

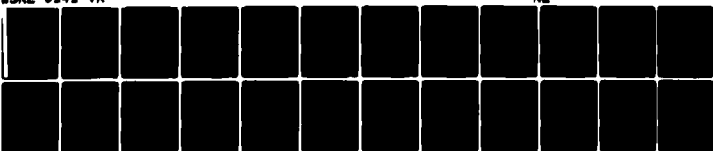
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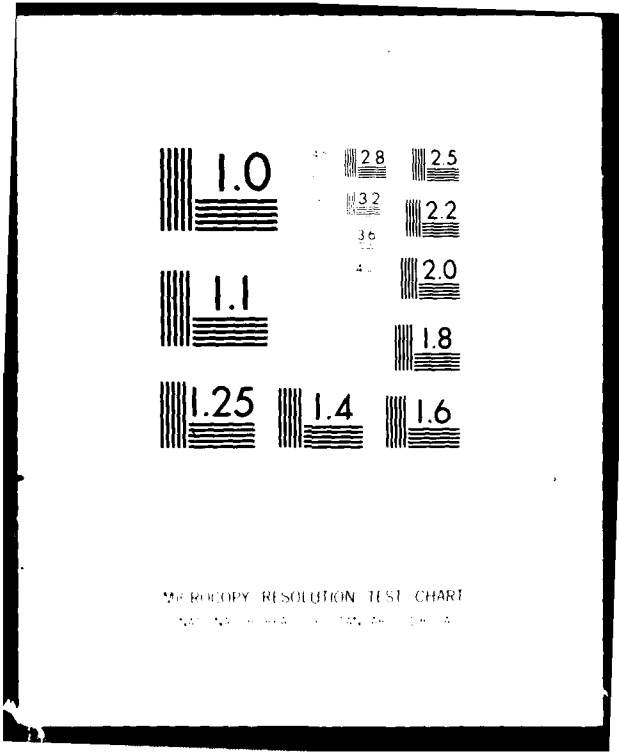
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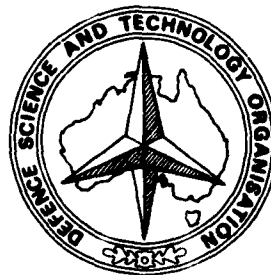


TECHNICAL REPORT

WSRL-0141-TR

AN ITERATIVE APPROACH TO THE DECONVOLUTION OF THE
NARROWBAND OUTPUTS OF A CONVENTIONAL BEAMFORMER

D.A. GRAY



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WSRL-0141-TR

AN ITERATIVE APPROACH TO THE DECONVOLUTION OF THE NARROWBAND
OUTPUTS OF A CONVENTIONAL BEAMFORMER

D.A. Gray

S U M M A R Y

In conventional beamforming the estimated noise power wave-number spectrum is the convolution of the incident distribution with the array response. By removing the effect of the array response an improved estimate of the incident wave-number spectrum may be obtained. In particular the method of progressive substitutions has been widely used to deconvolve the array window. By assuming all the incident signals to be a linear superposition of a finite number of plane waves this method is extended to deconvolve the narrowband beam outputs of a conventional beamformer. Two methods are used, the first to deconvolve the complex beam amplitudes and the second to deconvolve the beam powers. The convergence of these proposed iterative solutions is proved and the relationship of the limits to some recently proposed quadratic estimators is shown.

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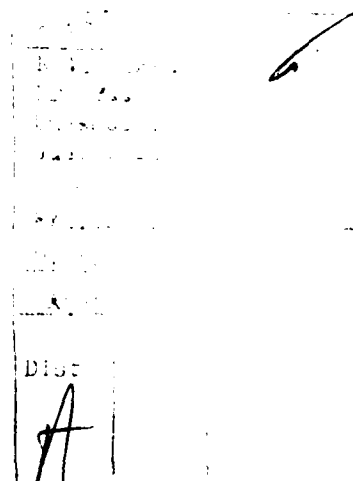
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Figure 1. N beams steered in the physical region at $d = \frac{\lambda}{4}$



1. INTRODUCTION

In conventional beamforming it is well known that the estimated complex wave-number spectrum of the incident noise distribution is the convolution of the true wave-number spectrum with the Fourier transform of the array window. Since the array window is zero outside a finite region, a consequence of this convolution is to limit the wave-number (and hence angular) resolution. Deconvolution techniques can be used to remove (or minimise) the effect of the array window and consequently improve the estimate of the wave-number spectrum. Unfortunately deconvolution methods do not always provide a unique estimate of the wave-number spectrum and constraints need to be imposed to obtain a unique solution.

As early as 1954 Bracewell and Roberts(ref.1) proposed an iterative deconvolution technique which progressively sharpens the estimated distribution and ultimately converges to a unique solution - termed the 'principal solution'. This solution has the additional property that it contains no wave-number components which give a zero output when convolved with the array response. Although this 'principal solution' has greater resolution than the conventional beamformer estimates, its resolution is still limited and it also suffers from ringing, i.e. an increase in the sidelobe levels. Its effect in terms of the spatial autocorrelation function can readily be seen. For a continuous line array the method replaces the Bartlett (or triangular) window, multiplying the spatial autocorrelation distribution by the square window. The effect of this is well known; the resolution is doubled at the expense of increased sidelobe levels.

However, the technique has received considerable attention. Axelrod et al (ref.1) have shown how the method can alternatively be formulated in terms of a least squares expansion of the wave-number power spectrum using a set of Fourier coefficients. Anderson and Tittle(ref.1), by defining the principal solution mathematically have shown that the principal solution only retains a finite number of terms in the expansion. Furthermore all these terms correspond to components which oscillate in wave-number (i.e. essentially angle) below a certain frequency.

Further work by Wilson(ref.1) and in particular McDonough(ref.1) has developed the method and applied it to some numerical examples. McDonough has also generalised the iterative method of Bracewell and Roberts to an array of arbitrary geometry.

The techniques discussed all suffer from one common limitation; i.e. the deconvolution is effected using the beam powers. Thus the relative phase information of either beams or receiver outputs is lost. A method overcoming this limitation has been proposed by Nuttall(ref.1) whereby the incident field (i.e. the noise field), in a similar manner to Axelrod et al, is expanded in a finite set of Fourier coefficients. However, instead of forming a least squares approximation using beam powers, Nuttall's method uses the crosspower spectral matrix of the receiver outputs. Yen(ref.2) has also adopted this approach, applied it to a linear array and obtained an expression for the coefficients of the Fourier expansion in terms of the relative phase delays of the receivers.

The expansion of the noise field in a Fourier series in wave-number (i.e. $\frac{d}{\lambda} \cos \theta$) also deserves some comments. In this approach the lowest order term represents the isotropic noise component and to represent anisotropic noise fields higher order Fourier terms are required. In the limit a single plane wave, represented by a delta function distribution of the noise field, requires an infinite number of coefficients for exact representation. The basis of this paper is to adopt what can be conceptually thought of as being almost the exact opposite to the above approach, that is, the incident field at the frequency of

interest is assumed to be composed of N independent plane waves (i.e. N delta functions) from predetermined directions and the field is assumed homogeneous. This reduces the estimation problem to one of estimating only the complex beam amplitudes or powers using an array of K discrete receivers.

This assumption has been successfully exploited by d'Assumpcao (ref.2) to derive some new quadratic estimators based on maximum likelihood and minimum variance criteria.

Under these assumptions the phase delays corresponding to the selected directions may be incorporated in the deconvolution process which is represented in matrix formulation since both the number of receivers and arrivals is finite. The method, although formulated for a two-dimensional noise field may readily be extended to three dimensions. In Section 2 an iterative deconvolution technique, which is a matrix extension of the method used by Bracewell and Roberts, is formulated under the above assumptions. Two similar techniques are proposed which use the array response at the frequency of interest to deconvolve either the complex beam outputs or the beam powers. The assumption that the directions of the N plane waves are known enables the total leakage in any beam to be estimated and subtracted out. The iterative method is used to successively refine and subtract out the leakage in all beams from the other beams. It is then shown in Section 3 that the limits of these iterative methods converge to a generalization of some particular quadratic estimators derived in reference 2

Recently, Yen (ref.3) using a similar assumption for the incident noise field has used the Prony method to estimate the direction and powers of the incident N plane waves for a line array. The expressions derived in this report together with the examples of Section 5 can be used to show an equivalence of the power deconvolution method to the linearized part of the Prony method used by Yen.

In Section 4 the suitability of these estimators in the presence of noise with an unknown covariance matrix is discussed and some general results are proved.

Finally some examples of the application of the method to a line array of K equispaced receivers are given.

2. FORMULATION OF THE ITERATIVE METHOD

The assumption that the incident field consists of N independent plane waves enables x_j , $j = 1, 2, \dots, K$ (the complex spectral amplitude of the output of the j -th receiver at a frequency f) to be expressed as a linear combination of y_k (the complex spectral amplitude of the k -th plane wave signal at some arbitrary reference point). The physical geometry of the array determines this particular combination. It follows that:

$$x_j = \sum_{k=1}^N V_{jk} y_k \quad (1)$$

where $V_{jk} = \exp(i\phi_{jk})$ and ϕ_{jk} are the phase delays corresponding to a signal from the k -th direction at the j -th receiver. For the example of a line array of equispaced receivers $\phi_{jk} = 2\pi f j \frac{\sin \phi_k}{c} d$ where d is the separation

of adjacent receivers. Denoting*

$$x^T = (x_1, x_2, \dots, x_K) ,$$

$$y^T = (y_1, y_2, \dots, y_N) ,$$

and

$$V = \left\{ v_{jk} \right\} ,$$

a $K \times N$ complex matrix, equation (1) can be rewritten as:

$$x = Vy . \quad (2)$$

2.1 Deconvolution of complex beam outputs

The output of a conventional beamformer, denoted as $y^{(0)}$ is then given as*:

$$y^{(0)} = \frac{V^H x}{NK} .$$

In order to deconvolve the $y_j^{(0)}$'s the method of progressive substitutions as discussed by Bracewell and Roberts is used. However, since the x_i 's are assumed related to the y_j 's by equation (2) it is now possible to deconvolve not just with respect to the array window but also with respect to the phase delays ϕ_{ij} .

If the initial estimate, $y^{(0)}$, was the incident noise field distribution then the receiver outputs would be

$$x = Vy^{(0)} .$$

Hence the output of the conventional beamformer would be $Wy^{(0)}$ where

$$W = \frac{V^H V}{NK} .$$

A measure of the error $\epsilon^{(0)}$ in the original estimate $y^{(0)}$ is

then defined by:

$$\epsilon^{(0)} = y^{(0)} - Wy^{(0)} .$$

* T denotes the transpose and H the complex transpose of either a vector or a matrix.

* The choice of NK rather than K as the normalizing factor ensures convergence of the iterative series (see Appendix II) at the expense of biasing the power spectral estimates. The reduction of this bias is discussed in Section 3.

A new approximation to the incident distribution is obtained by correcting for this error, viz:

$$y^{(1)} = y^{(0)} + \epsilon^{(0)} .$$

In general, let $y^{(n)}$ represent some n^{th} order approximation to the incident distribution. If this represented the true distribution then the output of a conventional beamformer would be $Wy^{(n)}$. The error term between this output and the observed output of the beamformer, $y^{(0)}$, is denoted as $\epsilon^{(n)}$ and is defined by:

$$\epsilon^{(n)} = y^{(0)} - Wy^{(n)} . \quad (3)$$

The method of progressive substitutions then implies that a new approximation, $y^{(n+1)}$, to the true incident distribution can be chosen as:

$$y^{(n+1)} = y^{(n)} + \epsilon^{(n)} .$$

This reduces to:

$$y^{(n+1)} = y^{(0)} + (I - W)y^{(n)} \quad (4)$$

by virtue of equation (3).

An alternative formulation of the iteration which gives the same results but has a direct physical significance will now be given.

Consider the ij^{th} element of W for $i \neq j$; it represents the (biased) response of the i^{th} beam to a plane wave from the j^{th} direction. Thus a given row of NW is the amplitude polar response of the array evaluated at the wave-numbers (or angles) corresponding to $j = 0, 1, \dots, N-1$. Now suppose $y^{(n)}$ is some approximation to the incident distribution. The components which distort, through leakage, an estimate of i^{th} beam will be given by:

$$\sum_{i \neq j} W_{ij} y_j^{(n)} .$$

This, for all beams, reduces to

$$(W - I)y^{(n)}$$

since, $W_{ii} = 1/N$. In order to attempt to cancel the effect of these sidelobes the 'leaked' components can be subtracted from $y^{(0)}$; the conventional estimate of the spectrum. This leads to a better approximation of the wave-number spectrum denoted as $y^{(n+1)}$ which is given

by equation (3). The convergence rate of the iteration will be increased by ensuring that the off-diagonal elements of W are small. This is alternatively interpreted as requiring the array polar diagram to have narrow beamwidth and low sidelobes.

2.2 Deconvolution of beam powers

In this section the iterative technique is used to deconvolve the beam powers. The conventional estimate of the N beam powers $s^{(0)}$ is defined by:

$$s_i^{(0)} = \frac{\langle (V^H_{xx} H_V)_{ii} \rangle}{\{NK\}^2} \quad (5)$$

where $\langle \rangle$ denotes the ensemble average (in practice replaced by a time average).

Define the vector m by:

$$m = \langle x \otimes x^* \rangle$$

where \otimes denotes the direct product⁵ and

$$A = V \odot V^H \quad (6)$$

where \odot is the Khatri-Rao product (i.e. $(A \odot B)_{ij} = a_{kj} b_{lj}$ where $i = (k - 1)K + l$) and the dimension of A is $K^2 \times N$. Then equation (5) for \hat{s}_i can be written as:

$$\hat{s}^{(0)} = \frac{A^H m}{\{NK\}^2}$$

Also defining:

$$S = \langle yy^H \rangle$$

and substituting in equation (5) for x it follows that:

$$\begin{aligned} \hat{s}_i &= \frac{(V^H V S V^H)_{ii}}{\{NK\}^2} \\ &= (WSW)_{ii} \end{aligned}$$

Furthermore the assumption that the signals from the differing directions are uncorrelated implies that S is diagonal and so the above equation reduces to:

$$s^{(0)} = W \square W^T s$$

where*

$$s_i = S_{ii}$$

and

$$(W \boxtimes W^T)_{ij} = \frac{(V^H V)_{ij} (V^H V)_{ij}^*}{\{NK\}^2} . \quad (7)$$

Thus an initial approximation to s can be $s^{(0)}$ which would result in an error term of:

$$\epsilon^{(0)} = s^{(0)} - W \boxtimes W^T s^{(0)} \quad (8)$$

which can be used to correct the original estimate, i.e.

$$s^{(1)} = s^{(0)} + \epsilon^{(0)} .$$

This process is now repeated and the general iterative equation becomes:

$$s^{(n+1)} = s^{(0)} + (I - W \boxtimes W^T) s^{(n)} .$$

As in the previous section the iteration matrix, i.e. $W \boxtimes W^T$, has a physical interpretation. From equation (7) it follows that $N^2 W \boxtimes W^T$ is simply the polar diagram of the beam powers and:

$$\sum_{i \neq j} (W \boxtimes W^T)_{ij} s_j^{(n)}$$

is the leakage into the i^{th} beam of the powers from the $N-1$ other directions. Thus the technique may be considered as reducing leakage of powers (either because of a broad beamwidth or high side lobe levels) from one beam to another by using the 'a priori' knowledge of the array's polar response.

* The Hadamard product, \boxtimes , of two matrices A and B, is defined by:

$$(A \boxtimes B)_{ij} = A_{ij} B_{ij} .$$

3. SOLUTION AND CONVERGENCE OF THE ITERATIVE METHOD

In this section and in the remainder of this paper, let U denote either of the $N \times N$ matrices, i.e. W or $W \square W^T$ and let $z^{(n)}$ represent either $y^{(n)}$ or $s^{(n)}$. From either equation (4) or (7) it follows that:

$$z^{(n)} = (I + (I - U) + \dots + (I - U)^n) z^{(0)} \quad (9)$$

and (see Appendix I) this can be shown to reduce to:

$$z^{(n)} = (I - (I - U)^{n+1}) U^- z^{(0)},$$

where U^- is any generalized inverse(ref.4) of the matrix U . Unfortunately convergence of this series does not hold in general since the eigenvalues of $(I - U)$ are not all less than 1.

3.1 Convergence

A simple modification of the recurrence relation allows convergence of the iteration. Replace equation (9) by:

$$z^{(n)} = z^{(0)} + (\lambda I - U) z^{(n-1)}.$$

The effect of the λ is to modify the recurrence formula to:

$$z^{(n)} = \lambda z^{(n-1)} + \epsilon^{(n-1)}$$

where $\epsilon^{(n-1)}$ is the error term defined by either equation (3) or equation (8). As shown in Appendix B the restriction $0 < \lambda < 1$ will guarantee convergence of the series. This requirement also has a heuristic physical interpretation since the condition that $\lambda < 1$ can be considered as automatically allowing for the fact that for $N > K$ there must always be leakage from one beam to another. This is a reflection of the fact that it is impossible, for $N > K$, to steer more than $K-1$ nulls. As a special case, for $N = K$ and V non-singular, convergence of the series for $\lambda = 1$ is possible since it is now possible to steer $K-1$ nulls and so prevent any leakage into a selected direction.

3.2 Limit (beam amplitudes)

The matrix identity

$$(I - (\lambda I - U)) (I + (\lambda I - U) + (\lambda I - U)^2 + \dots + (\lambda I - U)^n) = (I - (\lambda I - U)^{n+1}) \quad (10)$$

always holds. Since $\lambda < 1$ and $W = (V^H V)/(KN)$ it follows that $(1-\lambda)I+W$ is always non-singular. It then follows that:

$$y^{(n)} = \left\{ I - (\lambda I - W)^{n+1} \right\} \left\{ (1 - \lambda)I + W \right\}^{-1} y^{(0)}.$$

However, since the eigenvalues of $\lambda I - W$ are always strictly less than 1 (see Appendix B) it follows that:

$$\lim_{n \rightarrow \infty} (\lambda I - W)^{n+1} = 0.$$

Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} y^{(n)} &= \left\{ (1 - \lambda)I + W \right\}^{-1} y^{(0)} \\ &= \left\{ (1 - \lambda)I + \frac{V^H V}{KN} \right\}^{-1} \frac{V^H x}{KN} \end{aligned}$$

which, in general will give different estimates for differing values of λ . However, from reference 3, the Moore-Penrose pseudoinverse of V , denoted as V^+ is defined as:

$$V^+ = \lim_{\delta \rightarrow 0} (\delta I + V^H V)^{-1} V^H.$$

It then follows that:

$$\lim_{\lambda \rightarrow 1} \lim_{n \rightarrow \infty} y^{(n)} = V^+ x.$$

Thus, as a generalization of conventional beamforming V^H is replaced by V^+ the Moore-Penrose pseudoinverse.

It is worth noting that any choice of scaling for $y^{(0)}$ which ensures convergence gives rise to the same limiting solution $V^+ x$. However as discussed in Section 2 the above solution is biased. A sensible constraint to further impose is that all beams have the same maximum response in their look directions. Since the output of the inverse processor in the n^{th} direction due to a plane wave incident from that direction is:

$$(V^+ v_n)_n,$$

where v_n is the n^{th} column of V , it follows that for unity response in the look direction the appropriate form is:

$$y_i = \frac{(V^+ x)_i}{(V^+ v_i)_i}$$

or

$$y_i = \frac{(V^+ x)_i}{(V^+ V)_{ii}}.$$

3.3 Limit (beam powers)

Since the matrix identity (equation (10)) always holds it follows that:

$$s^{(n)} = \left\{ I - (\lambda I - W \square W^H)^{n+1} \right\} \left\{ (1 - \lambda)I + W \square W^H \right\}^{-1} s^{(0)}$$

provided the inverse of $\{(1 - \lambda)I + W \square W^H\}$ exists. The further matrix identity(ref.5):

$$(V^H V) \square (V^T V^*) = (V \circ V^*)^H (V \circ V^*) \quad (11)$$

guarantees that an inverse exists for $\lambda < 1$. Consequently since the eigen-values of $(\lambda I - W)$ are always strictly less than 1 it follows that:

$$\lim_{n \rightarrow \infty} (\lambda I - W \square W^T)^{n+1} = 0$$

and so

$$\lim_{n \rightarrow \infty} s^{(n)} = \left\{ (1 - \lambda)I - W \square W^H \right\}^{-1} s^{(0)} .$$

Using equation (11) this reduces to

$$\lim_{n \rightarrow \infty} s^{(n)} = \left\{ (1 - \lambda)I - \frac{A^H A}{\{KN\}^2} \right\}^{-1} \frac{Am}{\{KN\}^2}$$

where $A = V \circ V^*$. As in the previous case the limiting form as $\lambda \rightarrow 1^-$ is the pseudoinverse and so an 'optimum' processor is:

$$s = A^+ m$$

and substituting for A and m this becomes:

$$s = (V \circ V^*)^+ \langle x \circ x^* \rangle .$$

As before the limiting solution is independent of the scaling of $s^{(0)}$ and a solution which allows unity response in the look directions is given by:

$$s_j = \frac{(A^+ m)_j}{(A^+ A)_{jj}} .$$

4. DISCUSSION

4.1 Complex beam outputs

The estimator $\hat{y} = V^+x$, termed the inverse estimator, has a number of interesting properties which relate it to some quadratic processors which have recently been proposed.

- (a) If V is square and non-singular (i.e. $K = N$) then $V^+ = V^{-1}$. For a chosen direction the processor $V^{-1}x$ steers nulls in the remaining $(K - 1)$ directions and thus forms an unbiased estimator of the wave-number spectrum. This estimator has been derived in reference 2 using a maximum likelihood technique.
- (b) If for an arbitrary N , the K rows of V are linearly independent then $V^+ = V^H(VV^H)^{-1}$. Apart from the scaling factor, $(V^+V)_{nn}$, this is the minimum bias estimator derived in reference 2.
- (c) In general a class of solutions of the equation:

$$x = Vy \quad (12)$$

where $V\bar{V}y = y$, is given by:

$$\bar{y} = V^-x$$

where V^- is any generalized inverse. The solution derived here, V^+ , has the additional property that of all the \bar{y} 's which satisfy equation (12), it is the one with the minimum norm(ref.4). That

is $\|\bar{y}\|^2 = \sum_i |y_i|^2$ is also minimized. Alternatively if the

receiver outputs, i.e. the x_i , are uncorrelated then the pseudo-inverse is the one which also minimizes the norm of the weight vectors and thus limits the superdirectivity of the array.

- (d) In general the effect of noise (and also signals arriving from directions not accounted for by the v_{jk} 's) will be such x is only approximately equal to Vy . The choice of $\hat{y} = V^+x$ as an estimator is still an appropriate one since it is shown in reference 4 that it is the vector of minimum norm which minimizes $\|x - Vy\|^2$. That is, the estimator $\hat{y} = V^+x$ is the best (in a least squares sense) plane wave solution to the problem subject to the constraints imposed by the assumed source directions.
- (e) The total power output from the receivers is x^Hx . If \hat{y} is defined as $\hat{y} = Ux$ then, when U is any unitary matrix ($U^H U = I$), the total power in the \hat{y} 's, $\hat{y}^H \hat{y}$, equals that in the x 's. This is analogous to Parseval's theorem in Fourier analysis and is an expression of the conservation of energy. If the V 's are not unitary then this equality does not hold but a sensible criterion would be to require that it holds as an approximation. This then implies that:

$$\|x^H x - \hat{y}^H \hat{y}\|^2 \quad (13)$$

should be minimized.

The general solution \hat{y} which minimizes:

$$\|x - Vy\|^2$$

is given by:

$$\hat{y} = V^+x + (I - V^+V)z$$

where z is an arbitrary vector. Substituting for y , expression (13) reduces to:

$$x^H x - x^H V^+ H V^+ x - z^H z + z^H V^+ H V^+ z \quad (14)$$

where use has been made of two identities for the Moore-Penrose pseudoinverse:

$$V^+ V V^+ = V^+$$

and

$$(V^+ V)^H = V^+ V.$$

Differentiating the above with respect to z and equating to zero then implies that equation (14) is minimized when:

$$z = V^+ V z.$$

Hence the expression for \hat{y} reduces to:

$$\hat{y} = V^+ x.$$

Thus the Moore-Penrose pseudoinverse is that solution minimizing:

$$\|x^H x - \hat{y}^H \hat{y}\|^2$$

i.e., it most closely conserves the power.

An important consequence of this result is that if there is a strong source in the x 's with a wave-number not accounted for by the V 's then there will be considerable leakage of the power of this source into the estimated y_i . The (dis)advantage of this will depend on whether this strong source is either a desired signal or an interference.

(f) Idempotency of solution

Suppose the inverse estimates, i.e. $y = V^+ x$, are the true distribution of sources. The receiver outputs, \tilde{x} , become:

$$\tilde{x} = V V^+ x$$

and the output of the inverse estimator is:

$$\begin{aligned} V^+ \tilde{x} &= V^+ V V^+ x \\ &= V^+ x . \end{aligned}$$

This property of the inverse estimator is termed idempotency and it is precisely the lack of this property for the conventional processor that allows the iteration to be effected.

4.2 Beam powers

(a) If V is square and non-singular (i.e. $K = N$) and since:

$$W \boxtimes W^T = \frac{(V \circ V^*)^H (V \circ V^*)}{\{NK\}^2} ,$$

then $W \boxtimes W^T$ is positive definite and hence non-singular. Thus the series is convergent for $\lambda = 1$ and the limit becomes:

$$(W \boxtimes W^T)^{-1} s(0)$$

which on substitution of equations (6) and (11) becomes:

$$\{NK\}^2 (A^H A)^{-1} s(0)$$

which has been derived in reference 3 as a least squares estimator (see also below) under some more general conditions.

(b) From Section 3 the approximation:

$$m \approx V S V^H$$

where S is diagonal can be rewritten as

$$m \approx A s$$

where s is the vector of the diagonal elements of S . As in the previous section a particularly suitable approximation is:

$$\hat{s} = A^+ m$$

since in addition to minimizing:

$$\|m - A \hat{s}\|^2 \quad (\text{or } \|m - V S V^H\|^2)$$

it also minimizes:

$$\|\hat{s}\|^2$$

and hence limits the superdirectivity of the estimator.

5. EXAMPLES

Consider a linear array of K equispaced receivers separated by a distance d . If the incident distribution is assumed to be two-dimensional and composed of N plane waves with wave numbers k_j ($= (2\pi \sin \theta_j)/\lambda$) then the matrix of phase delays, V , has the form:

$$V_{ij} = z_j^i$$

where

$$z_j = e^{2\pi i d k_j}$$

This is a Vandermonde matrix(ref.6) and use may be made of its special properties. Furthermore, if the k_j are chosen to correspond to N arrival directions, equispaced in $\sin \theta$, and lying between $\pm\pi/2$ then V is simplified to:

$$V_{ij} = z^{ij}$$

where

$$z = e^{4\pi i d/\lambda N}$$

For such a V , W , defined as $(V^H V)/(KN)$ and the matrix to be used in the amplitude iteration, is given by:

$$\begin{aligned} W_{\rho\mu} &= \frac{1}{KN} \frac{(z^{-(\rho-\mu)K} - 1)}{(z^{-(\rho-\mu)} - 1)} \\ &= w_\nu \end{aligned}$$

where $\nu = \rho - \mu$. The matrix $W \square W^T$ can be readily reduced to:

$$\frac{1}{(KN)^2} \frac{\sin^2 \frac{2\pi d K}{\lambda} (\rho - \mu)}{\sin^2 \frac{2\pi d}{\lambda} (\rho - \mu)}$$

which, apart from the $\frac{1}{N^2}$ factor, is the polar response of a line array of K equispaced receivers. Since both W and $W \square W^T$ are Toeplitz it then follows that the term:

$$\sum_{v=0}^{N-1} U_{\rho v} z_v^{(n)}$$

in the iteration becomes:

$$\sum_{v=0}^{N-1} u_{v-\rho} z_v^{(n)}$$

This is simply the result of the first N terms of the cyclic convolution of the vector:

$$(u_0, u_1, \dots, u_{N-1}, u_{-N+1}, u_{-N+2}, \dots, u_{-1}, 0)^T$$

with the vector:

$$(z_0^{(n)}, z_1^{(n)}, \dots, z_{N-1}^{(n)}, 0, \dots, 0)^T.$$

As a result, if N is chosen to be a power of two, then the convolution may be efficiently evaluated by use of the fast Fourier transform. Unfortunately the $z_j^{(n+1)}$ for $j = N, \dots, 2N-1$ produced by this cyclic convolution are not zero and so the iteration cannot be effected completely in the transform space. This can alternatively be realized by observing that U^2 is not in general, Toeplitz. (Aside: the method proposed by Bracewell and Roberts assumed that the iteration could be effected completely in the transform domain. This amounts to assuming the array distribution to be cyclic and so is why they only succeeded in deconvolving the array shading).

V is generally of full rank and so V^+x , the limit of the amplitude iteration, may be written as:

$$V^H(VV^H)^{-1}x.$$

In particular:

$$(VV^H)_{\rho\mu} = \frac{(z^{(\rho-\mu)N} - 1)}{(z^{(\rho-\mu)} - 1)},$$

and is a complex Toeplitz matrix.

Efficient algorithms exist for the inversion of this matrix enabling V^+ to be rapidly calculated. For the particular case of $K = N$ then $V^+ = V^{-1}$ and once again, since V is a Vandermonde matrix, it may be efficiently inverted.

Similarly $A^H A$ can be shown to be a Toeplitz matrix and is non-negative since all its elements are positive.

Some examples of these for various values of N , K and d/λ will be given.

5.1 K receivers and 1 plane wave

In this case V is the $K \times 1$ column vector and V^+ trivially reduces to $\frac{V^H}{K}$ which is the conventional processor. The iterative equation for the amplitudes reduces to

$$y^{(n)} = y^{(0)} + (\lambda - 1)y^{(n-1)},$$

which for $\lambda = 1$ becomes $y^{(n)} = y^{(0)}$ as would be expected.

5.2 K receivers and N sources at half wavelength spacing

When $d = \lambda/2$ it follows that $z^N = 1$ with the result that:

$$VV^H = NI_K,$$

where I_K is the $K \times K$ identity matrix. As a consequence the first step in the iteration equation:

$$y^{(1)} = y^{(0)} + \left(\lambda I - \frac{V^H V}{NK} \right) \frac{V^H x}{NK}$$

reduces to:

$$y^{(1)} = y^{(0)} + \left(\lambda - \frac{1}{K} \right) y^{(0)}.$$

For $\lambda = \frac{1}{K}$ the first, and consequently all successive iterations, reduce to the conventional beamformer. It also follows that, for $\lambda = 1$, either as a consequence of the iteration or the fact that $V^+ = \frac{V^H}{N}$, the limit y reduces to the biased estimator, $\frac{V^H x}{N}$. Furthermore since $(V^+ V)_{ii} = \frac{K}{N}$ the standard solution $\frac{V^H x}{K}$, corresponding to unity response in the look direction, is obtained.

5.3 K receivers and K sources at quarter wavelength spacing

The number of independent beams lying between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ for a conventional processor at a quarter wavelength is $\frac{N}{2}$. The first of these is plotted in heavy lines in figure 1. The extra $\frac{N}{2}$ beams incorporated in V are redundant beams spaced half way between the adjacent independent

beams; the first redundant beam is plotted in dashed lines in figure 1. Now $w_\rho (=W_{\mu+\rho, \mu})$, essentially the array polar diagram is given by:

$$\begin{aligned} w_\rho &= \frac{-2}{N^2 (z^\rho - 1)} && \text{for } \rho \text{ odd} \\ &= 0 && \text{for } \rho \text{ even } \neq 0 \\ &= \frac{1}{N} && \text{for } \rho = 0. \end{aligned}$$

Consider the leakage of y_0 into the other beams; from figure 1 zero power will leak into the even-number beams since they correspond to nulls in the polar diagram (this is reflected in the fact that $w_\rho = 0$ for ρ even). The power leakage into the odd beams is given by $w_\rho y_0$ where w_ρ is given by above and y_0 is the output of the zeroth beam. Thus the form of w_ρ enables the leakage from y_0 (and in general all the beams) to be removed from the other beams as discussed in Section 2.

Some other points follow from this interpretation of the deconvolution technique. In figure 1 the side lobes of any beam are shown extending into what is termed the 'non-physical region'. This region corresponds to plane wave disturbances either generated in the array of receivers or propagating across the array with a velocity less the one assumed in calculating the phase delays. It is an inherent assumption of these examples that the amplitude of any of these effects is zero or at least very small. Some of these wave-number beams could be incorporated in V but unfortunately any attempt to account for the full $\frac{N}{2}$ (at a quarter wavelength) independent ones would, since the matrix W becomes circular, reduce V^+ to V^H ; the conventional processor. Once again this can be intuitively seen from a consideration of the zeros of the polar diagram.

6. CONCLUSIONS

Estimates of the angular distribution of the power incident on an array made by using the outputs of a conventional frequency-domain beamformer are distorted by leakage from one beam to another. Two techniques have been proposed in this paper which use a knowledge of the polar response of the array to minimise this leakage. The techniques are conceptually similar, the difference being that one uses the complex beam outputs whereas the other uses the beam powers. Both techniques can be effected as a series of iterative deconvolutions using either the narrowband array amplitude or polar response and the narrowband beam outputs or powers respectively of a conventional beamformer. Alternatively either the receiver outputs or the receiver crosspower spectral matrix may be used directly to evaluate the limits of the two iterative methods. In practice the choice would be determined by implementation requirements.

A particular attraction is that the techniques are linear and so the concept of a polar diagram is useful. The convergence of the iteration as a function of the parameter λ and the stability of the matrices to be inverted are areas warranting further investigation.

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APPENDIX I

A VECTOR IDENTITY

Given:

$$z^{(0)} = Uz \quad (I.1)$$

and $z^{(n)}$ is defined by the recursive relation:

$$z^{(n)} = z^{(0)} + (I - U)z^{(n-1)} \quad (I.2)$$

it is required to show that:

$$z^{(n)} = (I - (I - U)^{n+1})U^-z^{(0)} \quad (I.3)$$

where U^- is any generalized inverse of U .

Before proceeding with an inductive proof, it follows directly from equation (I.1) and the property of $UU^-U = U$ for any generalized inverse that:

$$UU^-z^{(0)} = z^{(0)} \quad (I.4)$$

where

$$z^{(0)} = y^{(0)} \text{ or } s^{(0)} .$$

Assuming that (I.3) holds for $z^{(n-1)}$ and then substituting for $z^{(n-1)}$, equation (I.2) reduces to:

$$\begin{aligned} z^{(n)} &= z^{(0)} + (I - U)(I - (I - U)^n)U^-z^{(0)} \\ &= z^{(0)} - UU^-z^{(0)} + (I - (I - U)^{n+1})U^-z^{(0)} \\ &= (I - (I - U)^{n+1})U^-z^{(0)} \end{aligned}$$

where the last step follows directly from (I.4). To complete the induction it is necessary to prove equation (I.3) when $n = 0$. For $n = 0$ equation (I.3) reduces to:

$$z^{(0)} = UU^-z^{(0)}$$

which follows directly from (I.4).

The proof is completed by showing that $z^{(n)}$ defined by equation (I.3) is unique for any choice of U^- .

Now any generalized inverse of U , U_1^- , can be written as

$$U_1^- = U^- + (I-U^-U)X + Y(I-UU^-) .$$

for arbitrary X and Y.
Thus

$$\begin{aligned} U_1^- z^{(0)} &= U^- z^{(0)} + (I-U^-U)Xz^{(0)} + Y(I-UU^-)z^{(0)} \\ &= U^- z^{(0)} + (I-U^-U)Xz^{(0)} \end{aligned}$$

by virtue of equation (I.4).
Hence equation (I.3) is unique if

$$(I - (I-U)^n) (I - U^-U) = 0 .$$

But

$$(I - (I-U)^n) (I - U^-U) = (I + (I-U) + \dots + (I-U)^{n-1})U (I - U^-U) = 0 .$$

Thus $z^{(n)}$ defined by equation (I.3) is unique.

APPENDIX II

CONVERGENCE OF A SERIES

In order to show that the geometric series $I + (\lambda I - W) + \dots + (\lambda I - W)^n$ converges as $n \rightarrow \infty$ it is necessary to show that $\lim_{n \rightarrow \infty} (\lambda I - W)^n$ is zero. A necessary and sufficient condition for this is that all the eigenvalues of $\lambda I - W$ are less than 1.

Let $P(\leq K)$ be the rank of W a $K \times K$ matrix defined by $W = \frac{V^H V}{NK}$ where V is any $K \times K$ matrix. It always holds (ref.5) that:

$$W = Q^H \Lambda Q \quad (\text{II.1})$$

where $\Lambda = \text{diag} \{ \lambda_1^2, \lambda_2^2, \dots, \lambda_p^2, 0, 0, \dots, 0 \}$

and $Q = \{q_{ij}\}$ is a unitary $N \times N$ matrix.

Convergence of the geometric series $I + (\lambda I - W) + \dots + (\lambda I - W)^n$ is guaranteed provided:

$$|\lambda - \lambda_i| < 1 \quad i = 1, 2, \dots, P$$

and

$$|\lambda| < 1 .$$

These two conditions can be seen to be satisfied provided $0 < \lambda < 1$ and $|\lambda_i| \leq 1$. The first condition can easily be satisfied by an appropriate choice of λ .

If the eigenvalues of the matrix $V^H V$ are μ_i it follows that the eigenvalues of $\frac{1}{\mu_{\max}} V^H V$ are less than or equal to unity. Since $V^H V$ is positive definite, i.e. $\mu_i = \lambda_i^2$ it follows that:

$$\mu_{\max} \leq \sum_{i=1}^P \mu_i .$$

However, it holds that:

$$\sum_{i=1}^P \mu_i = \text{Tr}(V^H V)$$

and substituting for V the above equation reduces to:

$$\sum \mu_i = NK .$$

Thus

$$\mu_{\max} \leq NK ,$$

with the equality only holding when $V^H V$ has one non-zero eigenvalue. For $N > 1$ this pathological case corresponds to all columns of V being identical and can be disregarded. Thus the choice:

$$\frac{V^H V}{NK}$$

always ensures convergence when V is of rank greater than unity although it should be realized that any normalizing factor greater than λ_{\max} will also guarantee convergence. For example if $V^H V$ is such that all eigenvalues are equal then:

$$\frac{V^H V}{N + 1}$$

satisfies the convergence requirements.

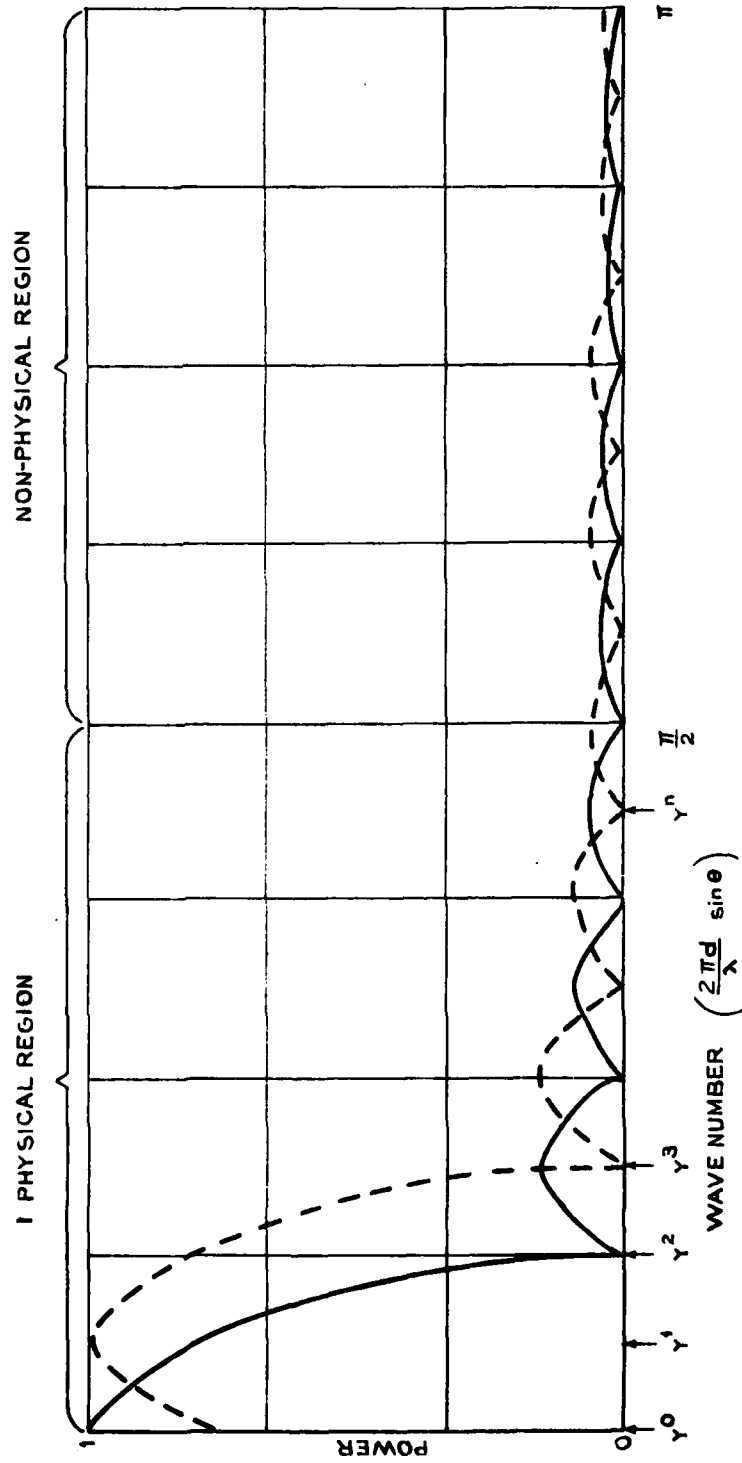


Figure 1. N beams steered in the physical region at $d = \frac{\lambda}{4}$

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