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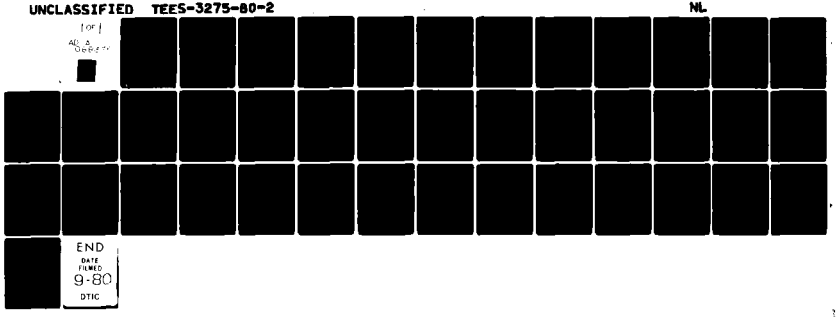

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A SURVEY OF CURRENT TEMPERATURE DEPENDENT ELASTIC-PLASTIC-CREEP CONSTITUTIVE LAWS FOR APPLICABILITY TO FINITE ELEMENT COMPUTER CODES

AEROSPACE ENGINEERING DEPARTMENT

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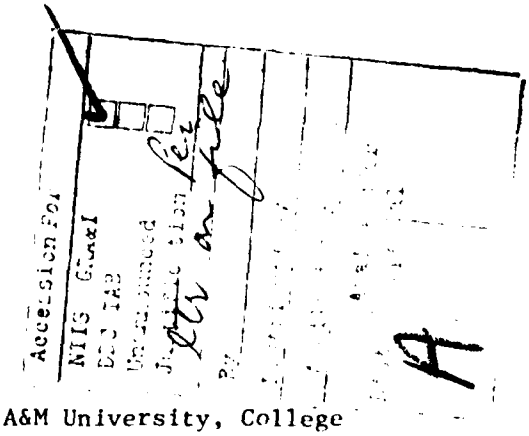
ABSTRACT

The volume of the research effort currently under way to characterize the constitution of thermoelastic-plastic-creep materials is ample evidence of its importance. Although significant work has been done to obtain working constitutive models, in many cases the theory has not been cast with the remaining field equations into a working continuum model. In this paper the author will consider models from classical plasticity and nonlinear viscoelasticity theory for applicability to the conservation of momentum. Based on physical accuracy as well as computational efficiency the author will assess the feasibility of further development of each model.



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A Survey of Current Temperature Dependent Elastic-Plastic-Creep Constitutive Laws for Applicability to Finite Element Computer Codes

David H. Allen*

INTRODUCTION

Current growth in computer capability has brought a parallel expansion in the research area dealing with the predicted response of materials whose constitution is nonlinear. One result of this tremendous explosion of technology has been the need for increasing specialization among scientists and engineers. This specialization has caused the division of constitutive theorists into two distinct and often noninteracting groups: the materials scientists and the mechanics engineers. Each group employs a different approach. Whereas the materials scientist develops a theory based on microstructure, the engineering mechanist tends to formulate a model from continuum theory. The result is that many microstructurally based theories are computationally weak, while continuum theories fail to satisfy microstructural constraints.

A sound constitutive theory must be both physically and computationally effective. In order to establish both of these criteria it is necessary to understand the problem at hand. Therefore, in the first section of this

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paper the author will define the constraints precisely. Following this two models for analyzing thermoelastic-plastic solids will be proposed. Finally, the author will consider strengths and weaknesses of each in order to assess which model is more suitable for further research.

PHYSICAL AND COMPUTATIONAL REQUIREMENTS OF THE MODEL

There are four requirements which a useful constitutive theory must satisfy: thermodynamics, physics, mathematics, and experimental data requirements. In this section each of these requirements will be outlined for the material models to be considered herein.

Thermodynamics

Thermodynamics should be used as the starting point for any constitutive theory. From this constraint will come a general mathematical framework for the model.

Recall that the result of the first and second laws applied to a general system in the neighborhood of the equilibrium state is the following N nonlinear equations of motion [1, 2, 3]:

$$\frac{\partial H}{\partial q_i} + b'_{ij} \frac{dq_j}{dt} = Q_i \ , \tag{1}$$

where H, the Helmholtz free energy, is given by:

$$H = H(q_i, T) \equiv U - TS \ , \tag{2}$$

U is defined by the first law of thermodynamics to be internal energy, and S and T are defined by the second law of thermodynamics to be entropy and temperature, respectively. In addition, Q_i and q_i are generalized forces and displacements, respectively. The nonlinearity in equation (1) arises from the functionality of F in equation (2) and the form of $b'_{ij} = b'_{ij}(q_k, T)$. Under suitably small displacements, Q_m may be regarded as the infinitesimal stress tensor, and q_n as the corresponding strain tensor, where m,n = 1,2...,6.

The remaining coordinates are called hidden coordinates due to the fact that the generalized forces for these indices are identically zero.

The specific mathematical form of the constitutive law will come from the form of the Helmholtz free energy suitable for modelling the material under consideration. For the thermoelastic-plastic material considered herein, it is assumed that the free energy may be expanded about any equilibrium configuration in a second order Taylor series expansion with respect to hidden coordinates similar to that previously proposed [1, 4]:

$$H = H_e + d_r q_r + \tfrac{1}{2} a'_{rs} q_r q_s \ , \tag{3}$$

where

$$H_e = H_e\ (q_m,\ T) \equiv \text{Helmholtz free energy in state e,}$$

$$d_r = d_r\ (q_m,\ T) \equiv (\partial H/\partial q_r)_e, \text{ and}$$

$$a'_{rs} = a'_{rs}\ (q_m,\ T) = a'_{sr} \equiv (\partial^2 H/\partial q_r \partial q_s)_e,$$

the subscript e denotes any equilibrium state for which all $Q_i = 0$, and subscripts r, s, and t range only over indices corresponding to hidden coordinates. Substitution of equation (3) into equations (1) will yield the following equations of motion:

$$\frac{\partial H_e}{\partial q_m} + \frac{\partial d_r}{\partial q_m} q_r + a'_{rs} q_s + b'_{mj} \frac{dq_j}{dt} = Q_m , \qquad (4)$$

for indices corresponding to observed coordinates, and

$$d_r + a'_{rs} q_s + b'_{rj} \frac{dq_j}{dt} = 0, \qquad (5)$$

for indices corresponding to hidden coordinates, where terms of second order in q_r have been neglected.

Specific constitutive equations will come from physical constraints applied to equations (4) and (5).

Physical Constraints

The thermodynamic framework postulated in equation (1) is insufficient to describe a usable constitutive theory. One must in addition define the type of material behavior that the theory should predict. The material to be considered herein is a so-called thermoelastic-plastic-creep material such as aluminum or steel. Mathematically, this type of material may be represented by:

$$S_{ij} = S_{ij} \{E_{mn}, T\}, \qquad (6)$$

where S_{ij} is the stress tensor, E_{mn} is the strain tensor, and { } signify history dependence. The functional dependence of the stress tensor on the strain history alleviates the necessity to include strain rate explicitly on the right hand side of equations (6). At this point no restriction is

necessary regarding the mathematical definitions applied to stress and strain.

Uniaxial tests on thermoelastic-plastic-creep materials will yield the general macrophenomenological behavior shown in Figures (1) and (2). It is seen from the data that several phenomena which should be modelled by the constitutive theory are: (1) the Bauschinger effect for reverse loading, (2) the nonunique mapping from strain to stress, (3) the pronounced effects of temperature and strain-rate, and (4) the general nonlinear behavior of the material. It may also be noted from the figures that the entire load history must be known in order to evaluate the state of strain, and that the effects of temperature are quite similar to those caused by strain rate. All of the above phenomena can be modelled using equations (6).

The microphenomenological behavior of the material is quite complex. The material behavior is characterized by an elastically reversible deformation component which is due to stretching of crystalline bonds, as well as a partially reversible deformation due to sliding of grain boundaries and certain complex phenomena such as dislocation glide and diffusion of molecules within grains. Elevated temperature causes molecular excitation with resulting bulk expansion. The Bauschinger effect is primarily due to microscopic stresses called back stresses which are built up during deformation.

The most important physical restriction imposed by micromechanics is that there is no physically observable mechanical difference between so-called rate independent plastic strain and rate dependent creep strain. These phenomena have often been described separately by macromechanians even though they may be attributed to the same processes.

Incorporating the above physical phenomena into equations (6) and subjecting

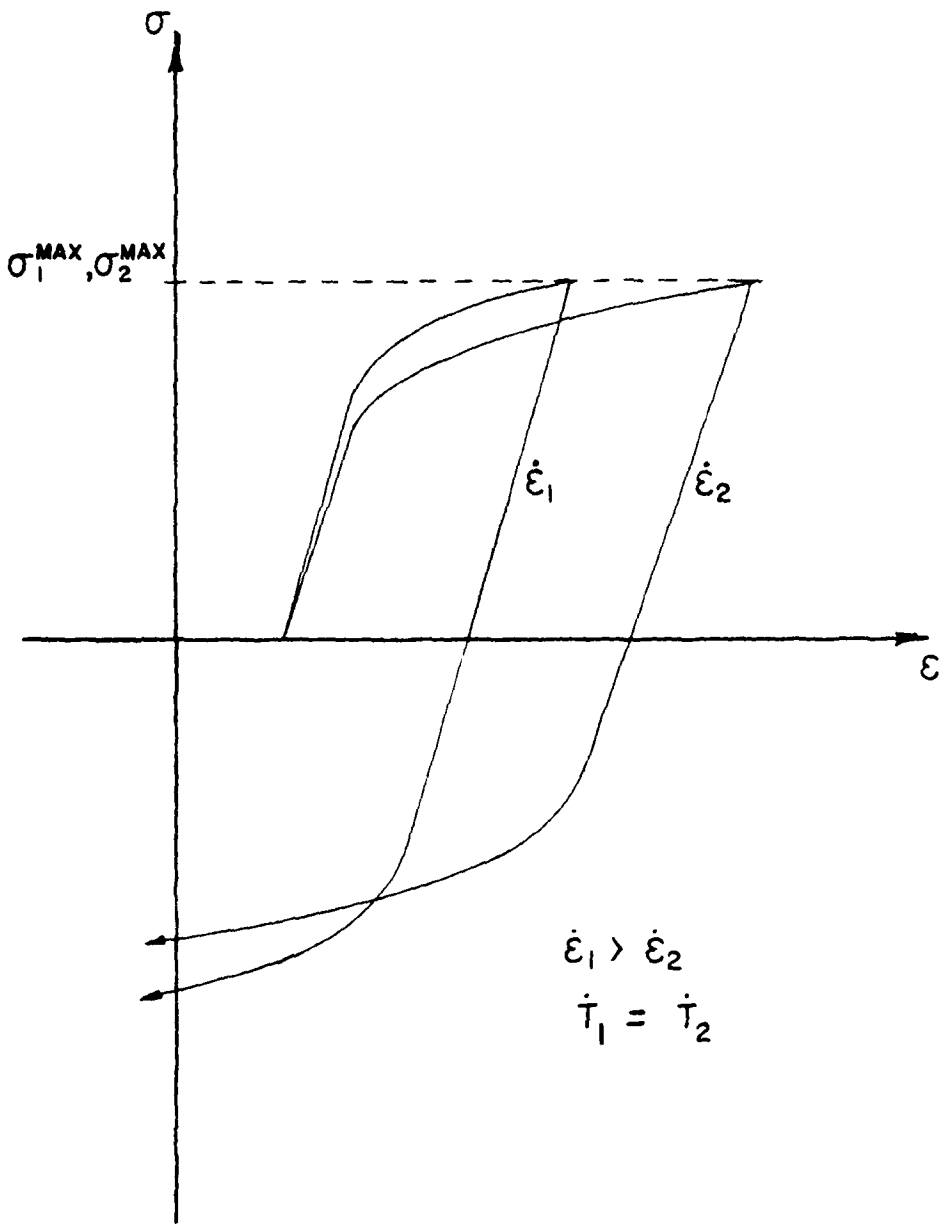


Figure 1. Uniaxial Behavior of Thermoplastic Material at Constant Temperature

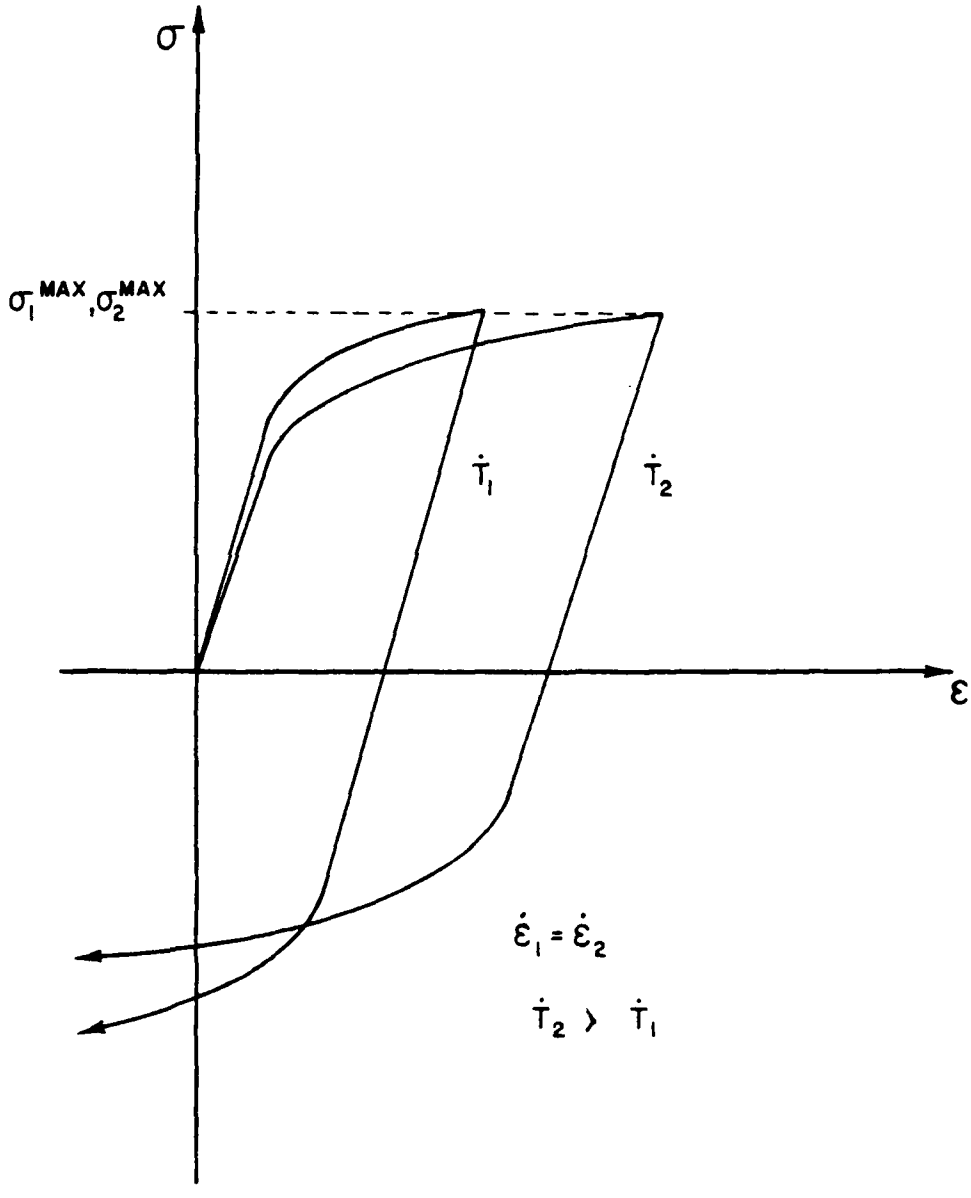


Figure 2. Uniaxial Behavior of Thermoplastic Material at Constant Strain Rate

them to the thermodynamic constraints imposed by equations (4) and (5) will result in a physically based constitutive theory for thermoelastic-plastic materials.

Computational Considerations

It is important that the constitutive theory founded on physical grounds also be computationally usable in solving continuum problems wherein the state of stress and strain vary spacially. This will require that the application to kinematics and conservation of momentum can be performed efficiently. To perform this portion of the model development it is necessary to be familiar with the various methods for satisfying conservation of momentum. Since a detailed explanation of those methods currently in use is beyond the scope of this paper, only one method will be considered herein.

The procedure chosen here is the finite element method applied to the virtual work variational principle. The impetus for choosing this method is that it is already widely in use in computer codes such as AGGIE I [5], NONSAP [6] and ADINA [7]. Thus, development of a new constitutive theory will require only modification of a relatively small portion of the code.

In order to determine the constraints imposed by this computational scheme, it is appropriate to review briefly the method used in the computer codes mentioned above. Recall that momentum is conserved by the theorem of virtual work [6], given by

$$\int_V S_{ij}^{t+\Delta t} \delta E_{ij}^{t+\Delta t} dV = \delta W^{t+\Delta t} \quad , \tag{7}$$

where δ represents a variation, $t+\Delta t$ denotes the time at the end of a load increment, and W represents the external work due to surface tractions and body

forces, both inertial and gravitational. The constitutive law and kinematic equations are then inserted in equation (7) to obtain a working variational principle. In order to obtain a usable principle, however, it is necessary to incrementalize the constitutive and kinematic equations using

$$S_{ij}^{\,t+\Delta t} = S_{ij}^{\,t} + \frac{\Delta S_{ij}}{\Delta t}\,\Delta t \quad , \tag{8}$$

and

$$E_{ij}^{\,t+\Delta t} = E_{ij}^{\,t} + \frac{\Delta E_{ij}}{\Delta t}\,\Delta t \quad . \tag{9}$$

Thus, the constitutive model must be input in incremental or rate form. This restriction results due to two computational difficulties. First, the solution must be obtained in small incremental load steps in order to guarantee convergence of the nonlinear equations resulting from constitution and kinematics. Second, the fact that stress and strain are not uniquely related in the constitutive theory requires that the constitutive theory be written in rate form during single signed load increments in order to guarantee a unique solution.* Thus, equations (6) are replaced by

$$\dot{S}_{ij} = \dot{S}_{ij}\,(\dot{E}_{mn}, \ddot{E}_{mn}, T, k_1, \ldots, k_n) \quad , \tag{10}$$

*Theoretically this restriction may be circumvented with proper state variables, that is, equation (6) may be replaced with $S_{ij} = S_{ij}\,(E_{mn}, \dot{E}_{mn}, T, k_1, \ldots k_n)$, where again k_i are state variables. However, to this author's knowledge, no theories of this type are currently available with sufficient accuracy to model the thermoelastic-plastic material.

where k_i are state variables obtained from the strain history.

It is to be noted that the above incrementation allows for stress to be written as a function of strain and not vice versa due to the form of the variational principle given in equation (7). This requires that displacement conditions be specified on the boundary of the continuum, and that the computational procedure will result in a solution scheme for obtaining unknown displacements as opposed to stresses.

The resulting variational principle is linearized and then discretized using the finite element method [6, 7] to obtain a set of coupled linear differential equations in the displacements at nodal points. The nonlinearities in the variational principle are accounted for by using a modified Newton-Raphson iterative scheme [7].

In summary, then, it is necessary to obtain a constitutive law which satisfies both the computational constraint given by equations (10) and the thermodynamic restrictions in equations (4) and (5), as well as the physical considerations previously noted.

Experimental Data Requirements

The amount of experimental data required to determine a working constitutive model will significantly affect the usefulness of the model. Therefore, consideration should be given to this final constraint in determining the most useful constitutive theory.

INVESTIGATION OF AVAILABLE THEORIES

The available theories for modelling response of thermoplastic materials are of three distinct types: (1) the microphenomenological or equation of state theories [8, 9, 10], (2) the classical plasticity theories [7, 12, 13], and (3) the nonlinear viscoelasticity theories [2, 13, 14, 15]. The first group is derived primarily from studies of the microphysical characteristics of the material. Many so-called unified theories, wherein creep and plasticity are considered to be a single inelastic term, fall within this group. The plasticity theories are primarily macrophenomenological in nature and are generally based on the notion of a yield surface in stress space. The viscoelastic theory derives mainly from thermodynamics, with the constitutive relation usually written in convoluted form.

Although the equation of state theories are receiving considerable attention at this time, their extreme variation in formulation makes it an impossible task to consider them here. It is hoped, however, that valuable information about the remaining two groups may be obtained from microphenomenology. Thus, the theories considered here will be plasticity and nonlinear viscoelasticity. For convenience only uniaxial theory is presented, although three-dimensional theory is analogous.

Classical Plasticity Theory

The author has previously proposed an extension to isothermal plasticity theory to account for thermomechanical and creep behavior [11, 16, 17] and similar to theories proposed by Snyder and Bathe [7] as well as Yamada and Sakurai [12]. That theory is first reviewed briefly.

Recall that in plasticity theory one postulates the existence of a yield surface in stress space, which may be extended to include temperature by

$$F(\sigma - \alpha) = k^2(\int d\bar{\epsilon}^P, T) \quad , \tag{11}$$

where σ is the state of stress, α is the center of the yield surface in stress and temperature space, and k represents the yield surface size as a function of the history of uniaxial plastic strain, $\int d\bar{\epsilon}^P$, and temperature, T. Inside the surface F all action is elastic, whereas on this surface the element undergoes some permanent deformation. The parameters α and $\int d\bar{\epsilon}^P$ may be interpreted as state variables representing the dislocation arrangement and dislocation density, respectively. Both may be conveniently obtained from an appropriate hardening rule and phenomenological uniaxial test data. The yield function [equation (11)] is supplemented with a flow rule, usually given by the normality condition:

$$d\bar{\epsilon}^P = d\lambda \frac{\partial F}{\partial \sigma} \quad , \tag{12}$$

where $d\lambda$ is a scalar to be determined and the superscript P denotes a rate independent instantaneous component of inelastic strain. In addition, it is assumed that the state of stress is dependent on the elastic strain only:

$$\sigma = E (\epsilon - \epsilon^P - \epsilon^C - \epsilon^T) \quad , \tag{13}$$

where the term in parentheses is the elastic recoverable strain caused by bond stretching, and E is the elastic modulus. In addition, the superscripts C and T denote rate dependent creep and temperature dependent thermal strains, respectively. Note that equation (13) satisfies physical constraints only if the sum of plastic and creep strain gives the total inelastic and nonrecoverable strain.

A combination of equations (11) through (13) in rate form will result in the following incremental constitutive law during plastic loading [16]:

$$\frac{d\sigma}{dt} = \left[E^{t+dt} - \frac{E^{t+dt} \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma} E^{t+dt}}{H' \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma} + E^{t+dt} \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma}} \right] \left(\frac{d\varepsilon}{dt} - \frac{d\varepsilon^C}{dt} - \frac{d\varepsilon^T}{dt} \right)$$

$$+ \left[\frac{dE}{dt} - \frac{E^{t+dt} \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma} \frac{dE}{dt}}{H' \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma} + E^{t+dt} \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma}} \right] (\varepsilon^t - \varepsilon^{Pt} - \varepsilon^{Ct} - \varepsilon^{Tt})$$

$$+ \left[\frac{\frac{\partial F}{\partial \sigma} E^{t+dt} \frac{\partial F}{\partial \sigma} \frac{\partial \sigma}{\partial T}}{H' \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma} + E^{t+dt} \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma}} \right] \left(\frac{dT}{dt} \right) , \qquad (14)$$

where H' is the slope of the uniaxial stress-plastic strain diagram, and superscripts t and t+Δt denote parameters at the start of the time step and the end of the time step, respectively. The time dependence in the elastic modulus is caused by a temperature change during the load step. Note that the form of equation (14) implies that all parameters on the right hand side may be determined a priori, that is, without recourse to the current stress increment. Note also that the above constitutive law in rate form is not rotationally invariant [18, 19]. Therefore, its use is restricted to infinitesimal strains only.

The above constitutive law has been implemented to the finite element code AGGIE I [5], along with a nonisothermal combined isotropic-kinematic hardening rule in order to account for the Bauschinger effect during nonisothermal cyclic loading. The model has been shown to give results comparable

to experiment for some thermomechanical load histories [16, 17]. Two significant shortcomings of the theory should be pointed out herein. First, the above theory was obtained without recourse to thermodynamics. Second, the assumption that creep and plasticity may be uncoupled violates the observation that they are physically indistinguishable phenomena.

Nonlinear Viscoelasticity Theory

The nonlinear viscoelasticity thoery is established directly from the thermodynamic equations of motion [equations (4) and (5)]. These equations are first specialized by assuming that:

$$\beta'_m = -\frac{\partial d_r}{\partial q_m}\frac{q_r}{\Theta} \quad ,$$

and

$$\beta'_r = -\frac{d_r}{\Theta} \quad , \tag{15}$$

where $\Theta \equiv T - T_R$ and T_R is the reference temperature. It follows that equations (4) and (5) may be rewritten as:

$$\frac{\partial H_e}{\partial q_m} + b'_{mj}\frac{dq_j}{dt} = Q_m + \beta'_m\Theta \quad , \tag{16}$$

and

$$a'_{rs}\,q_s + b'_{rj}\frac{dq_j}{dt} = \beta'_r\Theta \quad . \tag{17}$$

Now assume further that

$$a'_{rs} = a_F a_{rs};\ a_F = a_F(q_m, T) > 0 \quad , \tag{18}$$

$$b'_{ij} = a_D b_{ij};\ a_D = a_D(q_m, T) > 0 \quad , \text{ and} \tag{19}$$

$$\beta_i' = a_S \beta_i ; a_S = a_S(q_m, T) > 0 \quad , \tag{20}$$

where a_{rs}, b_{ij}, and β_{ij} are constants. The resulting equations will be:

$$a_{ij} q_j + b_{ij} \frac{dq_i}{d\psi} = \tilde{Q}_i + \beta_i \tilde{\Theta} \quad , \tag{21}$$

where

$$\tilde{Q}_i \equiv \frac{1}{a_F} \left(Q_i + a_F a_{in} q_n - \frac{\partial H_e}{\partial q_i} \right) \quad , \tag{22}$$

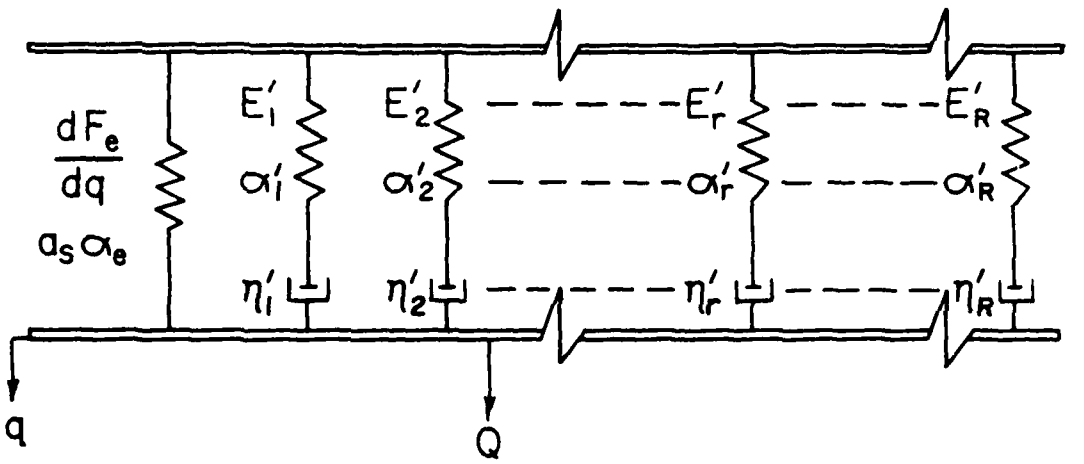
$$\tilde{\Theta} = \frac{a_S}{a_F} \Theta \quad , \tag{23}$$

and the reduced time parameter,

$$\psi = \int_0^t \frac{a_F}{a_D} \, dt \quad . \tag{24}$$

These equations are identical to those obtained by Schapery with the exception of the thermal dependence of a_F and a_S. Schapery has shown that using the second order free energy expansion in the equations of motion (21) is analogous to a generalized Maxwell model with isothermal elastic and thermal expansion coefficients. This analogy, shown in Figure 3, is also true in equations (21) except that these coefficients are now temperature dependent through the terms a_F and a_S.

Now consider a single Maxwell element, as shown in Figure 4. If one considers small displacements only, then the generalized force Q may be replaced by the uniaxial stress σ, and the generalized displacement q is replaced with the strain ε. The differential equation relating stress to strain can be



$$E'_r = E_r a_F \quad , \quad \eta'_r = \eta_r a_D \quad , \quad \alpha_r = \alpha_r \frac{a_S}{a_F}$$

$$a_F = a_F(q_m, T) \ , \ a_D = a_D(q_m, T) \ , \ a_S = a_S(q_m, T)$$

$$E_r, \eta_r, \alpha_r = \text{CONSTANT}$$

E'_r = Spring coefficient for the r^{th} element

α'_r = Coefficient of thermal conductivity for r^{th} element

η'_r = Dashpot viscosity for r^{th} element

Figure 3. Generalized Maxwell Model with Temperature Dependent Elements

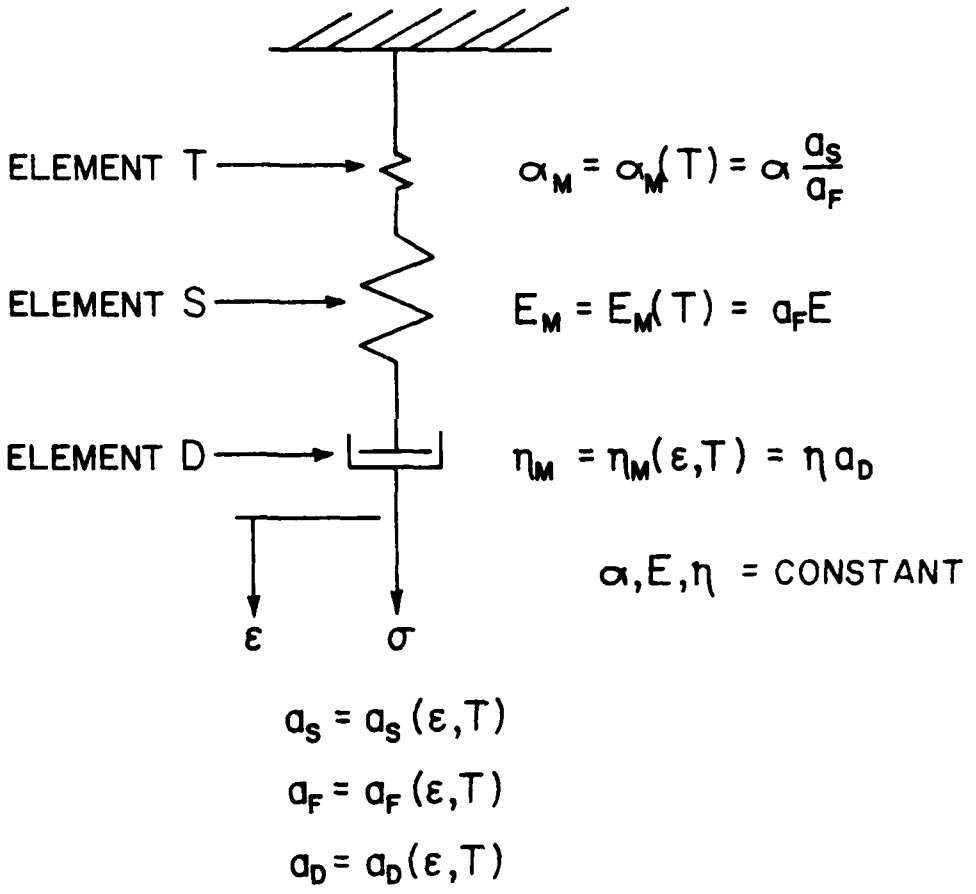


Figure 4. Single Nonlinear Maxwell Element

found to be

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon^T}{dt} + \frac{1}{E_M^{t+dt}} \frac{d\sigma}{dt} - \frac{1}{E_M^{t+dt}} \frac{dE_M}{dt} \frac{\sigma^t}{E_M^t} + \frac{\sigma}{\eta a_D} \quad , \tag{25}$$

where for convenience E_M is used rather than the equivalent form Ea_D, and ε^T denotes the strain in the spring due to a temperature change. Now assume that the entropy production coefficient a_D may be uncoupled as follows:

$$\frac{1}{a_D} = f(\sigma,\varepsilon,T) + g(\sigma,\varepsilon,T) \frac{d\bar{\varepsilon}^P}{dt} \quad , \tag{26}$$

where $d\bar{\varepsilon}^P/dt$ is the uniaxial plastic strain rate and the nonnegativity property imposed by thermodynamics still holds. Since the uniaxial plastic strain rate is both stress and temperature dependent, equation (26) may be rewritten in the following form:

$$\frac{1}{a_D} = f + g\left[\left(\frac{\partial\bar{\varepsilon}^P}{\partial T}\right)_t \frac{dT}{dt} + \left(\frac{\partial\bar{\varepsilon}^P}{\partial\sigma}\right)_t \frac{d\sigma}{dt}\right] \quad . \tag{27}$$

A further modification gives:

$$\frac{1}{a_D} = f + g\left[-\left(\frac{\partial\bar{\varepsilon}^P}{\partial\sigma}\right)_t \left(\frac{\partial\sigma}{\partial T}\right)_t \frac{dT}{dt} + \left(\frac{\partial\bar{\varepsilon}^P}{\partial\sigma}\right)_t \frac{d\sigma}{dt}\right] \quad . \tag{28}$$

Substituting (28) into (25) gives

$$\begin{aligned} \frac{d\varepsilon}{dt} = {} & \frac{d\varepsilon^T}{dt} + \frac{1}{E_M^{t+dt}} \frac{d\sigma}{dt} - \frac{1}{E_M^{t+dt}} \frac{dE_M}{dt} \frac{\sigma^t}{E_M^t} \\ & + \frac{\sigma}{\eta}\left\{f + g\left[-\left(\frac{\partial\bar{\varepsilon}^P}{\partial\sigma}\right)_t \left(\frac{\partial\sigma}{\partial T}\right)_t \frac{dT}{dt} + \left(\frac{\partial\bar{\varepsilon}^P}{\partial\sigma}\right)_t \frac{d\sigma}{dt}\right]\right\} \quad . \end{aligned} \tag{29}$$

Now define

$$\frac{d\epsilon^P}{dt} \equiv \frac{\sigma}{\eta} \; g \left[- \left(\frac{\partial \bar{\epsilon}^P}{\partial \sigma} \right)_t \left(\frac{\partial \sigma}{\partial T} \right)_t \frac{dT}{dt} + \left(\frac{\partial \bar{\epsilon}^P}{\partial \sigma} \right)_t \frac{d\sigma}{dt} \right] \quad , \text{ and} \tag{30}$$

$$\frac{d\epsilon^C}{dt} = \frac{\sigma}{\eta} f \quad , \tag{31}$$

so that the strain components in the Maxwell model are:

$$\epsilon^T = \alpha_M (T - T_R) \quad , \tag{32}$$

$$\epsilon^E = \sigma / E_M \quad , \tag{33}$$

$$\epsilon^P = - \int_{T_R}^{T} \frac{\sigma}{\eta} g \left(\frac{\partial \bar{\epsilon}^P}{\partial \sigma} \right)_t \left(\frac{\partial \sigma}{\partial T} \right)_t dT + \int_0^{\sigma} \frac{g}{\eta} \left(\frac{\partial \bar{\epsilon}^P}{\partial \sigma} \right)_t \sigma d\sigma \quad , \text{ and} \tag{34}$$

$$\epsilon^C = \int_0^t \frac{\sigma}{\eta} f \; dt \quad , \tag{35}$$

where T_R is the reference temperature at which the slider block undergoes zero strain. It can be seen that the viscous terms have been uncoupled into a rate independent plastic term (ϵ^P) and a rate dependent creep term (ϵ^C). Conceptually, this can be described as shown in Figure 5. The plastic slider block is seen to slide according to the stress and temperature inputs, but independently of time. Physically, this analog describes the total dislocation movement in a polycrystal as being composed of two segments: a portion of dislocation movement which occurs instantaneously, or at least as fast as the loading is applied, and a portion such as diffusion which occurs in a delayed

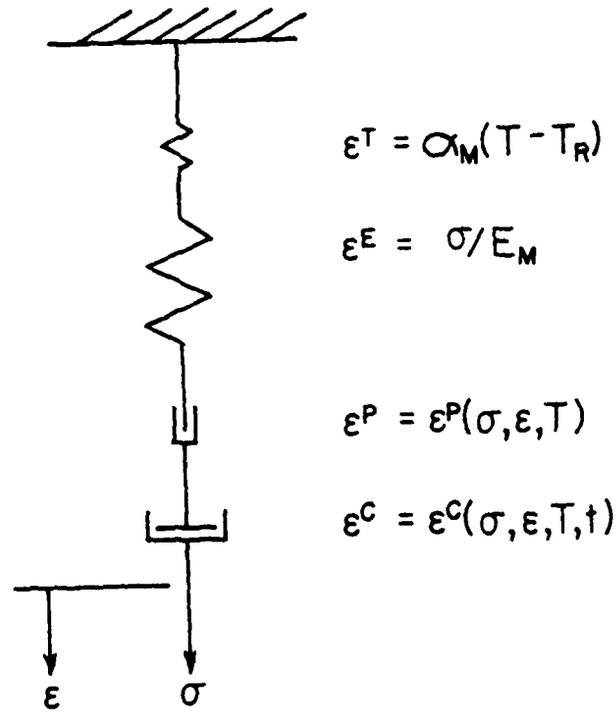


Figure 5. Specialized Nonlinear Maxwell Element with Rate Independent Slider Block

fashion. However one defines these terms, the end result must be that the sum of the strains in the plastic slider block and dashpot must equal the total inelastic strain.

If one considers strain as the input or independent variable it can be seen that equation (29) is a nonlinear first order differential equation in the dependent variable σ. There are several accepted procedures for solving the equation for the stress. Among these are closed form techniques such as a Fredholm or convolution integral, and approximate methods such as Runge-Kutta. Let us first consider a first order Runge-Kutta, sometimes called Euler's method. In this method the differential equation is solved for the stress rate, and the stress terms on the right hand side are assumed at time t. Thus, solving equation (29), one obtains

$$\frac{d\sigma}{dt} = \left[E_M^{t+dt} - \frac{E_M^{t+dt}\, g\sigma^t\, E_M^{t+dt}}{\left(\frac{\partial \sigma}{\partial \bar{\varepsilon}^P}\right)_t g\sigma^t + E_M^{t+dt}\, g\sigma^t} \right] \left(\frac{d\varepsilon}{dt} - \frac{d\varepsilon^C}{dt} - \frac{d\varepsilon^T}{dt} \right)$$

$$+ \left[\frac{dE_M}{dt} - \frac{\frac{dE_M}{dt}\, E_M^{t+dt}\, g\sigma^t}{\left(\frac{\partial \sigma}{\partial \bar{\varepsilon}^P}\right)_t g\sigma^t + E_M^{t+dt}\, g\sigma^t} \right] \left(\varepsilon^{Et} \right)$$

$$+ \left[\frac{E_M^{t+dt}\, g\sigma^t \left(\frac{\partial \sigma}{\partial T}\right)_t}{\left(\frac{\partial \sigma}{\partial \bar{\varepsilon}^P}\right)_t g\sigma^t + E_M^{t+dt}\, g\sigma^t} \right] \left(\frac{dT}{dt} \right), \qquad (36)$$

where, in this case

$$\frac{d\varepsilon^C}{dt} \equiv \left(\frac{d\varepsilon^C}{dt}\right)_t = \frac{\sigma^t}{\eta} \; f\,(\sigma^t, \varepsilon^t, T^t) \quad , \tag{37}$$

and the dashpot coefficient η is found from equation (26) and (30) to be

$$\eta = \sigma^t \; g(\sigma, \varepsilon, T) \quad . \tag{38}$$

Inspection of equation (14) reveals that if one defines

$$g = \frac{1}{\sigma^t} \; \frac{\partial F}{\partial \sigma} \; \frac{\partial F}{\partial \sigma} \quad , \tag{39}$$

where F is as defined in equation (11) and its partial derivative is evaluated at $\sigma = \sigma(t)$, then equation (39) is identical to the plasticity relation given by equation (14). To complete the theory it is necessary only to assume that g and f are zero for unloading. The impetus for using a third order expansion of free energy in conjunction with equations (15), (16), and (26), as well as the first order Runge-Kutta approximation is therefore justified in that introduction of these assumptions into the thermodynamic constraints [equations (4) and (5)] will bring about the recovery of the nonisothermal plasticity theory proposed by this author. The conclusion from this derivation is twofold. First, the classical plasticity theory is thermodynamically consistent under nonisothermal conditions, and the entropy generation term a_D may now be defined precisely for the nonisothermal plasticity theory proposed by the author. Second, the nonlinear viscoelasticity theory may be used to extend the current theory to model more complex phenomena.

The latter statement is illustrated by the fact that the definition of the dashpot coefficient η given by equation (38) requires that

$$\frac{1}{a_D} = g\left(\frac{d\varepsilon^C}{dt} + \frac{d\varepsilon^P}{dt}\right) \quad . \tag{40}$$

Therefore, the recombination of the plastic and creep strain rates into a single inelastic strain produces a unified theory which is accommodated quite simply using reduced time.

Now consider a comparison of an exact solution to equation (29), as opposed to the first order Runge-Kutta approximation given by the plasticity theory [equation (14)]. For this purpose we introduce reduced time [equation (24)] into the differential equation to obtain

$$\frac{\sigma}{\eta} + \frac{1}{E_M^{t+dt}}\frac{d\sigma}{d\psi} = \frac{d\varepsilon}{d\psi} - \frac{d\varepsilon^T}{d\psi} + \frac{\varepsilon^{E^t}}{E_M^{t+dt}}\frac{dE_M}{d\psi} \quad . \tag{41}$$

Equation (41) may be Laplace transformed in reduced time to give

$$\sigma = E_M^{t+dt}\int_0^{\psi} e^{\frac{-E_M^{t+dt}}{\eta}(\psi-\psi')}\frac{d(\varepsilon-\varepsilon^T)}{d\psi'}\,d\psi' \quad ,$$
$$+ \varepsilon^{Et}\int_0^{\psi} e^{\frac{-E_M^{t+dt}}{\eta}(\psi-\psi')}\frac{dE_M}{d\psi'}\,d\psi' \quad , \tag{42}$$

where ψ' is the dummy variable of integration, and ψ is the value of reduced time at time t+dt. Valanis has shown that a single Maxwell element is insufficient for modelling the response of metals [15]. However, for moderate load increments equation (41) is adequate since the generalized Maxwell model necessary to characterize metal response consists of nonoverlapping exponentials.

Thus, a single exponential is predominant at any strain level, with the appropriate Maxwell element described by the current material data in equation (42).

The strain rate and elastic modulus rate during a load step may be considered to be constant inputs:

$$\frac{d(\epsilon - \epsilon^T)}{dt'} = \dot{\epsilon} = \text{constant} \quad , \tag{43}$$

and

$$\frac{dE_M}{dt'} = \dot{E}_M = \text{constant} \quad , \tag{44}$$

so that equation (42) may be reduced to the following form

$$\sigma = E_M^{t+dt}\dot{\epsilon} \int_0^{t+dt} e^{\frac{-E_M^{t+dt}}{\eta}(\psi-\psi')} dt' + \epsilon^t \dot{E}_M \int_0^{t+dt} e^{\frac{-E_M^{t+dt}}{\eta}(\psi-\psi')} dt' \quad . \tag{45}$$

Therefore, the stress increment during a finite load step is

$$d\sigma = \sigma(t+dt) - \sigma(t) = \left(E_M^{t+dt}\dot{\epsilon} + \epsilon^t \dot{E}_M\right) \int_0^{\psi} a_D e^{\frac{-E_M^{t+dt}}{\eta}(\psi - \psi')} d\psi' - \sigma(t) \quad . \tag{46}$$

In general equation (46) must be solved numerically because the entropy generation term a_D is a function of time. It should be noted that the functional form of the nonlinear viscoelastic law proposed above is similar to that proposed in the classical plasticity theory [equation (14)]. In addition, its application to a finite element method should be straightforward due to the proper rate form.

A COMPUTATIONAL COMPARISON

For purposes of comparing the computational efficiency of the two models an example problem has been chosen which is traditionally quite difficult to solve in finite element codes. The problem involves a material not unlike aluminum with isothermal zero time stress-strain curves as shown in Figure 6. At elevated temperatures the creep strain rate may be approximated by a power law:

$$\frac{d\epsilon^C}{dt} = \frac{9 \times 10^{12}}{T(°K)} \left(\frac{\sigma}{E}\right)^5 \quad . \tag{47}$$

The presence of the extremely high creep rates given by equation (47) requires that most finite element codes solve this problem using very small thermo-mechanical load steps, thus consuming considerable computer time. This circumstance is due to the fact that since the stress increment is not known a priori, the creep strain rate must be estimated using the stress at the start of the step.

The material described above has been subjected to two uniaxial thermo-mechanical load histories, as shown in Figure 7. Since the total strain for this problem can be determined by integrating equation (47) and adding to the elastic and plastic strains, the solution may be obtained without resort to the constitutive laws presented herein. However, a measure of the accuracy of the constitutive theory is to apply the exact total strain increment to the constitutive theories and determine which better predicts the proper stress response. This problem becomes particularly interesting when the thermomechanical load is applied at slower and slower rates, thus inducing rapidly increasing creep strain.

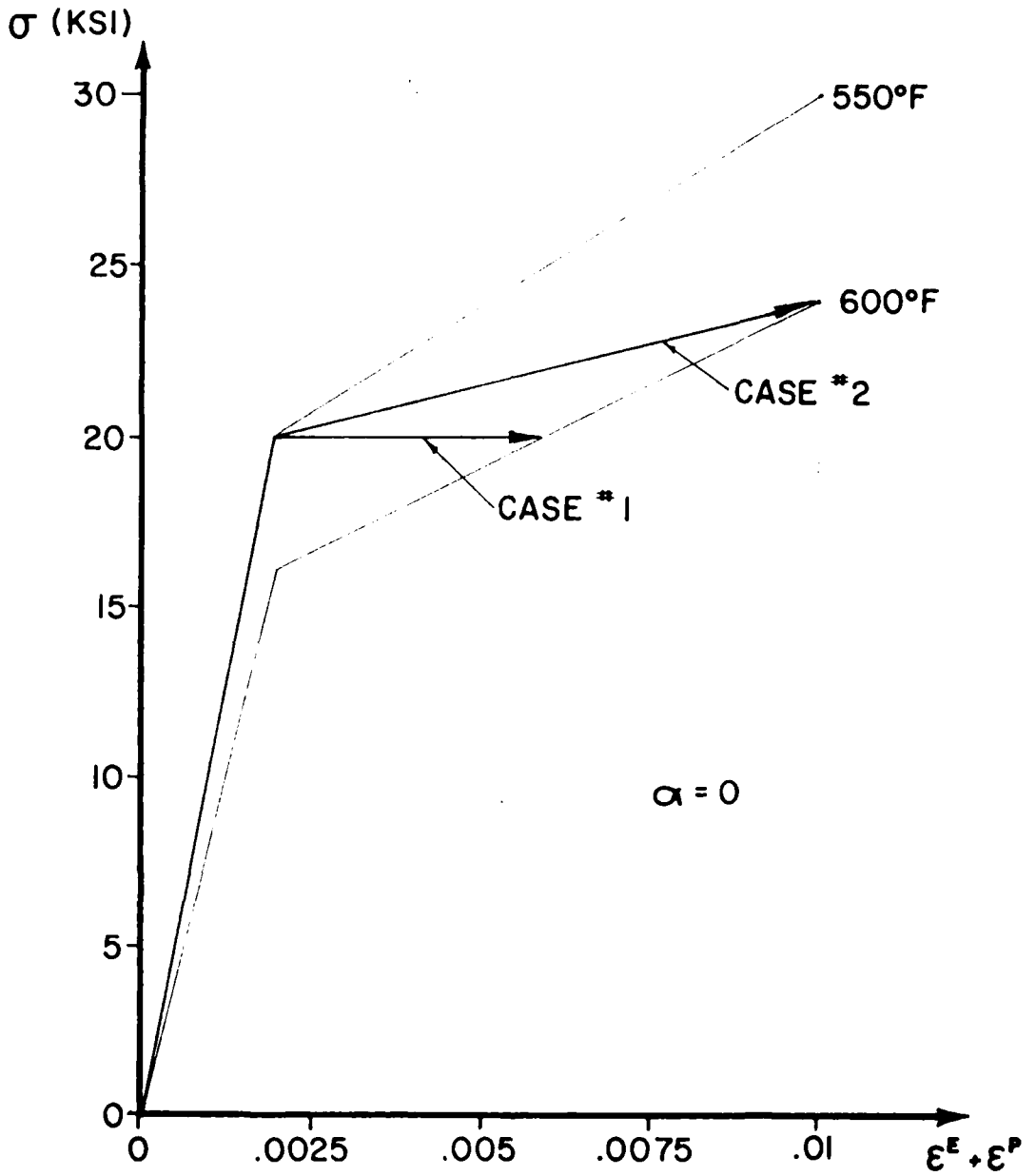


Figure 6. Material Data for Computational Test Cases

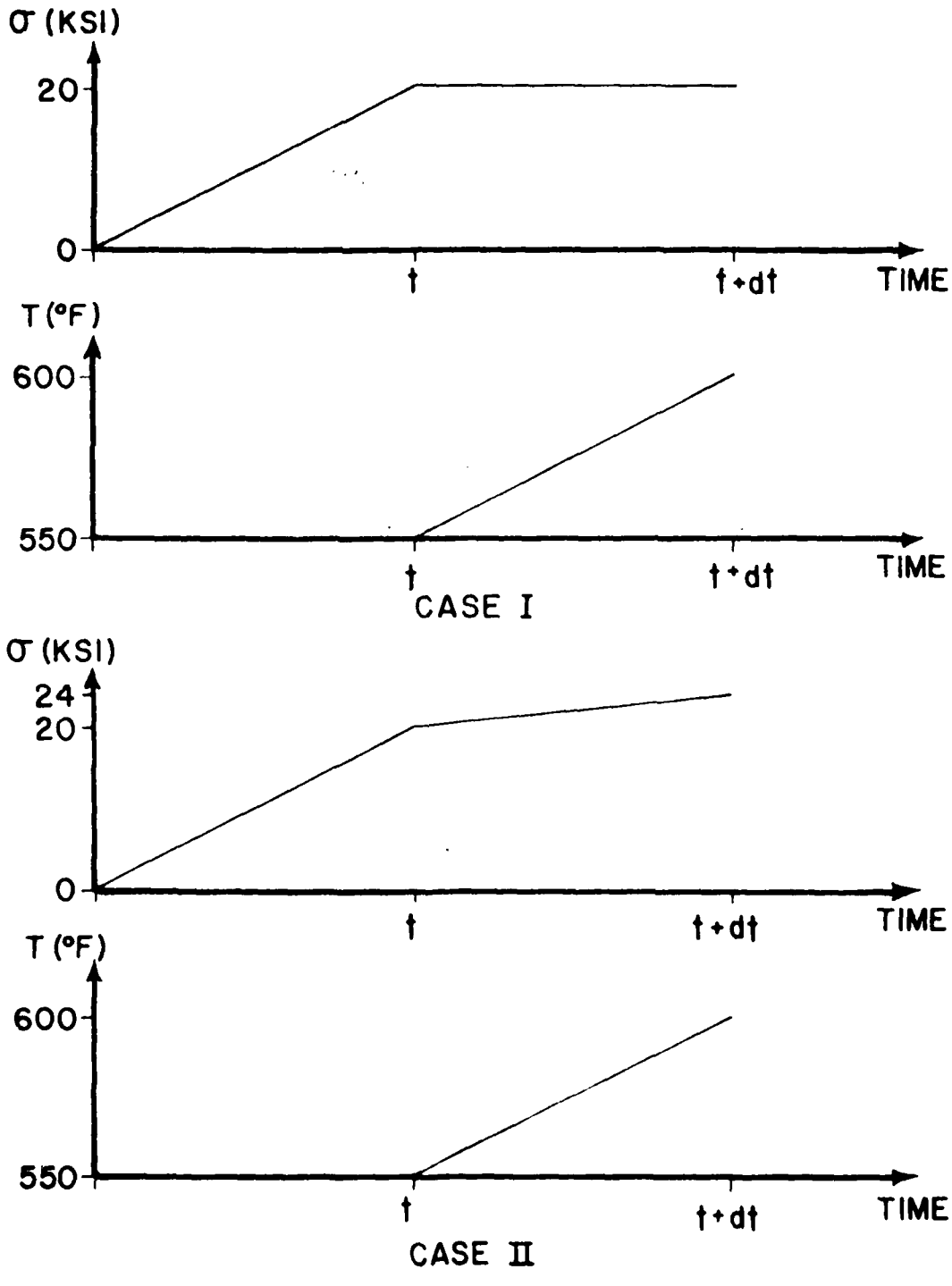


Figure 7. Load Histories for Test Cases I and II

Under the conditions noted in this example, if one assumes that the entropy generation term a_D is constant during the load step, which is identical to plasticity theory, then equation (46) may be integrated in closed form to give

$$d\sigma = \left[\left(\dot{\epsilon} E_M^{t+dt} + \epsilon^{E^t} \dot{E}_M\right) \frac{\eta a_D}{E_M^{t+dt}} - \sigma(t)\right] \left[1 - e^{\frac{-E_M^{t+dt}}{\eta} \psi}\right] . \qquad (48)$$

Valanis [14] has succeeded in characterizing the reduced time ψ for isothermal loadings. However, since the plastic strain rate [equation (30)] in this case is both temperature and stress rate dependent, this author has not yet found a way of determining this parameter. Therefore, in order to perform this computational comparison it was necessary to assume the plastic strain rate in equation (26) could be determined, and the exact values were utilized in solving equation (48) for the examples presented herein.

In addition, two other methods of solution were tested in this study. First, a fourth order Runge-Kutta (Kutta-Simpson 1/3 rule [20]), was tested in conjunction with equation (14). Second, the subincrementation approach [21] was used with equation (14). In the latter method one cycles through the constitutive law several times using subincremental loadings on each load step. For these tests the subincrement was limited to a total strain subincrement of .0005 IN/IN and was not to exceed one hundred subincrements during the load steps.

Results of the various solution procedures are shown in Figures 8 and 9, wherein all analyses except subincrementation were performed in a single time step. The results were obtained by first applying the thermomechanical loading in the time period shown on the abscissa and determining the exact strain increment by integrating equation (47) and adding to the elastic and

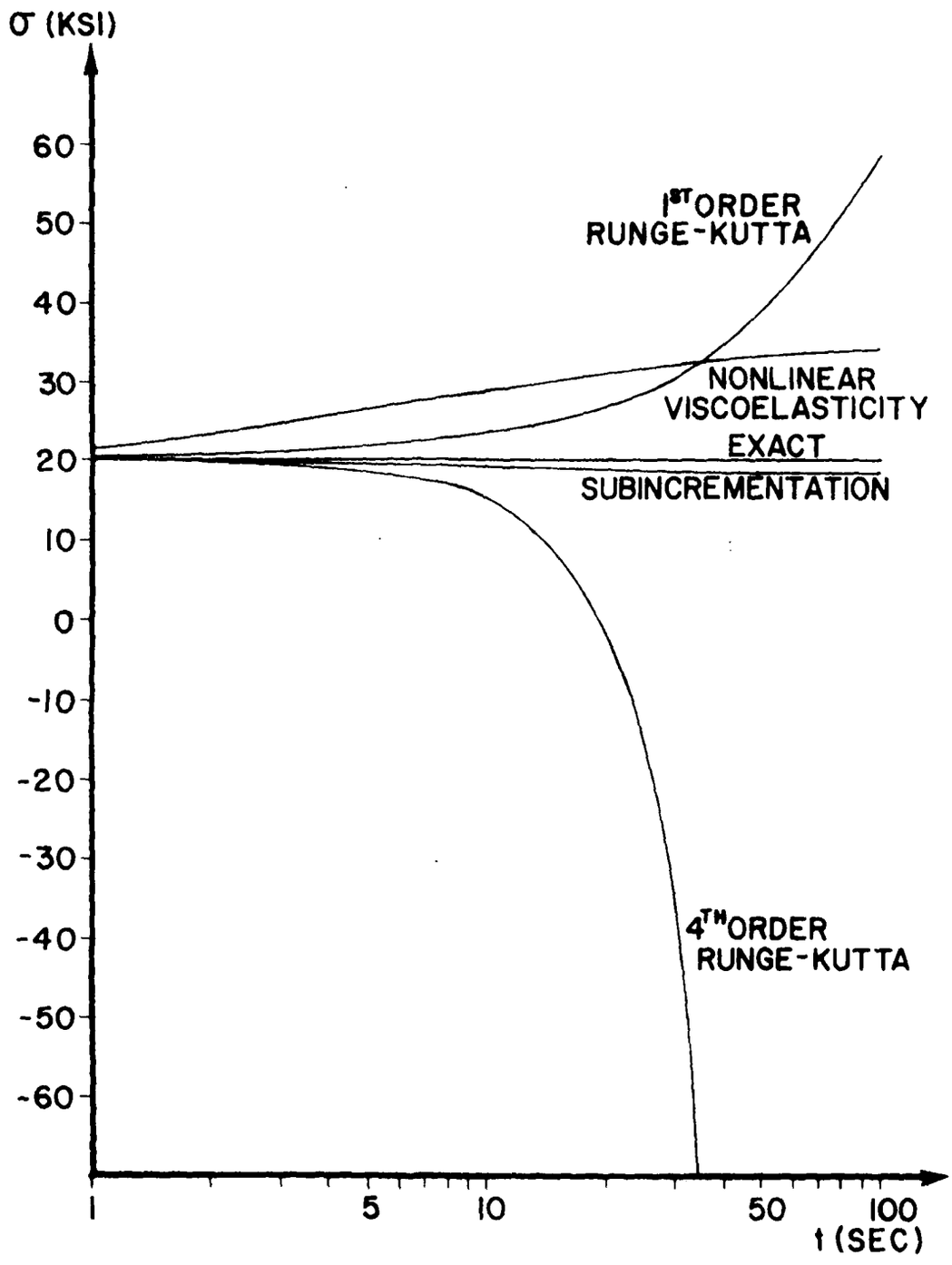


Figure 8. Computational Results for Test Case I

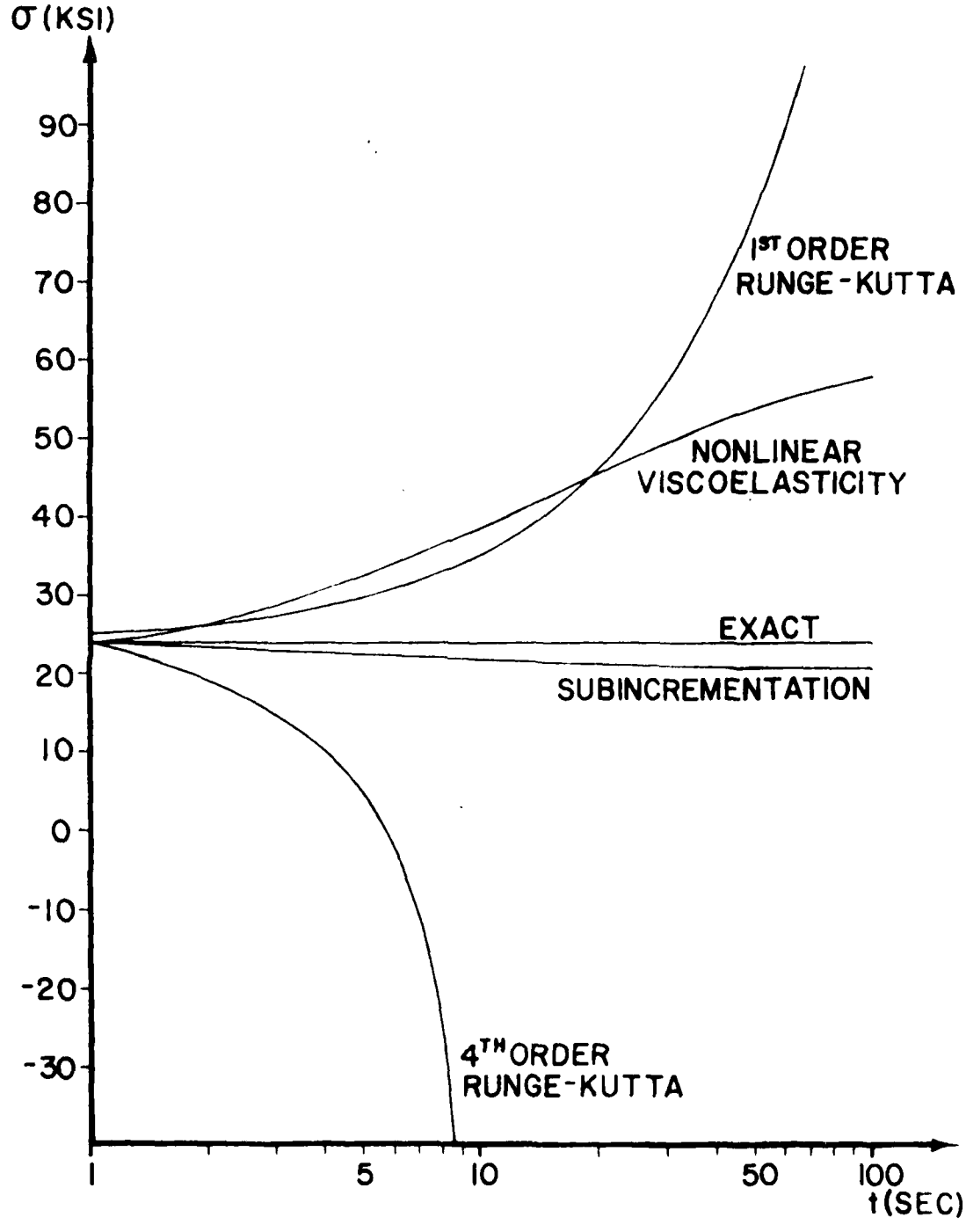


Figure 9. Computational Results for Test Case II

plastic increments shown in Figure 6. This exact strain increment was then used to determine the predicted stress increment in each of the four models discussed above. All of the models are seen to diverge from the exact solution for decreasing load rate. It is seen that the fourth order Runge-Kutta is extremely unstable and therefore unusable. This is due to the fact that the differential equation becomes increasingly "stiff" [22, 23] for increasing time. It is apparent that the first order Runge-Kutta and nonlinear viscoelasticity are at least stable, and will lead to an over approximation or safe estimate of the stress. The subincrementation method is demonstrated to be the best computational scheme for the examples considered here.

CONCLUSION

The author has compared two constitutive theories in this paper on the basis of physical and computational accuracy. It is seen that both models satisfy thermodynamic as well as physical constraints. In addition, both may be constructed in a rate form suitable for application to finite element computer codes. Preliminary evidence indicates that subincrementation used with the classical plasticity theory better satisfies computational requirements than does the nonlinear viscoelasticity. However, this author has considered only a specific type of nonlinear viscoelastic model, and final judgment on the issue is therefore not possible. In addition, there are certain aspects of the viscoelastic model such as the single integral form and the characterization of state variables via entropy generation terms which warrant further study.

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The volume of the research effort currently under way to characterize the constitution of thermoelastic-plastic-creep materials is ample evidence of its importance. Although significant work has been done to obtain working constitutive models, in many cases the theory has not been cast with the remaining field equations into a working continuum model. In this paper the author will consider models from classical plasticity and nonlinear viscoelasticity theory for applicability to the conservation of momentum.

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Based on physical accuracy as well as computational efficiency the author will assess the feasibility of further development of each model.