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FINITE ELEMENT MODELING OF THE COUPLED THERMAL AND MOMENTUM DIF--ETC(U)
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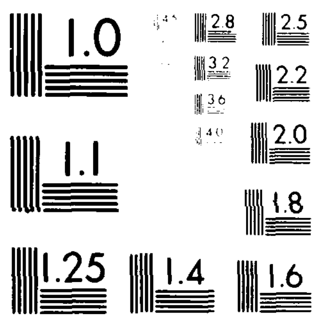

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The objective of this study is to develop the basic equation for a finite element formulation which can be used for solving problems related to coupled thermomechanical behavior.

The formulation is based on the introduction of a new quantity, defined as heat displacement which is related to temperature in the same manner as the mechanical displacement is related to strain. The introduction of such a quantity into the thermal equation leads to a form similar to the equation of motion, so that, the governing equation for heat conduction can be regarded as an equilibrium equation.

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20. Abstract (Continued)

Using this definition for the heat displacement the coupled thermal and momentum equations are written in a unified form with the thermomechanical coupling terms appearing explicitly in the constitutive relation and not in the equation of motion or heat conduction. This presentation is more appropriate since the behavior of a material is expressed through its constitutive relation.

The governing equations are used to write a variational formulation which, together with the concept of generalized coordinates, yields a set of differential equations with the time as the independent variable. These equations are presented in a form equivalent to that of the Lagrangian equations in mechanics.

Following this formulation, the basic finite element approach is used to derive two element models, one which is represented by a linear approximation of the displacements, and a second one which is represented by a higher order approximation of the displacements.

An application of the derived formulation for the two element models is given for the problem of induced thermal waves in a medium which is subjected to temperature changes at its boundary surface. Results obtained for this problem are for the coupled and uncoupled thermomechanical behavior.

The results obtained by the present formulation are compared with existing results of analytical solutions.

CONTENTS

A

LIST OF SYMBOLS

x_i	Cartesian coordinates
t	*time*
T_0	reference temperature
T	absolute temperature
θ	temperature change per unit temperature
H_i	heat displacement
s	increment of entropy per unit volume
c	heat capacity per unit volume
ρ	mass density per unit volume
$\bar{\sigma}$	heat stress
k_{ij}	thermal diffusion coefficient
λ_{ij}	thermal resistivity
v	volume
B	surface of volume v
n_i	unit vector
V	thermal potential energy
D	dissipation function
q_i, p_i	generalized coordinates
Q_i	thermal force vector
f_{ij}, g_{ij}	shape functions
β_{ij}	thermal moduli
e_{ij}	strain components
C_{ijkl}	mechanical moduli
U_i	mechanical displacement
σ_{ij}	stress tensor
K	kinetic energy
FF_i, FQ_i	thermomechanical forces
l	length of an element
A	cross sectional area of an element
(LE)	linear element model
(CE)	cubic element model
L	dimensionless characteristic length
ξ	dimensionless pace coordinate
$\bar{l}$	dimensionless length of an element
τ	dimensionless time
σ_1	dimensionless normal stress
σ_2	dimensionless lateral stress
U	dimensionless mechanical displacement
$\underline{H}$	dimensionless heat displacement
$\bar{\theta}$	dimensionless temperature
τ_0	*dimensionless rising time*
λ, μ	material constants
ν	Poisson's ratio
α	thermal coefficient of linear expansion
δ	thermomechanical coupling parameter
NE	number of elements between $\xi = 0.0$ and $\xi = 1.0$
TNE	total number of elements for L
w	inverse of $\bar{l}$

FINITE ELEMENT MODELING OF THE COUPLED THERMAL AND MOMENTUM DIFFUSION EQUATIONS

INTRODUCTION

The behavior of solids under temperature changes has long been studied and a large number of publications exist in both the fields of thermal and mechanical deformations. Developments in both of these fields during the last two decades has produced a variety of solutions and techniques of direct engineering application.

The basic theory for thermal effects on mechanical deformations has been well-established and analytical methods of solution exist for the governing equations. However, the practical application of analytical methods presents difficulties for bodies of complex geometries or bodies under complicated boundary conditions and cannot be relied upon for the solution of any problems of practical importance. Consequently, numerical analyses have been extensively studied and various approaches based on variational methods have been developed. Such methods and solutions can be found for a variety of applications. A variational principle for the equations of coupled thermoelasticity, introduced by Biot, and applications to this variational formulation are given in [1], [2] and [3] Boley, in [4]-[6], refers to solutions of thermoelastic problems and a problem on thermoelastic wave propagation is solved by Dunn in [19]. The thermal stresses induced in a semi-infinite body due to suddenly applied step heat input are discussed by Grimado in [14]. In addition, a variational formulation is given by Nickell and Sackman in [15] for the coupled linear thermoelasticity and an application of this formulaton to a boundary value problem can be found in [16].

The majority of the numerical analyses employ the finite element method because of its superiority over finite-difference techniques. The finite element method has been well-established in recent

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years and the basic formulation for this method is given under the name of stiffness and deflection analysis. An introduction to the finite element method can be found in the texts by Desai and Abel [11] and by Zienkiewicz [21], with applications to engineering related subjects. The method has also been employed in solving thermomechanical problems and applications of the method are given in [17] and [18]. In most of these studies, the effect of thermomechanical coupling is included in the formulation with applications to various problems of practical interest.

Existing finite element solutions are based on formulations where the coupled equations are treated as two separate equations with the coupling appearing explicitly in the thermal diffusion equation. As a result, existing solutions usually involve a time-consuming iterative process. A different approach which uses finite-difference methods to treat the heat equation and finite elements for the momentum equation was also suggested. Again, it requires a lengthy numerical iteration. Obviously, a more appropriate formulation will be one which describes, by a single discrete element, both thermal and mechanical deformations. This motivates the research reported in this study. Essentially, the purpose of this study is to formulate both the problems of heat transfer and thermal deformation under a unified formulation. In order to achieve this, a unified variational formulation is introduced for the coupled thermal momentum diffusion equations. The key to this is the introduction of a new quantity defined as heat displacement, which is equivalent to the mechanical displacement and has the units of length. As a result of this definition, the change in temperature is treated as a thermal deformation and it is equivalent to a mechanical deformation [22].

In the first chapter, the basic definitions are introduced and the coupled equations are written in a form such that the equation of motion and the equation of equilibrium for heat transfer are of similar form, with the coupling term appearing explicitly in the constitutive relations. Such formulations are more appropriate since the behavior of a material is expressed through its constitutive relations and not through the equation of motion or equilibrium. A variational equation is derived, based on the principle of virtual work in mechanics, which, with the use of generalized coordinates, leads to a unified equation for the general coupled case. Since this equation is expressed in terms of generalized coordinates, which can represent both mechanical and heat displacements, it can also be used for deriving a finite element formulation. This is done in the second chapter, where the basic finite element method is used to derive two finite element models for solving initial/boundary value problems. The first model is based on a linear approximation of the displacements and the second on a higher order approximation. The matrix equation for the first model is expressed in terms of nodal displacements and for the second model, in terms of nodal displacements and nodal deformation.

Each element model can be used:

a) for coupled diffusion problems,

b) for uncoupled ones by setting the thermomechanical coupling parameter equal to zero,

c) for pure thermal diffusion problems by setting the mechanical displacements equal to zero, and

d) for either dynamic or quasi-static solutions by retaining or neglecting the inertia term, respectively.

Numerical solutions of the matrix differential equations are obtained through a third-order backward finite difference scheme. This implicit integration technique has been found to produce stable results and its accuracy has been tested on a variety of problems.

The derived finite element formulation is used in the last chapter to solve an inital/boundary value problem of transient thermomechanical behavior, when uniform heating is applied on the boundary of a semi-infinite elastic medium. The results obtained are for both coupled and uncoupled cases and are compared to existing analytical solutions for their accuracy. A comparison is also given between the present and other numerical solutions.

I. ANALYTICAL FORMULATION

1. Basic Equations and Definitions

Consider a medium subjected to external heating and assume a cartesian coordinate system ($x_i, i = 1,2,3$). It is assumed that the medium is initially at uniform temperature T_0, which will be referred to as the reference temperature, and the state at this temperature as the reference state. It is also assumed that initially the medium is free from mechanical deformations.

The instantaneous absolute temperature is denoted by T and the difference $T - T_0$ defines the instantaneous relative temperature $\Delta\theta$, which is a function of the coordinate x_i, and the time t. Let

$$\theta = \frac{T - T_0}{T_0} \tag{1}$$

be defined as the temperature change per unit temperature T_0, or the instantaneous relative temperature per unit temperature. In the following it will be referred to as the temperature θ.

In the formulation of this study, when repeated indices in equations are present, the summation convention is assumed. We now define a vector field $H_i(x_i,t)$ as the heat displacement vector such that

$$\theta = \frac{\partial H_i}{\partial x_i} = H_{i,i} \tag{2}$$

In the above definition the temperature θ represents a thermal strain similar to mechanical strain. Note that H_i has the dimensions of displacement. Thus, there is a one-to-one correspondence between heat displacement-mechanical displacement and temperature-strain.

The increment of entropy s, per unit volume, for the case of zero strain is given by

$$s = c\,\frac{T - T_0}{T_0}$$

and due to (1) the entropy is $s = c\,\theta$, where c is the heat capacity per unit volume and related to the heat capacity per unit mass c_v, by $c = \rho c_v$ with ρ as the mass density per unit volume.

By introducing a quantity $\bar{\sigma}$ as the heat stress, the entropy s is related to this stress by

$$\bar{\sigma} = cT_0\theta = T_0 s \tag{3}$$

Equation (3) can be considered as a constitutive relation in the same fashion as the stress-strain relation.

The temperature field θ satisfies the thermal diffusion equation

$$\frac{\partial}{\partial x_i}\left[k_{ij}(x_i)\ \frac{\partial\theta(x_i,t)}{\partial x_j}\right] = c\ \frac{\partial\theta(x_i,t)}{\partial t} \tag{4}$$

wher k_{ij} is the thermal diffusion coefficient with the following symmetric property:

$$k_{ij} = k_{ji}.$$

Using Eq. (2), Eq. (4) is expressed as follows

$$\frac{\partial}{\partial x_i}\left\{k_{ij}\frac{\partial\theta}{\partial x_j}\right\} = c\frac{\partial}{\partial x_i}\left(\frac{\partial H_i}{\partial t}\right)$$

or

$$k_{ij}\frac{\partial\theta}{\partial x_j} = c\ \frac{\partial H_i}{\partial t},$$

and, after substituting Eq. (3) in the last equation, yields

$$k_{ij}\ \frac{\partial\bar{\sigma}}{\partial x_j} = c^2 T_0\ \frac{\partial H_i}{\partial t} \tag{5}$$

or

$$\frac{\partial\bar{\sigma}}{\partial x_j} = c^2\ T_0\,\lambda_{ij}\ \frac{\partial H_i}{\partial t}$$

Using the same definitions for the temperature θ as were described above, and under the assumption of small mechanical displacements U_i for every particle, the kinematic relations are given by

$$e^{*}_{ij} = \frac{1}{2}\ (U_{i,j} + U_{j,i}) - \beta_{kl}\ T_0\ [C_{ijkl}]^{-1}\theta, \tag{6}$$

$$\theta = H_{i,i} - \frac{1}{c}\ \beta_{ij}\ e_{ij}, \tag{7}$$

where e^{*}_{ij} are the components of the infinitesimal mechanical strain, which is equal to the difference between the total strain and the thermal strain. Similarly, the thermal strain θ is equal to the difference between the total heat strain and the strain induced by the mechanical deformation. The coefficients C_{ijkl} and β_{ij} are the mechanical and thermal moduli, respectively, which are symmetric in the sense that

$$C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk},$$
$$\beta_{ij} = \beta_{ji}.$$

For simple one-dimensional displacements in the x-direction, Eqs. (6) and (7) can be written as follows

$$e^{*}_{x} = \frac{\partial U_x}{\partial x} - \frac{T_0}{E}\beta\,\theta$$
$$\theta = \frac{\partial H_x}{\partial x} - \frac{1}{c}\beta e_x \qquad (8)$$

The constitutive relations for coupled thermal deformations can be expressed in terms of Eqs. (6) and (7) as

$$\sigma_{ij} = C_{ijkl}\, e^{*}_{kl} = C_{ijkl}\, e_{kl} - \beta_{ij} T_0\, \theta, \qquad (9)$$
$$\bar{\sigma} = cT_0\theta = cT_0 H_{i,i} - \beta_{ij} T_0 e_{ij}, \qquad (10)$$

where σ_{ij} are the components of the mechanical stress tensor and $\bar{\sigma}$ is the heat stress. The above relations clearly demonstrate the similarity between the stresses σ_{ij} and $\bar{\sigma}$.

The terms involving β_{ij} in Eqs. (6) to (10) represent thermomechanical coupling and for the uncoupled case one usually eliminates the coupling effect in Eq. (7) but retains the thermal expansion effect in Eq. (6), since the temperature variations due to mechanical deformations are negligibly small in many cases. The temperature then is evaluated independently of the strain. The momentum equation is given by

$$\rho \ddot{U}_i = \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad (11)$$

and the thermal diffusion equation by

$$c^2 T_0 \lambda_{ij} \dot{H}_i = \frac{\partial \bar{\sigma}}{\partial x_i} \qquad (12)$$

where ρ is the mass density per unit volume, λ_{ij} the thermal resistivity, and c the heat capacitance per unit volume. Body forces and heat sources have been neglected from the above equations.

An important feature of Eqs. (11) and (12) is that the thermomechanical coupling term does not appear explicitly in these equations, but only in the stress-strain relations, which is more appropriate since the stress-strain relations are those which describe the material behavior. The equations of equilibrium together with the kinematic relations and stress-strain relations will be used next to derive a variational formulation of these equations.

2. Variational Formulation

Choosing as unknown variables the mechanical displacement U_i and the heat displacement H_i to describe a physical system, a fundamental variational principle is derived by applying the principle of virtual work to the equilibrium equations.

Consider a variation δU_i of the mechanical displacement U_i with the corresponding variation δe_{ij} given by Eq. (6) and a variation δH_i of the heat displacement with the corresponding variation $\delta\theta$ given by Eq. (7). These variations are assumed to be virtual displacements, and by multiplying Eq. (11) by δU_i and Eq. (12) by δH_i , and integrating over a volume v of the medium, one obtains

$$\int_v (\rho \ddot{U}_i - \sigma_{ij,j})\ \delta U_i\, dv + \int_v\ (c^2 T_0 \lambda_{ij} \dot{H}_j\ - \bar{\sigma}_{,i})\ \delta H_i\, dv = 0 \tag{13}$$

Employing the divergence theorem, the first term in Eq. (13) becomes

$$\begin{aligned}\int_v (\rho \ddot{U}_i - \sigma_{ij,i})\delta U_i dv &= \int_v \rho \ddot{U}_i \delta U_i dv - \int_v\ (\sigma_{ij}\ \delta U_i)_{,j} dv\ +\ \int_v\ \sigma_{ij} (\delta U_i)_{,j}\ dv \\ &= \int_v \rho \ddot{U}_i \delta U_i dv - \int_B\ \sigma_{ij} n_j \delta U_i dB\ +\ \int_v\ \sigma_{ij} \delta e_{ij} dv \\ &= \int_v\ \rho \ddot{U}_i \delta U_i dv + \int_v\ \ \sigma_{ij} \delta e_{ij} dv \\ &\quad - \int_B\ \ F_i \delta U_i d\end{aligned} \tag{14}$$

where F_i is the surface traction applied on the boundary surface B of volume v.

Similarly, the second term of Eq. (13) yields

$$\int_v (c^2 T_0 \lambda_{ij} \dot{H}_j - \bar{\sigma}_{,i}) \delta H_i dv = \int_v c^2 T_0 \lambda_{ij} \dot{H}_j \; \delta H_i dv - \int_v (\bar{\sigma} \delta H_i)_{,i} dv$$

$$+ \int_v \bar{\sigma} (\delta H_i)_{,i} dv \tag{15}$$

$$= \int_v c^2 T_0 \lambda_{ij} \dot{H}_j \delta H_i dv + \int_v c T_0 \theta \delta \theta dv$$

$$+ \int_v T_0 \beta_{ij} \theta \delta e_{ij} dv - \int_B \bar{\sigma} n_i \delta H_i dB$$

Hence, Eq. (13) becomes

$$\int_v \rho \ddot{U}_i \delta U_i dv + \int_v c^2 T_0 \lambda_{ij} \dot{H}_j \delta H_i dv + \int_v c T_0 \theta \delta \theta \, dv + \int_v \sigma_{ij} \delta e_{ij} dv$$

$$+ \int_v T_0 \beta_{ij} \theta \, \delta \, e_{ij} dv$$

$$= \int_B F_i \delta U_i dB + \int_B \bar{\sigma} n_i \delta H_i dB$$

or

$$\int_v \rho \ddot{U}_i \delta U_i dv + \int_v c^2 T_0 \lambda_{ij} \dot{H}_j \delta H_i dv + \int_v C_{ijkl} \, e_{kl} \, \delta e_{ij} \, dv$$

$$+ \int_v c T_0 \, \theta \, \delta \, \theta \, dv \tag{16}$$

$$= \int_B \left[F_i \delta U_i + \bar{\sigma} n_i \delta H_i \right] dB$$

Now, if one considers an energy function which is expressed as the product of stress and strain

$$w = \frac{1}{2} \sigma_{ij} e_{ij} + \frac{1}{2} \bar{\sigma} \bar{\theta} \tag{17}$$

then the function satisifies

$$\frac{\partial w}{\partial e_{ij}} = \sigma_{ij} \text{ and } \frac{\partial w}{\partial \bar{\theta}} = \bar{\sigma} \tag{18}$$

where $\bar{\theta}$ is the total heat strain and equals $H_{i,i}$.

Alternatively, writing w in terms of strains

$$w = \frac{1}{2}(C_{ijkl}e_{kl} - \beta_{ij}T_0\theta)e_{ij} + \frac{1}{2}cT_0\theta\left[\theta + \frac{1}{c}\beta_{ij}e_{ij}\right]$$

$$= \frac{1}{2}C_{ijkl}e_{ij}e_{kl} + \frac{1}{2}cT_0\theta^2 \qquad (19)$$

$$\delta w = C_{ijkl}e_{ij}\delta e_{kl} + cT_0\theta\delta\theta \qquad (20)$$

A potential function is now defined as

$$V = \frac{1}{2}\int_v \left[C_{ijkl}e_{ij}e_{kl} + cT_0\theta^2\right]dv, \qquad (21)$$

which for isothermal deformation ($\theta = 0$) has the physical meaning of a strain energy function and for zero strain it reduces to a thermal potential.

Variations of V give

$$\delta V = \int_v C_{ijkl}e_{ij}\delta e_{kl}\,dv + \int_v cT_0\theta\,\delta\theta\,dv. \qquad (22)$$

Hence, Eq. (16) takes the form

$$\int_V \rho\ddot{U}_i\delta U_i\,dv + \int_v c^2T_0\lambda_{ij}\dot{H}_j\delta H_i\,dv + \delta V = \int_B \left[F_i\delta U_i + \bar{\sigma}\delta H_i n_i\right]dB.$$

The second term of Eq (23) is related to dissipation of heat energy and can be expressed as a variational invariant δD, given by

$$\delta D = \int_v c^2T_0\lambda_{ij}\dot{H}_j\delta H_i\,dv \qquad (24)$$

and Eq. (23) is written as

$$\int_v \rho\ddot{U}_i\delta U_i\,dv + \delta D + \delta V = \int_B \left[F_i\delta U_i + \bar{\sigma}n_i\delta H_i\right]dB = Q \qquad (25)$$

Equation (25) may be viewed as the variational form of the coupled thermal-momentum diffusion equations, with Q regarded as a generalized force.

3. Generalized Coordinates

Consider the mechanical and heat displacements represented by the following forms

$$U_i = U_i(q_1, q_2, \ldots, q_n, x_i, t)$$

and $\qquad i = 1, 2, 3 \qquad (26)$

$$H_i = H_i(q_i, q_2, \ldots, q_n, x_i, t)$$

which are given functions of the space coordinate x_i, of the time t and of the parameters q_i. These parameters are unknown functions of time and will be considered as the generalized coordinates. The variations of the fields U_i and H_i with respect to the generalized coordinates q_i are given by

$$\delta U_i = \frac{\partial U_i}{\partial q_j}\delta q_j, \quad \delta H_i = \frac{\partial H_i}{\partial q_j}\delta q_j, \qquad \begin{matrix} i = 1,2,3 \\ j = 1,n \end{matrix} \tag{27}$$

The variation of the potential V is

$$\delta V = \frac{\partial V}{\partial q_i}\delta q_i \quad i = 1,n \tag{28}$$

and the generalized forces Q_i have the form

$$\int_B \left[F_i \frac{\partial U_i}{\partial q_j} + \bar{\sigma} n_i \frac{\partial H_i}{\partial q_j} \right] \delta q_j dB = Q_i \delta q_i$$

and

$$Q_i = \int_B \left[F_j \frac{\partial U_j}{\partial q_i} + \bar{\sigma} n_j \frac{\partial H_j}{\partial q_i} \right] dB \tag{29}$$

Using the same procedure as in Lagrangian mechanics, the first term in Eq. (25) can be written in terms of the kinetic energy function as follows

$$\int_v \rho \ddot{U}_i \delta U_i dv = \int_v \rho \ddot{U}_i \frac{\partial U_i}{\partial q_j} \delta q_j dv.$$

From differential calculus one derives

$$\ddot{U}_i \frac{\partial U_i}{\partial q_i} = \frac{d}{dt}\left[\dot{U}_i \frac{\partial U_i}{\partial q_i}\right] - \dot{U}_i \frac{d}{dt}\left[\frac{\partial U_i}{\partial q_i}\right] \tag{30}$$

and from Eq. (26) one derives

$$\dot{U}_i = \frac{\partial U_i}{\partial q_j} \dot{q}_j + \frac{\partial U_i}{\partial t}.$$

Taking the partial derivative of the above with respect to q_i, yields

$$\frac{\partial \dot{U}_i}{\partial \dot{q}_j} = \frac{\partial U_i}{\partial q_j}, \tag{31}$$

and with respect to q_k

$$\frac{\partial \dot{U}_i}{\partial q_k} = \frac{\partial^2 U_i}{\partial q_j \partial q_k} \dot{q}_j + \frac{\partial^2 U_i}{\partial q_k \partial t} = \frac{d}{dt} \left(\frac{\partial U_i}{\partial q_k} \right). \tag{32}$$

Substituting Eqs (31) and (32) into Eq. (30) yields

$$\ddot{U}_i \frac{\partial U_i}{\partial q_j} = \frac{d}{dt} \left(\dot{U}_i \frac{\partial \dot{U}_i}{\partial \dot{q}_j} \right) - \dot{U}_i \frac{\partial \dot{U}_i}{\partial q_i}. \tag{33}$$

If the kinetic energy is expressed as

$$K = \frac{1}{2} \int_v \rho \dot{U}_i \dot{U}_i dv, \tag{34}$$

then combining Eqs. (30) - (34) one derives

$$\int_v \rho \ddot{U}_i \delta U_i dv = \left[\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} \right] \delta q_i. \tag{35}$$

To simplify δD, consider

$$\dot{H}_i = \frac{\partial H_i}{\partial q_j} \dot{q}_j + \frac{\partial H_i}{\partial t}$$

and

$$\frac{\partial \dot{H}_i}{\partial \dot{q}_j} = \frac{\partial H_i}{\partial q_j}.$$

Thus

$$\delta D = \frac{\partial D}{\partial \dot{q}_k} \delta q_k$$

where

$$\frac{\partial D}{\partial \dot{q}_k} = \int_v c^2 T_0 \lambda_{ij} \dot{H}_j \frac{\partial \dot{H}_i}{\partial \dot{q}_k} dv, \tag{36}$$

with D given by

$$D = \frac{1}{2} \int_v c^2 T_0 \lambda_{ij} \dot{H}_i \dot{H}_j dv. \tag{37}$$

Substituting Eqs. (28), (29), (35) and (36) in Eq. (25) one obtains

$$\left[\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_i} \right) - \frac{\partial K}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} \right] \delta q_i = Q_i \delta q_i \tag{38}$$

The system of Eqs. (38) represents the Lagrangian representation of the governing equations. They constitute a system of n differential equations corresponding to the generalized coordinates $q_i(t)$, $i = 1, n$.

Consider now a special case where the displacements are approximated by a linear combination of the generalized coordinates

$$U_i(x_j, t) = q_k(t)\, f_{ki}(x_j) \qquad i,j = 1,2,3 \tag{38}$$

$$H_i(x_j, t) = p_k(t)\, f_{ki}(x_j) \qquad k = 1,n \tag{39}$$

When Eqs. (39) are differentiated with respect to time and space variables, they yield

$$\dot{U}_i = \dot{q}_k f_{ki};$$

$$\dot{H}_i = \dot{p}_k g_{ki}; \tag{40}$$

$$U_{i,j} = q_k f_{ki,j};$$

$$H_{i,j} = p_k g_{ki,j}. \tag{41}$$

Then the strains e_{ij} and θ are given by

$$e_{ij} = \frac{1}{2} q_k \, (f_{ki,j} + f_{kj,i})$$

$$\theta = p_k g_{ki,i} - \frac{\beta_{ij}}{c} e_{ij} \tag{42}$$

The potential energy V can be expressed as

$$V = \frac{1}{2} v^1_{ij} q_i q_j - v^2_{ij} q_i p_j + \frac{1}{2} v^3_{ij} p_i p_j, \tag{43}$$

where

$$v^1_{kl} = \int_v \frac{1}{4} \left[C_{ijkl} + \frac{T_0}{c} \beta_{ij} \beta_{kl} \right] (f_{ki,j} + f_{kj,i}) (f_{lm,n} + (f_{ln,m}) dv, \tag{44}$$

$$v^2_{kl} = \int_v T_0 \beta_{ij} (f_{ki,j} + f_{kj,i}) g_{lm,m} dv,$$

$$v^3_{kl} = c T_0 \int_v g_{ki,i} \, g_{lj,j}.$$

(44)

Similarly, the dissipation and kinetic energy functions are

$$D = \frac{1}{2} d_{kl} \dot{p}_k \dot{p}_l,$$

$$K = \frac{1}{2} m_{kl} \dot{q}_k \dot{q}_l, \tag{45}$$

where

$$d_{kl} = \int_v c^2 T_0 \lambda_{ij} g_{ki} g_{lj} dv,$$

$$m_{kl} = \int_v \rho f_{ki} f_{li} dv. \tag{46}$$

The corresponding Lagrangian equations are then given by

$$m_{ij} \ddot{q}_j + v^1_{ij} q_j - v^2_{ij} p_j = FF_i$$

$$i,j = 1,n$$

$$d_{ij} \dot{p}_j + v^3_{ij} p_j - v^{2}{}_{ji} q_j = FQ_i \tag{47}$$

where FF_i and FQ_i are the mechanical and thermal forces corresponding to the generalized coordinates q_i and p_i respectively, and given by

$$FF_i = \int_B \sigma_{jk} n_k f_{ij} dB, \quad FQ_i = \int_B \bar{\sigma} n_j g_{ij} dB. \tag{48}$$

Eqs. (47) represent a system of $2n$ differential equations for the unknown field parameters q_i and p_i which describe the mechanical and heat displacement fields. The solution of this system then gives the thermal and mechanical deformations.

Note that the thermomechanical coupling is included in the coefficient v^2_{ij}. For the uncoupled case, the term $v^2_{ij} q_j$ is neglected and the second equation may be solved independently without the knowledge of q_i.

II. FINITE ELEMENT FORMULATION

The formulation in this study includes two kinds of one dimensional finite element models: a) a linear element (LE), with the minimum number of degrees of freedom, and b) a higher-order element with four degrees of freedom, two for each nodal point, which will be referred to as the cubic element (CE). The disadvantage of the linear element is that the value of strain throughout the element depends only on time. To achieve good results, one has to use a large number of elements, particularly for problems which involve transient behavior and discontinuities.

It should be pointed out that existing finite element analyses are usually based on the approximation of displacements. For pure thermal diffusion problems the temperature may be treated as a generalized displacement to apply finite element approximations. However, when other types of behavior are involved, such as mechanical deformations, similar treatments fail to yield a compatible formulation. This is the main reason that the present formulation is given in terms of the heat displacement field.

1. The Linear Element Analysis

The displacement fields U_i and H_i are approximated by

$$U(x,t) = a_1(t) + a_2(t)x$$

and

$$H(x,t) = b_1(t) + b_2(t)x \tag{49}$$

The conditions at the two nodal points are

$$\text{at } x = 0\text{: } U = U_1,\ H = H_1$$
$$\text{at } x = l\text{: } U = U_2,\ H = H_2$$

where (U_1, U_2) and (H_1, H_2) are the nodal displacements. Applying the above conditions in Eq. (49) one obtains

$$U = U_1 + (U_2 - U_1)\frac{x}{l}$$

$$H = H_1 + (H_2 - H_1)\frac{x}{l}$$

or, in an alternate form

$$\begin{aligned} U &= \left|1 - \frac{x}{l}\right| U_1 + \frac{x}{l} U_2 \\ H &= \left|1 - \frac{x}{l}\right| H_1 + \frac{x}{l} H_2 \end{aligned} \tag{50}$$

One can identify the shape functions f_{ij}, g_{ij} and the generalized coordinates q_i and p_i as

$$\begin{aligned} g_{11} = f_{11} = \left|1 - \frac{x}{l}\right|, \quad g_{12} = f_{12} = \frac{x}{l}; \\ q_1 = U_1, \quad q_2 = U_2; \\ p_1 = H_1, \quad p_2 = H_2. \end{aligned} \tag{51}$$

The only non-vanishing strain component is

$$e_{11} = \frac{1}{l}(U_2 - U_1), \tag{52}$$

and the stress components are then expressed as

$$\sigma_{ij} = C_{ij11} e_{11} - \beta_{ij} T_0 \theta.$$

If an isotropic elastic material is considered, shear stresses are zero and the axial stress components are

$$\begin{aligned} \sigma_{11} &= (\lambda + 2\mu) e_{11} - \beta T_0 \theta \\ &= \beta T_0 \left[\frac{\lambda + 2\mu}{\beta T_0} e_{11} - \theta\right] \\ \sigma_{22} = \sigma_{33} &= \beta T_0 \left[\frac{\lambda}{\beta T_0} e_{11} - \theta\right]. \end{aligned} \tag{53}$$

The kinematic relation gives

$$\theta = \frac{1}{l}\left[(H_2 - H_1) - \frac{\beta}{c}(U_2 - U_1)\right]. \tag{54}$$

The matrix coefficients from Eqs. (44) and (45) can be evaluated and the results are

$$
\begin{aligned}
v_{ij}^1 &= \left\{\lambda + 2\mu + \frac{\beta^2 T_0}{c}\right\} a_{ij}; \\
v_{ij}^2 &= \beta T_0 a_{ij}, \\
v_{ij}^3 &= c T_0 a_{ij}, \\
d_{ij} &= \frac{c T_0}{k} b_{ij}, \\
m_{ij} &= \rho b_{ij},
\end{aligned} \tag{55}
$$

where $a_{ij} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{A}{l}$ and $b_{ij} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \frac{Al}{6}$. Then the governing equations for calculating the nodal displacements are

$$
\begin{aligned}
&\rho b_{ij} \ddot{U}_j + \left\{\lambda + 2\mu + \frac{\beta^2 T_0}{c}\right\} a_{ij} U_j - \beta T_0 a_{ij} H_j = FF_i, \\
&\frac{c T_0}{\kappa} b_{ij} \dot{H}_j + c T_0 a_{ij} H_j - \beta T_0 a_{ij} U_j = FQ_i,
\end{aligned} \tag{56}
$$

with

$$FF_i = A \beta T_0 \sigma_i n \tag{57}$$

and

$$FQ_i = A c T_0 \theta_i n,$$

where σ_i is equal to σ_{11} for the nodal point 1. For the particular case of uncoupled thermoelasticity, the temperature variation due to coupling is very small and the $\beta T_0 a_{ij} U_j$ term in the second equation may be neglected.

2. The Cubic Element Analysis

Assume the following approximations for the displacement fields

$$U = a_0 + a_1 x + a_2 x^2 + a_3 x^3,$$
$$H = b_0 + b_1 x + b_2 x^2 + b_3 x^3; \tag{58}$$

then

$$e = \frac{\partial U}{\partial x} = a_1 + 2a_2 x + 3a_3 x^2$$

and

$$\theta = \frac{\partial H}{\partial x} - \frac{1}{c}\beta e = b_1 + 2b_2 x + 3b_3 x^2 - \frac{1}{c}\beta e \tag{59}$$

The conditions at the two nodal points are

$$\text{at } x = 0\text{: } U = U_1,\ e = e_1,\ H = H_1 \text{ and } \theta = \theta_1 - \frac{1}{c}\beta e_1$$
$$\text{at } x = l\text{: } U = U_2,\ e = e_2,\ H = H_2 \text{ and } \theta = \theta_2 - \frac{1}{c}\beta e_2$$

where (U_1, U_2) and (H_1, H_2) are the nodal displacements and (e_1, e_2), (θ_1, θ_2) are the nodal mechanical and thermal strains. The coefficients a_i and b_i are identified as

$$\begin{aligned}
a_0 &= U_1,\quad a_1 = e_1,\\
a_2 &= -\frac{1}{l^2}\left[(e_2 + 2e_1)l - 3(U_2 - U_1)\right],\\
a_3 &= \frac{1}{l^2}\left[(e_2 + e_1)l - 2(U_2 - U_1)\right],\\
b_0 &= H_1,\quad b_1 = \theta_1,\\
b_2 &= -\frac{1}{l^2}\left[(\theta_2 + 2\theta_1)l - 3(H_2 - H_1)\right],\\
b_3 &= \frac{1}{l^3}\left[(\theta_2 + \theta_1)l - 2(H_2 - H_1)\right].
\end{aligned} \tag{60}$$

Hence, the displacement and strain components have the form

$$\begin{aligned}
U &= f_{11}U_1 + f_{12}e_1 + f_{13}U_2 + f_{14}e_2,\\
e &= g_{11}U_1 + g_{12}e_1 + g_{13}U_2 + g_{14}e_2,\\
H &= f_{11}H_1 + f_{12}\theta_1 + f_{13}H_2 + f_{14}\theta_2,\\
\theta &= g_{11}H_1 + f_{12}\theta_1 + g_{13}H_2 + f_{14}\theta_2 - \frac{1}{c}\beta e.
\end{aligned} \tag{61}$$

One may identify the shape functions f_{1i} as

$$f_{11} = \left[1 - \frac{3x^2}{l^2} + \frac{2x^3}{l^3}\right], \quad f_{13} = \frac{3x^2}{l^2} - \frac{2x^3}{l^3},$$

$$f_{12} = \left[x - \frac{2x^2}{l} + \frac{x^3}{l^2}\right], \quad f_{14} = -\frac{x^2}{l} + \frac{x^3}{l^2}, \tag{62}$$

and g_{1i} is given by

$$g_{1i} = \frac{\partial f_{1i}}{\partial x}$$

The generalized coordinates q_i and p_i can be identified as

$$q_1 = U_1, \; q_2 = e_1, \; q_3 = U_2, \; q_4 = e_2,$$
$$p_1 = H_1, \; p_2 = \theta_1, \; p_3 = H_2, \; p_4 = \theta_2.$$

The expression for v_{ij}^k, d_{ij}, and m_{ij} can be written as

$$v_{ij}^1 = \left[(\lambda + 2\mu) + \frac{\beta^2 T_0}{c}\right] a_{ij}, \; v_{ij}^2 = \beta T_0 a_{ij}, \; v_{ij}^3 = c T_0 a_{ij},$$

$$d_{ij} = \frac{c T_0}{\kappa} b_{ij}, \; m_{ij} = \rho b_{ij}, \tag{64}$$

where a_{ij} and b_{ij} are given by

$$a_{ij} = \frac{A}{30l}\begin{bmatrix} 36 & 3l & -36 & 3l \\ 3l & 4l^2 & -3l & -l^2 \\ -36 & -3l & 36 & -3l \\ 3l & -l^2 & -3l & 4l^2 \end{bmatrix} \tag{65}$$

and

$$b_{ij} = \frac{Al}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \tag{66}$$

The governing equations in terms of the generalized coordinates are

$$\rho b_{ij}\ddot{q}_j + \left[(\lambda + 2\mu) + \frac{\beta^2 T_0}{c}\right] a_{ij} q_j - \beta T_0 a_{ij} p_j = FF_i$$

$$\frac{cT_0}{\kappa} b_{ij}\dot{p}_j + cT_0 a_{ij} p_j - \beta T_0 a_{ij} q_j = FQ_i \qquad (67)$$

with the generalized forces FF_i and FQ_i given by

$$FF_i = A\beta T_0 \sigma_{,i} n, \quad i = 1,3,$$

$$FQ_i = AcT_0 \theta_{,i} n, \quad i = 1,3, \qquad (68)$$

and

$$FF_2 = FF_4 = FQ_2 = FQ_4 = 0.$$

The differential Eqs. (67) constitutes a system of eight equations for the nodal displacements and nodal strains.

3. The Overall Problem and Boundary Conditions

In the previous sections, the equations for the basic linear and cubic element models were derived. Consider now a one-dimensional case with the medium modeled by either of the element models. Since the heat and mechanical displacements are compatible, the connection of the elements follows the rules and the assembly process analogous to the one used in the matrix analysis of structures. The most commonly used assembling method is the direct stiffness method, according to which the overall stiffness matrix results from the stiffness matrices of the individual elements by simple addition at the nodal points.

For the elements introduced in this analysis, the matrix equation for the element can be put in the form

$$\begin{bmatrix}\rho b_{ij} & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}\ddot{q}_i\\ \ddot{p}_i\end{Bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \dfrac{cT_0}{k} b_{ij}\end{bmatrix}\begin{Bmatrix}\dot{q}_i\\ \dot{p}_i\end{Bmatrix}$$

$$+ \begin{bmatrix}\left[(\lambda + 2\mu) + \dfrac{\beta^2 T_0}{c}\right] a_{ij} & -\beta T_0 a_{ij}\\ -\beta T_0 a_{ij} & cT_0 a_{ij}\end{bmatrix}\begin{Bmatrix}q_i\\ p_i\end{Bmatrix} = \begin{Bmatrix}FF_i\\ FQ_i\end{Bmatrix} \tag{69}$$

Then the assembled matrix equation has the following schematic form

$$\begin{bmatrix}[A_1] & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}\{\ddot{q}\}\\ \{\ddot{p}\}\end{Bmatrix} + \begin{bmatrix}0 & 0\\ 0 & [A_2]\end{bmatrix}\begin{Bmatrix}\{\dot{q}\}\\ \{\dot{p}\}\end{Bmatrix} + \begin{bmatrix}[A_3] & [A_4]\\ [A_5] & [A_6]\end{bmatrix}\begin{Bmatrix}\{q\}\\ \{p\}\end{Bmatrix} = \begin{Bmatrix}\{FF\}\\ \{FQ\}\end{Bmatrix} \tag{70}$$

where the square matrices $[A_i]$ are derived through the direct stiffness method. The vectors $\{q\}$ and $\{p\}$ represent nodal degrees of freedom and the force vectors $\{FF\}$ and $\{FQ\}$ have zero components at the connecting nodal points and are non-zero on the boundary. The solution of the above matrix equation gives the nodal displacements. Then the distribution of stresses can be evaluated by the stress-displacement relations and the kinematic relations.

The formulation of the overall problem is not complete unless boundary conditions are taken into consideration. The procedure is identical to the one in traditional matrix structural analysis, where partitioning of the equations eliminates the singularity of the matrix coefficients.

III. APPLICATION OF THE FINITE ELEMENT FORMULATION TO THE COUPLED DIFFUSION PROBLEM

1. Problem Formulation

In applying the derived finite-element formulation, the problem of the linear elastic half-space subjected to a time dependent temperature change on its traction-free boundary plane, is considered. The initial boundary value problem assuming a sudden heating of the boundary plane is known as Danilovskaya's problem and an extension of this problem is given by Sternberg and Chakravorty in Ref. [20].

The specific notations and descriptions of the problem are depicted in Fig. 1. Let (x,y,z) be the Cartesian coordinate system and consider an elastic medium occupying the half-space $x \geqslant 0$, with the boundary plane at $x = 0$ assumed to be free of traction at all times. For uniform boundary heating, it can be assumed that the medium is restricted to a uniaxial motion and hence

$$\begin{aligned} U_x &= U_x(x,t), \\ U_y &= U_z = 0. \end{aligned} \tag{71}$$

For the displacements in Eq. (71), the Cartesian components of shear stress vanish identically and the stress-strain relations for a linear material are given by

$$\begin{aligned} \sigma_x &= (\lambda + 2\mu)e_x - \beta(T - T_0), \\ \sigma_y &= \sigma_z = \lambda e_x - \beta(T - T_0), \end{aligned} \tag{72}$$

where $(\sigma_x, \sigma_y, \sigma_z)$ are the components of normal stress, λ, μ are the material constants, β is the thermal modulus and $T(x,t)$ is the time dependent temperature field and $T_0 = T(x,0)$.

If the body forces are absent, then the momentum equation reduces to

$$\frac{\partial \sigma_x}{\partial x} = \rho \frac{\partial^2 u_x}{\partial t^2}, \tag{73}$$

in which ρ denotes the constant mass density, and the temperature field is governed by the thermal diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \tag{74}$$

where $\kappa = k/c$ is the thermal diffusivity. The initial condition for the temperature is

$$T(x,0) = T_0, \tag{75}$$

while the uniform ramp heating yields the boundary conditions

$$\begin{aligned} T - T_0 &= 0 \quad \text{for} \quad t \leqslant 0; \\ T - T_0 &= \frac{t}{t_0}(T_1 - T_0) \quad \text{for} \quad 0 \leqslant t \leqslant t_0; \\ &= T_1 - T_0 \quad \text{for} \quad t \geqslant t_0; \end{aligned} \tag{76}$$

where t_0 represents the time required for the temperature to reach a constant value T_1.

If the medium is initially at rest, then the initial conditions for the displacement are

$$U_x(x,0) = 0, \quad \frac{\partial U_x}{\partial t}(x,0) = 0, \tag{77}$$

while the stress boundary condition is

$$\sigma_x(0,t) = 0, \quad t \geqslant 0. \tag{78}$$

To satisfy the regularity requirements, one also has

$$(T(x,t) - T_0), \quad U_x(x,t), \quad \sigma_x(x,t), \quad \sigma_y(x,t) \to 0, \quad \text{as} \quad x \to \infty \tag{79}$$

At this stage it is expedient to relate the dimensionless variables to the physical ones as follows:

$$\begin{aligned} &\xi = \frac{ax}{\kappa}, \quad \tau = \frac{a^2 t}{\kappa}, \\ &\sigma_1 = \frac{\sigma_x}{\beta(T_1 - T_0)}, \quad U = \frac{a(\lambda + 2\mu)U_x}{\beta(T_1 - T_0)\kappa}, \\ &\sigma_2 = \frac{\sigma_y}{\beta(T_1 - T_0)}, \quad H = \frac{a\, T_0 H_x}{\kappa(T_1 - T_0)}, \\ &\bar{\theta} = \frac{\theta}{\theta_1} = \frac{T - T_0}{T_1 - T_0} = \theta \frac{T_0}{T_1 - T_0}, \end{aligned} \tag{80}$$

where $\kappa = k/c$, $a^2 = (\lambda + 2\mu)/\rho$ and $\beta = \alpha(3\lambda + 2\mu)$, with α being the thermal coefficient of linear expansion. H is the dimensionless heat displacement in the present formulation.

Eqs. (73) and (74) can be written as

$$\frac{\partial \sigma_1}{\partial \xi} = \frac{\partial^2 U}{\partial \tau^2}, \tag{81}$$

and

$$\frac{\partial \bar{\theta}}{\partial \tau} = \frac{\partial^2 \bar{\theta}}{\partial \xi^2}.$$

The initial conditions are

$$U(\xi, 0) = \frac{\partial U(\xi, 0)}{\partial \tau} = \bar{\theta}(\xi, 0) = 0 \tag{82}$$

and the boundary conditions are

Type 1. Sudden heating of the boundary

$$\sigma_1(0, \tau) = 0 \text{ for all } \tau;$$
$$\bar{\theta}(0, \tau) = \begin{cases} 0 & \tau < 0 \\ 1 & \tau \geqslant 0 \end{cases} \tag{83}$$

Type 2. Ramp-heating of the boundary

$$\sigma_1(0, \tau) = 0 \text{ for all } \tau;$$
$$\bar{\theta}(0, \tau) = \begin{cases} 0 & \tau < 0 \\ \tau/\tau_0 & 0 \leqslant \tau \leqslant \tau_0. \\ 1 & \tau_0 \leqslant \tau \end{cases} \tag{84}$$

Applying the finite-element formulation developed in the previous chapters, the equations for the two element models can be written in the following dimensionless forms given below

Linear element (LE) model:

$$\begin{bmatrix} [b_l][0] \\ [0]\,[0] \end{bmatrix} \begin{Bmatrix} \{\ddot{U}\} \\ \{\ddot{H}\} \end{Bmatrix} + \begin{bmatrix} [0] & [0] \\ [0] & [b_l] \end{bmatrix} \begin{Bmatrix} \{\dot{U}\} \\ \{\dot{H}\} \end{Bmatrix} + 6w^2 \begin{bmatrix} (1+\delta)[a_l] & -[a_l] \\ -\delta\,[a_l] & [a_l] \end{bmatrix} \begin{Bmatrix} \{U\} \\ \{H\} \end{Bmatrix} = 6w \begin{Bmatrix} \{\overline{FF}\} \\ \{\overline{FQ}\} \end{Bmatrix} \tag{85}$$

Here, $w = \kappa/al$ with l as the element length and $\overline{FF}_i = (FF_i)/[Ac(T_1 - T_0)]$, $\overline{FQ}_i = (FQ_i)/[Ac(T_1 - T_0)]$. The two constant matrices $[a_l]$ and $[b_l]$ are given by

$$[a_l] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } [b_l] = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{86}$$

Cubic element (CE) model:

$$\begin{bmatrix} [b_c] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{U}\} \\ \{\ddot{e}\} \\ \{\ddot{H}\} \\ \{\ddot{\theta}\} \end{Bmatrix} + \begin{bmatrix} [0] & [0] \\ [0] & [b_c] \end{bmatrix} \begin{Bmatrix} \{\dot{U}\} \\ \{\dot{e}\} \\ \{\dot{H}\} \\ \{\dot{\theta}\} \end{Bmatrix}$$

$$+ 14w^2 \begin{bmatrix} (1+\delta)\,[a_c] & -\,[a_c] \\ -\delta\,[a_c] & [a_c] \end{bmatrix} \begin{Bmatrix} \{U\} \\ \{e\} \\ \{H\} \\ \{\theta\} \end{Bmatrix} = 420w \begin{Bmatrix} \{\overline{FF}\} \\ \{0\} \\ \{\overline{FQ}\} \\ \{0\} \end{Bmatrix} \tag{87}$$

Here the parameter δ is defined as the thermomechanical coupling parameter, given by

$$\delta = \frac{\beta^2 T_0}{c(\lambda + 2\mu)}, \tag{88}$$

and the matrices $[a_c]$ and $[b_c]$ are given by

$$[a_c] = \begin{bmatrix} 36 & -36 & 3/w & 3/w \\ -36 & 36 & -3/w & -3/w \\ 3/w & -3/w & 4/w^2 & -1/w^2 \\ 3/w & -3/w & -1/w^2 & 4w^2 \end{bmatrix};$$

$$[b_c] = \begin{bmatrix} 156 & 54 & 22/w & -13/w \\ 54 & 156 & 13/w & -22/w \\ 22/w & 13/w & 4/w^2 & -3/w^2 \\ -13/w & -22/w & -3/w^2 & 4/w^2 \end{bmatrix}. \tag{89}$$

The assembly of the above equations for the overall problem and their modifications due to the boundary conditions was coded in the computer program.

In the case of the (LE) model, after solving for the displacements, the temperature for the i th element can be obtained from the relation

$$\theta_i = w[H_{i+1} - H_i] - \delta w[U_{i+1} - U_i], \tag{90}$$

and the stresses for the i th nodal point,

$$\sigma_{1_i} = w(U_{i+1} - U_i) - \theta_i, \tag{91}$$
$$\sigma_{2_i} = \nu\sigma_{1_i} - (1 - 2\nu)\theta_i.$$

For the (CE) model, the solution of the equations will directly give the nodal displacements and strains, and thus the temperature and strain distributions are obtained directly from the solution of the matrix equation. Eqs. (85) through (91) can be applied to the coupled and uncoupled cases by simply setting $\delta \neq 0$ or $\delta = 0$, respectively. These equations can be used for solving the problem stated previously in this section.

2. Numerical Results

A numerical solution of Eqs. (85) and (87) for the finite-element analyses is obtained by using a third order, backward finite-difference scheme. A closed form analytical solution is given in Ref. [20] for the uncoupled equations and it will be used here to represent the exact solution. An analytical solution of the coupled case is given by Nickell and Sackman in Ref. [16], for the ramp-heating type of the boundary condition.

For the finite-element solution the total number of elements used, was TNE = 30 for the (LE) model and TNE = 20 for the (CE) one. The temperature boundary condition is characterized by the parameter τ_0 as follows:

Type 1. Sudden heating, $\tau_0 = 0$

$$\bar{\theta}(0,\tau) = \begin{cases} 0 & \tau < 0 \\ 1 & \tau \geqslant 0 \end{cases} \tag{V.2.1}$$

Type 2. Ramp-heating, $\tau_0 = 1.0$

$$\bar{\theta}(0,\tau) = \begin{cases} 0 & \tau < 0 \\ \tau/\tau_0 & 0 \leq \tau \leq 1.0 \\ 1 & \tau \geq 1.0 \end{cases} \qquad \text{(V.2.2)}$$

Two values of the coupling parameter were used, $\delta = 0$ for the uncoupled case and $\delta = 1$ for the coupled one. The time step sizes used for solving the matrix differential equations were $D\tau = 0.005$ for the (LE) and $D\tau = 0.01$ for the (CE). For evaluating the lateral stress σ_2, the value of $\nu = 1/3$ was used for Poisson's ratio.

The results obtained are illustrated in Figs. (1) through (12) for both element models and they are compared to the exact solution. For the case of $\tau_0 = 0$, $\delta = 1$, numerical data for the exact solution were not available, thus only the results from the present formulation are given. Figs. 1 to 4 depict the temperature, mechanical displacement and stresses as a function of time at $\xi = 1.0$ and for $\tau_0 = 0$. The results for the temperature, Fig. 1, show a very good agreement with the exact solution for both element models. For the displacement, Fig. 2, the (CE) model gives less error than the (LE), with very small instability around the point of discontinuity. For the stresses, Figs. 3 and 4, again the results for the (CE) are better than those of the (LE). The instability around the discontinuity point $\tau = 1.0$, is due partly to the numerical scheme and partly to the small number of elements between $\xi = 0$ and $\xi = 1.0$. This instability can be corrected by refining the mesh around the point of discontinuity.

The results for $\tau_0 = 1.0$, are illustrated in Figs. 5 to 8. The temperature as a function of time at $\xi = 1.0$, is given in Fig. 5, the displacement in Fig. 6, and the normal and lateral stresses in Figs. 7 and 8, respectively. As one can see from these figures, the results obtained through the CE solution show excellent agreement with the exact solution even around the discontinuity point. For the (LE) solution, the results for the displacement and stresses show an error which is due to the numerical approximation and can be corrected by refining the mesh around the point $\xi = 1.0$. In order to show the propagation of discontinuities in the half-space, the temperature is plotted in Fig. 9, the displacement in

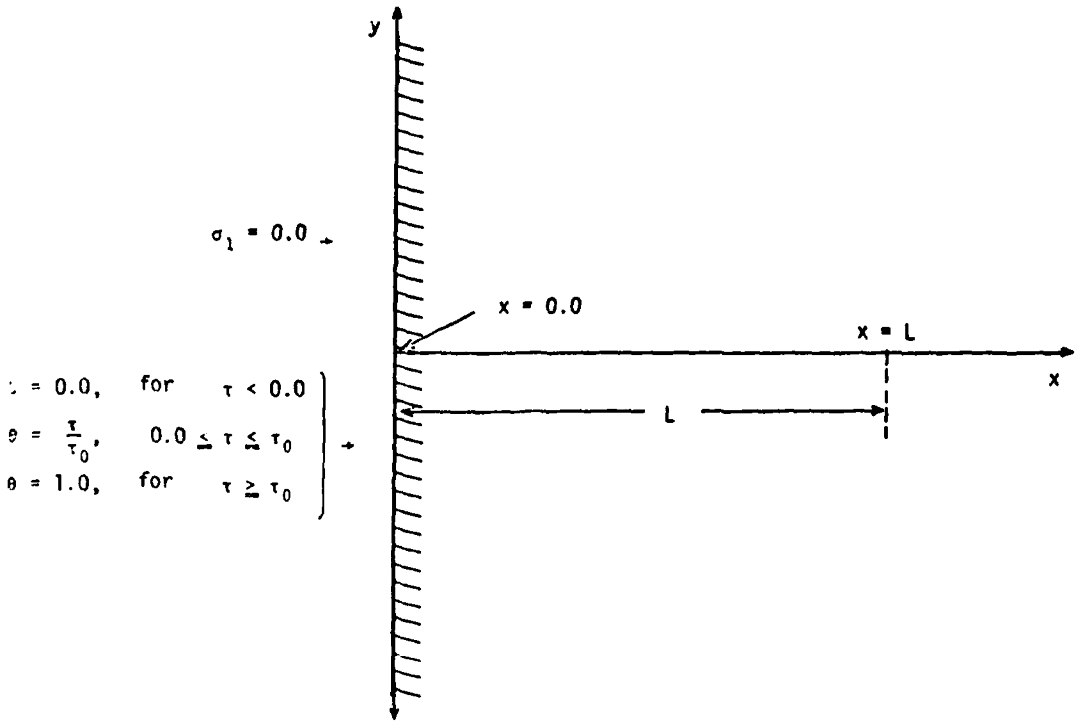

Figure 1 — Illustration of the semi-infinite space.

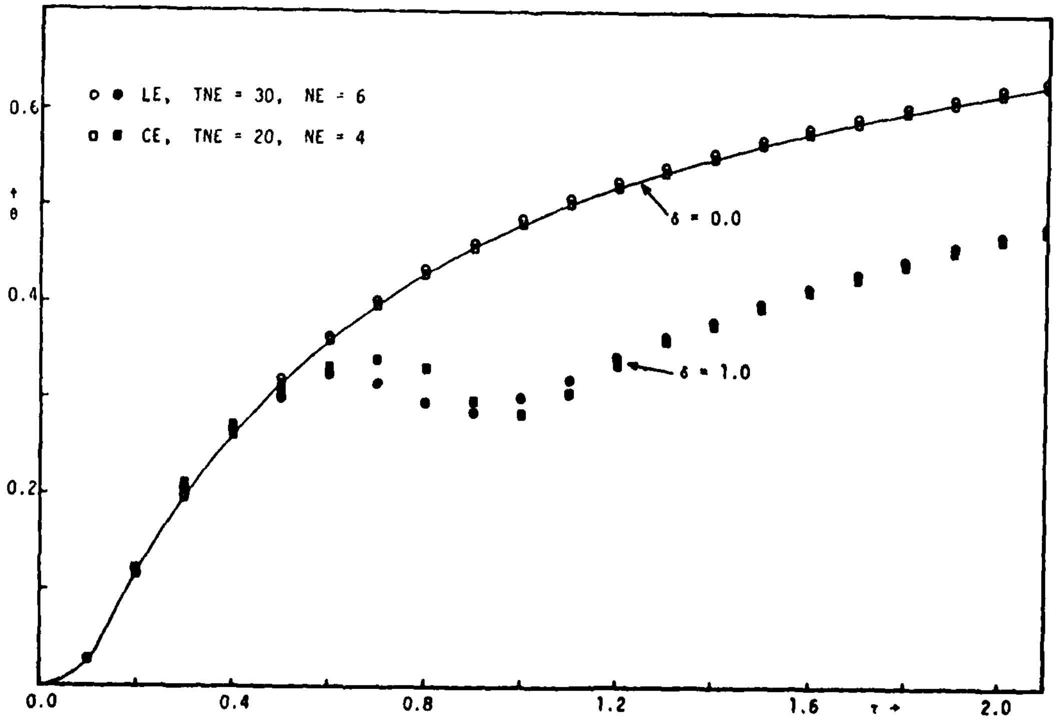

Figure 2 — Temperature distribution at $\xi = 1$, as a function of time, with $\tau_0 = 0.0$.

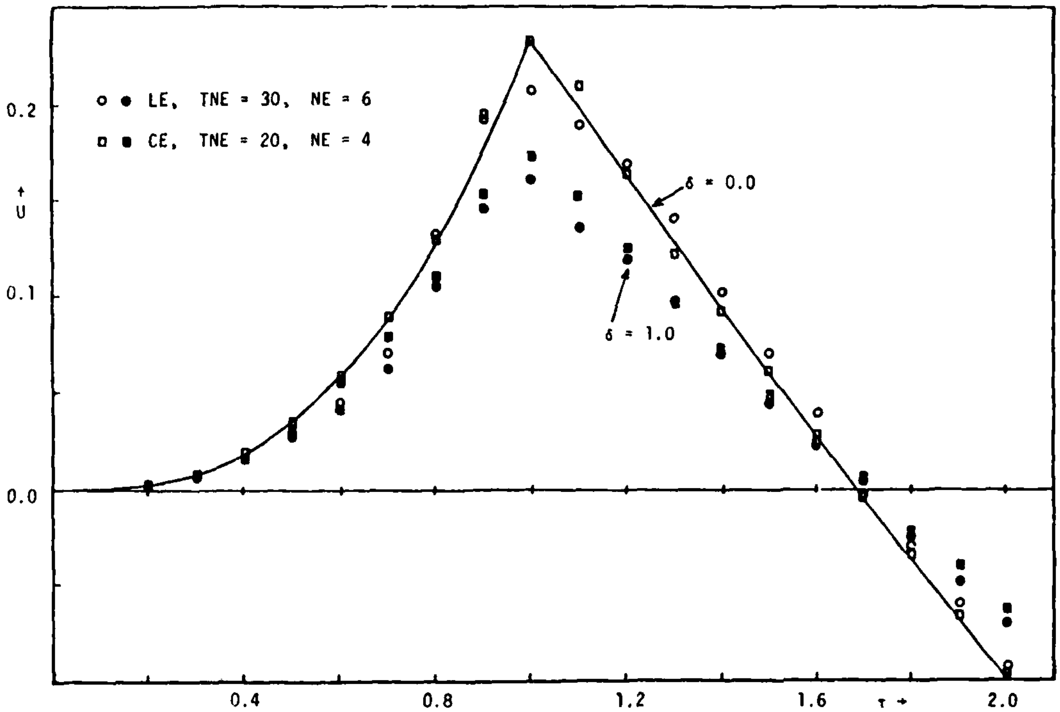

Figure 3 — Displacement distribution at $\xi = 1$, as a function of time, with $\tau_0 = 0.0$, dynamic solution

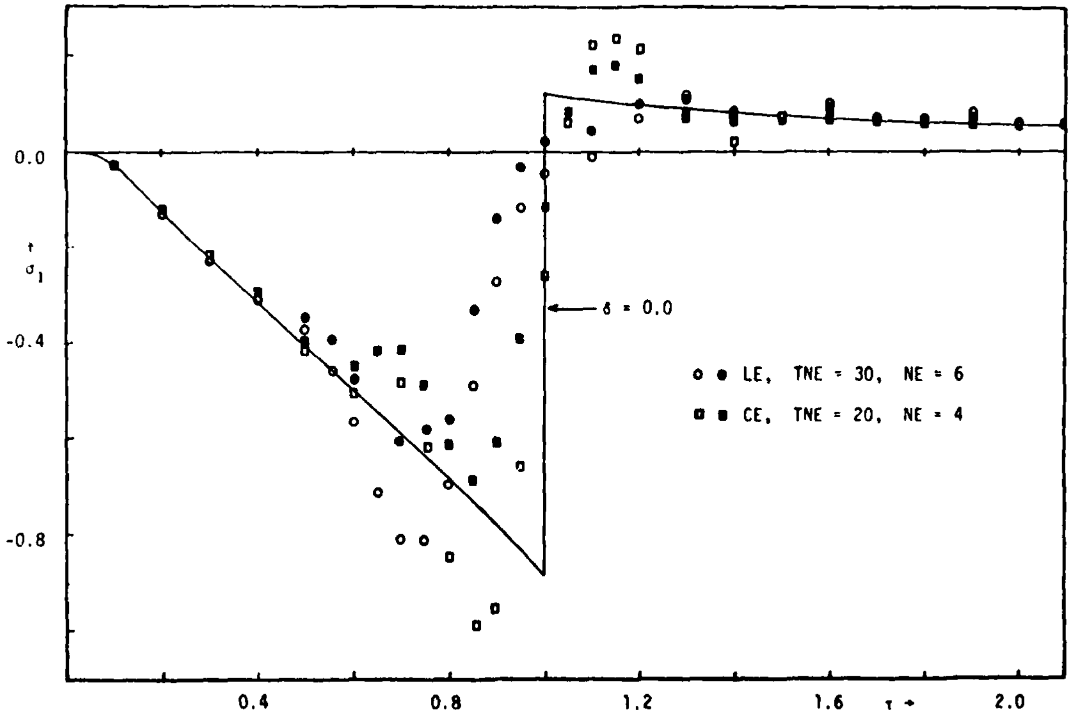

Figure 4 — Normal stress distribution at $\xi = 1$, as a function of time, with $\tau_0 = 0.0$, dynamic solution

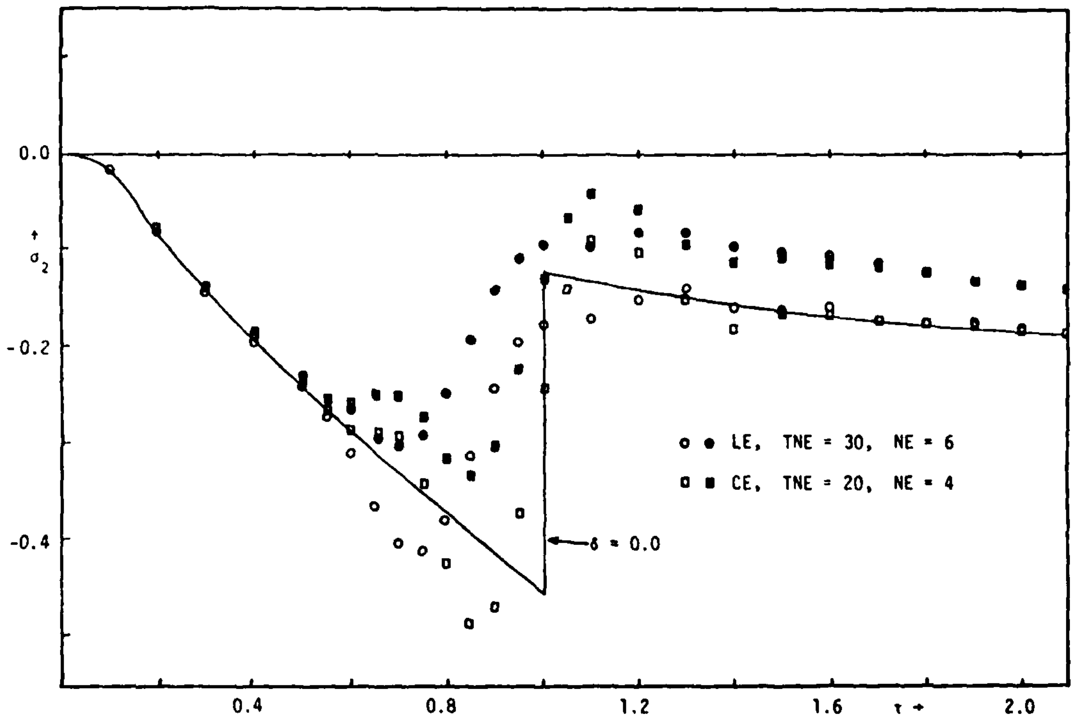

Figure 5 — Lateral stress distribution at $\xi = 1$, as a function of time, with $\tau_0 = 0.0$, dynamic solution.

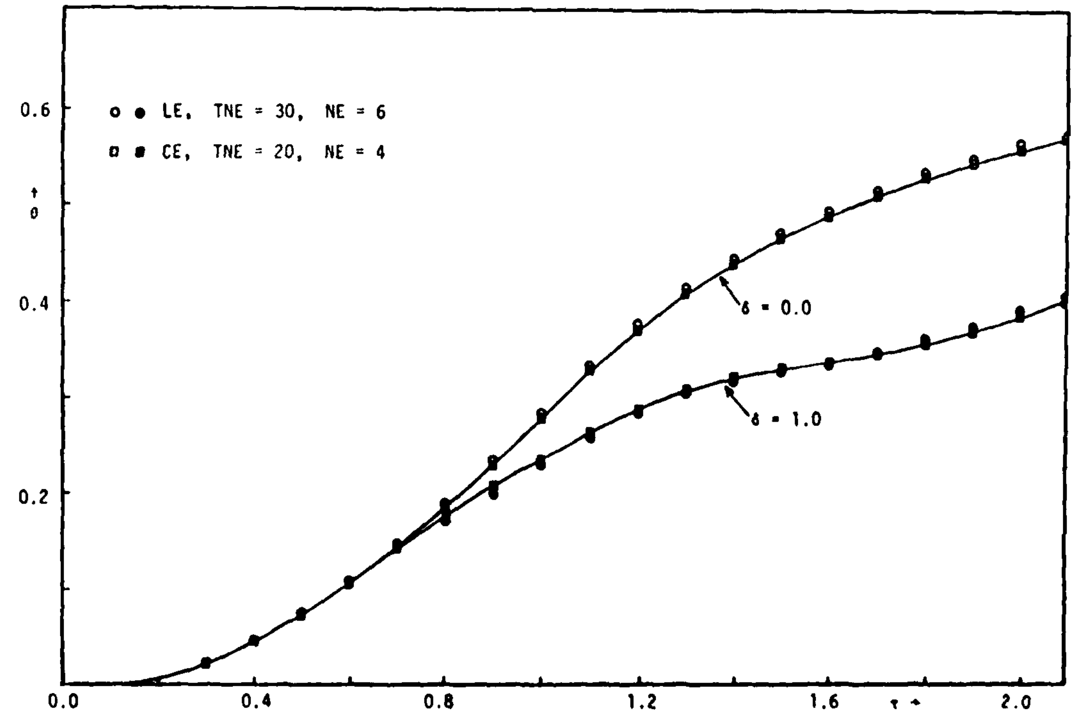

Figure 6 — Temperature distribution at $\xi = 1$, as a function of time, with $\tau_0 = 1.0$, dynamic solution.

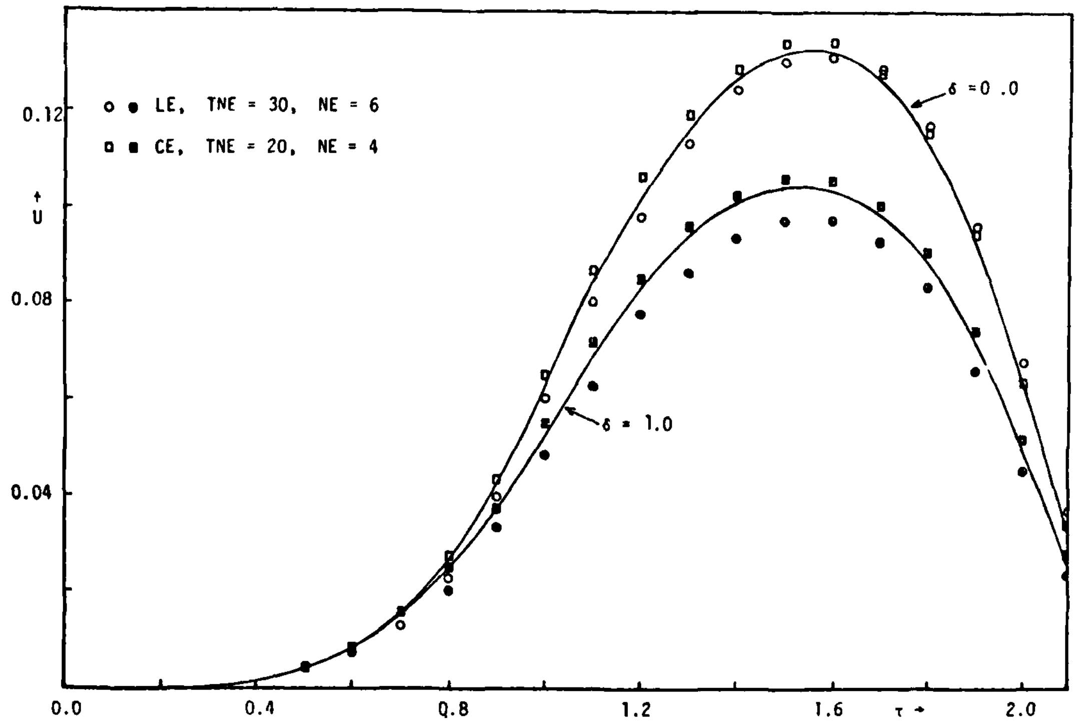

Figure 7 — Displacement distribution at $\xi = 1$, as a function of time, with $\tau_0 = 1.0$, dynamic solution.

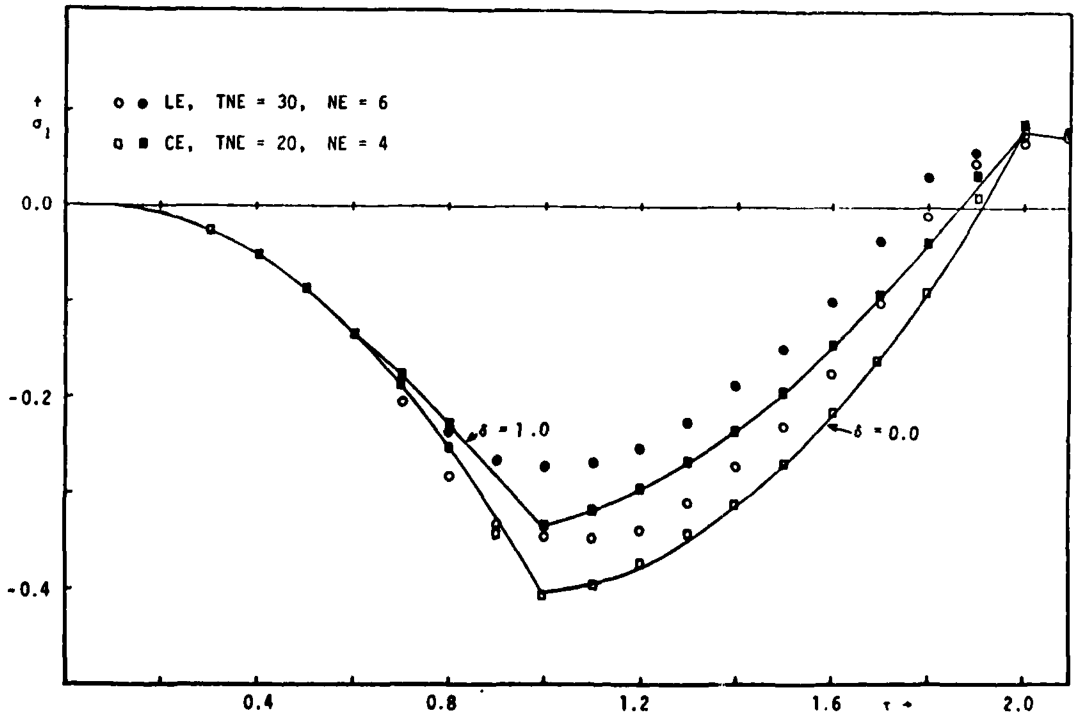

Figure 8 — Normal stress distribution at $\xi = 1$, as a function of time, with $\tau_0 = 1.0$, dynamic solution

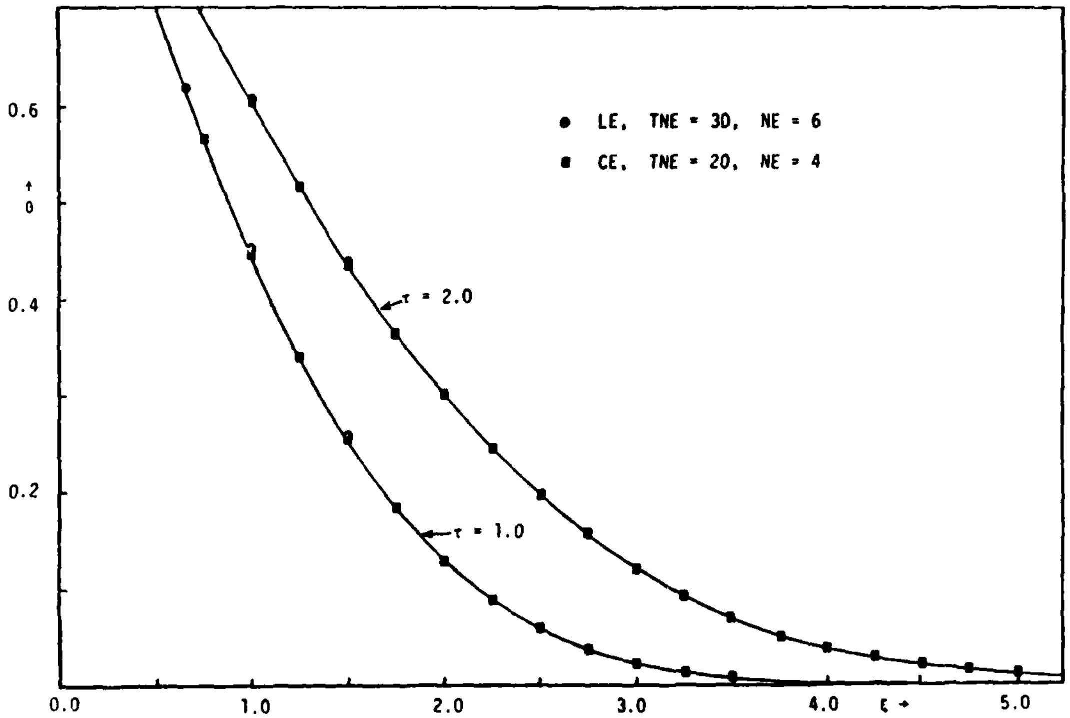

Figure 9 — Lateral stress distribution at $\xi = 1$, as a function of time, with $\tau_0 = 1.0$, dynamic solution.

Fig. 10, and the normal stress in Fig. 11 as a function of the space coordinates, for $\tau_0 = 0.25$ and $\tau = 1.0$, $\tau = 2.0$. These figures show how the thermal induced wave propagates into half-space. As one can see from these figures, the (CE) gives excellent agreement with the exact solution, and the LE gives satisfactory results but with a small instability around the discontinuity points.

SUMMARY AND CONCLUSIONS

A unified formulation for the coupled thermal-momentum diffusion equations was introduced in this report and, based on this formulation, two finite element models were developed for the purpose of solving problems on thermal and mechanical deformations.

The introduction of a new quantity, defined as heat displacement, is the basis for a unified presentation of the governing equations with one-to-one correspondence between thermal and mechanical quantities. This presentation is used to develop a successful displacement formulation for the finite element analysis. The resulting matrix differential equations for the finite element models are given in terms of nodal displacements and/or strains. Due to the nature of this formulation, boundary conditions, given in terms of temperature, strain, mechanical displacement and/or heat flow, can be easily handled.

In the present formulation a thermal force was also introduced, for which one should point out its significance as a boundary force. The regularity condition for Danilovskaya's problem requires that $\theta \to 0$ as $\xi \to \infty$. Since the last nodal point in the finite element approximation of the half-space represents infinity one should impose the above condition at this point, and then the thermal force is zero due to zero temperature. This assumption is not the correct one since the temperature at the last nodal point will increase as the thermal wave propagates and the time τ increases. If we consider the last nodal point as a boundary point and the thermal force as a boundary force which is equal to the temperature at that point, then the conditions of the boundary point are properly adjusted. The importance of this thermal force is shown in Fig. 12 , where the temperature is given as a function of ξ for

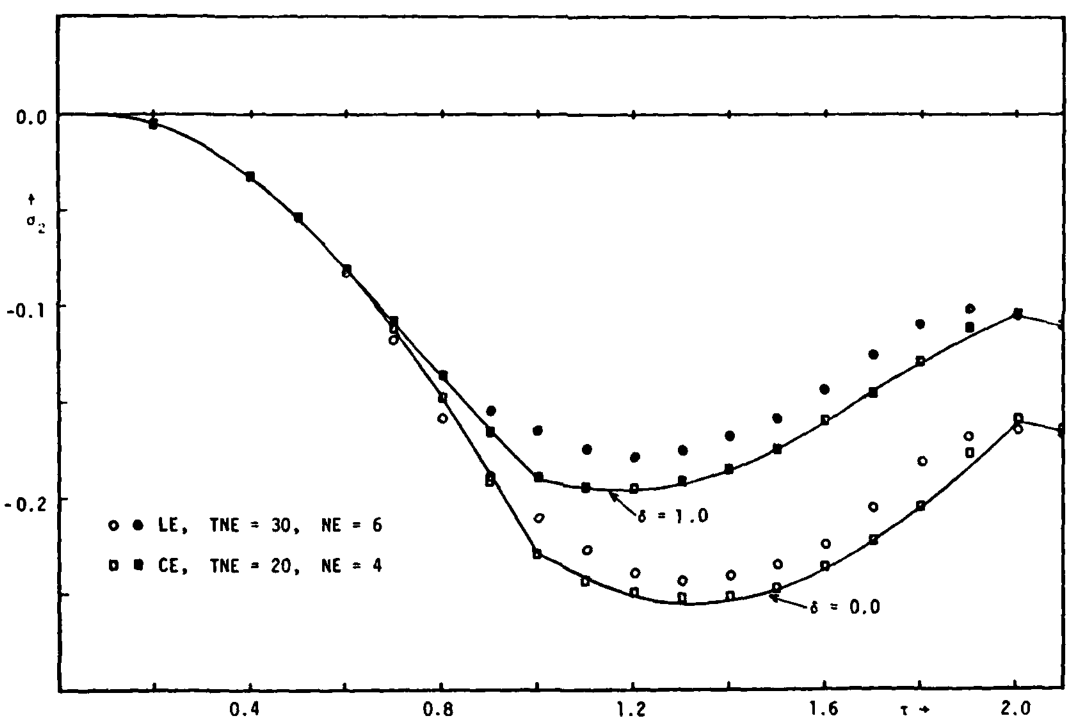

Figure 10 — Temperature distribution as a function ξ, dynamic solution with $\tau_0 = 0.25$.

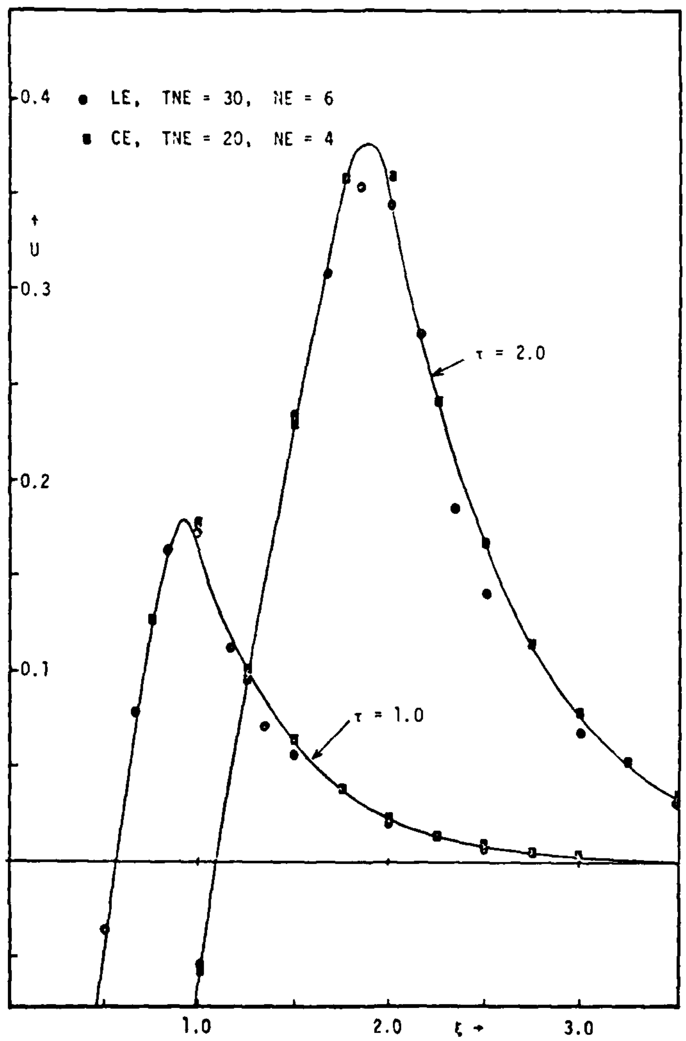

Figure 11 — Displacement distribution as a function ξ, dynamic solution with $\tau_0 = 0.25$, as a function of time, with $\tau_0 = 0.0$, dynamic solution

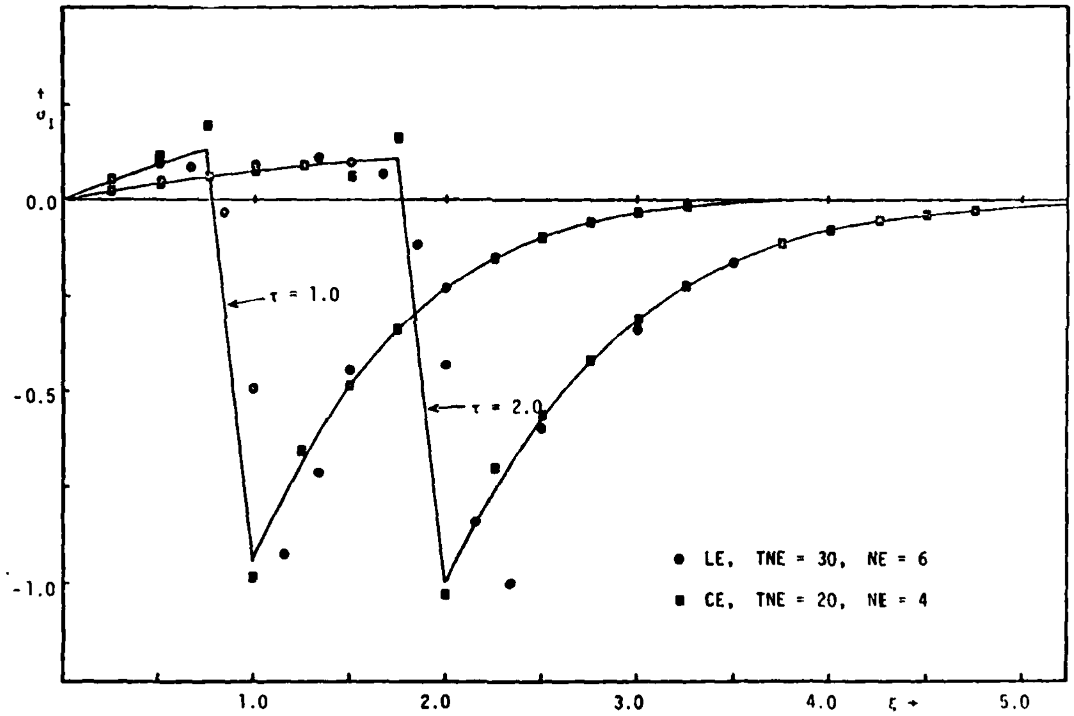

Figure 12 — Normal stress distribution as a function ξ, dynamic solution with $\tau_0 = 0.25$.

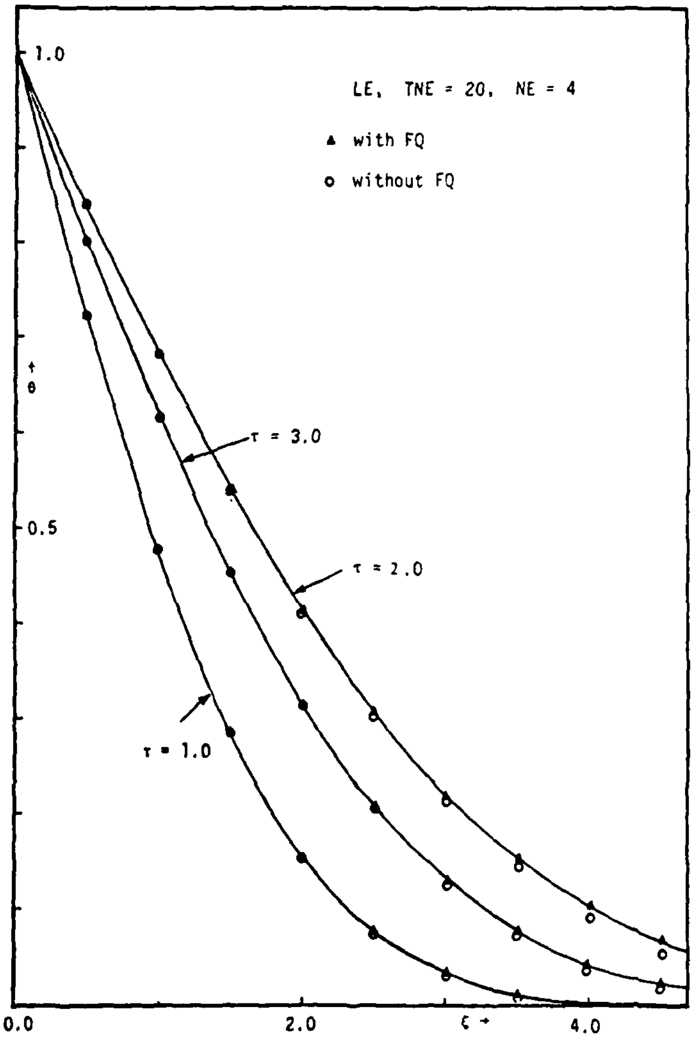

Figure 13 — Temperature distribution as a function of ξ, with $\tau_0 = 0.0$. The effect of the thermal force on the finite element solution.

the three different times τ and for $\tau_0 = 0.0$. The results illustrated show how the finite element solution behaves by neglecting or retaining the thermal forces. When the thermal force is neglected the solution is lower than the exact solution for $\xi \geq 3$ and as τ increases, but this is not true if the thermal force is retained in the formulation.

A solution of the matrix differential equations can be obtained by any standard integration technique. Here a third order backward finite-difference scheme was adopted for the solution.

The two finite elements developed in the present study were used successfully to model the propagation of thermally induced waves in a semi-infinite medium, with prescribed conditions for the stress and temperature on the boundary plane. For the temperature, the boundary conditions are of two types; one is sudden heating and the other is ramp heating is the boundary plane. Results obtained by the present finite element analysis are for the cases both of coupled and uncoupled behavior. These are compared to existing analytical solutions. The cubic element model gives an excellent representation of the exact solution and the linear element also gives satisfactory results within acceptable limits of accuracy.

A comparison of the present solution with existing numerical solutions of similar problems is appropriate in order to evaluate the efficiency of the present formulation. Two of the numerical solutions which solve the same thermoelastic problem are given by Nickell and Sackman in [16] and by Oden [17].

The solution given by Nickell and Sackman is based on the Ritz method and their results give satisfactory agreement with the exact solution. In order for the authors to achieve a stable solution, with small error, they used a very fine mesh for the characteristic length L, (TNE = 47, NE = 20), and a time step of 0.005-0.01. The computing time for such solutions was about five minutes. The present solution, with a less fine mesh, produces results of similar accuracy and with less computing time required to solve the same problem. For example, the linear element model solution with TNE = 30 requires about 70 seconds and the cubic one with TNE = 20, requires about 100 seconds. Thus, the present solution is more efficient for the same degree of accuracy. Another advantage of the present formulation over the one given by Nickell and Sackman is that it can be used for any geometric configuration and with any kind of boundary condition.

The second solution, given by Oden, is based on the finite element method. In reference [17], Oden gives results only for the temperature and displacement. For some of his results, Oden used up to TNE = 48 with NE = 10 in order to achieve a satisfactory solution. Comparing the two solutions it is apparent that the one proposed here requires a smaller number of elements for the same accuracy as Oden's. Oden also mentions that his results for the temperature are lower than the exact solution for $\xi > 3$ and as the time τ increases, and he suggests an increase of the characteristic length L in order to obtain improved results. This kind of correction becomes unnecessary to this formulation since this is corrected by the thermal force of the last nodal point as was shown in Figure 12.

In conclusion, the unified form of the present formulation should be emphasized, as well as its efficiency on handling various types of boundary conditions. The superiority of the cubic element model over the linear type should be also noted. Both element models can be used for solving diffusion problems and the choice between them depends on the particular needs of a problem. An extension of this formulation to other problems or to two dimensions is a next step in applying the finite element model.

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