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TECHNICAL REPORT

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ON A THIRD ORDER LINEAR
PREDICTOR-CORRECTOR DIGITAL FILTER

JULY 1980

FINAL REPORT

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1. INTRODUCTION

This paper is concerned with a class of linear predictor-corrector digital filters of the third order with three parameters. This class includes the four digital filters that have been or are being used at White Sands Missile Range (WSMR) of the United States Army. Two of them, namely the McCool's QD filter [3] and Shepherd's third order parabolic spline filter [4], were devised at WSMR for missile trajectory smoothing, reconstruction and differentiation. All of them have applications in real time radar tracking digital servomechanisms. The theory for this paper was developed largely from internal notes of WSMR by Chui [1] and Shepherd [4] although some laborious detail in [1] and [4] is not given here. The class of linear predictor-corrector digital filters to be discussed here can be defined as in the following. Set

$$\underline{w} = [w_1, w_2, w_3]^T. \quad (1.1)$$

Here, and throughout, the superscript T indicates the transpose of a matrix. Hence, $\underline{w}$ in (1.1) is a three-dimensional constant column vector. If $\{x_i\}$, $i = 0, 1, \dots$, denotes an input signal, h a positive constant, and the sequence of three-dimensional vectors

$$y_i = [y_i, y'_i, y''_i]^T, \quad (1.2)$$

$i = 0, 1, \dots$, denotes the output response corresponding to the input signal $\{x_i\}$ and subject to the initial condition $y_{-1} = [y_{-1}, y'_{-1}, y''_{-1}]^T$, then the class of digital filters we consider is defined by

$$y_{p+1} = \underline{u}_p(ph + h) + (x_{p+1} - \underline{u}_p(ph + h))\underline{w} \quad (1.3)$$

for $p = -1, 0, 1, \dots$, where

$$u_p(t) = y_p + y'_p(t - ph) + \frac{1}{2}y''_p(t - ph)^2, \text{ and} \quad (1.4)$$

$$\underline{u}_p(t) = [u_p(t), u'_p(t), u''_p(t)]^T.$$

The real-valued function $u_p(t)$ and the vector-valued function $\underline{u}_p(t)$ are both called predictor functions. Also, $\underline{u}_p(ph + h)$ is called the prediction at the time $t = ph + h$ based on the output response y_p , while $(x_{p+1} - \underline{u}_p(ph + h))\underline{w}$ is called the correction at the time $t = ph + h$. With $\underline{w} = [\alpha, \beta/h, \gamma/h^2]^T$ this filter appears as the general α - β - γ filter (cf. Steelman [5] and the references therein).

It is easy to verify that the filter (1.3), (1.4) is equivalent to the filter

$$y_{p+1} = Ay_p + x_{p+1}\underline{w}, \quad (1.5)$$

$p = -1, 0, 1, \dots$, where A is the 3×3 matrix

$$A = \begin{bmatrix} 1 - w_1 & (1 - w_1)h & \frac{1}{2}(1 - w_1)h^2 \\ -w_2 & 1 - w_2h & h - \frac{1}{2}w_2h^2 \\ -w_3 & -w_3h & 1 - \frac{1}{2}w_3h^2 \end{bmatrix}. \quad (1.6)$$

This matrix formulation exhibits the filter as a third order linear difference equation. With input $\{x_i\}$ where $x_i = 0$ for $i < 0$, output $\{y_i\}$, $i = 0, 1, \dots$, and initial value y_{-1} the general solution can be shown by induction to be

$$y_{p+1} = A^{p+2}y_{-1} + A^{p+1}x_0\underline{w} + A^p x_1\underline{w} + \dots + Ax_p\underline{w} + x_{p+1}\underline{w}. \quad (1.7)$$

If the response due to arbitrary initial value y_{-1} is to damp out, it is apparent from (1.7) that

$$\lim_{n \rightarrow \infty} A^n \underline{x} = 0 \text{ for all } \underline{x} \text{ in } \mathbb{R}^3. \quad (1.8)$$

The relationship between this observation and the stability of the filter (1.5), (1.6) is developed in section 2. More precisely, we will show that if all the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the matrix A lie in the open unit disc $|z| < 1$, then the digital filter (1.3) - (1.4), or equivalently (1.5) - (1.6), is stable. Furthermore, the condition $|\lambda_j| < 1$, for $j = 1, 2, 3$, is satisfied if and only if the condition (1.8) is satisfied. We will study this eigenvalue problem via z-transforms. This technique will be extended to "uncouple" the system (1.3) - (1.4) solving for each of the output response sequences $\{y_p\}$, $\{y'_p\}$, and $\{y''_p\}$ in terms of the input signal $\{x_p\}$ and the initial conditions y_{-1} , y'_{-1} , and y''_{-1} respectively in recursive forms. These three recursive formulas will allow the studying of each of the three output-response sequences $\{y_p\}$, $\{y'_p\}$, and $\{y''_p\}$ individually without referring to the other two. In particular, design criteria can be formulated.

In section 3, we will consider the special case of Shepherd's "parabolic spline predictor corrector filter", which is obtained from (1.5) and (1.6) by setting

$$\underline{w} = \left[\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2} \right]^T \quad (1.9)$$

where δ is considered as a design parameter.

In section 4 we consider somewhat more briefly McCool's "QD" filter [3], for which (with $\underline{w} = [\alpha, \beta/h, \gamma/h^2]^T$)

$$[\alpha, \beta, \gamma] = \left[\frac{60M^2}{10M^3 + 33M^2 + 23M - 6}, \beta = \frac{2\alpha}{M}, \gamma = \frac{2\alpha}{M^2} \right],$$

Morrison's "fading memory polynomial filter of degree 2" [5], for which

$$[\alpha, \beta, \gamma] = [1 - \theta^3, \frac{3}{2}(1 - \theta)^2(1 + \theta), (1 - \theta)^3],$$

and an $\alpha - \beta$ filter (for which one can set $w_3 = 0$ and $y_{-1}'' = 0$ in the beginning) studied by Gonzales in 1968 (cf. [2]).

2. STABILITY AND UNCOUPLING OF THE FILTERS

We will use the technique of z-transforms to study the class of digital filters defined by (1.3) - (1.4), or equivalently (1.5) - (1.6).

If $\{b_j\}$ is a bi-infinite sequence of complex numbers, then the z-transform of the sequence $\{b_j\}$, $j = \dots, -1, 0, 1, \dots$, is the formal Laurent series

$$B(z) = \sum_{j=-\infty}^{\infty} b_j z^{-j}.$$

Let $\{x_p\}$, $\{y_p\}$, $\{y_p'\}$ and $\{y_p''\}$ be defined as in section 1. We will set $x_p = 0$ if $p < 0$, and $y_p = y_p' = y_p'' = 0$ if $p < -1$. Hence, the z-transforms of these sequences are given by

$$\begin{aligned} X &= X(z) = \sum_{p=0}^{\infty} x_p z^{-p}, \\ Y_1 &= Y_1(z) = \sum_{p=-1}^{\infty} y_p z^{-p}, \\ Y_2 &= Y_2(z) = \sum_{p=-1}^{\infty} y_p' z^{-p}, \\ Y_3 &= Y_3(z) = \sum_{p=-1}^{\infty} y_p'' z^{-p} \end{aligned}$$

respectively. If we re-write the matrix system (1.5) in the form of a system of three simultaneous difference equations and take the z-transforms of each of these three equations, we obtain the following system of simultaneous linear algebraic equations

$$\begin{cases} (z-1+w_1)Y_1 - (h-hw_1)Y_2 - \left(\frac{1}{2}h^2 - \frac{1}{2}h^2w_1\right)Y_3 = w_1zX \\ w_2Y_1 + (z-1+hw_2)Y_2 - (h-\frac{1}{2}h^2w_2)Y_3 = w_2zX \\ w_3Y_1 + hw_3Y_2 + (z-1+\frac{1}{2}h^2w_3)Y_3 = w_3zX \end{cases} \quad (2.1)$$

In matrix representation, (2.1) can be written as

$$(A - zI_3)[Y_1, Y_2, Y_3]^T = -zXw \quad (2.2)$$

where I_3 is the 3×3 identity matrix. Let $H_1(z)$, $H_2(z)$, and $H_3(z)$ be the transfer functions of this digital filter; that is,

$$Y_j(z) = H_j(z)X(z) \quad (2.3)$$

for $j = 1, 2, 3$. Then $H_1(z)$, $H_2(z)$, and $H_3(z)$ can be obtained by solving the linear system (2.2) using Cramer's rule. Hence, they are rational functions in z^{-1} with the same denominator $\det(A - zI_3)$. This shows that if $\det(A - zI_3) \neq 0$ for all z with $|z^{-1}| \leq 1$ or $|z| \geq 1$, or equivalently all the eigenvalues of A lie inside the unit circle $|z| = 1$, then the filter (1.3) - (1.4) is stable. Let $\lambda_1, \lambda_2, \lambda_3$ be the (not necessarily distinct) eigenvalues of A and let $\underline{x}_1, \underline{x}_2, \underline{x}_3$ be three corresponding linearly independent eigenvectors. Then for any $\underline{x} \in \mathbb{R}^3$, $\underline{x} = \alpha_1\underline{x}_1 + \alpha_2\underline{x}_2 + \alpha_3\underline{x}_3$ for some constants α_1, α_2 , and α_3 . Hence, for any positive integer n , we have

$$A^n \underline{x} = \alpha_1 \lambda_1^n \underline{x}_1 + \alpha_2 \lambda_2^n \underline{x}_2 + \alpha_3 \lambda_3^n \underline{x}_3.$$

This shows that if $|\lambda_j| < 1$ for $j = 1, 2, 3$, then $A^n \underline{x} \rightarrow \underline{0}$ as $n \rightarrow \infty$.

Conversely, if $A^n \underline{x} \rightarrow \underline{0}$ for all $\underline{x} \in \mathbb{R}^3$, we can pick $\underline{x} = \underline{x}_j$ ($j = 1, 2, 3$) to yield $\lambda_j^n \underline{x}_j = A^n \underline{x}_j \rightarrow \underline{0}$ as $n \rightarrow \infty$. Since $\underline{x}_j \neq 0$, we must have $|\lambda_1|, |\lambda_2|, |\lambda_3| < 1$. That is, we have established the following

THEOREM 2.1. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix A given in (1.6) and $\Lambda = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$. Then the digital filter (1.3) - (1.4) is stable provided $\Lambda < 1$. Furthermore this stability condition $\Lambda < 1$ holds if and only if

$$\lim_{n \rightarrow \infty} A^n \underline{x} = \underline{0}$$

for all $\underline{x} \in \mathbb{R}^3$.

We next determine the transfer functions $H_1(z)$, $H_2(z)$, and $H_3(z)$.

To do this, we first compute $\det(zI_3 - A)$. It is given by

$$\begin{aligned} \det(zI_3 - A) = & z^3 + \{(w_1 - 3) + w_2 h + \frac{1}{2} w_3 h^2\} z^2 \\ & + \{(3 - 2w_1) - w_2 h + \frac{1}{2} w_3 h^2\} z + (w_1 - 1). \end{aligned} \quad (2.4)$$

Hence, by Cramer's rule, we obtain

$$Y_1(z) = \frac{zX(z)}{\det(zI_3 - A)} \begin{vmatrix} w_1 & (w_1 - 1)h & \frac{1}{2}(w_1 - 1)h^2 \\ w_2 & z + w_2 h - 1 & \frac{1}{2}w_2 h^2 - h \\ w_3 & w_3 h & z + \frac{1}{2}w_3 h^2 - 1 \end{vmatrix} \quad (2.5)$$

$$= \frac{zX(z)}{\det(zI_3 - A)} \{w_1 z^2 + (\frac{1}{2}w_3 h^2 + w_2 h - 2w_1)z + (\frac{1}{2}w_3 h^2 - w_2 h + w_1)\}.$$

Similarly, we obtain

$$Y_2(z) = \frac{zX(z)}{\det(zI_3 - A)} (z-1)(w_2 z + hw_3 - w_2), \quad (2.6)$$

$$Y_3(z) = \frac{zX(z)}{\det(zI_3 - A)} (z-1)^2 w_3. \quad (2.7)$$

Hence, the transfer functions $H_1(z)$, $H_2(z)$, and $H_3(z)$ as defined in (2.3) can be written as

$$H_1(z) = \frac{w_1 + (\frac{1}{2}w_3 h^2 + w_2 h - 2w_1)z^{-1} + (\frac{1}{2}w_3 h^2 - w_2 h + w_1)z^{-2}}{1 + \{(w_1-3) + w_2 h + \frac{1}{2}w_3 h^2\}z^{-1} + \{(3-2w_1) - w_2 h + \frac{1}{2}w_3 h^2\}z^{-2} + (w_1-1)z^{-3}} \quad (2.8)$$

$$H_2(z) = \frac{w_2 + (hw_3 - 2w_2)z^{-1} + (w_2 - hw_3)z^{-2}}{1 + \{(w_1-3) + w_2 h + \frac{1}{2}w_3 h^2\}z^{-1} + \{(3-2w_1) - w_2 h + \frac{1}{2}w_3 h^2\}z^{-2} + (w_1-1)z^{-3}} \quad (2.9)$$

and

$$H_3(z) = \frac{w_3 - 2w_3 z^{-1} + w_3 z^{-2}}{1 + \{(w_1-3) + w_2 h + \frac{1}{2}w_3 h^2\}z^{-1} + \{(3-2w_1) - w_2 h + \frac{1}{2}w_3 h^2\}z^{-2} + (w_1-1)z^{-3}} \quad (2.10)$$

If we put the expressions (2.8), (2.9), and (2.10) back into (2.3), multiply the denominator of $H_j(z)$ to $Y_j(z)$, and take the inverse z-transforms of each of the expressions for $j = 1, 2$, and 3 , we obtain the following

THEOREM 2.2. The digital filter given by (1.3) - (1.4) can be written as three uncoupled recursive digital filters:

$$\begin{aligned} y_p = & -\{(w_1-3) + w_2 h + \frac{1}{2}w_3 h^2\}y_{p-1} - \{(3-2w_1) - w_2 h + \frac{1}{2}w_3 h^2\}y_{p-2} \\ & - (w_1-1)y_{p-3} + w_1 x_p + (\frac{1}{2}w_3 h^2 + w_2 h - 2w_1)x_{p-1} + (\frac{1}{2}w_3 h^2 - w_2 h + w_1)x_{p-2}, \end{aligned} \quad (2.11)$$

$$\begin{aligned} y'_p = & -\{(w_1-3) + w_2 h + \frac{1}{2}w_3 h^2\}y'_{p-1} - \{(3-2w_1) - w_2 h + \frac{1}{2}w_3 h^2\}y'_{p-2} \\ & - (w_1-1)y'_{p-3} + w_2 x_p + (hw_3 - 2w_2)x_{p-1} + (w_2 - hw_3)x_{p-2}, \end{aligned} \quad (2.12)$$

and

$$y_p'' = -\{(w_1-3) + w_2h + \frac{1}{2}w_3h^2\}y_{p-1}'' - \{(3-2w_1) - w_2h + \frac{1}{2}w_3h^2\}y_{p-2}'' \\ - (w_1-1)y_{p-3}'' + w_3x_p - 2w_3x_{p-1} + w_3x_{p-2}, \quad (2.13)$$

where $p = 0, 1, 2, \dots$ with initial conditions $y_{-1}, y_{-1}',$ and y_{-1}'' , and
where $x_p = 0$ for $p < 0$ and $y_p = y_p' = y_p'' = 0$ for $p < -1$.

We now return to study the stability of the filter (1.3) - (1.4) a little closer via the transfer functions $H_1(z)$, $H_2(z)$, and $H_3(z)$. From (2.4), putting $z = 0$, we see that $\det A = 1 - w_1$. This says that $\lambda = 0$ is an eigenvalue of A if and only if $w_1 = 1$. If $w_3 = 0$, then the transfer function $H_3(z)$ is identically zero. On the other hand, if $w_3 \neq 0$, then the value $z^{-1} = 1$ is not a zero of $H_1(z)$, while it is a double zero of $H_3(z)$ and at least a simple zero of $H_2(z)$. Hence, in this case, all the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of A are essential in the consideration of stability. More precisely, we have the following.

COROLLARY 2.1. Let $w_3 \neq 0$. Then the digital filter (1.3) - (1.4) is stable if and only if $\Lambda := \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) < 1$.

3. A PARABOLIC SPLINE PREDICTOR-CORRECTOR DIGITAL FILTER

This filter is the special case $\underline{w} = \left[\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2} \right]^T$. It was devised by Shepherd [4] and has the following property: Set

$$v_p(t) = y_p + y'_p(t-ph) + \frac{1}{2} \left[-\frac{2\delta}{h^2} y_p - \frac{2\delta}{h} y'_p + (1-\delta) y''_p + \frac{2\delta}{h^2} x_{p+1} \right] (t-ph)^2, \quad (3.1)$$

and define $v(t)$ on $[0, \infty)$ by

$$v(t) = v_p(t) \quad \text{for} \quad ph \leq t < ph + h, \quad p = 0, 1, 2, \dots \quad (3.2)$$

Here, the input signal $\{x_p\}$ and the output response $y_p = [y_p, y'_p, y''_p]^T$ satisfy the filter relationship (1.3) - (1.4) with $\underline{w} = [\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$.

Using this relationship, it is easy to verify that the function v defined by (3.2) is indeed a parabolic (or third order C^1) spline on $[0, \infty)$ with knots at $\{0, h, 2h, \dots\}$. Since $v(ph) = y_p$ and $v'(ph) = y'_p$, the spline function v together with its derivative v' interpolate the output response $[y_p, y'_p]^T$ at the time $t = ph$, $p = 0, 1, \dots$. We like to think of the input signal $\{x_p\}$ as a noisy measurement of $\{f(ph)\}$

where f is a fairly smooth function to be reconstructed, and the output response $y_p = [y_p, y'_p, y''_p]^T$ as an approximation to $[f(ph), f'(ph), f''(ph)]^T$.

The approximation improves as δ tends to 1 as can be seen in (3.3)

below. Hence, the spline function v can be considered as an approximation to the function f which we wish to reconstruct from the noisy data $\{x_p\}$.

This particular filter is a one-parameter digital filter, parametrized

by δ . In fact, if we put $\underline{w} = [\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$ in (1.3), then (1.3) - (1.4) becomes

$$\begin{cases} y_{p+1} = (1-\delta)y_p + (1-\delta)hy'_p + \frac{1}{2}(1-\delta)h^2y''_p + x_{p+1}, \\ y'_{p+1} = -\frac{2\delta}{h}y_p + (1-2\delta)y'_p + (1-\delta)hy''_p + \frac{2\delta}{h}x_{p+1}, \\ y''_{p+1} = -\frac{2\delta}{h^2}y_p - \frac{2\delta}{h}y'_p + (1-\delta)y''_p + \frac{2\delta}{h^2}x_{p+1}, \end{cases} \quad (3.3)$$

and in matrix form, we have $y_{p+1} = A_\delta y_p + x_{p+1} [\delta, \frac{\delta}{h}, \frac{2\delta}{h^2}]^T$, with

$$A_\delta = \begin{bmatrix} 1 - \delta & (1 - \delta)h & \frac{1}{2}(1 - \delta)h^2 \\ -\frac{2\delta}{h} & 1 - 2\delta & (1 - \delta)h \\ -\frac{2\delta}{h^2} & -\frac{2\delta}{h} & 1 - \delta \end{bmatrix} \quad (3.4)$$

If we put $\delta = 1$, then $y_{p+1} = x_{p+1}$ in (3.3). Hence, if δ is close to 1, the spline function v is indeed an "approximation" to the noisy input data $\{x_p\}$. However, a digital filter must be stable. Therefore, we will discuss how close can δ approach 1 so that the digital filter defined by (3.3) remains stable. Since $w_3 = 2\delta/h^2 \neq 0$, we conclude from Corollary 2.1 that the digital filter is stable if and only if

$\Lambda_\delta := \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) < 1$ where λ_1, λ_2 , and λ_3 are the eigenvalues of A_δ . If z is one of the three eigenvalues λ_1, λ_2 , and λ_3 , then by (2.4), z must satisfy the equation

$$z^3 + (4\delta - 3)z^2 + 3(1 - \delta)z - (1 - \delta) = 0. \quad (3.5)$$

This equation does not contain the time increment h . Hence, Λ_δ is independent of h . For $\delta = 1$, the solutions of (3.5) are 0, 0, -1, so that $\Lambda_1 = 1$. Hence, the filter is not stable if $\delta = 1$. Let

$$\delta := \delta_n = 1 - \frac{1}{n}, \quad n = \pm 2, \pm 3, \dots \quad (3.6)$$

so that $\delta_n \rightarrow 1$. The following tables indicate the stability of the filter for different values of n . Note that for $n = 3$, the roots of (3.5) are

$\frac{1}{3}$, i , $-i$, so that $\Lambda_{\delta_3} = 1$ and the filter is unstable. Indeed, Steelman [5] pointed out that this filter is stable if and only if $2/3 < \delta < 1$ although a proof is not given in [5]. These tables verify the truth of the statement. Note that in Table 1A, for $n = 3$, δ_n should be $2/3$ instead of $.667$ and the filter should be unstable as mentioned above.

TABLE 1A. STABILITY VS. MAXIMUM MODULUS OF THE EIGENVALUES OF A_1

<u>N</u>	<u>DELTA = 1 - 1/N</u>	<u>MAXIMAL MOD OF THE ROOT</u>	<u>STABILITY</u>
2	0.500	1.1228020	NO
3	0.667	0.9999995	NO
4	0.750	0.9158811	YES
5	0.800	0.8543339	YES
6	0.833	0.8067909	YES
7	0.857	0.7685731	YES
8	0.875	0.7369311	YES
9	0.889	0.7101327	YES
10	0.900	0.6870247	YES
11	0.909	0.6668116	YES
12	0.917	0.6489261	YES
13	0.923	0.6329178	YES
14	0.929	0.6184893	YES
15	0.933	0.6054116	YES
16	0.938	0.5934409	YES
17	0.941	0.5824399	YES
18	0.944	0.5722803	YES
19	0.947	0.5627684	YES
20	0.950	0.5539731	YES
21	0.952	0.5458879	YES
22	0.955	0.5380396	YES
23	0.957	0.5307868	YES
24	0.958	0.5239464	YES
25	0.960	0.5174800	YES
26	0.962	0.5113528	YES
27	0.963	0.5055345	YES
28	0.964	0.5000002	YES
29	0.966	0.5835004	YES
30	0.967	0.6170683	YES
31	0.968	0.6419301	YES
32	0.969	0.6621327	YES
33	0.970	0.6793841	YES
34	0.971	0.6942258	YES
35	0.971	0.7074718	YES
36	0.972	0.7193651	YES
37	0.973	0.7301440	YES

TABLE 1A (Cont)

<u>N</u>	<u>DELTA = 1 - 1/N</u>	<u>MAXIMAL MOD OF THE ROOT</u>	<u>STABILITY</u>
38	0.974	0.7399907	YES
39	0.974	0.7490339	YES
40	0.975	0.7573915	YES
41	0.976	0.7651453	YES
42	0.976	0.7723694	YES
43	0.977	0.7791147	YES
44	0.977	0.7854404	YES
45	0.978	0.7913876	YES
46	0.978	0.7969885	YES
47	0.979	0.8022742	YES
48	0.979	0.8072767	YES
48	0.980	0.8120122	YES
50	0.980	0.8165183	YES

TABLE 1B. STABILITY VS. MAXIMUM MODULUS OF THE EIGENVALUES OF A_1

<u>N</u>	<u>DELTA - 1 + 1/N</u>	<u>MAXIMAL MOD OF THE ROOTS</u>	<u>STABILITY</u>
2	1.500	3.4733100	NO
3	1.333	2.7423110	NO
4	1.250	2.3622870	NO
5	1.200	2.1263980	NO
6	1.167	1.9643850	NO
7	1.143	1.8455800	NO
8	1.125	1.7543650	NO
9	1.111	1.6819070	NO
10	1.100	1.6228290	NO
11	1.091	1.5736540	NO
12	1.083	1.5320190	NO
13	1.077	1.4962740	NO
14	1.071	1.4652290	NO
15	1.067	1.4379890	NO
16	1.063	1.4138780	NO
17	1.059	1.3923700	NO
18	1.056	1.3730690	NO
19	1.053	1.3556360	NO
20	1.050	1.3398040	NO
21	1.048	1.3253690	NO
22	1.045	1.3121390	NO
23	1.043	1.2999750	NO
24	1.042	1.2887430	NO
25	1.040	1.2783470	NO
26	1.038	1.2686840	NO
27	1.037	1.2596920	NO
28	1.036	1.2512910	NO
29	1.034	1.2434240	NO
30	1.033	1.2360490	NO

TABLE 1B (Cont)

<u>N</u>	<u>DELTA - 1 + 1/N</u>	<u>MAXIMAL MOD OF THE ROOTS</u>	<u>STABILITY</u>
31	1.032	1.2291180	NO
32	1.031	1.2225870	NO
33	1.030	1.2164250	NO
34	1.029	1.2105970	NO
35	1.029	1.2050840	NO
36	1.028	1.1998570	NO
37	1.027	1.1948880	NO
38	1.026	1.1901720	NO
39	1.026	1.1856750	NO
40	1.025	1.1813930	NO
41	1.024	1.1773080	NO
42	1.024	1.1734020	NO
43	1.023	1.1696650	NO
44	1.023	1.1660900	NO
45	1.022	1.1626620	NO
46	1.022	1.1593790	NO
47	1.021	1.1562260	NO
48	1.021	1.1531930	NO
49	1.020	1.1502780	NO
50	1.020	1.1474740	NO

NOTE: To give a more accurate picture, we also use the values $\delta = N/50$, $N = 0, \dots, 100$ as in Table 1C.

TABLE 1C. Stability VS. MAXIMUM MODULUS OF THE EIGENVALUES OF A_1

<u>N</u>	<u>DELTA = N/50</u>	<u>MAXIMAL MOD OF THE ROOT</u>	<u>STABILITY</u>
0	0.0000	1.0040010	NO
1	0.0200	1.1561870	NO
2	0.0400	1.1873490	NO
3	0.0600	1.2053820	NO
4	0.0800	1.2170370	NO
5	0.1000	1.2247450	NO
6	0.1200	1.2296930	NO
7	0.1400	1.2325600	NO
8	0.1600	1.2337690	NO
9	0.1800	1.2336110	NO
10	0.2000	1.2325780	NO
11	0.2200	1.2299390	NO
12	0.2400	1.2266840	NO
13	0.2600	1.2226040	NO
14	0.2800	1.2177650	NO
15	0.3000	1.2122150	NO
16	0.3200	1.2060020	NO

TABLE 1C (Cont)

<u>N</u>	<u>DELTA = N/50</u>	<u>MAXIMAL MOD OF THE ROOT</u>	<u>STABILITY</u>
17	0.3400	1.1991330	NO
18	0.3600	1.1916550	NO
19	0.3800	1.1835760	NO
20	0.4000	1.1749050	NO
21	0.4200	1.1656540	NO
22	0.4400	1.1558210	NO
23	0.4600	1.1454680	NO
24	0.4800	1.1344070	NO
25	0.5000	1.1227820	NO
26	0.5200	1.1105690	NO
27	0.5400	1.0977530	NO
28	0.5600	1.0842650	NO
29	0.5800	1.0700990	NO
30	0.6000	1.0552270	NO
31	0.6200	1.0396080	NO
32	0.6400	1.0232000	NO
33	0.6600	1.0059480	NO
34	0.6800	0.9877929	YES
35	0.7000	0.9686793	YES
36	0.7200	0.9484389	YES
37	0.7400	0.9270499	YES
38	0.7600	0.9043519	YES
39	0.7800	0.8801807	YES
40	0.8000	0.8543406	YES
41	0.8200	0.8265516	YES
42	0.8400	0.7964933	YES
43	0.8600	0.7638065	YES
44	0.8800	0.7275256	YES
45	0.9000	0.6870258	YES
46	0.9200	0.6407076	YES
47	0.9400	0.5859586	YES
48	0.9600	0.5176454	YES
49	0.9800	0.8165164	YES
50	1.0000	0.9999986	NO
51	1.0200	1.1474740	NO
52	1.0400	1.2783470	NO
53	1.0600	1.3992770	NO
54	1.0800	1.5134960	NO
55	1.1000	1.6228290	NO
56	1.1200	1.7284450	NO
57	1.1400	1.8311170	NO
58	1.1600	1.9314120	NO
59	1.1800	2.0297310	NO
60	1.2000	2.1263980	NO
61	1.2200	2.2216460	NO
62	1.2400	2.3156780	NO
63	1.2600	2.4086440	NO
64	1.2800	2.5006810	NO
65	1.3000	2.5918890	NO

TABLE 1C (Cont)

<u>N</u>	<u>DELTA = N/50</u>	<u>MAXIMAL MOD OF THE ROOT</u>	<u>STABILITY</u>
66	1.3200	2.6823650	NO
67	1.3400	2.7721790	NO
68	1.3600	2.8614030	NO
69	1.3800	2.9500870	NO
70	1.4000	3.0382880	NO
71	1.4200	3.1260400	NO
72	1.4400	3.2133900	NO
73	1.4600	3.3003620	NO
74	1.4800	3.3869950	NO
75	1.5000	3.4733050	NO
76	1.5200	3.5593280	NO
77	1.5400	3.6450760	NO
78	1.5600	3.7305680	NO
79	1.5800	3.8158280	NO
80	1.6000	3.9008630	NO
81	1.6200	3.9856960	NO
82	1.6400	4.0703320	NO
83	1.6600	4.1547900	NO
84	1.6800	4.2390740	NO
85	1.7000	4.3232030	NO
86	1.7200	4.4071750	NO
87	1.7400	4.4910080	NO
88	1.7600	4.5747040	NO
89	1.7800	4.6582750	NO
90	1.8000	11.1541500	NO
91	1.8200	4.7706570	NO
92	1.8400	4.9082810	NO
93	1.8600	4.9914050	NO
94	1.8800	5.0744260	NO
95	1.9000	4.6608080	NO
96	1.9200	5.2401970	NO
97	1.9400	5.3229540	NO
98	1.9600	5.4056270	NO
99	1.9800	5.4882250	NO

The transfer functions $\underline{H}_\delta(z) := [H_1(z), H_2(z), H_3(z)]^T$ can also be obtained by substituting $\underline{w} = [\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$ into (2.8) - (2.10). We have

$$H_1(z) = \frac{\delta + \delta z^{-1}}{1 + (4\delta - 3)z^{-1} + 3(1 - \delta)z^{-2} + (\delta - 1)z^{-3}}, \quad (3.7)$$

$$H_2(z) = \frac{\frac{2\delta}{h}(1 - z^{-1})}{1 + (4\delta - 3)z^{-1} + 3(1 - \delta)z^{-2} + (\delta - 1)z^{-3}}, \quad (3.8)$$

and

$$H_3(z) = \frac{\frac{2\delta}{h^2}(1 - z^{-1})^2}{1 + (4\delta - 3)z^{-1} + 3(1 - \delta)z^{-2} + (\delta - 1)z^{-3}}, \quad (3.9)$$

Also, by applying Theorem 2.2, we can uncouple the system (3.3) to yield:

$$y_p = -(4\delta - 3)y_{p-1} - 3(1 - \delta)y_{p-2} + (1 - \delta)y_{p-3} + \delta x_p + \delta x_{p-1}, \quad (3.10)$$

$$y'_p = -(4\delta - 3)y'_{p-1} - 3(1 - \delta)y'_{p-2} + (1 - \delta)y'_{p-3} + \frac{2\delta}{h}x_p - \frac{2\delta}{h}x_{p-1} \quad (3.11)$$

and

$$y''_p = -(4\delta - 3)y''_{p-1} - 3(1 - \delta)y''_{p-2} + (1 - \delta)y''_{p-3} + \frac{2\delta}{h^2}x_p - \frac{4\delta}{h^2}x_{p-1} + \frac{2\delta}{h^2}x_{p-2}, \quad (3.12)$$

for $p = 0, 1, 2, \dots$ with initial conditions $x_{-1} = x_{-2} = 0$ and $y_{-2} = y_{-3} = 0$ and $\underline{y}_{-1} = [y_{-1}, y'_{-1}, y''_{-1}]^T$ preassigned. Equations (3.10) - (3.12) enable the user to plot the output response $\underline{y}_p = [y_p, y'_p, y''_p]^T$ in three different graphs, where the graph of $\{y_p\}$ shows how the filter smooths the input signal $\{x_p\}$, $p = 0, 1, \dots$. It is also interesting to study how close $\{y_p\}$ is to $\{x_p\}$ when δ is close to but different from 1. This can be done using the transfer function $H_1(z)$ given in (3.7) as in the following. Since the filter is stable for $\frac{2}{3} < \delta < 1$, we always pick such values of δ . For simplicity, we set

$$\epsilon = 1 - \delta > 0$$

so that we have

$$1 - H_1(z) = \epsilon \frac{(1 - z^{-1})^3}{1 + (1 - 4\epsilon)z^{-1} + 3\epsilon z^{-2} - \epsilon z^{-3}} \quad (3.13)$$

By applying Parseval's identity, we have

$$\begin{aligned} \sum_{p=-1}^{\infty} |x_p - y_p|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{-i\omega}) - Y_1(e^{-i\omega})|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 - H_1(e^{-i\omega})|^2 |X(\omega)|^2 d\omega \end{aligned} \quad (3.14)$$

where we have used the common notation $X(\omega) = X(e^{-i\omega})$ and $x_{-1} = 0$.

Hence, from (3.13), we have

$$\begin{aligned} |y_{-1}|^2 + \sum_{p=0}^{\infty} |x_p - y_p|^2 &= \frac{\epsilon^2}{2\pi} \int_{-\pi}^{\pi} \left| \frac{(1 - e^{i\omega})^3}{1 + (1 - 4\epsilon)e^{i\omega} + 3\epsilon e^{i2\omega} - \epsilon e^{i3\omega}} \right|^2 |X(\omega)|^2 d\omega \\ &= \frac{4\epsilon^2}{\pi} \int_{-\pi}^{\pi} \frac{(1 - \cos\omega)^3}{(1 + \rho_{\epsilon} \cos\omega + 3\epsilon \cos 2\omega - \epsilon \cos 3\omega)^2 + (\rho_{\epsilon} \sin\omega + 3\epsilon \sin 2\omega - \epsilon \sin 3\omega)^2} |X(\omega)|^2 d\omega, \end{aligned} \quad (3.15)$$

where $\rho_{\epsilon} := 1 - 4\epsilon$.

The last expression allows us to design Shepherd's parabolic spline predictor-corrector digital filter in an optimal way. Depending on the input spectrum $X(\omega)$, one can pick $\epsilon = 1 - \delta$ (numerically) such that $0 < \epsilon < 1/3$ and such that the last expression above is minimized. In particular, if we have faith on the data $\{x_p\}$, then we can pick $\epsilon > 0$ very close to zero. This gives

$$|y_{-1}|^2 + \sum_{p=0}^{\infty} |x_p - y_p|^2 = \frac{4\epsilon^2}{\pi} \int_{-\pi}^{\pi} \frac{(1 - \cos \omega)^3}{1 + 2\rho_c \cos \omega + \rho_c^2} |X(\omega)|^2 d\omega \quad (3.16)$$

where $\rho_c = 1 - 4\epsilon$ as above. This integral can be used to design the filter efficiently.

4. THE α - β - γ TRACKERS

The α - β - γ tracking equations discussed by Steelman [5] can be obtained from (1.3) - (1.4) by putting $\underline{w} = [\alpha, \beta/h, \gamma/h^2]^T$. This shows that each α - β - γ tracker is a third order predictor-corrector filter, and conversely (e.g. the filter of section 3 is an α - β - γ tracker with $\beta = \gamma = 2\delta$). From (2.8) - (2.10), the transfer functions of this filter can be found to be

$$H_1(z) = \frac{\alpha + (-2\alpha + \beta + \frac{1}{2}\gamma)z^{-1} + (\alpha - \beta + \frac{1}{2}\gamma)z^{-2}}{1 + (\alpha + \beta + \frac{1}{2}\gamma - 3)z^{-1} + (-2\alpha - \beta + \frac{1}{2}\gamma + 3)z^{-2} + (\alpha - 1)z^{-3}} \quad (4.1)$$

$$H_2(z) = \frac{\frac{\beta}{h} + (-2\frac{\beta}{h} + \frac{\gamma}{h})z^{-1} + (\frac{\beta}{h} - \frac{\gamma}{h})z^{-2}}{1 + (\alpha + \beta + \frac{1}{2}\gamma - 3)z^{-1} + (-2\alpha - \beta + \frac{1}{2}\gamma + 3)z^{-2} + (\alpha - 1)z^{-3}} \quad (4.2)$$

and

$$H_3(z) = \frac{\frac{\gamma}{h^2} - 2\frac{\gamma}{h^2}z^{-1} + \frac{\gamma}{h^2}z^{-2}}{1 + (\alpha + \beta + \frac{1}{2}\gamma - 3)z^{-1} + (-2\alpha - \beta + \frac{1}{2}\gamma + 3)z^{-2} + (\alpha - 1)z^{-3}} \quad (4.3)$$

These equations were also obtained in [5] in a different form. However, using the formulation in (4.1) - (4.3), we can immediately uncouple the filter as in section 2, yielding

$$\begin{aligned} y_p = & -(\alpha + \beta + \frac{1}{2}\gamma - 3)y_{p-1} + (2\alpha + \beta - \frac{1}{2}\gamma - 3)y_{p-2} - (\alpha - 1)y_{p-3} \\ & + \alpha x_p - (2\alpha - \beta - \frac{1}{2}\gamma)x_{p-1} + (\alpha - \beta + \frac{1}{2}\gamma)x_{p-2} \end{aligned} \quad (4.4)$$

$$y_p' = -(\alpha + \beta + \frac{1}{2}\gamma - 3)y_{p-1}' + (2\alpha + \beta - \frac{1}{2}\gamma - 3)y_{p-2}' - (\alpha - 1)y_{p-3}' + \frac{\beta}{h}x_p - (\frac{2\beta}{h} - \frac{\gamma}{h})x_{p-1} + (\frac{\beta}{h} - \frac{\gamma}{h})x_{p-2}, \quad (4.5)$$

and

$$y_p'' = -(\alpha + \beta + \frac{1}{2}\gamma - 3)y_{p-1}'' + (2\alpha + \beta - \frac{1}{2}\gamma - 3)y_{p-2}'' - (\alpha - 1)y_{p-3}'' + \frac{\gamma}{h^2}x_p - 2\frac{\gamma}{h^2}x_{p-1} + \frac{\gamma}{h^2}x_{p-2}. \quad (4.6)$$

where $p = 0, 1, 2, \dots$, $y_p = y_p' = y_p'' = 0$ if $p < -1$ and $x_{-1} = x_{-2} = 0$.

The case where $\alpha = 1 - \theta^3$, $\beta = \frac{3}{2}(1 - \theta)^2(1 + \theta)$, and $\gamma = (1 - \theta)^3$ is called a "fading memory polynomial filter of degree 2" by Morrison (cf. [5]). The uncoupled recursive filters (4.4) - (4.6) can be simplified to be

$$y_p = 3\theta y_{p-1} - 3\theta^2 y_{p-2} + \theta^3 y_{p-3} + (1 - \theta^3)x_p - 3\theta(1 - \theta^2)x_{p-1} + 3\theta^2(1 - \theta)x_{p-2}, \quad (4.7)$$

$$y_p' = 3\theta y_{p-1}' - 3\theta^2 y_{p-2}' + \theta^3 y_{p-3}' + \frac{3}{2h}(1 - \theta)^2(1 + \theta)x_p - \frac{2}{h}(1 - \theta)^2(1 + 2\theta)x_{p-1} + \frac{1}{2h}(1 - \theta)^2(1 + 5\theta)x_{p-2}, \quad (4.8)$$

and

$$y_p'' = 3\theta y_{p-1}'' - 3\theta^2 y_{p-2}'' + \theta^3 y_{p-3}'' + \frac{(1 - \theta)^3}{h^2}(x_p - 2x_{p-1} + x_{p-2}), \quad (4.9)$$

for $p = 0, 1, \dots$ with $y_p = y_p' = y_p'' = 0$ for $p < -1$ and $x_{-1} = x_{-2} = 0$.

Note that the feed-back coefficients are particularly simple. The stability of this filter is also particularly easy to check. In fact, it is stable if and only if $|\theta| < 1$.

As another special case of the α - β - γ filter, let us set $\gamma = 0$. This is the so-called α - β tracker, studied in WSMR by Gonzales in 1968 (cf. [2]). In this case, the factor $1 - z^{-1}$ can be cancelled in each of the expressions in (4.1), (4.2), and (4.3); and the transfer functions of the α - β tracker simply become

$$H_1(z) = \frac{\alpha - (\alpha - \beta)z^{-1}}{1 + (\alpha + \beta - 2)z^{-1} - (\alpha - 1)z^{-2}}, \quad (4.10)$$

$$H_2(z) = \frac{\frac{\beta}{h} - \frac{\beta}{h}z^{-1}}{1 + (\alpha + \beta - 2)z^{-1} - (\alpha - 1)z^{-2}}, \quad (4.11)$$

and

$$H_3(z) \equiv 0. \quad (4.12)$$

Since the denominator is a quadratic polynomial, we know immediately that this α - β filter is stable if and only if $\Lambda_{\alpha, \beta} < 1$ where

$$\Lambda_{\alpha, \beta} := \max \left[\left| \frac{\alpha + \beta}{2} - 1 - \sqrt{\left(\frac{\alpha + \beta}{2} \right)^2 - \beta} \right|, \left| \frac{\alpha + \beta}{2} - 1 + \sqrt{\left(\frac{\alpha + \beta}{2} \right)^2 - \beta} \right| \right] \quad (4.13)$$

Hence, it follows easily that for $\beta \geq [(\alpha + \beta)/2]^2$, we have $\Lambda_{\alpha, \beta} < 1$ if and only if $\alpha > 0$. That is, a stability condition for the α - β tracker is:

$$\alpha > 0, \quad (\alpha + \beta)^2 \leq 4\beta. \quad (4.14)$$

Another sufficient condition for stability of the α - β filters is

$$\beta > 0, \quad \alpha + \beta \leq 2. \quad (4.15)$$

Of course, the other sufficient condition for stability

$$\alpha > 0, \quad \beta > 0, \quad 2\alpha + \beta < 4, \quad (4.16)$$

which also follows easily, was obtained by Gonzales (cf. [2,5]). Using an inverse z-transform as in section 2, the α - β tracker equations can be uncoupled into the form:

$$y_p = -(\alpha + \beta - 2)y_{p-1} + (\alpha - 1)y_{p-2} + \alpha x_p - (\alpha - \beta)x_{p-1}, \quad (4.17)$$

$$y'_p = -(\alpha + \beta - 2)y'_{p-1} + (\alpha - 1)y'_{p-2} + \frac{\beta}{h}(x_p - x_{p-1}), \quad (4.18)$$

and $y''_p = 0$, $p = 0, 1, \dots$, where $y_p = y'_p = 0$ if $p < -1$ and $x_p = 0$ if $p < 0$. The condition (4.15) is very useful. It says that if the first feed-back coefficient is nonnegative and $\beta > 0$, then the α - β filter (4.17) - (4.19) is always stable.

Finally, let us discuss McCool's QD filter [cf. 3], namely:

$$\alpha = \frac{60M^2}{10M^3 + 33M^2 + 23M - 6}, \quad (4.19)$$

$$\beta = \frac{2\alpha}{M} \quad \text{and} \quad \gamma = \frac{2\alpha}{M^2}, \quad (4.20)$$

where M is a natural number. The transfer functions of this filter can be obtained by substituting (4.20) into (4.1) - (4.3) and the uncoupled recursive input-output relationship can be obtained from (4.4) - (4.6). We now study the stability of the QD filter. Set

$$f(z) = z^3 + [(M+1)\rho_M - 3]z^2 + [(-2M - 2M+1)\rho_M + 3]z + (M^2\rho_M - 1)$$

with $\rho_M = 60/(10M^3 + 33M^2 + 23M - 6)$. If λ_1 , λ_2 , and λ_3 are the roots of $f(z) = 0$ and $\Lambda_M = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$, then by Corollary 2.1, since $w_3 = \gamma/h^2 = 2\rho_M \neq 0$, we note that the QD digital filter is stable if and only if $\Lambda_M < 1$. The following table indicates the stability of this filter for $M = 0, \pm 1, \pm 2, \dots, \pm 20$.

TABLE 2. STABILITY FOR McCOOL'S QD FILTER

M	RO = P _M	MAXIMAL MOD OF THE ROOT	STABILITY
-20	-0.0009	1.2024780	NO
-19	-0.0011	1.2146490	NO
-18	-0.0012	1.2282810	NO
-17	-0.0015	1.2437600	NO
-16	-0.0018	1.2615480	NO
-15	-0.0022	1.2820790	NO
-14	-0.0028	1.3061450	NO
-13	-0.0036	1.3346260	NO
-12	-0.0047	1.3689580	NO
-11	-0.0063	1.4111290	NO
-10	-0.0087	1.4641570	NO
-9	-0.0124	1.5328290	NO
-8	-0.0188	1.6252010	NO
-7	-0.0303	1.7560380	NO
-6	-0.0538	1.9554370	NO
-5	-0.1099	2.2953960	NO
-4	-0.2857	3.0000000	NO
-3	-1.2500	5.2570590	NO
-2*			
-1	-10.0000	4.8910190	NO
0	-10.0000	13.5226500	NO
1	1.0000	0.9999990	YES
2	0.2381	0.5061458	YES
3	0.0952	0.6546553	YES
4	0.0478	0.7329382	YES
5	0.0275	0.7819132	YES
6	0.0172	0.8155725	YES
7	0.0115	0.8401833	YES
8	0.0081	0.8589627	YES
9	0.0059	0.8737922	YES
10	0.0044	0.8857723	YES
11	0.0034	0.8956649	YES
12	0.0027	0.9040021	YES
13	0.0022	0.9110962	YES
14	0.0018	0.9172200	YES
15	0.0014	0.9225416	YES
16	0.0012	0.9272274	YES
17	0.0010	0.9313622	YES
18	0.0009	0.9350671	YES
19	0.0007	0.9383648	YES
20	0.0006	0.9413764	YES

*Coefficients of the polynomials are undefined; division by zero.

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