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RELATION BETWEEN THE DOUGALL AND MUKI POTENTIALS

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US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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General solutions of the equations of elastostatics have been given by Dougall in terms of one set of potentials and by Muki in terms of another set of potentials. It is shown how the potentials of Muki are related to the potentials of Dougall.

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I. INTRODUCTION

In 1914, Dougall[1] treated the problem of equilibrium of an elastic rod with the use of a general solution of the equations of elasticity. Dougall's solution is expressed in terms of three arbitrary harmonic potentials. In 1960, Muki[2] treated elastic equilibrium problems with the use of a general solution expressed in terms of another set of potentials. Muki's potentials comprise an arbitrary biharmonic function and an arbitrary harmonic function. Our purpose here is to establish the relations between the Dougall potentials and the Muki potentials.

II. THE EQUATIONS OF ELASTIC EQUILIBRIUM-NOTATION

Our analysis will be in terms of cartesian tensors. (For a further discussion, see Appendix A.) We deal with a cartesian coordinate system x_1, x_2, x_3. We shall adopt the convention that lower case Latin indices will take the values 1, 2, 3. A comma followed by an index, say i, denotes partial differentiation with respect to the coordinate corresponding to that index, say x_i.

Dougall writes the equations of elastic equilibrium in the form

$$\mu \nabla^2 u_i + (\lambda + \mu) \Delta,_i = 0, \tag{1}$$

where u_i is the displacement vector, ∇^2 is the laplacian operator; i.e.,

$$\nabla^2 u_i = u_{i,jj} \quad , \tag{2}$$

Δ denotes the volumetric expansion; i.e.,

$$\Delta = u_{i,i} \tag{3}$$

and λ and μ are the Lamé constants.

Muki, on the other hand, wrote Eqs. (1) in the form

$$\nabla^2 u_i + \frac{1}{1 - 2\nu} \Delta,_i = 0 \tag{4}$$

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[1] *J. Dougall, "An Analytical Theory of the Equilibrium of an Isotropic Elastic Rod of Circular Cross-section", Trans. Roy. Soc. Edinburgh, XLIX, Part IV, (No. 17), pp 895-978 (1914).*

[2] *R. Muki, "Asymmetric Problems of the Theory of Elasticity for a Semi-Infinite Solid and a Thick Plate", Progress in Solid Mechanics, I, pp 401-439 (1960).*

where ν is the Poisson ratio, and Eqs. (2) and (3) hold. In order to compare their solutions, we should express them both in terms of the same material constants. We find it notationally convenient to write the equations of equilibrium given by Eqs. (1) and (4) as

$$\nabla^2 u_i - \frac{2A}{2A - 1} \Delta,_i = 0 \tag{5}$$

where

$$A = \frac{\lambda + \mu}{2 (\lambda + 2\mu)} = \frac{1}{4(1 - \nu)} . \tag{6}$$

In the forms of solution given both by Dougall and by Muki, one of the axes plays a distinguished role. We shall take this to be the x_3 axis. This is done because, for problems involving cylinders, it is convenient to take a coordinate axis, usually called the z-axis, to be in the direction of the cylinder axis. Thus we shall identify x_1, x_2, and x_3, respectively, with the x, y and z axis of a cartesian coordinate system. We shall then adopt the convention that lower case Greek indices will take the values 1, 2 and we shall identify the index 3 with the index z. Thus,

$$x_3 = z \quad ,$$

$$u_3 = u_z \ ,$$

$$u_{i},_3 = u_{i},_z \ ,$$

etc. We then write our equations in expanded form. For Eqs. (5) we write

$$\nabla^2 u_\alpha - \frac{2A}{2A - 1} \Delta,_\alpha = 0 \quad , \tag{7}$$

$$\nabla^2 u_z - \frac{2A}{2A - 1} \Delta,_z = 0 \quad . \tag{8}$$

For Eqs. (2) we write

$$\nabla^2 u_\alpha = u_{\alpha,\beta\beta} + u_{\alpha,zz} \quad ,$$

$$\nabla^2 u_z = u_{z,\beta\beta} + u_{z,zz} \quad .$$

For Eq. (3) we write

$$\Delta = u_{\alpha,\alpha} + u_{z,z} \quad . \tag{9}$$

We are now ready to introduce the potentials.

III. THE DOUGALL POTENTIALS

Dougall introduces three potentials* which we shall denote as $\tilde{\phi}$, $\tilde{\theta}$ and $\tilde{\psi}$. These potentials are arbitrary harmonic functions, i.e.,

$$\nabla^2\tilde{\phi} = 0, \ \nabla^2\tilde{\theta} = 0, \ \nabla^2\tilde{\psi} = 0.$$

Dougall states his general solution as an arbitrary linear combination of three basic solutions, which are as follows: one solution is given by

$$u_\gamma = x_\gamma \tilde{\phi}_{,zz}, \ u_z = - x_\gamma \tilde{\phi}_{,\gamma z} - \frac{2(\lambda + 2\mu)}{\lambda + \mu} \tilde{\phi}_{,z} \tag{10}$$

which are his Eqs. (2). A second basic solution is given by

$$u_i = \tilde{\theta}_{,i}$$

which is his Eq. (3). Dougall's third basic solution is given by

$$u_\alpha = e_{\alpha\beta}\psi_{,\beta} \quad , u_z = 0 \quad ,$$

**Please note that lower case Greek letters ϕ, θ, ψ will be used for the Dougall potentials and upper case Φ, Ψ for the Muki potentials. Such notation is intended to conform with that in References 1 and 2, and thus makes it easy for the reader to reconcile the results here with those in the references. However, the reader is advised to be careful in distinguishing the upper and lower case letters.*

which are his Eqs. (4). As explained in Appendix B, $e_{\alpha\beta} = e_{\beta\alpha}$ and $e_{12} = 1$.

In Section 5 below, we shall give a slightly simplified form of the Dougall potentials.

IV. THE MUKI POTENTIALS

Muki expresses his potentials in terms of cylindrical coordinates r, Θ, z. He writes

$$u_r = -\frac{\partial^2 \hat{\Phi}}{\partial r \partial z} + \frac{2}{r}\frac{\partial \hat{\Psi}}{\partial \Theta} , \qquad (11)$$

$$u_\Theta = -\frac{1}{r}\frac{\partial^2 \hat{\Phi}}{\partial \Theta \partial z} - 2\frac{\partial \hat{\Psi}}{\partial r} , \text{ and} \qquad (12)$$

$$u_z = 2(1 - \nu)\nabla^2 \hat{\Phi} - \frac{\partial^2 \hat{\Phi}}{\partial z^2} , \qquad (13)$$

where $\hat{\Phi}$ is an arbitrary biharmonic function and $\hat{\Psi}$ is an arbitrary harmonic function, i.e.,

$$\nabla^4 \hat{\Phi} = 0, \qquad \nabla^2 \hat{\Psi} = 0 ,$$

and u_r, u_Θ, u_z are the physical components of displacement. Our Eqs. (11), (12), and (13) are Muki's Eqs. (3)[2]. Writing

$$\Phi = -\frac{\hat{\Phi}}{2A} , \qquad \Psi = \frac{2\hat{\Psi}}{A}$$

and using Eq. (6), we can express the Muki solution in cartesian coordinates as follows:

$$u_\alpha = 2A\,\Phi,_{\alpha z} + e_{\alpha\beta}\,A\Psi,_\beta , \qquad (14)$$

$$u_z = 2A\,\Phi,_{zz} - \nabla^2\Phi . \qquad (15)$$

We shall find this slight modification of the form of the equations to be useful in our comparisons.

V. HARMONIC ANTI-DERIVATIVES OF HARMONIC FUNCTIONS - A SIMPLIFIED FORM OF THE DOUGALL POTENTIALS

We shall demonstrate here two closely related results which we shall need below:

a. If g is a function such that $g,_z$ is harmonic, then there exists a harmonic function $\tilde{g}$ such that $\tilde{g},_z = g,_z$.

b. If $f,_\alpha$ are harmonic functions, $\alpha = 1,2$, then there exists a harmonic function $\tilde{f}$ such that $\tilde{f},_\alpha = f,_\alpha$, $\alpha = 1,2$

We shall first establish b. Given that $f,_\alpha$ is harmonic, it follows that

$$(\nabla^2 f),_\alpha = 0$$

so that $\nabla^2 f$ is a function of z alone. Let $h(z)$ satisfy

$$\nabla^2 h(z) = \frac{d^2 h}{dz^2} = \nabla^2 f$$

then $\nabla^2(f - h) = 0$. (16)

Let $\tilde{f} = f - h$, (17)

then it follows from Eq. (16) that $\tilde{f}$ is harmonic. Furthermore, since h is a function of z alone, $h,_\alpha = 0$, whence from Eq. (17)

$$\tilde{f},_\alpha = f,_\alpha \quad ,$$

which establishes b.

The proof of a is quite similar, but we shall go through it anyway.

Since $g,_z$ is harmonic, $(\nabla^2 g),_z = 0$ and thus $\nabla^2 g$ is a function only of x_1 and x_2. Let $\ell(x_1, x_2)$ satisfy

$$\nabla^2 \ell = \ell,_{\alpha\alpha} = \nabla^2 g \quad ,$$

then

$$\nabla^2 (g - \ell) = 0 \quad . \tag{18}$$

Define $\tilde{g}$ by

$$\tilde{g} = g - \ell \quad ; \tag{19}$$

then by Eq. (18), $\tilde{g}$ is harmonic. Moreover, since ℓ does not depend on z; i.e., $\ell,_z = 0$, it follows from Eq. (19) that

$$\tilde{g},_z = g,_z \quad ,$$

which establishes a.

We are now ready for a modified and somewhat simplified form of the Dougall potentials. First, let ϕ be given by

$$\phi = \frac{1}{A} \tilde{\phi},_z \tag{20}$$

Then it follows that ϕ is harmonic. Moreover, given any harmonic ϕ, it follows from the immediately preceding argument that one may find a harmonic $\tilde{\phi}$ satisfying Eq. (20). Thus we may replace Dougall's first solution given by Eqs. (10) by

$$u_\gamma = Ax_\gamma \phi,_z \qquad u,_z = - Ax_\gamma \phi,_\gamma - \phi$$

where ϕ is an arbitrary harmonic function. For convenience, we shall write

$$\theta = \frac{1}{A} \tilde{\theta}, \quad \psi = \frac{1}{A} \tilde{\psi} \quad ,$$

so that the general Dougall solution will be given by

$$u_\alpha = Ax_\alpha \phi,_z + A\theta,_\alpha + Ae_{\alpha\beta} \psi,_\beta \tag{21}$$

$$u_z = - Ax_\alpha \phi,_\alpha - \phi + A\theta,_z \quad . \tag{22}$$

Now Eqs. (21) and (22) are written in such a way as to make comparison with the Muki solution, Eqs. (14) and (15), easier.

VI. FINDING THE DOUGALL POTENTIALS GIVEN THE MUKI POTENTIALS

Suppose that a solution is given in terms of the Muki potentials. We address ourselves now to the question of determining a corresponding set of Dougall potentials.

Because the second term in Eqs. (14) has the same form as the third term term in Eqs. (21), we see that we need merely find the Dougall potentials ϕ, θ, ψ, which satisfy

$$Ax_\alpha \phi,_z + A\theta,_\alpha + Ae_{\alpha\beta}\psi,_\beta = 2A\Phi,_{\alpha z} \quad , \tag{23}$$

$$-Ax_\alpha \phi,_\alpha - \phi + A\theta,_z = 2A\Phi,_{zz} - \nabla^2\Phi \quad , \tag{24}$$

given a biharmonic Muki potential Φ. Once we have found these, we may add to the Dougall solution terms of the form $Ae_{\alpha\beta}\Psi,_\beta$ to get agreement with the general Muki solution.

If we calculate Δ from the Dougall solution with the use of Eqs. (21), (22) and (9), we obtain

$$\Delta = (2A - 1)\ \phi,_z \quad ,$$

where we have made use of $\nabla^2\phi = \nabla^2\theta = 0$. Similarly if we calculate Δ from the Muki solution, as given by Eqs. (14) and (15), we obtain

$$\Delta = (2A - 1)\ (\nabla^2\Phi),_z \quad ,$$

where we have used Eq. (9) and $\nabla^4\Phi = 0$. Thus ϕ must satisfy

$$\phi,_z = \nabla^2\Phi,_z \quad . \tag{25}$$

We shall see that it will suffice to take

$$\phi = \nabla^2\Phi \quad . \tag{26}$$

We shall discuss below how the Dougall solution will be altered by replacing ϕ as given by Eq. (26) by another solution of Eq. (25). But first let us pursue the implications of Eq. (26).

Substituting Eq. (26) into Eqs. (23) and (24), we see that we must now find θ and ψ such that

$$x_\alpha(\nabla^2\Phi)_{,z} + \theta_{,\alpha} + e_{\alpha\beta}\psi_{,\beta} = 2\Phi_{,\alpha z} \quad , \tag{27}$$

$$-x_\alpha(\nabla^2\Phi)_{,\alpha} + \theta_{,z} = 2\Phi_{,zz} \quad . \tag{28}$$

We start with Eq. (28). Let $\hat{\theta}$ be any solution of Eq. (28) for θ, i.e., let $\hat{\theta}_{,z}$ satisfy

$$\hat{\theta}_{,z} = 2\Phi_{,zz} + x_\alpha\nabla^2\Phi_{,\alpha} \quad . \tag{29}$$

We shall show; then, that $\hat{\theta}_{,z}$ is harmonic. To this end we differentiate Eq. (29) with respect to x_β. (Remember $x_{\alpha,\beta} = \delta_{\alpha\beta}$). We obtain

$$\begin{aligned}\hat{\theta}_{,z\beta} &= 2\Phi_{,zz\beta} + \delta_{\alpha\beta}\nabla^2\Phi_{,\alpha} + x_\alpha\nabla^2\Phi_{,\alpha\beta} \\ &= 2\Phi_{,zz\beta} + \nabla^2\Phi_{,\beta} + x_\alpha\nabla^2\Phi_{,\alpha\beta} \quad .\end{aligned}$$

We differentiate again with respect to β and sum on β to get

$$\begin{aligned}\hat{\theta}_{,z\beta\beta} &= 2\Phi_{,zz\beta\beta} + \nabla^2\Phi_{,\beta\beta} + \delta_{\alpha\beta}\nabla^2\Phi_{,\alpha\beta} + x_\alpha\nabla^2\Phi_{,\alpha\beta\beta} \\ &= 2\Phi_{,zz\beta\beta} + 2\nabla^2\Phi_{,\beta\beta} + x_\alpha\nabla^2\Phi_{,\alpha\beta\beta} \quad .\end{aligned} \tag{30}$$

Now we differentiate Eq. (29) twice with respect to z to get

$$\hat{\theta},_{zzz} = 2\Phi,_{zzzz} + x_\alpha \nabla^2\Phi,_{\alpha zz} \quad . \tag{31}$$

Adding Eq. (30) to Eq. (31), we arrive at

$$\nabla^2\hat{\theta},_z = \hat{\theta},_{z\beta\beta} + \hat{\theta},_{zzz}$$

$$= 2\Phi,_{zz\beta\beta} + 2\Phi,_{zzzz} + 2\nabla^2\Phi,_{\beta\beta}$$

$$+ x_\alpha \, [\nabla^2\Phi,_{\beta\beta} + \nabla^2\Phi,_{zz}],_\alpha \quad . \tag{32}$$

Note that

$$\Phi,_{zz\beta\beta} + \Phi,_{zzzz} = \nabla^2\Phi,_{zz} \quad , \tag{33}$$

$$(\nabla^2\Phi),_{zz} + (\nabla^2\Phi),_{\beta\beta} = \nabla^2\nabla^2\Phi = \nabla^4\Phi \quad . \tag{34}$$

Putting Eqs. (33) and (34) into Eq. (32), we get

$$\nabla^2\hat{\theta},_z = 2\,\nabla^4\Phi + x_\alpha \nabla^4\Phi,_\alpha \quad . \tag{35}$$

But Φ is biharmonic. Thus Eq. (35) gives $\nabla^2\hat{\theta},_z = 0$.

According to Section 5 above, we can find a harmonic function θ such that $\theta,_z = \hat{\theta},_z$. This function will then satisfy Eq. (28) as well as $\nabla^2\theta = 0$. It will be our second Dougall potential.

We have only left now to find a harmonic ψ which satisfies Eqs. (27). We shall proceed as follows: first, let ψ_β be a vector (not necessarily the derivative of a scalar), which satisfies Eqs. (27). We shall show that $\psi_{\beta,\gamma} = \psi_{\gamma,\beta}$. This will then allow us to write $\psi_\beta = \hat{\psi}_\beta$ where $\hat{\psi}$ is a scalar. Then we shall show that $\hat{\psi},_\beta$ is harmonic. Using the results of Section 5 above, we can then assert that we can find a harmonic function ψ which satisfies Eqs. (27).

We start then, with ψ_β satisfying Eqs. (27). (A cursory examination of Eqs. (27) shows that ψ_1 and ψ_2 are determined explicitly once Φ and θ are given.) Then let us differentiate Eqs. (27) with respect to x_α and sum on α. We get

$$\delta_{\alpha\alpha}\nabla^2\Phi,_z + x_\alpha\nabla^2\Phi,_{z\alpha} + \theta,_{\alpha\alpha} + e_{\alpha\beta}\psi_\beta,_\alpha = 2\Phi,_{\alpha\alpha z}$$

or

$$2\nabla^2\Phi,_z + x_\alpha\nabla^2\Phi,_{z\alpha} + \theta,_{\alpha\alpha} + e_{\alpha\beta}\psi_\beta,_\alpha = 2\Phi,_{\alpha\alpha z} \quad . \tag{36}$$

Differentiation of Eq. (28) with respect to z gives

$$-x_\alpha\nabla^2\Phi,_{\alpha z} + \theta,_{zz} = 2\Phi,_{zzz} \quad . \tag{37}$$

Adding Eq. (36) to Eq. (37), we get

$$2\nabla^2\Phi,_z + \nabla^2\theta + e_{\alpha\beta}\psi_\beta,_\alpha = 2\nabla^2\Phi,_z \quad . \tag{38}$$

Since, however, $\nabla^2\theta = 0$, Eq. (38) becomes simply $e_{\alpha\beta}\psi_\beta,_\alpha = 0$ or $\psi_{2,1} - \psi_{1,2} = 0$, whence, by Green's theorem in the plane, there exists a scalar $\hat{\psi}$ such that $\psi_\alpha = \hat{\psi},_\alpha$, and we have for Eqs. (27)

$$x_\alpha\nabla^2\Phi,_z + \theta,_\alpha + e_{\alpha\beta}\hat{\psi},_\beta = 2\Phi,_{\alpha z} \quad . \tag{39}$$

Our next step is to show that $\hat{\psi},_\beta$ is harmonic. We differentiate Eqs. (39) with respect to x_γ to obtain

$$\delta_{\alpha\gamma}\nabla^2\Phi,_z + x_\alpha\nabla^2\Phi,_{\gamma z} + \theta,_{\alpha\gamma} + e_{\alpha\beta}\hat{\psi},_{\beta\gamma} = 2\Phi,_{\alpha\gamma z} \quad .$$

We differentiate once more with respect to x_γ and sum on γ. We get

$$2\delta_{\alpha\gamma}\nabla^2\Phi,_{z\gamma} + x_\alpha\nabla^2\Phi,_{z\gamma\gamma} + \theta,_{\alpha\gamma\gamma} + e_{\alpha\beta}\hat{\psi},_{\beta\gamma\gamma} = 2\Phi,_{\alpha\gamma z\gamma}$$

or

$$2\nabla^2\Phi,_{\alpha z} + x_\alpha\nabla^2\Phi,_{z\gamma\gamma} + \theta,_{\alpha\gamma\gamma} + e_{\alpha\beta}\hat{\psi},_{\beta\gamma\gamma} = 2\Phi,_{\alpha z\gamma\gamma} \quad . \tag{40}$$

We now differentiate Eqs. (39) twice with respect to z to get

$$x_\alpha\nabla^2\Phi,_{zzz} + \theta,_{\alpha zz} + e_{\alpha\beta}\hat{\psi},_{\beta zz} = 2\Phi,_{\alpha zzz} \quad . \tag{41}$$

We add Eqs. (40) to Eqs. (41) to obtain

$$2\nabla^2\Phi,_{\alpha z} + x_\alpha\left[\nabla^2\Phi,_{\gamma\gamma} + \nabla^2\Phi,_{zz}\right],_z + \nabla^2\theta,_\alpha + e_{\alpha\beta}\nabla^2\hat{\psi},_\beta$$
$$= 2\left[\Phi,_{\gamma\gamma} + \Phi,_{zz}\right],_{\alpha z} \quad .$$

Since $\nabla^2\theta,_\alpha = 0$, we have

$$2\nabla^2\Phi,_{\alpha z} + x_\alpha\nabla^4\Phi,_z + e_{\alpha\beta}\nabla^2\hat{\psi},_\beta = 2\nabla^2\Phi,_{\alpha z} \quad ,$$

whence, since $\nabla^4\Phi = 0$,

$$e_{\alpha\beta}\nabla^2\hat{\psi},_\beta = 0 \quad . \tag{42}$$

For $\alpha = 1$, Eqs. (42) give $\nabla^2\hat{\psi},_2 = 0$. For $\alpha = 2$, Eqs. (42) give $\nabla^2\hat{\psi},_1 = 0$. Therefore

$$\nabla^2\hat{\psi},_\alpha = 0. \tag{43}$$

According to Section 5, then, Eqs. (43) imply that we can find a harmonic function ψ such that

$$\psi,_{\alpha} = \hat{\psi}_{\alpha} \quad .$$

Such a function will be our third Dougall potential and we are done.

<u>Remark</u>: There is a certain amount of non-uniqueness in the determination of the Dougall potentials from the Muki potentials. This non-uniqueness we shall now discuss.

To begin with, let us note that Eq. (26) is not the general solution of Eq. (25). Indeed, the general solution, which we denote by $\bar{\phi}$, is

$$\bar{\phi} = \nabla^2\Phi + h(x_{\alpha}) \quad , \tag{44}$$

where $h(x_{\alpha})$ is a function of x_1, x_2 such that

$$\nabla^2 h = h,_{\alpha\alpha} = 0 \quad . \tag{45}$$

We reserve the unbarred quantities ϕ, θ, ψ, for the solutions we already have. Suppose that $\bar{\theta}$ and $\bar{\psi}$ are the corresponding other Dougall potentials. We have from Eqs. (23), (24), (44) and (45)

$$x_{\alpha}\nabla^2\Phi,_z + \bar{\theta},_{\alpha} + e_{\alpha\beta}\bar{\psi},_{\beta} = 2\Phi,_{\alpha z} \quad , \tag{46}$$

$$-x_{\alpha}(\nabla^2\Phi),_{\alpha} - x_{\alpha}h,_{\alpha} + \bar{\theta},_z \frac{h}{A} = 2\Phi,_{zz} \quad . \tag{47}$$

Comparing Eqs. (46) and (47) with Eqs. (27) and (28), we get

$$(\bar{\theta} - \theta),_{\alpha} + e_{\alpha\beta}(\bar{\psi} - \psi),_{\beta} = 0 \quad , \tag{48}$$

$$-x_{\alpha}h,_{\alpha} + (\bar{\theta} - \theta),_z = 0 \quad . \tag{49}$$

It follows from Eq. (49) that

$$\bar{\theta} = \theta + zx_\beta h,_\beta + \zeta(x_\beta) \quad . \qquad (50)$$

Furthermore,

$$\nabla^2 z\ x_\beta h,_\beta = 2zh,_{\gamma\gamma} + zx_\beta h,_{\beta\gamma\gamma} = 0 \quad .$$

It follows then that $\zeta,_{\alpha\alpha} = 0$. We put this result into Eqs. (48) to obtain

$$(x_\beta h,_\beta),_\alpha z + \zeta,_\alpha + e_{\alpha\beta}(\bar{\psi} - \psi),_\beta = 0.$$

Now both $x_\beta h,_\beta$ and ζ are plane harmonic functions. Therefore they have harmonic conjugates. Let $\eta(x_\alpha)$ be the harmonic conjugate of $x_\beta h,_\beta$, i.e.,

$$(x,_\tau h,_\tau),_\alpha = e_{\alpha\beta}\eta,_\beta \quad .$$

Let ξ be the harmonic conjugate of ζ, then

$$e_{\alpha\beta}\ [z\eta + \xi + \bar{\psi} - \psi],_\beta = 0 \quad .$$

It follows that

$$\bar{\psi} = \psi - z\eta - \xi + az + b \quad , \qquad (51)$$

where a and b are arbitrary constants. This is true because $\bar{\psi} - \psi + z\eta - \xi$ is a harmonic function of z alone. Thus Eqs. (44), (50) and (51) give the arbitrariness in the Dougall potentials.

VII. FINDING THE MUKI POTENTIALS FROM THE DOUGALL POTENTIALS

Suppose that ϕ, θ and ψ are a set of Dougall potentials. Using Eqs. (14), (15), (21) and (22), we see that a corresponding set of Muki potentials Φ and Ψ must satisfy

$$Ax_\alpha \phi,_z + A\theta,_\alpha + Ae_{\alpha\beta}\psi,_\beta = 2A\Phi,_{\alpha z} + e_{\alpha\beta}A\Psi,_\beta \quad , \tag{52}$$

$$-Ax_\alpha \phi,_\alpha - \phi + A\theta,_z = 2A\Phi,_{zz} - \nabla^2\Phi \quad . \tag{53}$$

Again, we equate dilatations to obtain Eq. (25).

$$\nabla^2\Phi,_z = \phi,_z \quad .$$

Let Φ^o be a particular solution of

$$\nabla^2\Phi^o = \phi \quad .$$

Then Eqs. (25) and (26) give

$$\nabla^2(\Phi - \Phi^o),_z = 0 \quad ,$$

whence $\nabla^2(\Phi - \Phi^o) = Ah(x_\alpha)$, $\quad h,_{\alpha\alpha} = 0$.

Let $H,_{\alpha\alpha} = h$. Then $\nabla^2[\Phi - \Phi^o - AH] = 0$. Thus,

$$\Phi = \Phi^o + AH + F(x_i) \quad , \tag{54}$$

where $F(x_i)$ is harmonic. The general solution of Eq. (25), then is given by Eq. (54), where Φ^o is a particular solution of Eq. (25), F is an arbitrary harmonic function and H is an arbitrary biharmonic function of x_α. Let us put Eqs. (26) and (54) into Eqs. (52) and (53). We get

$$x_\alpha \nabla^2 \Phi^o{}_{,z} + \theta_{,\alpha} + e_{\alpha\beta}\psi_{,\beta} = 2\Phi^o{}_{,\alpha z} + 2F_{,\alpha z} + e_{\alpha\beta}\Psi_{,\beta} \quad ,$$

$$-x_\alpha \nabla^2 \Phi^o{}_{,\alpha} + \theta_{,z} = 2\Phi^o{}_{,zz} + 2F_{,zz} - h \quad ,$$

or

$$x_\alpha \nabla^2 \Phi^o{}_{,z} + \bar{\theta}_{,\alpha} - zh_{,\alpha} + e_{\alpha\beta}[\psi - \Psi]_{,\beta} = 2\Phi^o{}_{,\alpha z} \quad , \tag{55}$$

$$-x_\alpha \nabla^2 \Phi^o{}_{,\alpha} + \bar{\theta}_{,z} = 2\Phi^o{}_{,zz} \quad , \tag{56}$$

where

$$\bar{\theta} = \theta - 2F_{,z} + zh \quad . \tag{57}$$

Now let η be the conjugate harmonic function to h; i.e., the Cauchy Riemann equations

$$h_{,\alpha} = e_{\alpha\beta}\eta_{,\beta}$$

are satisfied.

Then Eqs. (55) and (56) become

$$x_\alpha \nabla^2 \Phi^o{}_{,z} + \bar{\theta}_{,\alpha} + e_{\alpha\beta}\bar{\psi}_{,\beta} = 2\Phi^o{}_{,\alpha z} \, , \tag{58}$$

$$-x_\alpha \nabla^2 \Phi^o{}_{,\alpha} + \bar{\theta}_{,z} = 2\Phi^o{}_{,zz} \quad , \tag{59}$$

where $\quad \bar{\psi} = \psi - \Psi - z\eta \quad .$ (60)

Then $\bar{\theta}$ and $\bar{\psi}$ are determined as in Section 6.

Note that Eq. (59) determines $\bar{\theta}$ to within an arbitrary harmonic function of x_α. If we denote this function by $p(x_\alpha)$ and a particular solution of Eq. (59) by $\bar{\theta}^o$

$$\bar{\theta} = \bar{\theta}^o + p(x_\alpha) \quad .$$

If q is the conjugate harmonic function of p; i.e., $p,_\alpha = e_{\alpha\beta} q,_\beta$, and if $\bar{\psi}^o$ is a particular solution of Eqs. (58) for p = 0, we have

$$\bar{\psi} = \bar{\psi}^o - q + az + b \quad ,$$

where a and b are arbitrary constants.

We now go back to Eqs. (57) and (60). We have

$$\bar{\theta}^o - \theta = - 2F,_z + zh - p \qquad (61)$$

$$\psi - \bar{\psi}^o = \Psi - q - az - b + z\eta \qquad (62)$$

For any arbitrary assignment of the plane harmonic function q, and any assignment of a plane biharmonic function H and any constants a and b we can determine harmonic functions $F,_z$ and Ψ from Eqs. (61) and (62). Then F is determined to within addition of an arbitrary plane harmonic function and thus Φ is determined from Eq. (59).

It follows then that we can find a harmonic function ψ such that $\psi,_\beta = \psi,_\beta$ and thus we have our third Dougall potential. In this way we see that one can always find harmonic functions ϕ, θ, ψ, which satisfy Eqs. (21) and (22) whenever u_i is a solution of Eqs. (1).

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APPENDIX A

NOTE ON CARTESIAN TENSOR NOTATION

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NOTE ON CARTESIAN TENSOR NOTATION

One expresses vectors and tensors in terms of their components in a cartesian coordinate system whose coordinates are x_1, x_2, x_3. One writes x_i for the general coordinate and understands that i will range over the values 1, 2, 3.

In case a term is written with a repeated index, summation over that index is understood. Thus,

$$x_i v_i \text{ means } \sum_{i=1}^{3} x_i v_i \text{ or } x_1 v_1 + x_2 v_2 + x_3 v_3 \; ,$$

$$\sigma_{ijj} \text{ means } \sigma_{i11} + \sigma_{i22} + \sigma_{i33} \; .$$

Generally speaking, the index on a vector or tensor indicates its cartesian component. Thus v_1, v_2, v_3 are respectively the x_1, x_2, and x_3 components of a vector $\underset{\sim}{v}$.

A comma denotes partial differentiation with respect to one of the coordinates. Thus,

$$\phi_{,i} = \frac{\partial \phi}{\partial x_i} \; , \quad v_{i,j} = \frac{\partial v_i}{\partial x_j}$$

The Kroneker delta δ_{ij} represents the components of the unit matrix:

$$\delta_{ij} = 1 \text{ if } i = j; \quad \delta_{ij} = 0 \text{ if } i \neq j \quad ; \text{ and}$$

we have $\delta_{ii} = 3$.

We adopt a special convention: small Greek letters range over the values 1, 2. Thus, u_α stands for u_1 and u_2 only.

$$x_\alpha u_\alpha = x_1 u_1 + x_2 u_2 .$$

Small Latin letters continue to range over the values 1,2,3.

The quantity $\delta_{\alpha\beta}$ is the two dimensional Kroneker delta. (It represents a 2 x 2 unit matrix). Thus, $\delta_{\alpha\alpha} = 2$.

We also make use of a two-dimensional alternating tensor $e_{\alpha\beta}$. It is defined by the following: $e_{\alpha\beta} = -e_{\beta\alpha}$, $e_{12} = 1$. Thus, $e_{11} = -e_{11}$, $e_{11} = 0$. Similarly $e_{22} = 0$. Also, $e_{21} = -1$. Its matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

To see the use of this, suppose given a vector v_α and consider $w_\alpha = e_{\alpha\beta}v_\beta$. Thus, $w_1 - v_2$, $w_2 = -v_1$. This is used, for example, in the condition that a two-dimensional vector be the gradient of a scalar. This condition is

$$\frac{\partial v_1}{\partial x_2} = \frac{\partial v_2}{\partial x_1} \qquad ,$$

or $v_{1,2} = v_{2,1}$. This is written $e_{\alpha\beta}v_{\beta,\alpha} = 0$.

Note that

$$e_{\alpha\beta}e_{\alpha\gamma} = \delta_{\beta\gamma}$$

This can be checked merely by trying all four combinations of β, γ. For example, if $\beta = 1$, $\gamma = 2$, we get

$$e_{\alpha 1}e_{\alpha 2} = e_{11}e_{12} + e_{21}e_{22} = 0 \qquad ,$$

since $e_{11} = e_{22} = 0$. If $\beta = 1$, $\gamma = 1$, we get

$$e_{\alpha 1}e_{\alpha 1} = e_{11}e_{11} + e_{12}e_{12} = 1, \text{ since}$$

$$e_{11} = 0,\ e_{12} = 1.$$

Also the Cauchy-Rieman equations are written using $e_{\alpha\beta}$. Thus if a function f(z) of the complex variable z is written in terms of its real and imaginary parts, $f(z)=U(x_1,x_2)+iV(x_1,x_2)$. Then the Cauchy-Riemann equations $U,_1 = V,_2$ and $U,_2 = V,_1$ become simply $U,_\alpha = e_{\alpha\beta}V,_\beta$.

APPENDIX B

DOUGALL POTENTIALS AS GENERAL SOLUTIONS OF THE ELASTICITY EQUATIONS

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DOUGALL POTENTIALS AS GENERAL SOLUTIONS OF THE ELASTICITY EQUATIONS

We shall show here that the Dougall potentials give the general solution of the elasticity equations as represented by Eqs. (7) and (8).

To verify that Eqs. (21) and (22) satisfy Eqs. (7) and (8), as long as ϕ, θ and ψ are harmonic, is a matter of straight-forward substitution. We shall now address ourselves to the more involved question, namely that of showing that every solution is of the form given by Eqs. (21) and (22).

Suppose then, that u_i is a solution to Eqs. (7) and (8), or simply Eqs. (5). Then,

$$(2A - 1)u_{i},_{jj} - 2A\, u_{j},_{ji} = 0 \quad .$$

Differentiate with respect to x_i and sum, to get

$$\nabla^2\Delta = u_{i},_{ijj} = 0 \quad .$$

Thus $\Delta = u_{i},_{i}$ is harmonic. Let

$$\phi,_{z} = \frac{\Delta}{2A - 1} \tag{B-1}$$

such that ϕ is harmonic, and let $\theta = \hat{\theta}$ be a solution of Eq. (22). Then differentiating Eq. (22), we get

$$\nabla^2 u_z = -\,2A\phi,_{\beta\beta} - A\nabla^2\hat{\theta},_{z} \quad .$$

But

$$\phi,_{\beta\beta} = -\phi,_{zz} = \frac{-\Delta_z}{2A - 1}$$

$$\nabla^2 u_z = \frac{2A\Delta,_{z}}{2A - 1} - A\nabla^2\hat{\theta},_{z}$$

But, using (8), we get

$$\frac{2A\Delta,_z}{2A - 1} = \frac{2A\Delta,_z}{2A - 1} - A\nabla^2\hat{\theta},_z \quad ,$$

$$\nabla^2\hat{\theta},_z = 0.$$

Thus we can find a harmonic solution θ of Eq. (22). Now let ψ_β satisfy

$$u_\alpha = Ax_\alpha\phi,_z + A\theta,_\alpha + Ae_{\alpha\beta}\psi_\beta \quad .$$

Differentiate with respect to α and sum to get

$$u_{\alpha},_{\alpha} = 2A\phi,_z + Ax_\alpha\phi,_{z\alpha} + A\theta,_{\alpha\alpha} + Ae_{\alpha\beta}\psi_\beta,_\alpha \quad .$$

Let us then differentiate Eq. (22) with respect to z and add the result to this equation. We get,

$$\Delta = (2A - 1)\phi,_z + Ae_{\alpha\beta}\psi_\beta,_\alpha \quad . \qquad \text{(B-2)}$$

Now from Eqs. (B-1) and (B-2), we get

$$\Delta = \Delta + Ae_{\alpha\beta}\psi_\beta,_\alpha \quad ,$$

or

$$e_{\alpha\beta}\psi_\beta,_\alpha = 0.$$

Thus, $\psi_\beta = \hat{\psi},_b$ where $\hat{\psi}$ is a scalar. Thus $\hat{\psi}$ satisfies Eqs. (21). We have just to show, now, that $\hat{\psi},_\beta$ is harmonic.

From Eqs. (21), we get

$$u_{\alpha,\gamma\gamma} = 2A\phi,_{z\alpha} + Ax_\alpha\phi,_{z\gamma\gamma} + A\theta,_{\alpha\gamma\gamma} + Ae_{\alpha\beta}\hat{\psi},_{\beta\gamma\gamma}$$

and

$$u_{\alpha,zz} = Ax_\alpha\phi,_{zzz} + A\theta,_{\alpha zz} + Ae_{\alpha\beta}\hat{\psi},_{\beta zz} \quad .$$

Adding the last two equations, we have

$$\nabla^2 u_\alpha = 2A\phi,_{z\alpha} + Ae_{\alpha\beta}\nabla^2\hat{\psi},_\beta \quad .$$

Using Eqs. (7) and (B-1), we get

$$\frac{2A}{2A-1}\Delta,_\alpha = \frac{2A}{2A-1}\Delta,_\alpha + Ae_{\alpha\beta}\nabla^2\hat{\psi},_\beta \quad ,$$

or

$$\nabla^2\hat{\psi},_\beta = 0 \quad .$$

It follows then that we can find a harmonic function ψ such that $\psi,_\beta = \hat{\psi},_\beta$ and thus we have our third Dougall potential.

In this way we see that one can always find harmonic functions ϕ,θ,ψ which satisfy Eqs. (21) and (22) whenever u_i is a solution of Eqs. (1).

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