. . . .

(N) w_N is a function of $r, \theta_1, \theta_2, \ldots, \theta_{N-1}$ with θ_{N-1} alone varying;

then the Jacobian can be expressed as the product of the several partial differential coefficients for those relations*

$$J = \frac{\partial w_1}{\partial r} \cdot \frac{\partial w_2}{\partial \theta_1} \cdot \frac{\partial w_3}{\partial \theta_2} \cdot \ldots \cdot \frac{\partial w_N}{\partial \theta_{N-1}} .$$

*See Edwards, J. (1954) A Treatise on the Integral Calculus, Vol. II, Chelsea Publishing Company.

As an illustration consider:

(x, y, z) and (r, θ, ϕ) where

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad \text{and } z = r\cos\theta$$

which we then write as

$$x = \sqrt{r^2 - y^2 - z^2} \qquad z = r\cos\theta \qquad y = r\sin\theta\sin\phi$$

each successive relation containing an additional member of the second set of variables. Then

$$J = \frac{\partial x}{\partial r} \cdot \frac{\partial z}{\partial \theta} \cdot \frac{\partial y}{\partial \phi} = -r^2 \sin\theta .$$

For the particular sets of variables with which we are concerned we can establish a similar series of relations:

(1) $w_2 = r\cos\theta_{N-1}$

(2) $w_3 = r\sin\theta_{N-1}\cos\theta_{N-2}$

(3) $w_4 = r\sin\theta_{N-1}\sin\theta_{N-2}\cos\theta_{N-3}$

.

.

.

(N-1) $w_N = r\sin\theta_{N-1}\sin\theta_{N-2}\cdots\sin\theta_2\cos\theta_1$

(N) $w_1 = r\sin\theta_{N-1}\sin\theta_{N-2}\cdots\sin\theta_2\sin\theta_1.$

This has the proper form if we rewrite the relation for w_1 in its equivalent form:

$$w_1 = \sqrt{r^2 - w_2^2 - w_3^2 - \cdots - w_N^2}.$$

Then the Jacobian for our variables is

$$J = (-1)^{N-1} r^{N-1} \sin^{N-2}\theta_{N-1} \sin^{N-3}\theta_{N-2} \cdots \sin^2\theta_3 \sin\theta_2 .$$

Appendix B

Evaluation of the Zeroth and Second Moment Integrals of a Multivariate Exponential Probability Density

The purpose of this section is to present the mathematical analysis required to evaluate some particular multiple integrals involving sinusoidal forms. Specifically, the integrations of interest are those required for the evaluation of the zeroth and second moment integrals for the selected N-variate exponential probability density.

The zeroth case, the simpler one, involves successive integrals of descending powers of sine terms. The result is also of value in the more complicated second moment evaluation.

Consider that $\int_0^{\pi} \sin^M \theta \; d\theta = 2 \int_0^{\pi/2} \sin^M \theta \; d\theta$ and that we have the relation:

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx$$

where $B(p, q)$ is the Beta function.

Then for $x = \sin^2 \theta$ and $dx = 2 \sin \theta \cos \theta \; d\theta$

$$B(p, q) = 2 \int_0^{\pi/2} (\sin \theta)^{2p-2} (\cos^2 \theta)^{q-1} \sin \theta \cos \theta \; d\theta$$

and

$$\int_0^{\pi} \sin^M \theta \; d\theta = B\left(\frac{M+1}{2}, \frac{1}{2}\right).$$

For the zeroth moment case we have:

$$I_o(\theta) = \int_0^{\pi} \sin^{N-2} \theta_{N-1} \, d\theta_{N-1} \int_0^{\pi} \sin^{N-3} \theta_{N-2} \, d\theta_{N-2} \cdots \int_0^{\pi} \sin^2 \theta_3 \, d\theta_3 \int_0^{\pi} \sin \theta_2 \, d\theta_2 .$$

So,

$$I_o(\theta) = \left[B\left(\frac{N-1}{2}, \frac{1}{2}\right) B\left(\frac{N-2}{2}, \frac{1}{2}\right) \cdots B\left(\frac{3}{2}, \frac{1}{2}\right) B\left(1, \frac{1}{2}\right) \right].$$

Now

$$B\left(\frac{M+1}{2}, \frac{1}{2}\right) = \Gamma\left(\frac{M+1}{2}\right) \Gamma\left(\frac{1}{2}\right) \Big/ \Gamma\left(\frac{M+2}{2}\right)$$

which leads to

$$I_o(\theta) = \left[\frac{(\pi)^{\frac{N-2}{2}} \; \Gamma\left(\frac{N-1}{2}\right) \; \Gamma\left(\frac{N-2}{2}\right) \cdots \Gamma\left(\frac{3}{2}\right) \Gamma(1)}{\Gamma\left(\frac{N}{2}\right) \; \Gamma\left(\frac{N-1}{2}\right) \cdots \Gamma\left(\frac{3}{2}\right) \quad .} \right] = \frac{(\pi)^{\frac{N-2}{2}}}{\Gamma\left(\frac{N}{2}\right)} .$$

When combined with the remaining terms the final value for the total zeroth moment is:

$$I_o = \frac{2\,(\pi)^{N/2} \, C_1 \, \Gamma(N)}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_N} \; C_2^{\,N} \, \Gamma(N/2)} = 1 .$$

The evaluation of the second moment is more complex. Since the form is

$$I_2 = \frac{C_1}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_N}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{W_1^{\,2}}{\lambda_1} + \frac{W_2^{\,2}}{\lambda_2} + \cdots + \frac{W_N^{\,2}}{\lambda_N} \right)$$

$$e^{-C_2\left[W_1^{\,2} + W_2^{\,2} + \cdots W_N^{\,2}\right]^{1/2}} \; dW_1 \cdots dW_N$$

we must make use of both the Jacobian and the connective relations between the two sets of variables as described in Appendix A. The new form is:

$$I_2 = \left[\frac{C_1}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_N}} \int_0^\infty r^{N+1} e^{-C_2 r} dr\right]$$

$$\cdot \left[\frac{1}{\lambda_2} \int_0^\pi \sin^{N-2} \theta_{N-1} \cos^2 \theta_{N-1} d\theta_{N-1} \int_0^\pi \sin^{N-3} \theta_{N-2} d\theta_{N-2} \cdots\right.$$

$$\int_0^\pi \sin^2 \theta_3 d\theta_3 \int_0^\pi \sin \theta_2 \theta_2 d\theta_2 \int_0^{2\pi} d\theta_1$$

$$+ \frac{1}{\lambda_3} \int_0^\pi \sin^N \theta_{N-1} d\theta_{N-1} \int_0^\pi \sin^{N-3} \theta_{N-2} \cos \theta_{N-2} d\theta_{N-2} \int_0^\pi \sin^{N-4} \theta_{N-3} d\theta_{N-3} \cdots$$

$$\int_0^\pi \sin \theta_2 \theta_2 d\theta_2 \int_0^{2\pi} d\theta_1$$

$$+ \cdots\cdot$$

$$+ \frac{1}{\lambda_N} \int_0^\pi \sin^N \theta_{N-1} d\theta_{N-1} \int_0^\pi \sin^{N-1} \theta_{N-2} d\theta_{N-2} \cdots \int_0^\pi \sin^3 \theta_2 d\theta_2 \int_0^{2\pi} \cos^2 \theta_1 d\theta_1$$

$$\left. + \frac{1}{\lambda_1} \int_0^\pi \sin^N \theta_{N-1} d\theta_{N-1} \int_0^\pi \sin^{N-1} \theta_{N-2} d\theta_{N-2} \cdots \int_0^\pi \sin^3 \theta_2 d\theta_2 \int_0^{2\pi} \sin^2 \theta_1 d\theta_1\right] .$$

In a similar fashion to the previous case we make use of the relations:

$$\int_0^\infty r^{N+1} e^{-C_2 r} dr = \frac{(N+1)!}{C_2^{N+2}} ,$$

$$\int_0^\pi \sin^M \theta \, d\theta = B\left(\frac{M+1}{2}, \frac{1}{2}\right) ,$$

and

$$\int_0^\pi \sin^M \theta \cos^2 \theta \, d\theta = B\left(\frac{M+1}{2}, \frac{3}{2}\right) .$$

These substitutions lead to:

$$I_2 = \left(\frac{2\, C_1\, (N+1)!}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_N}\, C_2^{N+2}} \right) I_2(\theta)$$

where

$$\begin{aligned} I_2(\theta) = \Bigg[& \frac{B(\frac{N-1}{2}, \frac{3}{2}) B(\frac{N-2}{2}, \frac{1}{2}) \cdots B(1, \frac{1}{2}) B(\frac{1}{2}, \frac{1}{2})}{\lambda_2} \\ & + \frac{B(\frac{N+1}{2}, \frac{1}{2}) B(\frac{N-2}{2}, \frac{3}{2}) \cdots B(1, \frac{1}{2}) B(\frac{1}{2}, \frac{1}{2})}{\lambda_3} \\ & + \frac{B(\frac{N+1}{2}, \frac{1}{2}) B(\frac{N}{2}, \frac{1}{2}) B(\frac{N-3}{2}, \frac{3}{2}) \cdots B(1, \frac{1}{2}) B(\frac{1}{2}, \frac{1}{2})}{\lambda_4} \\ & + \cdots\cdots \\ & + \frac{B(\frac{N+1}{2}, \frac{1}{2}) B(\frac{N}{2}, \frac{1}{2}) \cdots B(\frac{5}{2}, \frac{1}{2}) B(1, \frac{3}{2}) B(\frac{1}{2}, \frac{1}{2})}{\lambda_{N-1}} \\ & + \frac{B(\frac{N+1}{2}, \frac{1}{2}) B(\frac{N}{2}, \frac{1}{2}) \cdots B(\frac{5}{2}, \frac{1}{2}) B(2, \frac{1}{2}) B(\frac{1}{2}, \frac{3}{2})}{\lambda_N} \\ & + \frac{B(\frac{N+1}{2}, \frac{1}{2}) B(\frac{N}{2}, \frac{1}{2}) \cdots B(\frac{5}{2}, \frac{1}{2}) B(2, \frac{1}{2}) B(\frac{3}{2}, \frac{1}{2})}{\lambda_1} \Bigg] . \end{aligned}$$

$I_2(\theta)$ is thus a complicated sum of products of Beta functions. It is then necessary to evaluate this term. The procedure involves showing that all the individual products of Beta functions are equal. The last term ($\lambda_i = \lambda_1$) is a special case where none of the integrands contain a cosine term. This will be considered separately. In the remaining terms, there are (N - 1) integrands, one of which has a cosine-squared factor multiplying some power of a sine function and the others are expressable as powers of various sine functions. In each successive integrand one additional sine-squared factor is present compared to the preceding product. This suggests that by looking at the general K^{th} and $(K+1)^{st}$ terms and showing equivalence we can demonstrate that all are identical.

The general term is:

$$[I_2(\theta)]_i = \lambda_{i+1}^{-1} B(\frac{N-1}{2}, \frac{3}{2}) \prod_{j=i}^{i-1} B(\frac{N-j+2}{2}, \frac{1}{2}) \prod_{j=i+1}^{N-1} B(\frac{N-j}{2}, \frac{1}{2}) .$$

If we consider the differences in form for the K^{th} and $(K+1)^{st}$ terms we find two in each:

$$\Delta_K = B(\frac{N-K}{2}, \frac{3}{2}) B(\frac{N-K-1}{2}, \frac{1}{2})$$

and

$$\Delta_{K+1} = B(\frac{N-K+2}{2}, \frac{1}{2}) B(\frac{N-K-1}{2}, \frac{3}{2}) .$$

The remaining product terms are the same in both cases. Thus the two terms will be equivalent if we can show that $\Delta_K \overset{?}{=} \Delta_{K+1}$. To show this we write the Beta functions as Gamma functions:

$$\Delta_K = \frac{\Gamma(\frac{N-K}{2}) \Gamma(\frac{3}{2}) \Gamma(\frac{N-K-1}{2}) \Gamma(\frac{1}{2})}{\Gamma(\frac{N-K-3}{2}) \Gamma(\frac{N-K}{2})} = \frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{N-K-1}{2})}{\Gamma(\frac{N-K+1}{2})}$$

and

$$\Delta_{K+1} = \frac{\Gamma(\frac{N-K+2}{2}) \Gamma(\frac{1}{2}) \Gamma(\frac{N-K-1}{2}) \Gamma(\frac{3}{2})}{\Gamma(\frac{N-K+3}{2}) \Gamma(\frac{N-K+2}{2})} = \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{3}{2}) \Gamma(\frac{N-K-1}{2})}{\Gamma(\frac{N-K+3}{2})} .$$

So $\Delta_K = \Delta_{K+1}$ and, in general, consecutive terms will be equivalent. For the special case of the N^{th} ($\lambda_i = \lambda_1$) term, consider that for $K = (N-1)$ the general expression is:

$$[I_2(\theta)]_{N-1} = \frac{B(\frac{1}{2}, \frac{3}{2})}{\lambda_N} \prod_{j=1}^{N-2} B(\frac{N-j+2}{2}, \frac{1}{2})$$

while for K = N

$$[I_2(\theta)]_N = \frac{B(\frac{3}{2}, \frac{1}{2})}{\lambda_1} \prod_{j=1}^{N-2} B(\frac{N-j+2}{2}, \frac{1}{2}) .$$

These two expressions differ only in the form of the initial factors. Since the Beta function is symmetric with respect to its arguments (as is clearly seen when it is written using Gamma functions) these final terms are also equivalent and we have shown that all the products of Beta functions determining $I_2(\theta)$ are equivalent.

Since all the Beta function products are equivalent, we then have:

$$I_2 = \left(\frac{2\, C_1\, (N+1)!\, \sum_{i=1}^{N} \lambda_i^{-1}}{\sqrt{\lambda_1 \lambda_2 \cdots \lambda_N}\, C_2^{N+2}} \right) B(\frac{N-1}{2}, \frac{3}{2}) \prod_{j=1}^{N-2} B(\frac{N-j-1}{2}, \frac{1}{2}) .$$

To simplify we expand the Beta function and obtain:

$$B(\frac{N-1}{2}, \frac{3}{2}) \prod_{j=1}^{N-2} B(\frac{N-j-1}{2}, \frac{1}{2}) = (\pi)^{N/2} [2\, \Gamma(\frac{N+2}{2})]^{-1} = (\pi)^{N/2} [N\, \Gamma(N/2)]^{-1}$$

and hence:

$$I_2 = \frac{2\, C_1\, \Gamma(N+2) \sum_{i=1}^{N} \lambda_i^{-1} (\pi)^{N/2}}{N \sqrt{\lambda_1 \lambda_2 \cdots \lambda_N}\, C_2^{N+2}\, \Gamma(N/2)} = N\sigma^2 .$$

This completes the evaluation of the multiple integrals required to specify the unknown coefficients of the multivariate exponential probability distribution function for use with the data sets.

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