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Virginia Polytechnic Institute and State University

Industrial Engineering and Operations Research
BLACKSBURG, VIRGINIA 24061

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THE M/G/1 QUEUE WITH

DELAYED FEEDBACK

by

Robert Doyle Foley

VTR 8010

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Industrial and Operations Engineering)
in The University of Michigan
1979

Doctoral Committee:

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The M/G/1 queue with delayed feedback is a two node queueing network. External arrivals to node one from a Poisson process. Node one has a general server with an unlimited queue capacity and a first come, first serve queue discipline. After completing service a customer either departs from the network or feeds back to node two. Node two delays customers before allowing them to reenter node one. The delay is modelled as a ·/M/1 queue. After completing service at node two, a customer reenters node one. Thus the total input process to node one

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ABSTRACT (Con't)

consists of exogenous arrivals and endogenous reentries from node two. The probability of a customer feeding back is allowed to depend on the queue length at nodes one and two and on the service time of the customer at node one.

The system is analyzed by showing that $\{X_n, T_n\}$ forms a Markov renewal process, where T_n is the time of the nth output and X_n is an ordered triplet consisting of the two queue lengths at time T_n and a binary random variable indicating whether the nth output departs or feeds back to node two. The semi-Markov kernel is computed and conditions for $\{X_n, T_n\}$ to be an irreducible, aperiodic Markov renewal process with an infinite lifetime are determined. A necessary and sufficient condition and an easily checked sufficient condition for ergodicity are established.

Using the results about $\{X_n, T_n\}$, it is shown that the joint queue length process is a semi-regenerative process. The time-dependent joint queue length distribution is given. The busy period of the system is shown to form a renewal process and its interrenewal distribution is obtained.

The customer flows generate point processes. It is shown that the outputs from node one, the feedbacks to node two and the departures from the network are each Markov renewal point processes. It is also shown under general conditions that if the departure process is a renewal process, the departure process must be a Poisson process. In addition, if the departure process is a Poisson process, either the server at node one is also an exponential server or there are an uncountable number of initial distributions yielding a Poisson departure process.

In analyzing the flow processes, a result is derived which has interest independent or queues with feedback. Consider a right-continuous Markov process

ABSTRACT (Con't)

$\{X(t): t \geq 0\}$ with left hand limits and an invariant probability measure μ. It is shown that if $\{T_n\}$ forms a non-anticipating Poisson point process and if $\{X(T_n-)\}$ is a Markov chain, then is a stationary distribution for $\{X(T_n-)\}$. This result implies that the distribution seen by Poisson arrivals to a queueing network is the same as the distribution at an arbitrary point in time.

This report is an interim report of on-going research. It may be amended, corrected or withdrawn, if called for, at the discretion of the author.

To My Parents

ACKNOWLEDGEMENTS

I wish to thank all the members of my committee. Professor Shih taught one of the more valuable courses I took at The University of Michigan as well as pointing our some unnecessarily restrictive assumptions I had made in my work. Professors Wendel and Platzman provided many valuable comments on earlier versions of this work.

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Ms. Paula Myers is responsible for the excellent typing. It has been a pleasure working with Paula. Her attention to detail has eliminated many problems before they became serious problems.

I am especially grateful to Professor Disney. It would be impossible to give adequate credit to him for all that he has done for me. My research was partially supported by funds he obtained under National Science Foundation Grant ENG77-22757 and Office of Naval Research Contract N00014-75-C-0492 (NR042-296). The education he provided included not only technical knowledge of the field but also a sense of professionalism and an appreciation of scholarly work. He is an outstanding educator and an all-around fine human being.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Symbol	Definition	Page
T^a	Arrival process	13
T^d	Departure process	13
T^f	Feedback process	13
T^i	Input process	13
T^o	Output process	13
T^r	Reentry process	13
V_t	Forward recurrence time	39
X^o	Output state	19
X^d	Departure state	34
X^f	Feedback state	34
Y_n	Feedback switch	19
$\mathbb{N}$	$\{0,1,2,\cdots\}$	18
$\mathscr{B}$	Borel sets	12
$\mathscr{D}$	Discrete topology on F	18
$\mathscr{E}$	Discrete topology on E	19
$\mathscr{F}(\cdot)$	σ-algebra generated by $(\cdot)$	26
$(\Omega, \mathscr{F}, P)$	Probability space	11
α	Service rate at Q_2	12
γ	Probability measure	41
Γ	Set of probability measures	42
λ	Arrival rate	11
μ	$1/E[S_n^{(1)}]$	29
ν	Probability measure	36
π	Stationary distribution of X^o	37
ρ	λ/μ	31
ω	Sample path	12

Symbol	Definition	Page
Ω	Sample space	12

CHAPTER I

INTRODUCTION

In this dissertation we analyze a two server queueing network with feedback. Chapter I provides an overview of this subject and of this dissertation. In the first section of Chapter I, we supply some background on the subject of queueing networks and feedback in queueing networks which provides motivation for analyzing the M/G/1 queue with delayed feedback. Section two contains an informal description of the M/G/1 queue with delayed feedback. Section three reviews the literature on queueing systems with feedback. Section four describes the main results of this dissertation. In section five the organization of the dissertation is explained. It contains a description of the numbering scheme for chapters, theorems, lemmas, etc. and a brief account of the contents of the remaining chapters.

1. Background Remarks

The field of queueing network theory dates back at least twenty-five years to R. R. P. Jackson's (1954) paper on two M/M/1 queues in tandem. Recent interest in modelling computer systems and computer-communications networks has spurred growth in queueing network theory. Two different philosophies on the analysis of queueing networks have evolved. The first approach analyzes the queueing network as a whole. The second approach involves decomposing the network into subnetworks and analyzing each subnetwork separately. In order to do this, the departure process of each subnetwork must be characterized as well as the superposition and decomposition of flow processes in the network. This interest in the flow processes has gradually led to the realization that queueing networks with feedback are fundamentally different from queueing networks without feedback. Burke (1976) first exhibited this difference when he showed that the input process of the M/M/1 queue with instantaneous feedback is not a Poisson process. Since all flow processes in Jackson networks without feedback are Poisson processes, the introduction of feedback has fundamentally altered the input process. Subsequently, Foley (1977) proved that the output process of the M/M/1 queue with instantaneous feedback was not even a renewal process.

Despite the marked difference in queueing networks with feedback, the literature contains only a handful of theoretical papers analyzing queues with instantaneous feedback and even fewer studying queueing networks with delayed feedback. These queueing systems are important as evidenced by the many applied papers modelling systems as queues with feedback. Disney and Wyszewianski (1975) survey over 200 papers which model computer systems as queues with feedback.

Thus a detailed analysis of the M/G/1 queue with delayed feedback will be useful to practitioners modelling real world systems with delayed feedback. Furthermore, it will provide insight into understanding more general networks.

2. Informal Problem Description

A queue with delayed feedback consists of a queue in which customers completing service either depart from the system or feed back to a delay system. A fed back customer is delayed for a random length of time before reentering the queue (see fig. 1).

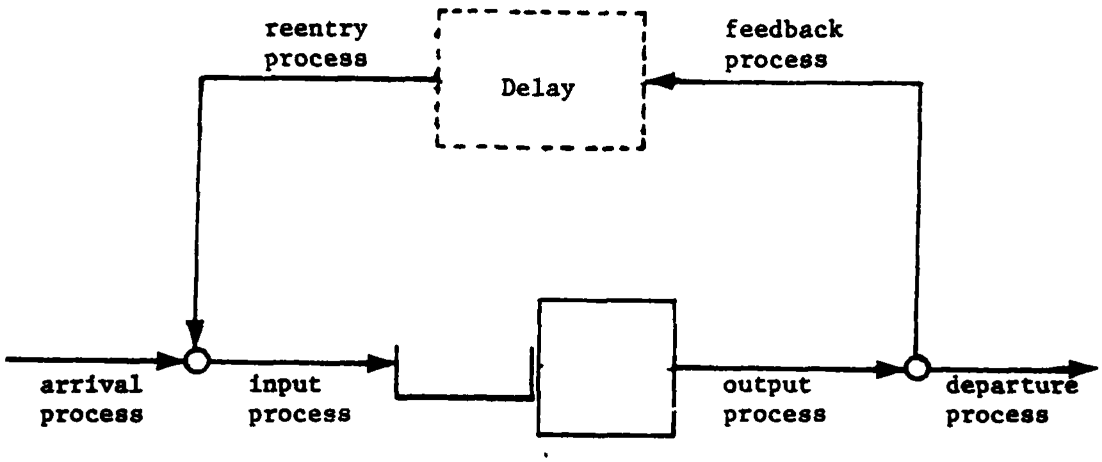

Fig. 1. A Queue with Delayed Feedback

We have chosen to model the delay mechanism as a single server queue with exponential service times and an infinite queue capacity. Other models are possible. Nakamura (1971) modelled the delay mechanism as an infinite server queue with exponential service times. In Nakamura's system the delay represented the time a user of a time-sharing system spent thinking before entering another command.

The M/G/1 queue with delayed feedback is a queue with delayed feedback in which the arrival process (see fig. 1) is a Poisson process and the lower server is a general server. The M/G/1 queue with delayed feedback as shown in figure 2 is the system analyzed in this paper.

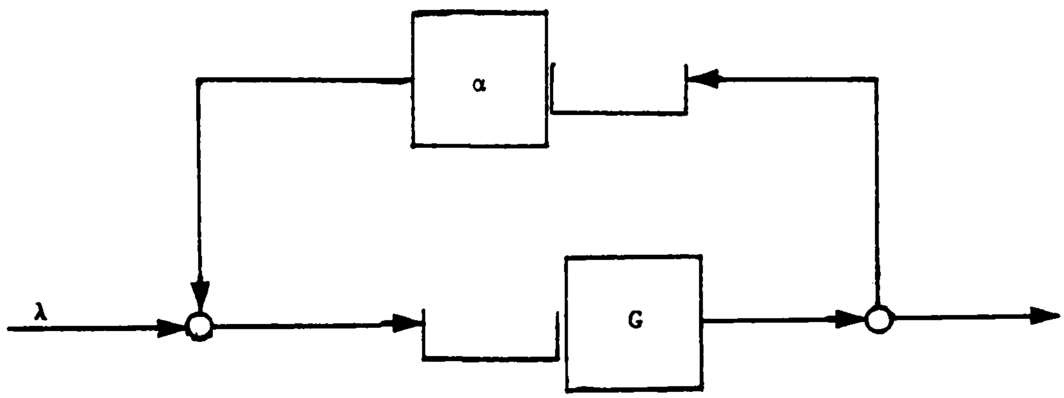

Fig. 2. The M/G/1 Queue with Delayed Feedback

We have not specified how a customer is selected to feed back. We will be more specific in Chapter II. In general the probability of feeding back will be allowed to depend on the queue lengths at both servers and the service time of the customer completing service. Bernoulli feedback is an important special case in which each customer feeds back with a fixed probability p independent of the past history of the system. Note that the M/M/1 queue with delayed Bernoulli feedback is a Jackson (1957) queueing network. We assume throughout the dissertation that customers are served using a first-come, first-serve queue discipline and that both queues have an infinite capacity waiting area.

3. Literature Review

In this section we will review the literature which pertains to queueing systems with feedback. However, we will not review the enormous literature on classical queueing systems which fall under the heading of G/G/c queues. For background literature in classical queueing theory, the reader is referred to Cohen (1969) and Syski (1960).

For a review of queueing network theory, the reader should consult Disney (1975), Lemoine (1977) or the review in the beginning of Boxma's (1977) dissertation. Disney and Wysewianski (1975) survey over 200 papers which model computer systems as queueing systems with feedback.

The M/G/1 queue with instantaneous Bernoulli feedback has been examined by several authors including Takacs (1963), Davignon (1974), Davignon and Disney (1976), Burke (1976), Foley (1977), and Disney and McNickle (1977). For a stationary system, Takacs calculates the queue length distribution and derives the Laplace-Stieltjes transform of the total time spent in the system by a customer. Even in the M/M/1 queue with instantaneous Bernoulli feedback the transform is quite complicated. Takacs was able to determine the first two moments of the distribution. Takacs analyzed the queue length process by finding an equivalent M/G/1 queue without feedback. The equivalent queue retains the same queue length characteristics as well as the same departure process characteristics.

Davignon and Disney (1977) analyze the M/G/1 queue with instantaneous state dependent feedback. The probability of a customer feeding back after completing service was allowed to depend on the change in the queue length since the last service completion, the length of service received, and whether the previous customer fed back or departed. For this system, the authors characterize the stationary queue length process, output process, and departure process. The busy period distribution is also computed.

Burke (1976) used the M/M/1 queue with instantaneous Bernoulli feedback as a counterexample to the belief that the random processes representing the flows on the arcs in Jackson (1957) networks are Poisson processes. Many queueing theorists and applied researchers believed that the flows on the arcs in Jackson networks are Poisson processes (e.g. Nakamura (1971)).

Subsequently Foley (1977) showed that the output process in the M/M/1 queue with instantaneous Bernoulli feedback in equilibrium is not even a renewal process. This result is surprising since even though the output process fails to be a renewal process, the departure process, which is obtained by randomly selecting points from the output process with a fixed probability q, is a Poisson process.

Disney and Hannibalsson (1977) analyze an M/M/1 queue with delayed Bernoulli feedback. The model analyzed is similar to the M/M/1 queue with delayed Bernoulli feedback as discussed in this paper except that each queue has a finite waiting room. The steady state distribution of the two dimensional Markov process $\{N(t), M(t)\}$ is determined where $N(t)$ is the queue length in the lower system and $M(t)$ is the number in the delay system. An algorithm was developed to numerically compute the steady state distribution for a specific system.

4. Main Results

Our first result shows that the state of the system at output points (see fig. 1), (X^o, T^o), is a Markov renewal process. T_n^o is the time of the nth putput and X_n^o is an ordered triplet consisting of the two queue lengths at time T_n^o and an indicator of whether the customer departs or feeds back. This provided the foundation for much of the following analysis. The semi-Markov kernel for (X^o, T^o) is computed and conditions for (X^o, T^o) to be an irreducible aperiodic Markov renewal process with an infinite lifetime are determined based on the governing sequences. The governing sequences are the interarrival times, service time, at the upper and lower queues, and the probability of a customer feeding back. A condition for (X^o, T^o) to be ergodic is determined which involves the invariant measure for the Markov chain X^o. Since in general the invariant measure is not explicitly known, an easily calculated sufficient condition is also determined by applying one of Foster's criteria (cf. Cohen (1969)).

In the following section, III.2 we examine the time dependent queue length process, $Z = \{Z_t\}$. Z is shown to be a semi-regenerative process with imbedded Markov renewal process (X^o, T^o). The probability $P\{Z_t = (j_1, j_2) \mid X_0^o = (i_1, i_2, i_3), T_0^o = 0\}$ is determined and the limiting distribution of Z is analyzed.

In III.3 the busy cycle is shown to be a renewal process. The busy cycle is the time between two successive returns to a completely empty system. Since the busy cycle is a renewal process, we need only describe the interrenewal distribution to characterize the process. The interrenewal distribution is found by using the fact that successive returns to an empty system correspond to a Markov renewal process obtained by filtering (X^o, T^o).

The point processes generated by customer flows are analyzed in III.4. The output process, departure process and feedback process are shown to be Markov renewal processes. In particular we are concerned with determining conditions for the departure process to be a renewal process. We show under general conditions that if the departure process is a renewal process it must be a Poisson process. A conjecture is stated which, if verified, implies that in order to have a renewal departure process, the lower server must be an exponential server.

In Chapter IV we prove some results which were used in Chapter

III and also have interest independent of queues with delayed feedback. We assume that we have a right continuous Markov process $\{X(t): t \geq 0\}$ with left limits and an invariant probability measure μ. We show that if a sequence of stopping times $\{T_n\}$ forms a non-anticipating Poisson point process and if $\{X(T_n -)\}$ is a Markov chain, then μ is a stationary distribution for $\{X(T_n -)\}$. The sequence of stopping times is non-anticipating if the forward recurrence time until the next stopping time after t is independent of X_t. This result can be used to show that the distribution seen by Poisson arrivals to a queueing network is the same as the distribution at an arbitrary point in time.

5. Organization

This dissertation is divided into five chapters and one appendix. Chapters are assigned a Roman numeral. Each chapter is subdivided into sections which are assigned an Arabic numeral. Hence, the third section of Chapter IV is referred to as IV.3. Within each chapter; definitions, lemmas, propositions and theorems are assigned a name, n_1, n_2, where n_1 is the section number and n_2 is assigned consecutively within the section. Within a chapter, the Roman numeral is suppressed. Hence in Chapter III we might refer to Lemma 2.1, but outside of Chapter III we would refer to Lemma III.2.1. Equations, whenever needed, are referenced by an Arabic numeral in parentheses to the right of the equation and numbered consecutively within a chapter. Figures are numbered consecutively throughout the dissertation. Bibliographic references consist of the author's name followed by the publication date enclosed in parentheses.

In Chapter II we normally describe the M/G/1 queue with delayed feedback in terms of the governing sequences. The governing sequences are the arrivals, service times at the upper and lower queues, and the probability of a customer feeding back. The processes of interest, e.g. queue length process, are defined.

Chapter III contains the analysis of the M/G/1 queue with delayed feedback. The system at output points is analyzed as a Markov renewal process. The time dependent queue length process is described and its limiting behavior studied. The busy cycle is shown to be a renewal process and an expression of the interrenewal distribution is obtained.

Chapter IV contains results for more general systems than the M/G/1 queue with delayed feedback and were used in Chapter III. We show that under certain conditions a Markov process and a Markov chain imbedded in the Markov process have the same stationary distribution. These results are useful in areas other than queues with delayed feedback. In Chapter V, we summarize the results in this dissertation and several areas of further research are discussed.

CHAPTER II

FORMAL PROBLEM DESCRIPTION

In Chapter II, we formally discuss the problem to be studied. Section one contains a description of the physical structure of the queueing network. In section two, the governing sequences and assumptions about the governing sequences are listed. Section three introduces notation for the processes of interest (e.g. queue length process), which are determined by the governing sequences and the structure of the network.

1. Queueing Network Structure

The queueing network contains two servers as shown in figure 3. The lower queue will be denoted Q_1, and the upper queue, Q_2. Q_1 and Q_2 are FIFO queues with infinite queue capacities. All external arrivals enter Q_1. After waiting and being serviced at Q_1, the customer outputs. Some outputs depart from the system and the remainder feed back to Q_2. After waiting and being serviced at Q_2, the customer reenters Q_1 (see figure 3).

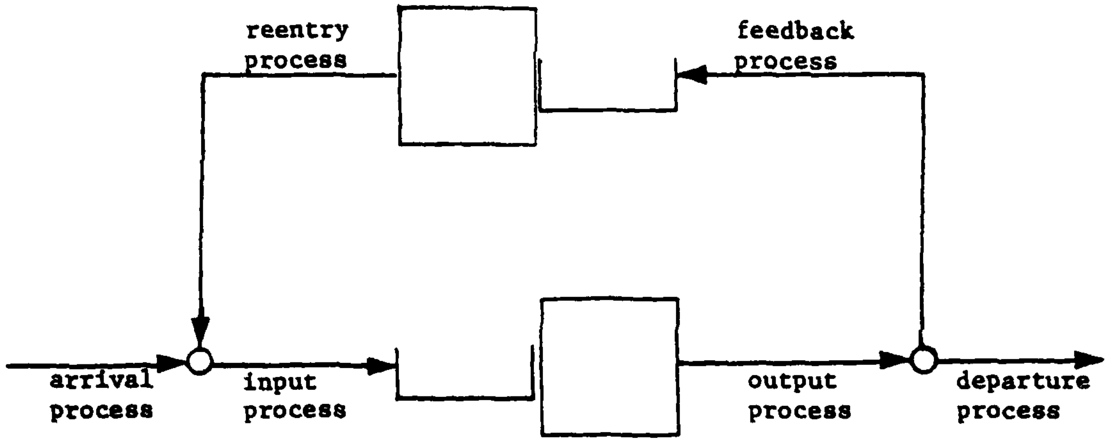

Fig. 3. Structure of the Queueing Network

2. Governing Sequences of Random Variables

The governing sequences are the basic random variables (e.g. service times) from which the processes of interest (e.g. queue length) are defined. Let $\{A_n: n=1,2,...\}$, $\{S_n^{(1)}: n=1,2,...\}$, $\{S_n^{(2)}: n=1,2,...\}$, and $\{Y_n: n=1,2,...\}$ be non-negative, real valued random variables on a complete probability space $(\Omega, \mathscr{F}, P)$. A_n will denote the nth interarrival time to the system; $S_n^{(1)}$, the nth service time at Q_1; $S_n^{(2)}$, the nth service time at Q_2; and Y_n will be either 0 or 1 depending on whether the nth customer completing service at Q_1 departs or feeds back, respectively. Note that the nth arrival does not necessarily receive the nth service time or the nth state of the decomposition switch. $\{A_n\}$, $\{S_n^{(1)}\}$, and $\{S_n^{(2)}\}$ are each i.i.d. sequences of random variables and, furthermore are independent sequences with the following properties:

$$P\{A_n \le t\} = 1 - e^{-\lambda t}, \quad 0 < \lambda < \infty, \quad t \ge 0,$$

$$P\{S_n^{(1)} \le t\} = G(t), \quad t \ge 0,$$

$$G(0+) = 0,$$

$$0 < E[S_n^{(1)}] < \infty,$$

$$P\{S_n^{(2)} \leq t\} = 1 - e^{-\alpha t}, \quad 0 < \alpha < \infty, \quad t \geq 0.$$

Thus, the arrival times form a Poisson process. Q_1 has a general server. Q_2 has an exponential server. Any sample path $\omega \in \Omega$ determines four sequences of random variables; $\{A_n(\omega)\}$, the interarrival times; $\{S_n^{(1)}(\omega)\}$, the service times at Q_1; $\{S_n^{(2)}(\omega)\}$, the service times at Q_2; and $\{Y_n(\omega)\}$, the states of the decomposition switch. Y_k will be allowed to depend on $\{A_n\}$, $\{S_n^{(1)}\}$, $\{S_n^{(2)}\}$, and $\{Y_n\}$ in a limited fashion. We will describe the probabilistic structure of Y_n in section three.

3. Processes of Interest

We begin section three with a brief introduction to point processes. Most of the following is taken from Daley and Vere-Jones (1972). Let X be any complete separable σ-compact metric space. Let $\mathcal{N}$ be the space of all non-negative integer-valued measures $N(\cdot)$ defined on the σ-algebra $\mathcal{B}(X)$ of all Borel sets of the state space X, with $N(A) < \infty$ for all bounded $A \in \mathcal{B}(X)$. Let $\mathcal{S}$ be the σ-algebra of subsets of $\mathcal{N}$ generated by all sets of the form $\{N: N(A) \leq k\}$, $(k \in \{0,1,2,\ldots\}$, $A \in \mathcal{B}(X))$.

Definition 3.1. A general (stochastic) point process is a measurable mapping from a probability space $(\Omega, \mathcal{F}, P)$ into $(\mathcal{N}, \mathcal{S})$.

Definition 3.2. A point process is a general point process with state space $X = \mathbb{R}_+ = (0,\infty]$.

Definition 3.3. A general point process with state space $X = Y \times \mathbb{R}_+$ is called a marked point process and Y is referred to as the mark space.

Marked point processes are sometimes called labelled point processes or point processes with an ancillary variable. At time 0, i_1 customers are in Q_1 and i_2 customers are in Q_2. Six point processes will be used to represent the flows of customers in the system.

N^a will be the point process representing the arrival process. N_t^a is a measurable mapping from $(\Omega, \mathscr{F}, P)$ into $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$. For any $\omega \in \Omega$, $N_t^a(\omega)$ is the number of arrivals in the interval $(0,t]$, i.e. $N_t^a = N^a(0,t]$. Similarly, N^i will represent the input process to Q_1, N^o will represent the output process from Q_1, N^d will represent the departure process from the system, N^f will represent the feedback process to Q_2, and N^r will represent the reentry process from Q_2 to Q_1. These six point processes are determined by the initial conditions and the governing sequences. It is easy to see that

$$N_t^i = N_t^a + N_r^r \text{ and } N_t^o = N_t^d + N_t^f.$$

From N_t^a we can determine T_n^a, the time of the nth arrival. Let $T^a = \{T_n^a\}$. Similarly we can determine T^i, T^o, T^d, T^f, and T^r.

In the following theorem we construct the point processes representing the customer flows and show that the point processes are measurable functions defined on $(\Omega, \mathscr{F}, P)$.

Theorem 3.4. N_t^a, N_t^i, N_t^o, N_t^d, N_t^f, N_t^r, T_n^a, T_n^i, T_n^o, T_n^d, T_n^f and T_n^o are measurable functions from $(\Omega, \mathscr{F}, P)$ into $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$.

Proof by induction. Initially there are i_1 customers in Q_1 and i_2 customers in Q_2. Define

$$T_0 = T_0^a = T_0^i = T_0^o = T_0^d = T_0^f = T_0^r = 0,$$
$${}^0T_n^a = {}^0T_n^i = {}^0T_n^o = {}^0T_n^d = {}^0T_n^f = {}^0T_n^r = 0,$$
$${}^0N_t^a = {}^0N_t^i = {}^0N_t^o = {}^0N_t^d = {}^0N_t^f = {}^0N_t^r = 0.$$

We will construct a sequence of measurable functions ${}^{i}T_{n}^{a} \uparrow T_{n}^{a}$ as $i \uparrow \infty$. Similarly for T_{n}^{i}, T_{n}^{o}, T_{n}^{d}, T_{n}^{f}, and T_{n}^{r}, there will be a sequence of measurable functions converging pointwise to each of the functions. Let T_n denote the time of the nth event. An event consists of a change in either one or both queue lengths. Thus an event corresponds to either an input or an output. Let $Z(T_0) = (i_1, i_2)$. $Z(T_n)$ will denote the queue length just after time T_n, the time of the nth event. Our proof is by induction. We assume that T_k, ${}^{k}T_{j}^{a}$, ${}^{k}T_{j}^{i}$, ${}^{k}T_{j}^{o}$, ${}^{k}T_{j}^{d}$, ${}^{k}T_{j}^{f}$, ${}^{k}T_{j}^{r}$, ${}^{k}N_{t}^{a}$, ${}^{k}N_{t}^{i}$, ${}^{k}N_{t}^{o}$, ${}^{k}N_{t}^{d}$, ${}^{k}N_{t}^{f}$, ${}^{k}N_{t}^{r}$, $Z(T_k)$ are measurable for all $k \leq n$. Then we show that T_{n+1}, ${}^{n+1}T_{j}^{a}$, ${}^{n+1}T_{j}^{i}$, ${}^{n+1}T_{j}^{o}$, ${}^{n+1}T_{j}^{d}$, ${}^{n+1}T_{j}^{f}$, ${}^{n+1}T_{j}^{r}$, ${}^{n+1}N_{t}^{a}$, ${}^{n+1}N_{t}^{i}$, ${}^{n+1}N_{t}^{o}$, ${}^{n+1}N_{t}^{d}$, ${}^{n+1}N_{t}^{f}$, ${}^{n+1}N_{t}^{r}$, and $Z(T_{n+1})$ are measurable. We will sometimes write ${}^{n}N^{a}(t)$ for ${}^{n}N_{t}^{a}$, and similarly for the other counting processes. Now T_{n+1} can be expressed as

$$
\begin{aligned}
T_{n+1} = I_{(0,\infty)\times(0,\infty)}(Z(T_n))\{\min[\,&{}^{n}T^{a}_{{}^{n}N^{a}(T_n)} + A_{{}^{n}N^{a}(T_n)+1}, \\
&I_{(0,\infty)\times[0,\infty)}(Z({}^{n}T^{o}_{{}^{n}N^{o}(T_n)}))({}^{n}T^{o}_{{}^{n}N^{o}(T_n)} + S^{(1)}_{{}^{n}N^{o}(T_n)+1}) \\
&+ I_{\{0\}\times[0,\infty)}(Z({}^{n}T^{o}_{{}^{n}N^{o}(T_n)}))({}^{n}T^{i}_{{}^{n}N^{i}({}^{n}T^{o}_{{}^{n}N^{o}(T_n)})+1} + S^{(1)}_{{}^{n}N^{o}(T_n)+1}), \\
&I_{[0,\infty)\times(0,\infty)}(Z({}^{n}T^{r}_{{}^{n}N^{r}(T_n)}))({}^{n}T^{r}_{{}^{n}N^{r}(T_n)} + S^{(2)}_{{}^{n}N^{r}(T_n)+1}) \\
&+ I_{[0,\infty)\times\{0\}}(Z({}^{n}T^{r}_{{}^{n}N^{r}(T_n)}))({}^{n}T^{f}_{{}^{n}N^{f}({}^{n}T^{r}_{{}^{n}N^{r}(T_n)})+1} + S^{(2)}_{{}^{n}N^{r}(T_n)+1})]\} \\
+ I_{\{0\}\times(0,\infty)}(Z(T_n))\{\min[\,&{}^{n}T^{a}_{{}^{n}N^{a}(T_n)} + A_{{}^{n}N^{a}(T_n)+1}, \\
&I_{[0,\infty)\times(0,\infty)}(Z({}^{n}T^{r}_{{}^{n}N^{r}(T_n)}))({}^{n}T^{r}_{{}^{n}N^{r}(T_n)} + S^{(2)}_{{}^{n}N^{r}(T_n)+1}) \; +
\end{aligned}
$$

$$+ I_{[0,\infty)\times\{0\}}(Z({}^nT^r_{{}^nN^r(T_n)}))({}^nT^f_{{}^nN^f({}^nT^r_{{}^nN^r(T_n)})+1} + S^{(2)}_{{}^nN^r(T_n)+1})]\}$$

$$+ I_{(0,\infty)\times\{0\}}(Z(T_n))\{\min[{}^nT^a_{{}^nN^a(T_n)} + A_{{}^nN^a(T_n)+1},$$

$$I_{(0,\infty)\times[0,\infty)}(Z({}^nT^o_{{}^nN^o(T_n)}))({}^nT^o_{{}^nN^o(T_n)} + S^{(1)}_{{}^nN^o(T_n)+1})$$

$$I_{\{0\}\times[0,\infty)}(Z({}^nT^o_{{}^nN^o(T_n)}))({}^nT^i_{{}^nN^i({}^nT^o_{{}^nN^o(T_n)})+1} + S^{(1)}_{{}^nN^o(T_n)+1})]\}$$

$$+ I_{\{0\}\times\{0\}}(Z(T_n))[{}^nT^a_{{}^nN^a(T_n)} + A_{{}^nN^a(T_n)+1}].$$

A little explanation of the previous equation is due. The equation is divided into four terms depending on the queue length at time T_n. In the first term the queue length at both queues is strictly positive, hence there is a customer in service at both queues. Consequently, the time of the next event is the minimum of three random variables: the time until the next arrival, the time until the next output, and the time until the next reentry. The time of the next arrival is the time of the last arrival plus the next interarrival time. The time until the next output must be broken into two cases. In the first case, the queue length at Q_1 at the time of the last output was greater than zero. Hence the time of the next output is the time of the last output plus the next service time. In the second case at the time of the last output, Q_1 became idle. Hence the time until the next output is the time until the first input after it became idle plus the next service time. Note that an input must have occurred by time T_n since Q_1 has a positive queue length at time T_n. The time until the next reentry is also divided into two cases analogously to the time until the next output except that we condition on whether Q_2 is busy or idle at the time of the last reentry. Now T_{n+1} is measurable since T_{n+1} is a combination of sums, products, mins and compositions of functions which are measurable either by the inductive hypothesis or by being a member of one of the governing sequences. We have implicitly assumed that ${}^{n}T^{a}_{{}^{n}N^{a}(T_n)}$ is the time of the last arrival in $(0, T_n)$ and similarly for ${}^{n}T^{o}_{{}^{n}N^{o}(T_n)}$, etc.. This will follow from the following recursive definition of ${}^{n}T^{a}_{k}$ and ${}^{n}N^{a}_{t}$. First note that $T_{n+1} > T_n$ a.s., since in every case the time until the next event is the sum of some past time plus an independent random variable, which is exponentially distributed in all cases except when the random variable is $S_n^{(1)}$. We know that an event has occurred. Now we need to decide whether it was an arrival, departure, feedback, or reentry. T_{n+1} has the following form

$$
\begin{aligned}
T_{n+1} = {} & I_{(0,\infty) \times (0,\infty)}(Z(T_n))\{\min[f_a, f_o, f_r]\} \\
& + I_{\{0\} \times (0,\infty)}(Z(T_n))\{\min[f_a, f_r]\} \\
& + I_{(0,\infty) \times \{0\}}(Z(T_n))\{\min[f_a, f_o]\} \\
& + I_{\{0\} \times \{0\}}(Z(T_n))f_a \, .
\end{aligned}
$$

Define

$$
\begin{aligned}
{}^{n+1}T^{a}_{k} = {} & {}^{n}T^{a}_{k} + I_{\{k-1\}}({}^{n}N^{a}(T_n)) \\
& \cdot [I_{(0,\infty) \times (0,\infty)}(Z(T_n)) I_{(0,\infty)}(f_o - f_a) I_{(0,\infty)}(f_r - f_a) \\
& + I_{\{0\} \times (0,\infty)}(Z(T_n)) I_{(0,\infty)}(f_r - f_a) +
\end{aligned}
$$

$$+ I_{(0,\infty)\times\{0\}}(Z(T_n))I_{(0,\infty)}(f_o-f_a)$$

$$+ I_{\{0\}\times\{0\}}(Z(T_n))]T_{n+1}$$

$${}^{n+1}T_k^d = {}^{n}T_k^d + I_{\{k-1\}}({}^{n}N^d(T_n))$$

$$\cdot\ [I_{(0,\infty)\times(0,\infty)}(Z(T_n))I_{(0,\infty)}(f_a-f_o)I_{(0,\infty)}(f_r-f_o)$$

$$+ I_{(0,\infty)\times\{0\}}(Z(T_n))I_{(0,\infty)}(f_a-f_o)](1 - Y_{{}^{n}N^o(T_n)+1})T_{n+1}$$

$${}^{n+1}T_k^o = {}^{n}T_k^o + I_{\{k-1\}}({}^{n}N^o(T_n))$$

$$\cdot\ [I_{(0,\infty)\times(0,\infty)}(Z(T_n))I_{(0,\infty)}(f_a-f_o)I_{(0,\infty)}(f_r-f_o)$$

$$+ I_{(0,\infty)\times\{0\}}(Z(T_n))I_{(0,\infty)}(f_a-f_o)]T_{n+1}$$

$${}^{n+1}T_k^f = {}^{n}T_k^f + I_{\{k-1\}}({}^{n}N^f(T_n))$$

$$\cdot\ [I_{(0,\infty)\times(0,\infty)}(Z(T_n))I_{(0,\infty)}(f_a-f_o)I_{(0,\infty)}(f_r-f_o)$$

$$+ I_{(0,\infty)\times\{0\}}(Z(T_n))I_{(0,\infty)}(f_a-f_o)]Y_{{}^{n}N^o(T_n)+1}T_{n+1}$$

$${}^{n+1}T_k^r = {}^{n}T_k^r + I_{\{k-1\}}({}^{n}N^r(T_n))$$

$$\cdot\ [I_{(0,\infty)\times(0,\infty)}(Z(T_n))I_{(0,\infty)}(f_a-f_r)I_{(0,\infty)}(f_o-f_r)$$

$$+ I_{\{0\}\times(0,\infty)}(Z(T_n))I_{(0,\infty)}(f_a-f_r)]T_{n+1}$$

$${}^{n+1}T_k^i = {}^{n}T_k^i + I_{\{k-1\}}({}^{n}N^i(T_n))$$

$$\cdot\ [I_{(0,\infty)\times(0,\infty)}(Z(T_n))(1 - I_{(0,\infty)}(f_a-f_o)I_{(0,\infty)}(f_r-f_o))$$

$$+ I_{\{0\}\times(0,\infty)}(Z(T_n)) \ +$$

$$+ I_{(0,\infty)\times\{0\}}(Z(T_n))I_{(0,\infty)}(f_o-f_a)$$

$$+ I_{\{0\}\times\{0\}}(Z(T_n))]T_{n+1}.$$

Thus ${}^{n+1}T_k^a$ is the time of the kth arrival if it is one of the first $n+1$ events. Otherwise, ${}^{n+1}T_k^a = 0$. The same observation is true about ${}^{n+1}T_k^i$, ${}^{n+1}T_k^o$, ${}^{n+1}T_k^d$, ${}^{n+1}T_k^f$, and ${}^{n+1}T_k^r$. Now define

$${}^{n+1}N^a(t) = \sum_k I_{(0,t]}({}^{n+1}T_k^a), \quad {}^{n+1}N^i(t) = \sum_k I_{(0,t]}({}^{n+1}T_k^i),$$

$${}^{n+1}N^o(t) = \sum_k I_{(0,t]}({}^{n+1}T_k^o), \quad {}^{n+1}N^d(t) = \sum_k I_{(0,t]}({}^{n+1}T_k^d),$$

$${}^{n+1}N^f(t) = \sum_k I_{(0,t]}({}^{n+1}T_k^f), \quad {}^{n+1}N^r(t) = \sum_k I_{(0,t]}({}^{n+1}T_k^r).$$

Thus, ${}^{n+1}N^a(t)$ is the number of arrivals in the interval $(0,t]$ which are also one of the first $n+1$ events. The same holds for the other counting processes. Now,

$$Z(T_{n+1}) = (i_1 + {}^{n+1}N^i(T_{n+1}) - {}^{n+1}N^o(T_{n+1}),\ i_2 + {}^{n+1}N^f(T_{n+1}) - {}^{n+1}N^r(T_{n+1})),$$

and we have completed the induction. Each of the functions defined above are measurable since we have expressed them as sums, products or compositions of measurable functions. Now as $n \to \infty$

$${}^nT_k^a \uparrow T_k^a, \quad {}^nT_k^i \uparrow T_k^i, \quad {}^nT_k^o \uparrow T_k^o,$$

$${}^nT_k^d \uparrow T_k^d, \quad {}^nT_k^f \uparrow T_k^f, \quad {}^nT_k^r \uparrow T_k^r,$$

$${}^nN_t^a \uparrow N_t^a, \quad {}^nN_t^i \uparrow N_t^i, \quad {}^nN_t^o \uparrow N_t^o,$$

$${}^nN_t^d \uparrow N_t^d, \quad {}^nN_t^f \uparrow N_t^f, \quad {}^nN_t^r \uparrow N_t^r.$$

Our proof is completed since if $\{f_n\}$ are measurable and $f_n \to f$, then f is measurable. □

Let $\mathbb{N} = \{0,1,2,\ldots\}$, $F = \mathbb{N} \times \mathbb{N}$ and $(F, \mathscr{D})$ be the discrete topological space. The stochastic process $Z = \{Z_t : t \geq 0\}$ will be the <u>joint queue length process</u>, where

$$Z_0(\omega) = (i_1, i_2)$$

and

$$Z_t(\omega) = (i_1 + N_t^i(\omega) - N_t^o(\omega),\ i_2 + N_t^f(\omega) - N_t^r(\omega)), \quad t > 0.$$

If $Z_t = (j_1, j_2)$ then at time t, Q_1 has j_1 customers and Q_2 has j_2 customers: Z_t is a measurable function from $(\Omega, \mathscr{F})$ to $(F, \mathscr{D})$, and is a right continuous function of t having left hand limits a.e..

We now characterize the probabilistic structure of $\{Y_n\}$, the decomposition switch. We assume that

$$\begin{aligned} P\{Y_{n+1} &= k \mid Z_t;\ t \leq T^o_{n+1}\} \\ &= P\{Y_{n+1} = k \mid Z_{T^o_{n+1}-} = (j_1, j_2),\ S^{(1)}_{n+1} = x\} \\ &= \begin{cases} h_k(j_1, j_2, x) & k = 0, 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $Y_1, \ldots, Y_n$ is determined by $\{Z_t;\ t \leq T^o_{n+1}\}$.

<u>Bernoulli feedback</u> is a special case in which

$$h_k(j_1, j_2, x) = \begin{cases} p & \text{if } k = 1, \\ 1-p & \text{if } k = 0. \end{cases}$$

With Bernoulli feedback $\{Y_n\}$ is a i.i.d. sequence of Bernoulli random variables and independent of the other governing sequences.

In much of our analysis, we will be examining the system at output points. Hence, X^o_n will represent the state of the system at time T^o_n. Let $E = \mathbb{N} \times \mathbb{N} \times \{0,1\}$, $(E, \mathscr{E})$ be the discrete topological space, $X^o_0 = (Z_0, 0)$ and $X^o_n = (Z_{T^o_n-} - (1, 0), Y_n)$. Thus X^o_n is an ordered triple and if $X_n = (i_1, i_2, i_3)$ then just after the nth output from Q_1, i_1 customers are in Q_1, i_2 customers are in Q_2 (excluding the nth output should it feedback) and i_3 is 0 or 1 depending on whether the nth output departs or feeds back, respectively. Let $X^o = \{X_n^o\}$. (X^o, T^o) will denote $\{(X_n^o, T_n^o)\}$ where X_n is the state at the nth output and T_n^o is the time of the nth output.

CHAPTER III

ANALYSIS

In Chapter III we study the processes of interest as defined in section three of Chapter II. Section one analyzes (X^o, T^o), the system embedded at outputs. We show that (X^o, T^o) is a Markov renewal process on a countable state space. This allows us, in section two, to analyze Z, the time dependent queue length process, as a semi-regenerative process. Section three analyzes the busy cycle of the system. Section four contains the analysis of the point processes representing the flows in the network.

1. The System at Output Points

Recall that T_n^o is the time of the nth output from Q_1, and if $X_n^o = (i_1, i_2, i_3)$, then at time T_n^o-, $i_1 + 1$ customers are in Q_1, i_2 customers are in Q_2, and i_3 is 0 or 1 depending on whether the nth output departs or feeds back, respectively. Our first theorem characterizes the structure of (X^o, T^o).

Theorem 1.1. (X^o, T^o) is a Markov renewal process with state space $E = \mathbb{N} \times \mathbb{N} \times \{0, 1\}$.

Proof. We need to show that

$$P\{X_{n+1}^o = (j_1,j_2,j_3),\ T_{n+1}^o - T_n^o \le t \mid X_o^o,\dots,X_n^o;\ T_o^o,\dots,T_n^o\}$$

$$= P\{X_{n+1}^o = (j_1,j_2,j_3),\ T_{n+1}^o - T_n^o \le t \mid X_n^o = (i_1,i_2,i_3)\}$$

for all $(j_1,j_2,j_3) \in E$, $(i_1,i_2,i_3) \in E$, $n \in \mathbb{N}$, and $t \geq 0$. Let $O_{n+1} = T^o_{n+1} - T^o_n$. If $i_1 > 0$, $O_{n+1} = S^{(1)}_{n+1}$. If $i_1 = 0$, then $O_{n+1} = S^{(1)}_{n+1} + I$. If $i_2 + i_3 = 0$, I is an exponential random variable with parameter λ. If $i_2 + i_3 > 0$, I is an exponential random variable with parameter $(\alpha + \lambda)$. Now the number of arrivals to Q_1 and number of departures from Q_2 depend only on X^o_n and O_{n+1}. Since Y_{n+1} depends only on j_1, j_2, and $S^{(1)}_{n+1}$, (X^o, T^o) has the desired property. $\square$

Let Q denote the semi-Markov kernel over E, that is

$$Q_{ij}(t) = P\{X^o_{n+1} = j,\ T^o_{n+1} - T^o_n \leq t \mid X^o_n = i\}$$

where $i = (i_1,i_2,i_3)$, $j = (j_1,j_2,j_3)$, and $i, j \in E$.

<u>Lemma</u> 1.2

$$Q_{ij}(t) = \begin{cases} \displaystyle\int_0^t h_{j_3}(j_1,j_2,x) M_\alpha^{i_2}(i_2-j_2,x) M_\lambda^\infty(j_1-i_1+1+j_2-i_2,x)\,G(dx) \\ \qquad \text{if } \ i_i > 0,\ \ i_3 = 0,\ \ j_1 - i_1 + 1 \geq i_2 - j_2 \geq 0, \\[2ex] \displaystyle\int_0^t \lambda e^{-\lambda s} Q_{mj}(t-s)\,ds, \quad m = (1,0,0), \\ \qquad \text{if } \ i_1 = 0,\ \ i_2 = 0,\ \ i_3 = 0,\ \ j_1 \geq 0,\ \ j_2 = 0, \\[2ex] \displaystyle\int_0^t (\lambda+\alpha) e^{-(\lambda+\alpha)s} \left[\frac{\lambda}{\lambda+\alpha} Q_{mj}(t-s) + \frac{\alpha}{\lambda+\alpha} Q_{nj}(t-s)\right] ds, \\ \qquad m = (1, i_2, 0), \quad n = (1, i_2 - 1, 0), \\ \qquad \text{if } \ i_1 = 0,\ \ i_2 > 0,\ \ i_3 = 0, \\[2ex] Q_{mj}(t), \quad m = (i_1, i_2+1, 0), \\ \qquad \text{if } \ i_3 = 1, \\[2ex] 0 \quad \text{otherwise,} \end{cases}$$

$$\text{where } M_\mu^k(j,x) = \begin{cases} e^{-\mu x}(\mu x)^j/j! & j < k, \\ \sum_{i=k}^{\infty} e^{-\mu x}(\mu x)^i/i! & j = k, \\ 0 \quad \text{otherwise.} \end{cases}$$

Proof.

Case 1. $i_1 > 0$, $i_3 = 0$. Since $i_1 > 0$, $T_{n+1}^o - T_n^o = S_{n+1}^{(1)}$, hence it has distribution $G(x)$. $i_2 - j_2$ customers must leave Q_2. $M_\alpha^{i_2}(i_2 - j_2, x)$ is the probability of this event occurring in time x. $j_1 - i_1 + 1$ customers must enter Q_1. Since $i_2 - j_2$ come from Q_2, $j_1 - i_1 + 1 + j_2 - i_2$ must arrive from the outside. $M_\lambda^\infty(j_1 - i_1 + 1 + j_2 - i_2, x)$ is the probability of this event occurring in time x. Clearly $j_1 - i_1 + 1 \geq i_2 - j_2 \geq 0$.

Case 2. $i_1 = i_2 = i_3 = 0$. Since the system is empty, the system waits an exponentially distributed length of time, s, for an arrival. The probability of now transferring from (1, 0, 0) to j in time $t - s$ is given in case 1.

Case 3. $i_1 = 0$, $i_2 > 0$, $i_3 = 0$. Since $i_1 = 0$ the system waits for an exponential length of time, s, until either an arrival occurs or a customer departs Q_2. An arrival occurs first with probability $\lambda/(\lambda+\alpha)$ and then the probability of transferring from $(1, i_2, 0)$ to j is given in case 1. Similarly an output from Q_2 occurs first with probability $\alpha/(\lambda+\alpha)$ and then the probability of transferring from $(1, i_2 - 1, 0)$ to j is given in case 1.

Case 4. $i_3 = 1$. The probability of transferring from $(i_1, i_2, 1)$ to j is the same as the probability of transferring from $(i_1, i_2 + 1, 0)$ to j. □

Hereafter, we denote $P\{A|X_0^o = i\}$ as $P_i\{A\}$. Let

$$Q^n(i, j, t) = P_i\{X_n^o = j,\ T_n^o \leq t\}$$

and

$$R(t) = \sum_{n=0}^{\infty} Q^n(t)$$

where $Q^n(t)$ is the matrix with $Q^n(i,j,t)$ as its i, jth element, and R(t) is the Markov renewal kernel with elements R(i,j,t).

$$Q(\infty) = \lim_{t\to\infty} Q(t)$$

is the one step transition matrix for the Markov chain X^o (see Çinlar (1975; p. 314)).

Criterion 1.3. (X^o, T^o) is an irreducible Markov renewal process with state space E if and only if

$$0 < \int_0^\infty h_0(j_1, j_2, x)\ G(dx) < 1, \quad j_1, j_2 \in \mathbb{N}. \tag{1}$$

Proof. If $\int_0^\infty h_0(j_1, j_2, x)\ G(dx) = 0$ then the state $(j_1, j_2, 0)$ is not reachable. Similarly, $(j_1, j_2, 1)$ is not reachable if $\int_0^\infty h_0(j_1, j_2, x)\ G(dx) = 1$. If (1) holds, then there is a positive probability of transferring to $(i_{1+1}, i_2, 0)$, $(i_1-1, i_2, 0)$, $(i_1, i_2-1, 0)$ or $(i_1, i_2, 1)$ from $(i_1, i_2, 0)$ in one step. Similarly there is a positive probability of transferring to $(i_1, i_2+1, 0)$ or $(i_1, i_2, 0)$ from $(i_1, i_2, 1)$. By using a finite number of these steps, there is a positive probability of reaching any state from any other state. □

Theorem 1.4. (X^o, T^o) is aperiodic.

Proof. First, some state $(0,\cdot,\cdot)$ is reachable, since in any state $(i_1,\cdot,\cdot)$, $i_1 > 0$, there is a positive probability of entering some state $(i_1-1,\cdot,\cdot)$.

Hence, we need only show that the distribution of the time between returns to $(0, i_2, i_3)$ is aperiodic. But this follows since the inter-return time is composed of a random number of service times and at least one idle time. The idle time is exponentially distributed, hence the inter-return time cannot be arithmetic with a fixed span δ.

Let L be the lifetime of (X^o, T^o). That is,

$$L = \sup_n T_n^o.$$

The following lemma shows that L is almost surely infinite.

Lemma 1.7.

$$L = \sup_n T_n^o = \infty, \quad \text{a.s..}$$

Proof. Since

$$T_n^o \geq S_1^{(1)} + S_2^{(1)} + \cdots + S_n^{(1)},$$

we have

$$L \geq S_1^{(1)} + S_2^{(1)} + \cdots + S_n^{(1)}.$$

By the strong law of large numbers

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} S_i^{(1)}}{n} \to E[S_1^{(1)}] \text{ a.s..}$$

Hence $\sum_{i=1}^{n} S_i^{(1)} \to \infty$ a.s., implying $L = \infty$, a.s..

In the remaining chapters, we will make use of our results about (X^o, T^o) in order to analyze other processes of interest.

2. The Time Dependent Queue Length Process

Z was defined in II.3 to be the joint queue length process. If $Z_t = (i_1, i_2)$, then at time t, Q_1 contains i_1 customers and Q_2 contains

i_2 customers. We use the results about (X^o, T^o) from section one to characterize Z as a semi-regenerative process (cf. Çinlar (1975) p. 343) and to determine the time dependent queue length process.

Lemma 2.1. $\{T_n^o\}$ are stopping times for Z.

Proof. Let $T_0^o = 0$. Now

$$T_n^o = \inf_{t > T_{n-1}^o} \{t: Z_{t-} - Z_t = (-1,\cdot)\}.$$

Let $B \in \mathscr{F}(Z_t;\ 0 \leq t \leq T_n^o)$. Clearly

$$B \cap \{T_{n+1}^o \leq t\} \in \mathscr{F}(Z_s;\ s \leq t). \qquad \square$$

Lemma 2.2. Given i then for all j we have

$$E_i[f(Z_{T_n^o + t_1},\dots,Z_{T_n^o + t_m}) | \mathscr{F}(Z_u;\ u \leq T_n^o)]$$

$$= E_j[f(Z_{t_1},\dots,Z_{t_m})] \text{ on } \{X_n^o = j\}.$$

Proof. By the Markov property of inter-arrival times and service times at Q_2, $\mathscr{F}(Z_u;\ u \leq T_n^o)$ and $\mathscr{F}(Z_u;\ u \geq T_n^o)$ are conditionally independent given X_n, since T_n^o is a point of service completion at Q_1. $\square$

Now we are in a position to characterize the structure of the queue length process, Z. Recall that $F = \mathbb{N} \times \mathbb{N}$.

Theorem 2.3. Z is a semi-regenerative process with state space F.

Proof. From II.3, we know that $Z = \{Z_t \geq 0\}$ is a stochastic process with the discrete topological space $(F,\mathscr{D})$ as its state space, that Z_t is a right continuous function of t having left hand limits a.e.. From section one, we know that (X^o, T^o) is a Markov renewal process with an infinite lifetime. Clearly, $\{X_n^o = i\} \in \mathscr{F}(Z_u;\ u \leq T_n^o)$. This combined with lemmas 2.1 and 2.2 shows that Z is a semi-regenerative process. $\square$

<u>Lemma</u> 2.4. Define $K_t(i, j) = P_i\{Z_t = j,\ T_1^o > t\}$, where $i = (i_1, i_2, i_3) \in E$ and $j = (j_1, j_2) \in F$. Then $K_t(i, j)$ is

$$K_t(i,j) = \begin{cases} M_\alpha^{i_2+i_3}(i_2+i_3-j_2,\ t)\, M_\lambda^\infty(j_1-i_1+j_2-i_2-i_3,\ t)[1-G(t)] \\ \qquad \text{if} \quad i_1 > 0 \\ e^{-(\lambda+\alpha)t} \\ \qquad \text{if} \quad i = (0, i_2, i_3), \quad j = (0, i_2+i_3), \quad i_2+i_3 > 0, \\ e^{-\lambda t} \\ \qquad \text{if} \quad i = (0,0,0), \quad j = (0,0), \\ \displaystyle\int_0^t e^{-\lambda s}[1-F(t-s)]\, \frac{e^{-\lambda(t-s)}[\lambda(t-s)]^{j_1-1}}{(j_1-1)!}\, ds, \\ \qquad \text{if} \quad i = (0,0,0), \quad j = (j_1, 0), \quad j_1 > 0, \\ \displaystyle\int_0^t (\lambda+\alpha)e^{-(\lambda+\alpha)s}[1-F(t-s)] \\ \quad \cdot\left[\frac{\lambda}{\lambda+\alpha} M_\alpha^{i_2+i_3}(i_2+i_3-j_2,\ t-s)M_\lambda^\infty(j_1+j_2-i_2-i_3-1,\ t-s)\right. \\ \quad + \frac{\lambda}{\lambda+\alpha} M_\alpha^{i_2+i_3-1}(i_2+i_3-1-j_2,\ t-s) \\ \qquad \left.\cdot\ M_\lambda^\infty(j_1+j_2-i_2-i_3,\ t-s)\right] ds \\ \qquad \text{if} \quad i = (0, i_2, i_3), \quad j = (j_1, j_2), \quad i_2+i_3 > 0, \end{cases}$$

$$\text{where } M_\mu^k(j,x) = \begin{cases} e^{-\mu x}(\mu x)^j/j! & j < k, \\ \displaystyle\sum_{i=k}^\infty e^{-\mu x}(\mu x)^i/i! & j = k, \\ 0 \quad \textit{otherwise}. \end{cases}$$

$K_t(i, j)$ describes the behavior of Z between semi-regeneration points (i.e. output points).

Proof. Since

$$P_i\{Z_t = (j_1, j_2),\ T_1^o > t\} = P_i\{T_1^o > t\}P_i\{Z_t = (j_1, j_2) \mid T_1^o > t\},$$

we first determine the distribution of T_1^o. Then conditioned on $T_1^o > t$, the event $Z_t = (j_1, j_2)$ will be realized if the number of people in Q_2 decreases from $i_2 + i_3$ to j_2 and the number of people in Q_1 increases from i_1 to j_1. □

Let $P_t(i, A) = P_i\{Z_t \in A\}$. By combining the results of section one with Theorem 2.3 and Lemma 2.4, we are able to obtain the time dependent solution to the queue length process.

Theorem 2.5. For any $i = (i_1, i_2, i_3) \in E$ and $j = (j_1, j_2) \in F$,

$$P_t(i, j) = \sum_{k \in E} \int_0^t R(i, k, ds) K_{t-s}(k, j).$$

Proof. Since Z is a semi-regenerative process with an embedded Markov renewal process (X^o, T^o), the theorem follows directly from Theorem 10.6.8 of Çinlar (1975; p. 346). □

To complete the analysis of the queue length process, we will examine the limiting distribution of Z_t as $t \to \infty$. First, let ν be an invariant measure for the Markov chain X^o and $m(j) = E_j[T_1^o] < \infty$.

Theorem 2.6. If (X^o, T^o) is an irreducible aperiodic recurrent process and $\nu m < \infty$, then

$$\lim_{t \to \infty} P_t(i, A) = \frac{1}{\nu m} \sum_j \nu(j) \int_0^\infty K_t(j, A)\, dt.$$

Remark: In section one, we analyzed conditions for (X^o, T^o) to be irreducible and aperiodic.

Proof. The result follows directly from Theorem 10.6.12 and from the proof of Theorem 10.7.5 case (b) of Çinlar (1975; p. 347, p. 351). □

Recall that $h_0(j_1, j_2, x)$ is the probability of a customer departing given that his service time was x and that he will leave behind j_1 customers in Q_1 and j_2 customers in Q_2. When

$$h_0(j_1, j_2, x) = h_0(j_1, j_2)$$

we can obtain some easily computed sufficient conditions for $\nu m < \infty$. Let $1/\mu$ denote the expected service time of a customer. Since $1/\mu < m_i < 1/\mu + 1/\lambda$ for all i, $\nu m < \infty$ if $\Sigma \nu_i < \infty$. If X^o is an irreducible aperiodic Markov chain this is equivalent to showing that X^o is ergodic.

Theorem 2.7. X^o is ergodic if X^o is an irreducible aperiodic process and if for all but finitely many (j_1, j_2)

$$\frac{\lambda}{\mu h_0(j_1, j_2)} < 1$$

and

$$\frac{\lambda(h_1(j_1,j_2) + \lambda/\mu)}{\alpha(h_0(j_1,j_2) - \lambda/\mu)} < 1.$$

Remark: In the M/M/1 queue with delayed Bernoulli feedback with $h_0(j_1,j_2) = q$, the sufficient conditions become

$$\frac{\lambda}{\mu q} < 1 \quad \text{and} \quad \frac{\lambda(p + \lambda/\mu)}{\alpha(q - \lambda/\mu)} < 1.$$

These conditions are slightly more conservative than the necessary and sufficient conditions in Jackson (1957) which are

$$\frac{\lambda}{\mu q} < 1 \quad \text{and} \quad \frac{\lambda p}{\alpha q} < 1.$$

<u>Proof</u>. We use one of Foster's criteria (cf. Cohen (1969) p. 25) for the ergodicity of a Markov chain. The condition is that we find some nonnegative function f such that

$$\left| E[f(X_{n+1}) \mid X_n] \right| < \infty$$

and for all but finitely many states X_n^o,

$$E[f(X_{n+1}^o) - f(X_n^o) \mid X_n^o] < 0.$$

Since $X_n^o = (i_1, i_2, i_3)$, let

$$f(X_n^o) = f(i_1, i_2, i_3) = i_1 + i_2 + i_3.$$

Thus $f(X_n^o)$ is the total number in the system. Now,

$$E[f(X_{n+1}^o) - f(X_n^o) | X_n^o = (i_1,i_2,i_3)] = \begin{cases} \frac{\lambda}{\mu} - h_0(j_1,j_2) & \text{if } i_1 > 0, \\ \frac{\alpha}{\lambda+\alpha}\left(\frac{\lambda}{\mu} - h_0(j_1,j_2)\right) & \\ + \frac{\lambda}{\lambda+\alpha}\left(1 + \frac{\lambda}{\mu} - h_0(j_1,j_2)\right) & \\ \qquad \text{if } i_1 = 0 \text{ and } i_2 > 0. & \end{cases}$$

Hence if

$$\frac{\lambda}{\mu} - h_0(j_1,j_2) < 0$$

and

$$\frac{\lambda}{\lambda+\alpha}\left(1 + \frac{\lambda}{\mu} - h_0(j_1,j_2)\right) + \frac{\alpha}{\lambda+\alpha}\left(\frac{\lambda}{\mu} - h_0(j_1,j_2)\right) < 0,$$

and since $|E[f(X_{n+1}^o)|X_n^o]| < \infty$, X_n^o is ergodic. The above simplify to the stated conditions. □

An example of an application of Theorem 2.7 is the M/G/1 queue with delayed feedback in which

$$h_1(j_1, j_2, x) = h_1(j_1, j_2) = p_1^{j_1} p_2^{j_2}, \quad 0 < p_1, p_2 < 1.$$

Using Theorem 2.7, a sufficient condition for ergodicity is that

$$\frac{\lambda}{\mu} < 1 \text{ and } \frac{\lambda\rho}{\alpha(1 - \rho)} < 1$$

where $\rho = \lambda/\mu$. This can be shown since there exists J such that for all (j_1, j_2) with $j_1 + j_2 > J$

$$\frac{\lambda}{\mu h_0(j_1,j_2)} < \frac{\lambda}{\mu[1 - \max(p_1,p_2)^J]} < 1$$

and

$$\frac{\lambda(h_1(j_1,j_2) + \lambda/\mu)}{\alpha(h_0(j_1,j_2) - \lambda/\mu)} < \frac{\lambda(\max(p_1,p_2)^J + \lambda/\mu)}{\alpha(1 - \max(p_1,p_2)^J - \lambda/\mu)} < 1.$$

It is interesting to note that the sufficient conditions do not depend on the specific values of p_1 and p_2. In fact the result does not even depend on the geometric form of $h_1(j_1,j_2)$. The only necessary property is that for each $\varepsilon > 0$, there exists a finite number of states (j_1,j_2) such that $h_1(j_1,j_2) > 1 - \varepsilon$.

By analogy to Jackson networks and M/G/1 queues, one might guess that a necessary and sufficient condition for ergodicity is that

$$\frac{\lambda}{\mu h_0(j_1,j_2)} < 1 \text{ and } \frac{\lambda h_1(j_1,j_2)}{\alpha h_0(j_1,j_2)} < 1$$

for all but a finite number of states. In the previous example this would reduce to $\lambda/\mu < 1$. Thus an interesting question is whether there exists a system with $\lambda/\mu < 1$, for each $\varepsilon > 0$, has only a finite number of states with $h_1(j_1,j_2) > 1 - \varepsilon$, and that fails to be ergodic.

3. The Busy Cycle

A busy cycle is comprised of an idle period followed by a busy period. During the idle period both Q_1 and Q_2 are idle. During the busy period at least one of the two queues has a customer in service. The busy period is the sum of i + j independent random variables, i service times at Q_1 with distribution G and j idle times at Q_2 which are exponentially distributed with parameter $\lambda + \alpha$. We will be able to examine the busy cycle by filtering the (X^o, T^o) process analyzed in section one. Let $D = \{(0, 0, 0)\}$ and $E = \mathbb{N} \times \mathbb{N} \times \{0, 1\}$. Assume $X_0^o = (0, 0, 0)$ and $N_0 = 0$. Let

$$N_n = \inf\{i \mid X_i \in D, \quad i > N_{n-1}\}, \qquad n \in \mathbb{N} - \{0\}.$$

Now, we define

$$\hat{X}_n = X^o_{N_n} \text{ and } \hat{T}_n = T^o_{N_n}, \qquad n \in \mathbb{N}.$$

Theorem 3.1. $(\hat{X}, \hat{T})$ is a Markov renewal process.

Proof. The result follows from Theorem 10.1.13 of Çinlar (1975).□

In general, the busy cycle of a queueing network is not necessarily a renewal process. For this network, however, we have the following:

Corollary 3.2. $\hat{T} = \{\hat{T}_i;\ i \in \mathbb{N}\}$ is a renewal process.

Proof. $(\hat{X}, \hat{T})$ is a Markov renewal process whose state space has only one point. By Theorem 10.1.11 of Çinlar (1975), $\hat{T}$ is a renewal process. □

Since $\hat{T}$ is a renewal process, we need only obtain the distribution of an arbitrary interval to characterize the process. First, we partition the semi-Markov kernel of (X, T). Let $D' = E - D$ and

$$Q = \begin{array}{c c} & \begin{array}{cc} D & D' \end{array} \\ \begin{array}{c} D \\ D' \end{array} & \begin{bmatrix} Q_{DD} & Q_{DD'} \\ Q_{D'D} & Q_{D'D'} \end{bmatrix} \end{array} .$$

Theorem 3.3. *The distribution of the busy cycle is given by*

$$Q_{DD} + Q_{DD'} * \left(\sum_{n=0}^{\infty} Q^n_{D'D'}\right) * Q_{D'D} .$$

where * denotes convolution (cf. Çinlar (1975)(10.3.1) p. 323).

Proof. *Follows directly* from theorem 8.15 of Çinlar (1969).

We could analyze Z_t, the joint queue length process, as a regenerative process with $\hat{T}$ as the set of regeneration points. However, it is easier to describe the behavior of Z_t between output points (as done in section two) than between departure points leaving an empty system.

4. Customer Flows

In section four, we analyze the flow of customers on the arcs of the network. The flow on each arc will be represented as either a point process or as a marked point process. (See Definitions II.3.1 through II.3.4 for these concepts). Recall from section III.3 that:

N^a is the arrival process,

N^i is the input process,

N^o is the output process,

N^d is the departure process,

N^f is the feedback process,

N^r is the reentry process.

Equivalently, we have

T_n^a is time of the nth arrival,

T_n^i is time of the nth input,

T_n^o is time of the nth output,

T_n^d is time of the nth departure,

T_n^f is time of the nth feedback,

T_n^r is time of the nth reentry.

First, we examine the customer flows as marked point processes. A marked point process (sometimes called a labelled point process or a point process with an ancillary variable) can be thought of as a point process in which each point T_n has an associated random variable X_n lying in the mark space. Marked point processes are useful in representing an arrival process to a queue with several types of customers. The basic point T_n would be the time of the nth arrival and the associated random variable X_n would be the type of the customer. Clearly the process (X^o, T^o) analyzed in section one is a marked point process. Our first result is a restatement of Theorem 1.1 and characterizes (X^o, T^o), the marked output process.

Theorem 4.1. The marked point process (X^o, T^o) is a Markov renewal process.

Now define

$$X_n^d = Z_{T_n^d-} - (1, 0), \qquad X^d = \{X_n^d\},$$

$$X_n^f = Z_{T_n^f-} - (1, 0), \qquad X^f = \{X_n^f\}.$$

We can now characterize the marked departure process (X^d, T^d) and the marked feedback process (X^f, T^f).

Theorem 4.2. The marked point process (X^d, T^d) is a Markov renewal process with state space $\mathbb{N} \times \mathbb{N}$.

Proof. Let $D = \{(i_1, i_2, 0) \mid (i_1, i_2, 0) \in E\}$. (X^d, T^d) can be obtained by filtering (X^o, T^o) with D (i.e. at departure points). By Theorem 10.1.13 of Çinlar (1975; p. 315), (X^d, T^d) is a Markov renewal process with state space $\mathbb{N} \times \mathbb{N} \times \{0\}$. Clearly we can allow $\mathbb{N} \times \mathbb{N}$ to be the state space. $\square$

Theorem 4.3. The marked point process (X^f, T^f) is a Markov renewal process with state space $\mathbb{N} \times \mathbb{N}$.

Proof. The proof is the same as the proof of Theorem 4.2 except that $D = \{(i_1, i_2, 1) \mid (i_1, i_2, 1) \in E\}$. $\square$

The semi-Markov kernels for (X^f, T^f) and (X^d, T^d) can be obtained from the semi-Markov kernel of (X^o, T^o). Thus, we can always characterize (X^o, T^o), the marked output process; (X^d, T^d), the marked departure process; and (X^f, T^f), the marked feedback process, as Markov renewal processes with a countable state space. The result does not follow for the marked input process, (X^i, T^i); the marked reentry process, (X^r, T^r) or the marked arrival process, (X^a, T^a). For example, let Q_1 have a deterministic server. X^i, X^r, and X^a will not have the Markov property at inputs, reentries, or arrivals unless the state space has the forward or backward service time of the customer in service at Q_1. But this requires an uncountable state space.

Queueing networks are often analyzed by decomposing the network and analyzing each component separately. The arrival process to each component is assumed to be a renewal process or a Poisson process. Since the arrival process to a component may be the departure process from some other component, the arrival process may not be a Poisson or even a renewal process. It would be useful to know conditions for the customer flow to be weakly lumpable to Poisson or renewal processes.

The term weakly lumpable has been used by Kemeny and Snell (1960) in connection with Markov chains. Serfozo (1969) defined weak lumpability analogously for Markov renewal processes. Simon (1978) has done further research in analyzing equivalances between Markov renewal processes. Our definition of weak lumpability relates marked point processes with renewal processes.

Definition 4.4. A marked point process (X, T) is weakly lumpable to a renewal process iff T is a renewal process. If in addition T is a Poisson point process, then (X, T) is weakly lumpable to a Poisson point process.

For example, the departure process from an M/M/1 queue can be modelled as a Markov renewal process (X, T) where T_n is the time of the nth output and X_n is the number in the system at time T_n (see Disney et al. (1972)). It is well known that if the distribution of X_0 is the equilibrium distribution of an M/M/1 queue, then T is a Poisson point process. Thus in equilibrium, (X, T) is weakly lumpable to a Poisson point process.

For a Markov renewal point process (X, T), the conditions for T to be a renewal process can be expressed in terms of ν, the distribution of X_0, and Q, the semi-Markov kernel of (X, T). Note that a Markov renewal process (X, T) may be weakly lumpable for some values of ν and not for others.

Proposition 4.5. (X, T) is weakly lumpable to a renewal process iff for all n, $t_1, t_2, \ldots, t_n$

$$\nu Q(t_1)Q(t_2)\cdots Q(t_n)u = \nu Q(t_1)u\nu Q(t_2)u\cdots \nu Q(t_n)u \tag{2}$$

where u is a column vector of ones and the row vector ν is the distribution of X_0.

Proof. Since $T_0 \leq T_1 \leq \cdots$, T is a renewal process iff $T_1 - T_0, T_2 - T_1, \cdots, T_n - T_{n-1}$, are i.i.d. random variables. That is for all n

$$P_\nu\{T_1 - T_0 \leq t_1, \ldots, T_n - T_{n-1} \leq t_n\} = \prod_{i=1}^{n} P_\nu\{T_1 - T_0 \leq t_i\}.$$

But this is (2). □

Simon (1979) has proven two useful results on the weak lumpability of an irreducible Markov renewal process to a renewal process. Assume that the Markov chain X has a *stationary probability distribution* π. First, the only renewal process which is a candidate has interrenewal distribution $\pi Q(t)u$. Second, if (X, T) is weakly lumpable to a renewal process when X_0 has distribution ν, then (X, T) is also weakly lumpable to the same renewal process when X_0 has distribution π. In summary, if (X, T) with initial distribution π is not weakly lumpable to the renewal process with interrenewal distribution $\pi Q(t)u$, then (X, T) cannot be lumped to any renewal process regardless of the distribution of X_0.

The following result shows that under most conditions of interest the only possible renewal departure process is a Poisson process with rate λ.

Theorem 4.6. Assume that X^d is an irreducible Markov chain with a stationary probability distribution π and that $Z_t u - Z_0 u$ (the difference between the total number of customers at time t and at time 0) converges in distribution to an almost surely finite random variable. If (X^d, T^d) is weakly lumpable to a renewal process then (X^d, T^d) is weakly lumpable to a Poisson process with rate λ.

Proof. Assume that X_0^d has distribution π, where π is the invariant probability distribution for the Markov chain X^d. From Simon's second result we know that if (X^d, T^d) is not weakly lumpable with distribution π, it is not weakly lumpable for any initial distribution. Recall that $X_0^d = (Z_0, 0)$. Now

$$Z_t u - Z_0 u = N_t^a - N_t^d . \tag{3}$$

Berman (1978, Theorem 7.4.2) has shown that if the moments of the interrenewal times characterize the interrenewal distribution of N_t^a and if the left hand side of (3) converges in distribution to an almost surely finite random variable then a necessary condition for N_t^d to be a renewal process is that it have the same interrenewal distribution. Since an exponential distribution is characterized by its moments we conclude that the only possible renewal departure process is a Poisson process. $\square$

In the remainder of this section we will restrict our attention to systems satisfying the hypothesis of Theorem 4.6. Hence determining if the departure process is a renewal process is equivalent to determining if the departure process is a Poisson process.

Now we also restrict our attention to queues in which the probability of a customer feeding back is independent of the customer's sojourn time. That is,

$$h_0(j_1, j_2, x) = h_0(j_1, j_2).$$

Define a new random variable V_t to be the remaining service time of the customer in service at time t in Q_1. If Q_1 is empty define $V_t = 0$. Now $(Z, V) = \{Z_t, V_t\}$ is a Markov process where Z_t is the joint queue length at time t. Let μ be the invariant and initial probability measure for (Z, V).

Kelly (1976) has called a network <u>quasi-reversible</u> if (a) departures of group i customers, for $i = 1,2,\ldots,I$ form independent Poisson processes; and (b) the state of the network at time t is independent of departures from the network up until time t. In our system we have one type of a customer, i.e. $I = 1$. The state of the network at time t is (Z_t, V_t). Thus condition (b) requires (Z_t, V_t) and $\{N_s^d; s \leq t\}$ to be independent. Melamed (1979a) has introduced a weaker condition called <u>pointwise independence</u> which requires (Z_t, V_t) and N_t^d to be independent for all t.

<u>Theorem</u> 4.7. If N_t^d is a Poisson process and (Z_t, V_t) is independent of the departure process up to and including time t (i.e. of $\{N_s^d; s \leq t\}$) then Q_1 must have an exponential server.

Remark: If the hypothesis of Theorem 4.7 is satisfied then the network is said to be quasi-reversible (cf. Kelly (1976)). The hypothesis also implies pointwise independence, though the converse is not necessarily true. Melamed shows that in a countable state regular Markov process, pointwise independence implies that N_t^d is a Poisson process, which is reassuring even though not immediately applicable unless the server at Q_1 has a special structure (e.g. Erlangian).

<u>Proof</u>. The proof relies on an important result appearing in the following chapter on general networks. Consider a non-anticipating

Poisson arrival process (i.e. a Poisson arrival process in which the time until the next arrival is independent of the current state of the network); then the distribution of the state of the network imbedded before arrivals is the same as the distribution of the state of the network at an arbitrary point in time.

Thus in our network the stationary distribution of (Z, V) imbedded just before arrivals is μ, the invariant probability measure for (Z, V). Now consider the behavior of the network in reverse time. The departure process after a time reversal becomes the arrival process. Our hypothesis implies that the time since the last departure before time t is independent of (Z_t, V_t). But after a time reversal this is the time until the next arrival after time t. Hence the departure process becomes a non-anticipating Poisson arrival process after a time reversal. Thus we conclude that the stationary distribution of (Z_t, V_t) imbedded just after departures (i.e. in reverse time just before arrivals) is also μ.

Now just after a departure, if $V_t > 0$ then $V_t = S_n^{(1)}$ for some n (i.e. V_t is an entire service time). At an arbitrary point in time, if $V_t > 0$ then V_t is the forward recurrence time of $S_n^{(1)}$. Since they have the same distribution, $S_n^{(1)}$ must be exponentially distributed (cf. Çinlar (1975) pp. 306-307).

In the hypothesis of Theorem 4.7 we require (Z_t, V_t) to be independent of $\{N_s;\ s \leq t\}$. We think this is a reasonable hypothesis, i.e. it holds in most, if not all, cases in which the M/G/1 queue with delayed feedback has Poisson departures. The reason follows from Conjecture 4.9.

Definition 4.8. The departure process is uniquely weakly lumpable to a Poisson process if

$$P_\mu\{T_1^d \le t_1,\ldots,T_n^d \le t_n\} = P_\gamma\{T_1^d \le t_1.,,,.t_n^d \le t_n\} = \prod_{i=1}^{n} (1-e^{-\lambda t_i})$$

implies $\mu = \gamma$, where μ and γ are initial distributions for (Z_0, V_0) and μ is the invariant probability distribution of (Z, V).

Intuitively, the definition says that if the network is not initially in equilibrium then the departure process is not a Poisson process.

The following conjecture would resolve the entire issue.

Conjecture 4.9. The M/G/1 queue with delayed feedback is uniquely weakly lumpable to a Poisson process.

There is evidence supporting Conjecture 4.9 but we have been unable to prove it.

Proposition 4.10. Unique weak lumpability to a Poisson process implies quasi-reversibility which in turn implies pointwise independence.

Proof. Assume that the system is uniquely weakly lumpable to a Poisson process. Assume that (Z_0, V_0) has distribution μ, hence N_t^d is a Poisson process. Thus at any time $t > 0$, (Z_t, V_t) has distribution μ independent of $\{N_s^d;\ 0 \le s \le t\}$, for otherwise $N_{t+s}^d - N_t^d$ would not have a Poisson distribution by unique weak lumpability. Hence the system is quasi-reversible. As noted earlier quasi-reversibility implies pointwise independence. □

By looking at the contrapositive to Proposition 4.10 we obtain the following result.

Corollary 4.11. If the system is not quasi-reversible and if in equilibrium the departure process is a Poisson process then there are two distinct (and hence an uncountable number of) initial distributions for (Z, V) yielding a Poisson departure process.

Proof. Let Γ denote the set of initial distribution for (Z, V) such that N_t^d is a Poisson process. $\mu \in \Gamma$. If the system is not quasi-reversible then the system is not uniquely weakly lumpable. Hence there exist $\gamma \neq \mu$ such that $\gamma \in \Gamma$. Alternatively if the system is not quasi-reversible there exists t and $B \in \mathscr{F}(N_s^d;\ s \leq t)$ such that

$$P(B) > 0,$$

$$P_\mu\{(Z_t, V_t) \in C \mid B\} \neq \mu(C). \tag{4}$$

If (4) does not hold for some t and B then the system would be quasi-reversible since (Z_t, V_t) and $\{N_s^d;\ s \leq t\}$ would be independent. Let

$$P_\mu\{(Z_t, V_t) \in C \mid B\} = \gamma(C).$$

Then $\gamma \in \Gamma$. Once we have two elements $\mu, \gamma \in \Gamma$, then

$$p\gamma + (1-p)\mu \in \Gamma,$$

for all p such that $p\gamma + (1-p)\mu$ is a probability measure. $p\gamma + (1-p)\mu$ is certainly a probability measure for all $p \in [0, 1]$. Thus we have an uncountable number of elements in Γ. It is possible to generate more elements. Take any $\gamma \in \Gamma$, t and $B \in \mathscr{F}(N_s^d;\ s \leq t)$, with $P(B) > 0$. Let

$$P_\gamma\{Z_t, V_t) \in C \mid B\} = \eta(C).$$

Then $\eta \in \Gamma$. And as before, any probability measure expressible as a linear combination of elements of Γ is also in Γ.

In Corollary 4.11, "if the system is not pointwise independent" could have replaced with "if the system is not quasi-reversible" and the result would follow. Before concluding this section we will briefly discuss the weak lumpability of the output process, (X^o, T^o), to a renewal process.

Proposition 4.12. In the M/G/1 queue with delayed Bernoulli feedback if N_t^o is a renewal process then N_t^o is a Poisson process with rate λ/q.

Proof. If N_t^o is a renewal process then N_t^d is a renewal process since N_t^d is obtained from N_t^o by selecting points with a fixed probability q. But N_t^d must be a Poisson process if it is a renewal process. Consequently N_t^o must be a Poisson process since no other renewal process when subjected to Bernoulli filtering yields a Poisson process. Clearly since N_t^d must have rate λ, N_t^o must have rate λ/q. □

Furthermore, if it could be established that the M/G/1 queue with delayed Bernoulli feedback is uniquely weakly lumpable to a Poisson process then N_t^o is not a Poisson process since the network must be a Jackson network $(G = M)$. From Melamed (1979b) it is known that the flow on a non-exit arc in a Jackson network is not a Poisson process.

CHAPTER IV

RESULTS FOR GENERAL SYSTEMS

In Chapter IV we look at some results for more general systems than the M/G/1 queue with delayed feedback. The notation in Chapter IV and the rest of the dissertation are distinct.

The term Markov process will be restricted to continuous time stochastic processes and the term Markov chain will be restricted to discrete time stochastic processes. We show that under certain conditions a Markov process $\{X(t): t \geq 0\}$ and an imbedded Markov chain $\{X(T_n^a-): n \geq 0\}$ have the same invariant probability distribution. The sequence $T_1^a, T_2^a, \ldots$ forms a Poisson process with rate α, but is not necessarily independent of the Markov process $\{X(t): t \geq 0\}$. Our proof uses an external observer who periodically samples the state of $\{X(t): t \geq 0\}$ at times $T_1^e, T_2^e, \ldots$. The sequence $T_1^e, T_2^e, \ldots$ also forms a Poisson process with rate α but is independent of $T_1^a, T_2^a, \ldots$ and $\{X(t): t \geq 0\}$. We first show that the Markov process $\{X(t): t \geq 0\}$ and the Markov chain $\{X(T_n^e): n \geq 0\}$ have the same invariant probability distribution. Next we show that the Markov chain $\{X(T_n^e): n \geq 0\}$ and $\{X(T_n^a-): n \geq 0\}$ have the same invariant probability distribution. Hence $\{X(t): t \geq 0\}$ and $\{X(T_n^a-): n \geq 0\}$ have the same invariant probability distribution.

The last section of the chapter is devoted to applications of the above results to several areas of applied probability including queueing network theory and stochastic control.

1. Assumptions and Notation

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(E, \mathscr{E})$ a complete separable metric space where $\mathscr{E}$ is the σ-algebra of Borel sets of E. Let $X = \{X(t): t \geq 0\}$ be a strong Markov process with an indecomposable state space $(E, \mathscr{E})$ over $(\Omega, \mathscr{F}, P)$ with right continuous sample paths having left hand limits a.e.. Furthermore assume that X has a temporally homogeneous transition function $P_t(x,A)$, $x \in E$, $A \in \mathscr{E}$, with an invariant probability measure μ and that $P_\mu\{X(t-) \neq X(t)\} = 0$ for all t.

We use P_t to denote the following operator induced by $P_t(x,A)$ on $(E, \mathscr{E})$ measurable functions,

$$P_t f(x) = \int P_t(x,dy)f(y).$$

We use μf to denote the following linear functional induced by μ on the $(E, \mathscr{E})$ measurable function $f(x)$,

$$\mu f = \int \mu(dx)f(x).$$

Since μ is an invariant probability measure in the above notation,

$$\mu P_t I_A(x) = \mu I_A = \mu(A), \quad A \in \mathscr{E}, \quad t \geq 0,$$

where $I_A(x)$ is the indicator function of the set A. We define one more operator U^α which is called the α-potential operator and defined to be

$$U^\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)dt = \int_0^\infty e^{-\alpha t} \int P_t(x,dy)f(y)dt, \quad \alpha > 0.$$

The sequence of external observation times $0 = T_0^e \leq T_1^e \leq \ldots$ is a Poisson process with rate α and is independent of the Markov process

$\{X(t): t \geq 0\}$. Let

$$\mathscr{F}_t = \sigma((X(s);\ s \leq t),\ (N_s^e;\ s \leq t))$$

where N_s^e is the counting process associated with the Poisson process $\{T_n^e: n \geq 0\}$. Intuitively, the σ-algebra $\mathscr{F}_t$ contains all the information of the Markov process $\{X(s): s \geq 0\}$ and the Poisson process $\{T_n^e: n \geq 1\}$ up to *time t*. Let

$$\mathscr{F}_{t-} = \sigma(\bigcup_{s<t} \mathscr{F}_s).$$

Similarly, $\mathscr{F}_{t-}$ can be interpreted as containing all the information up to but not including time t. Also define

$$\mathscr{G}_t = \sigma(X(s);\ s \leq t).$$

The σ-algebra $\mathscr{G}_t$ contains all the information in the process X up to time t. Clearly,

$$\mathscr{G}_t \subset \mathscr{F}_t$$

and X is a Markov process adapted to either $\mathscr{G}_t$ or $\mathscr{F}_t$.

Let $0 = T_0^a \leq T_1^a \leq \cdots$ be a sequence of $\mathscr{G}_t$ stopping times. Define $X_n^a = X(T_n^a-)$, $n \leq 0$. We will show that under certain conditions $\{X_n^a: n \geq 0\}$ is a Markov chain with the same invariant distribution as the Markov process X. Our proof uses the Markov chain $\{X_n^e: n \geq 0\}$ where $X_n^e = X(T_n^e)$. As stated in the introduction we first show that X and $\{X_n^e: n \geq 0\}$ have the same invariant probability distribution and then that $\{X_n^e: n \geq 0\}$ and $\{X_n^a: n \geq 0\}$ have the same invariant probability distribution.

2. $\{X(t): t \geq 0\}$ vs. $\{X_n^e: n \geq 0\}$

Proposition 2.1. $\{X_n^e: n \geq 0\}$ is a Markov chain with single step transition function

$$P(x,A) = \alpha U^\alpha I_A(x), \quad x \in E, \ A \in \mathscr{E}.$$

Proof. For each n, T_n^e is an $\mathscr{F}_t$ stopping time. Since X has the strong Markov property $\{X(T_n^e): n \geq 1\}$ forms a Markov chain.

Now, $T_{n+1}^e - T_n^e$ is exponentially distributed with parameter α and is independent of the Markov process X. Hence,

$$P(x,A) = \int_0^\infty \alpha e^{-\alpha t} P_t(x,A)\,dt$$

$$= \alpha \int_0^\infty e^{-\alpha t} P_t I_A(x)\,dt$$

$$= \alpha U^\alpha I_A(x). \qquad \square$$

The following theorem states that the Markov process X has invariant probability measure μ if and only if the Markov chain $\{X_n^e: n \geq 0\}$ has invariant probability measure μ.

Theorem 2.2. $\mu P_t I_A(x) = \mu(A) \iff \mu\alpha U^\alpha I_A(x) = \mu(A), \quad \alpha > 0.$

Proof. $(\Longrightarrow)$

$$\mu\alpha U^\alpha I_A(x) = \int \mu(dx)\ \alpha \int_0^\infty e^{-\alpha t} \int P_t(x,dy) I_A(y)\,dt.$$

By Tonelli's theorem (Royden (1964), p. 234) the right hand side becomes

$$= \int_0^\infty \alpha e^{-\alpha t} \int \mu(dx) \int P_t(x,dy) I_A(y) dt$$

$$= \int_0^\infty \alpha e^{-\alpha t} \mu(A)$$

$$= \mu(A).$$

$(\Longleftarrow)$ Since $\mu \alpha U^\alpha I_A(x) = \mu(A)$,

$$\mu(dx)\alpha \int_0^\infty e^{-\alpha t} \int P_t(x,dy) I_A(y) dt = \mu(A).$$

Hence,

$$\int_0^\infty e^{-\alpha t} \alpha \int \mu(dx) \int P_t(x,dy) I_A(y) dt = \int_0^\infty e^{-\alpha t} \alpha \mu(A) dt.$$

By the a.s. uniqueness of Laplace transforms (see Feller, Vol. II, p. 433) combined with the right continuity of $P_t(x,A)$

$$\int \mu(dx) \int P_t(x,dy) I_A(y) dt = \mu(A).$$

Thus $\mu P_t I_A(x) = \mu(A)$. □

The previous theorem can be intuitively interpreted as stating that in equilibrium the distribution seen by an external observer is the same as the distribution at an arbitrary point in time.

Theorem 2.2 has been proven for countable state Markov processes by Çinlar (1975), Chapter 8, Theorem 5.21 and for standard Markov processes by Nagasawa and Sato (1963). A standard Markov process is required to be quasi-left-continuous. That is, if $\{T_n\}$ is an increasing sequence of $\mathcal{F}_t$ stopping times with limit T, then $X(T_n) \to X(T)$ a.s.. It is easy to construct a system which has an uncountable state space which is not quasi-left-continuous. For example an M/D/1 queue with $X(t) = (Q_t, V_t)$, where Q_t is the queue length at time t and V_t is the remaining service time of the customer in service at time t, is a Markov process. The state space is uncountable and

$$T_0 = \inf\{t: V_t = 1\},$$

$$T_n = \inf\{t: T_n \geq T_{n-1}, V_t \leq \frac{1}{n}\},$$

$$T = \lim_n T_n,$$

is a sequence of stopping times with the $\lim_n X(T_n) > X(T)$ a.s..

3. <u>$\{X_n^e: n \geq 0\}$ vs. $\{X_n^a: n \geq 0\}$</u>

$\{X_n^a: n \geq 0\}$ is a discrete time stochastic process imbedded in the Markov process $\{X(t): t \geq 0\}$. We would like $\{X_n^a: n \geq 0\}$ to be a Markov chain. In general $\{X_n^a: n \geq 0\}$ may not be a Markov chain. In the appendix we construct an example in which $\{X_n^a: n \geq 0\}$ is not a Markov chain. The following criterion gives sufficient conditions for $\{X_n^a: n \geq 1\}$ to have the Markov property.

<u>Criterion</u> 3.1. If

$$P\{X(T_n^a) \in B \mid (X_t;\ t < T_n^a)\} = P\{X(T_n^a) \in B \mid X(T_n^a-)\} \text{ a.s.}$$

for all $B \in \mathscr{E}$, then $P\{X_{n+1}^a \in A \mid X_n^a, X_{n-1}^a, \ldots\} = P\{X_{n+1}^a \in A \mid X_n^a\}$.

<u>Proof</u>. Let the regular conditional distribution $P_n(B \mid x)$ be a version of $P\{X(T_n^a) \in B \mid X(T_n^a-) = x\}$, (Breiman (1968), p. 77).
Since T_n^a is a $\mathscr{G}_t$ stopping time

$$P\{X^a_{n+1} \in A \mid X^a_n = x_n,\ X^a_{n-1} = x_{n-1}, \ldots\}$$

$$= \int P\{X(T^a_{n+1}-) \in A \mid X(T^a_n) = y\}\ P_n(dy \mid x_n)$$

$$= P\{X^a_{n+1} \in A \mid X^a_n = x_n\}. \qquad \square$$

Let us define V^a_t to be the forward recurrence time until the next T^a_n after time t. Thus,

$$V^a_t = \inf\{T^a_n\colon T^a_n > t\} - t.$$

<u>Theorem</u> 3.2. If $\{X^e_n\colon n \geq 1\}$ and $\{X^a_n\colon n \geq 1\}$ are stationary Markov chains with indecomposable state spaces and if

$$P\{V^a_t \leq x \mid \mathscr{F}_t\} = 1 - e^{-\alpha x},\ x \geq 0, \tag{1}$$

then μ is an invariant probability measure for X^e_n iff μ is an invariant probability measure for X^a_n.

<u>Proof</u>. Let $T^a = \{T^a_n\}$, $T^e = \{T^e_n\}$ and $T = T^a \cup T^e = \{T_n\}$ where $T_1 = \min\{T^a_1, T^e_1\}$ and

$$T_1 \leq T_2 \leq \cdots.$$

Equation (1) implies that T^a is a Poisson process. T^e was defined to be a Poisson process and is independent of T^a. Hence T is a Poisson process with rate 2α. Let $X_n = X(T_n-)$. Note that $X(T^e_n-) = X(T^e_n)$ a.s..

Now,

$$E_\mu\left[\lim_{n\to\infty}\frac{\sum_{i=1}^{n} I_A(X_i)I_Te(T_i)}{1+\sum_{i=1}^{n} I_Te(T_i)}\right]=\mu^e(A) \tag{2}$$

and

$$E_\mu\left[\lim_{n\to\infty}\frac{\sum_{i=1}^{n} I_A(X_i)I_Ta(T_i)}{1+\sum_{i=1}^{n} I_Ta(T_i)}\right]=\mu^a(A), \tag{3}$$

where μ^a and μ^e are the stationary distributions of X_n^a and X_n^e, respectively. Equation (2) and (3) follow from the fact that a stationary Markov chains with an indecomposable state space is ergodic. Hence in both (2) and (3), the expression inside the brackets converges a.s. to the unique stationary distribution (Breiman (1968), Theorem 7.16). (2) and (3) are identical except that $I_Te(T_i)$ in (2) replaces $I_Ta(T_i)$ in (3). But the probability of a particular observation point being either an arrival or an external observer conditioned on the previous history is .5. That is,

$$P_\mu\{I_Te(T_n)=k \mid \mathscr{F}_{T_n^-}\}=P_\mu\{I_Ta(T_n)=k \mid \mathscr{F}_{T_n^-}\}=\begin{cases}1/2 & \text{if } k=0,1,\\ 0 & \text{otherwise.}\end{cases}$$

Consequently, $\mu^e(A)=\mu^a(A)$ for all $A \in \mathscr{E}$. □

By Theorem 2.2 $\mu=\mu^e$. Thus we have under the conditions of Theorem 3.2 that if any one of X_t, X_n^a and X_n^e has stationary distribution

μ then all three have stationary distribution μ.

4. Applications of Results

Work Sampling. In work sampling an employee is observed at random points in time. At each observation point, the particular task being performed by the employee is noted. From these observations, the proportion of time an employee devotes to each task is estimated. Under the assumptions of Theorem 2.2, if the observation points form an independent Poisson process, the estimate converges a.s. to the true value. In other sampling schemes, even other renewal processes, it is possible to obtain a biased result.

Poisson Arrivals to Queueing Networks. One important application of Theorem 3.2 is to the distribution of the state of a queueing network imbedded at arrival points. Let the queueing network be modelled as a Markov process $\{X(t): t \geq 0\}$ and T_n^a be the time of the nth arrival. If the arrival process is a Poisson process satisfying (1), then under the assumptions of Theorem 3.2., in equilibrium the distribution of the state of the queueing network at an arbitrary point in time is the same as the distribution imbedded just before arrivals. In order to apply Theorem 3.2 it is not necessary that all the arrival processes be Poisson processes. If an arrival process of a certain type of customer, or of customers to a certain queue form a Poisson process, we can imbed before these arrivals. The other arrival processes need not even be renewal processes. Theorem 3.2 is applicable to finite capacity systems (e.g. M/G/1/N). In finite capacity systems it is necessary to imbed before each of the Poisson arrivals whether it is blocked or not. It should be pointed out that even very complex queueing networks can be modelled as a Markov process satisfying the restrictions of section 2. The state space E is required to be a complete separable metric space which would allow R^∞, infinite dimensional Euclidean space. Hence even a queueing network with renewal exogenous arrivals and general multi-server queues can be modelled as a Markov process.

The above result is a generalization of a well known result that in equilibrium in an M/G/1 queue the distribution of the queue length seen by an arrival is the same as the distribution of the queue length at an arbitrary point in time, (Cooper (1972) p. 154).

Stochastic Control. Let T_n^a be the nth time we observe the system to be controlled and based on the observation reset some control parameter. For example consider an M/G/1 queue which we are attempting to control. Our Markov process might be $X_t = (Q_t, U_t, S_t)$ where Q_t is the queue length at time t, U_t is the length of service accumulated by the customer in service at time t and S_t is a parameter controlling the speed of the server. Let T_n^a be an independent Poisson process. Define $X_n^a = X(T_n^a-)$. Based on X_n^a we take some action on the system. Several possibilities are removing or adding customers, changing the service rate or turning the server on or off. We incur a cost for each observation and action taken. In addition there is a cost associated with the length of time in each state. Our objective is to minimize expected cost per unit time. Based on Theorem 3.2 we need only solve for the invariant distribution of X_n^a in order to determine our costs. From this it is possible to determine the optimal control strategy in order to minimize expected cost per unit time.

Poisson Departure Processes from Queueing Networks. If a departure process from a queueing network is a Poisson process, is the stationary distribution imbedded just after departures the same as the distribution at an arbitrary point in time? Theorem 3.2 appears to be inapplicable since (1) is violated. That is the time until the next departure is clearly dependent on the current state. Let T_n^a be the time of the nth departure and

$$V_t^a = t - \sup\{T_n^a : T_n^a < t\}.$$

Thus V_t^a is the backward recurrence time to the last departure before time t. Consider the reversed Markov process. The departure times are arrival times in reverse time. If

$$P\{V_t^a \leq x \mid (X(s);\ s \geq t)\} = 1 - e^{-\alpha x} \tag{4}$$

then the time since the last departure is independent of the future. Hence in reversed time (1) holds. If the reversed Markov process satisfies the conditions of section 2 and $\{X(T_n^a+): n \geq 0\}$ is a stationary Markov process with an indecomposable state space then $\{X(t): t \geq 0\}$ and $\{X(T_n^a+): n \geq 0\}$ have the same invariant probability distribution. It should be pointed out that (4) may be satisfied for judicious selections of a Markov process modelling a system and not for others. For example consider an M/M/1 queue with $X_t^{(1)} = (Q_t, A_t)$ and $X_t^{(2)} = (Q_t, B_t)$ where Q_t is the queue length at time t, A_t is the amount of service accumulated by the customer in service and B_t is the remaining service time of the customer in service. The Markov process $X_t^{(2)}$ satisfies (4) but $X_t^{(1)}$ does not. In $X_t^{(1)}$ the Markov chain imbedded after departures has

$A_{T_n} a_+ = 0$ a.s.. Thus $X_t^{(1)}$ does not have the property that the distribution of the state of the network imbedded just after departures is the same as the distribution at an arbitrary point in time, while $X_t^{(2)}$ does have the property (in equilibrium).

CHAPTER V

DISCUSSION

Davignon (1977) points out that in the queue with instantaneous feedback, published research reports outside of the computing literature are rare. The same is true only more so of the queue with delayed feedback. In the computing literature, the queue with delayed feedback is a useful and frequent model of the behavior of a computer system. However there are few theoretical results beyond expected values. The few results which are more general are usually special cases of results on Jackson networks. This dissertation analyzes the M/G/1 queue in order to provide results for practitioners modelling a system as an M/G/1 queue with delayed feedback and also to gain insight into queueing networks.

1. Summary

Chapter I informally described the topic of queues with feedback and reviewed pertinent literature. Chapter II formally described the M/G/1 queue with delayed feedback in terms of the governing sequences of random variables. The governing sequences of random variables are the interarrival times, the service times at the upper and at the lower queue, and the probability of a customer feeding back. The probability of a customer feeding back was allowed to depend on the past history to a limited extent. In section 3 the processes of interest were defined. The processes of interest are (generally complicated) functions of the governing sequences. The processes of interest are the flow processes: arrivals, inputs, outputs, departures, feedbacks, and reentries; and the queue length process in both continuous time and imbedded at outputs.

In Chapter III, we analyzed the processes of interest. Our first result characterized (X^o, T^o), the system at output points, as a Markov renewal process. This result provided the foundation for much of the analysis. The subsequent lemma exhibited the semi-Markov kernel of (X^o, T^o). With state dependent feedback, the possibility of a reducible state space arises. A simple necessary and sufficient condition for the irreducibility of (X^o, T^o) was determined. Irreducibility in turn implied that (X^o, T^o) was aperiodic. Under the assumptions of the model the lifetime $L = \sup_n T_n^o$ was almost surely infinite. The above conditions on (X^o, T^o) could be easily determined from the governing sequences. Ergodicity was more difficult to verify. In order to determine if X^o was ergodic it was necessary to determine if there existed a stationary probability vector for the Markov chain X^o. However an easily computed sufficient condition was established using Foster's criteria. Thus it is easy to determine whether (X^o, T^o) is an aperiodic irreducible Markov renewal process with an infinite lifetime. In addition there is an easily checked sufficient condition for ergodicity.

Z, the continuous time queue length process, was analyzed by using results about (X^o, T^o). Z was shown to be a semi-regenerative process with semi-regeneration times T^o.

In the next section, the busy cycle was shown to be a renewal process and an expression was found for the distribution function of the interrenewal times.

Section 4 analyzed the flow processes on the arcs. In particular, the output process, feedback process, and departure process were shown to be Markov renewal processes. It was also shown that, under general conditions, if the departure process was a renewal process then it must be a Poisson process. A conjecture was stated which, if verified, would in turn show that if the departure process was a Poisson process, Q_1 must have an exponential server. Thus the only M/G/1 queue with delayed Bernoulli feedback having renewal departures would be the M/M/1 queue with delayed Bernoulli feedback, which is a Jackson network. This would in turn imply that the M/G/1 queue with delayed Bernoulli feedback, never has a renewal output process.

Chapter IV discussed some general results which were used in Chapter III but are also useful in areas other than the queue with delayed feedback. It was shown that if $\{T_n\}$ is a sequence of stopping times for a Markov process $\{X(t): t \geq 0\}$ with invariant probability measure μ, and $X(T_n-)$ is a Markov chain, then μ is an stationary probability measure for $X(T_n-)$. This can be used to show that Poisson arrivals to a queueing network see the same distribution as at an arbitrary point in time.

2. Future Research

Several areas of further research come to mind as extensions of the present work. Foremost is proving or disproving Conjecture 4.9 on unique weak lumpability. Verification of Conjecture 4.9 would characterize all M/G/1 queues with delayed Bernoulli feedback having renewal departure processes or renewal output processes.

Another interesting question is whether there exists a system which fails to be ergodic but has

$$\frac{\lambda}{h_1(j_1,j_2)} < 1 \text{ and } \frac{\lambda}{\alpha}\frac{h_1(j_1,j_2)}{h_0(j_1,j_2)} < 1,$$

for all but a finite number of states. More substantial but along the same lines would be determining necessary and sufficient conditions, which are easily computable, for the system to be ergodic.

A third untouched area is the difficult but needed study of the sojourn time in queues with delayed feedback. In computer models the sojourn time, i.e. the departure time minus the arrival time of an arbitrary customer, is often the process of interest. This distribution is unknown even in the case of most Jackson (1957) networks (cf. Simon and Foley (1979)). In particular the sojourn time distribution is unknown even in the case of the M/M/1 queue with delayed Bernoulli feedback.

3. Closing Comments

Most of the work in this dissertation and in queueing network theory in general involves structural results and characterizations of processes. There is a need for accurate approximations and tight bounds on the processes of interest to supplement the structural results. Even more strongly, there is a need for computable accurate approximations and bounds.

APPENDIX

AN EXAMPLE IN WHICH $\{X_n^a: n \geq 1\}$ IS NON-MARKOVIAN.

Let $X = \{X(t): t \geq 0\}$ be a Markov process and $\{T_n^a; n \geq 0\}$ a sequence of stopping times for X. We construct an example in which $\{X_n^a: n \geq 1\}$ fails to be a Markov chain where $X_n^a = X(T_n^a-)$. Our state space E is the closed interval $[-1, 1]$. Let $Z_1, Z_2, Z_3, \ldots$ be a sequence of i.i.d. exponential random variables with parameter α.

$$T_0^a = 0,$$

$$T_n^a = \min\{1, Z_n\} + T_{n-1}^a, \quad n \geq 1,$$

$$X(T_0^a) = \begin{cases} 1 & \text{w.p.} \quad .5, \\ -1 & \text{w.p.} \quad .5, \end{cases}$$

$$X(T_n^a) = \begin{cases} X(T_{n-1}^a) & \text{if } Z_n < 1, \\ -X(T_{n-1}^a) & \text{if } Z_n \geq 1, \end{cases}$$

and for $T_n^a < t < T_{n+1}^a$

$$X(t) = \begin{cases} X(T_n^a) - (t - T_n^a) & \text{if } X(T_n^a) = 1, \\ X(T_n^a) + (t - T_n^a) & \text{if } X(T_n^a) = -1. \end{cases}$$

A particular realization of X(t) appears in figure 4.

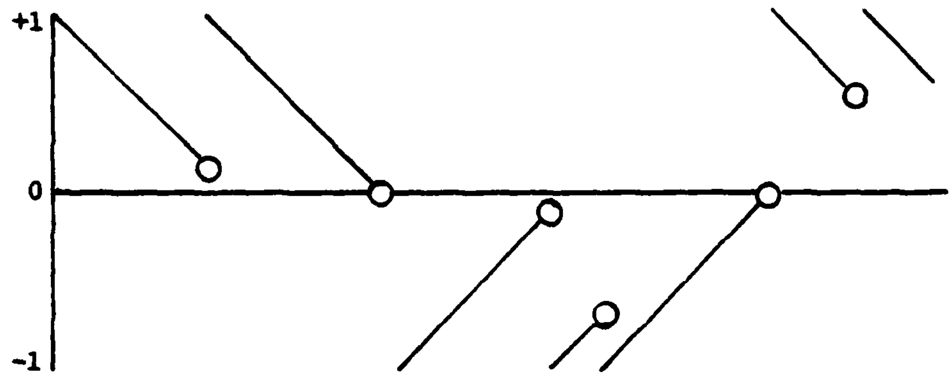

Fig. 4 - A Realization of X(t)

The points of discontinuity not on the zero axis form a Poisson process. At a point of discontinuity t^* not on the zero axis, $X(t^*) = 1$ if $X(t^*-) > 0$ and $X(t^*) = -1$ if $X(t^*-) < 0$. At a point of discontinuity t^* on the axis (i.e. $X(t^*-) = 0$), $X(t^*) = 1$ if X(t) approaches xero from below at t^*. Similarly, $X(t^*) = -1$ if X(t) approaches zero from above at time t^*.

$X = \{X(t): t \geq 0\}$ is a Markov process satisfying the assumptions in the first paragraph of section 2 and $\{T_n^a: n \geq 1\}$ is a sequence of stopping times for X. Let $X_n^a = X(T_n^a-)$. To show that X_n^a is not a Markov chain we need only note that

$$P\{X_{n+1}^a > 0 \mid X_n^a = 0\} = .5(1 - e^{-\alpha})$$

while

$$P\{X_{n+1}^a > 0 \mid X_n^a = 0, X_{n-1}^a > 0\} = 0.$$

Note that Criterion 3.1 does not hold for the above process. In addition (1) of Theorem 3.2 fails. It is conceivable that if (1) holds then $\{X_n^a: n \geq 1\}$ must be a Markov chain.

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