AD-A090 034   NAVAL RESEARCH LAB WASHINGTON DC   F/G 18/1
SPHEROMAK TILTING INSTABILITY IN CYLINDRICAL GEOMETRY.(U)
SEP 80 J M FINN, W M MANHEIMER, E OTT   DOE-EX-76-A-34-1006
UNCLASSIFIED NRL-MR-4316   NL

1 OF 1
AD A
090034

END
DATE
FILMED
11-80
DTIC

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: NRL Memorandum Report 4316
2. GOVT ACCESSION NO.: AD-A090 034
3. RECIPIENT'S CATALOG NUMBER: NRL-MR-43
4. TITLE (and Subtitle): SPHEROMAK TILTING INSTABILITY IN CYLINDRICAL GEOMETRY
5. TYPE OF REPORT & PERIOD COVERED: Interim report on a continuing NRL problem.
6. PERFORMING ORG. REPORT NUMBER:

7. AUTHOR(s): John M. Finn,* Wallace M. Manheimer and Edward Ott**

8. CONTRACT OR GRANT NUMBER(s):

9. PERFORMING ORGANIZATION NAME AND ADDRESS: Naval Research Laboratory, Washington, D.C. 20375

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 67-0896-0-0

11. CONTROLLING OFFICE NAME AND ADDRESS: U.S. Department of Energy, Washington, D.C. 20545

12. REPORT DATE: Sep[redacted]80

13. NUMBER OF PAGES: 27

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): DOE-

15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Present address: *Science Applications, Inc., McLean, VA 22102

**University of Maryland, College Park, MD 20742

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Spheromak
Tilting mode
Force free equilibria
Linearized MHD code

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The internal tilting instability in a force free spheromak plasma in cylindrical geometry is examined. It is found that this instability, originally found in spherical geometry, also occurs in cylindrical geometry. The analysis proceeds by first demonstrating that if no mode rational surface is present in the plasma, a necessary and sufficient condition for ideal magnetohydrodynamic instability is that there exist a solution to $\nabla \times \underset{\sim}{B}_m = \mu_m \underset{\sim}{B}_m$, where $\underset{\sim}{B}_m \sim \exp(im\theta)$, with $\mu_m < \mu_o$. Solutions to this equation are investigated using two approaches, by a series expansion and by a numerical solution of a modified set of linearized magnetohydrodynamic equations. The eigenvalue for the $m = 1$ mode satisfies $\mu_1 < \mu_o$ for $L/a > 1.67$. Since no mode rational surface exists for this elongation, an ideal magnetohydrodynamic mode, identified as the tilting mode, is unstable for these parameters. All modes with $m \neq 1$ are shown to be stable.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

CONTENTS

SPHEROMAK TILTING INSTABILITY IN CYLINDRICAL GEOMETRY

I. INTRODUCTION AND SUMMARY

The compact torus, or ~~more~~ specifically spheromak,[1] approach to magnetic fusion has numerous advantages as a reactor over tokamak designs. Most important of these are relative compactness, the fact that a toroidal blanket is not necessary, and the absence of external toroidal field coils. The last of these can be expressed by saying that the engineering beta (ratio of plasma pressure to magnetic pressure at the coils) can be quite high.

Recently, theoretical work by Rosenbluth and Bussac[1,2] has shown that a spheromak plasma in a tight fitting spheroidal shell is unstable to a magnetohydrodynamic mode which they call the internal tilting mode if the boundary is slightly elongated, i.e. prolate.

More recently, a spheromak-type configuration has been formed at the University of Maryland by a combination of Z-pinch and theta pinch technologies.[3] Although the spherical wall equilibrium of Ref. 2 describes this device in a qualitative fashion, a more appropriate theoretical model for the Maryland device is an equilibrium with cylindrical conducting walls and endplates. In addition, we feel that it is of value to show that the tilting instability is not specific to systems with conducting walls that are almost spherical. Detailed comparison of the

Manuscript submitted July 29, 1980

model with the experimental device and the observed tilting instability with the theory has been presented elsewhere.[4] In this paper we describe in more detail two of the theoretical approaches used and their justification.

Our first approach, discussed in Secs. II and III is that taken in Ref. 2 for the spherical geometry, namely the Woltjer-Taylor[5,6] approach of computing nonaxisymmetric force free equilibria. We show in Sec. II that the existence of nonaxisymmetric equilibria with eigenvalue less than that of the axisymmetric equilibrium is a necessary and sufficient condition for instability if no mode rational surface occurs in the plasma. Our geometry, in which the normal component of the magnetic field is required to be zero on cylindrical walls of radius $r = a$ and length $z = L$, is inherently more difficult than that of Ref. 2 , where expansions can be performed about an analytically tractable spherical boundary. Nevertheless, we are able to compute nonaxisymmetric equilibria for every value of elongation L/a. Our results show that an instability with $m = 1$ (perturbed quantities behave as $e^{im\theta}$, where θ is the toroidal angle) occurs for $L/a > 1.67 \pm 1\%$. We identify this as the tilting mode. Furthermore, it appears that a second $m = 1$ instability occurs for $L/a > 4.14$, but this mode appears to have a smaller growth rate than the tilt. We also find that the $m = 2$, 3, and 4 equilibria as well as those with $m \to \infty$ have higher energy than the axisymmetric equilibrium, implying that all modes except $m = 1$ are stable.

Our second approach, discussed in Sec. IV, is by means of a linearized time dependent modified magnetohydrodynamic code written for force free geometries. The results of this code, specifically the marginal stability point for the $m = 1$ tilting mode, agree to within a percent with the equilibrium analysis.

Nonaxisymmetric force free equilibria for this geometry have been computed by a somewhat different method by An, et al.[7] This paper and Ref. 7 were both motivated by results of Marklin[8] and of An.[9] Marklin utilized a linearized time dependent magnetohydrodynamic code which yielded tilting instability for $L/a > 1.67$ and also showed the existence of an unstable $m - 1$ internal kink mode when the safety factor q is greater than unity on mag axis. On the other hand, the work of An[9] showed that nonaxisymmetric force free equilibria corresponding to the tilting mode are not obtainable by the method of separation of variables. As shown in Sec. III and Ref. 7, however, nonaxisymmetric solutions yielding Marklin's tilting mode result[8] do, in fact exist, but are not separable.

II. THE TAYLOR-WOLTJER APPROACH

The Taylor-Woltjer approach is based upon extremizing the magnetic energy $W = \int \frac{1}{2} B^2 \, d^3x$ with $K = \int A \cdot B \, d^3x$ held constant. The quantity, K, which is an invariant of motion of ideal magnetohydrodynamics, is called the magnetic helicity. Introducing a Lagrange multiplier μ for the constraint, the first variation of $W - \mu K/2$ gives

$$\nabla \times B = \mu B \tag{1}$$

where μ is a constant. For this special class of force free equilibria, we have $\nabla \times A = \mu A + \nabla \chi$, from which we find $W = \mu K/2$. (We need not consider the complications that arise in a system in a toroidal container[10] because our system is simply connected.) Therefore, the solution to (1) with the smallest value of μ will have the lowest energy and might be expected to be stable. We shall see later that μ may be assumed to be positive. We cannot expect this analysis to extend into a vacuum region, where K is not conserved. Just as obviously, (1) cannot be satisfied in a vacuum. We therefore restrict our attention to internal modes, i.e. we assume that there is a conducting wall at the plasma boundary, $r = a$ and $z = 0$, $z = L$.

Indeed, the second variation of $W - \mu K/2$ is

$$\delta W^* = \int (\delta B^2 - \mu \delta A \cdot \delta B) \, d^3x. \tag{2}$$

It is easily seen that the usual potential energy δW of linearized ideal magnetohydrodynamics[11] for equilibria satisfying (1) and with zero pressure is exactly δW^*, but with the added condition that the perturbed vector potential must be expressed in the form of a plasma displacement as $\delta \underset{\sim}{A} = \underset{\sim}{\xi} \times \underset{\sim}{B}$. Therefore the ideal magnetohydrodynamic energy principle is expressed in terms of δW^* with the added constraint $\underset{\sim}{B} \cdot \delta \underset{\sim}{A} = 0$. This implies $\delta W \geq \delta W^*$, which guarantees that the condition $\delta W^* > 0$ is sufficient for ideal magnetohydrodynamic stability. Also, it is shown in the Appendix that the condition $\underset{\sim}{B} \cdot \delta \underset{\sim}{A} = 0$ can be satisfied by a gauge transformation except at a mode rational surface. Therefore, if no mode rational surface occurs in the plasma, $\delta W^* = \delta W$ and we have a necessary and sufficient condition for stability.

We shall see that all solutions of (1) may be considered to have only one Fourier component $e^{im\theta}$. [Incidentally, the transformation $m \to -m$, $B_\theta \to -B_\theta$, $\mu \to -\mu$ leaves (1) invariant, proving that μ can be assumed to be positive.] This property also holds for the normal modes in the system whose fields are given by the axisymmetric ($m = 0$) solution to (1). Therefore, if there exists a solution $\delta \underset{\sim}{B}$ of (1) with eigenvalue $\tilde{\mu}$, we have $\delta \underset{\sim}{B} = \tilde{\mu} \delta \underset{\sim}{A}$ (plus a gradient), from which we conclude that

$$\delta W^* = \tilde{\mu}(\tilde{\mu} - \mu) \int \delta \underset{\sim}{A}^2 \, d^3x \ . \qquad (3)$$

Therefore, if $\tilde{\mu}$ is less than the eigenvalue μ of the axisymmetric solution of (1), and if no mode rational surface (i.e. where mq is an integer, where q is the usual safety factor) exists in the plasma, the axisymmetric equilibrium is unstable. We prove in the Appendix that the usual boundary condition at a conducting wall at the edge of the plasma, $\hat{n} \times \delta \underset{\sim}{A} = 0$, is easily satisfied by a gauge transformation which is compatible with the gauge in which $\underset{\sim}{B} \cdot \delta \underset{\sim}{A} = 0$.

Finally we can prove that the solution of (1) with minimum eigenvalue μ_o is stable. If we assume for contradiction that this minimum energy state is unstable, it follows that there exists some trial function $\delta\underset{\sim}{A}$ for which δW^* is negative. Then, minimizing $\lambda = \delta W^* / \int \delta\underset{\sim}{B}^2 d^3x$, we find that $\nabla \times \delta\underset{\sim}{B} = \mu_o/(1 - \lambda)\delta\underset{\sim}{B}$, where $\lambda < 0$ by assumption. But $\mu_o/(1 - \lambda) < \mu_o$, which is a contradiction.

Taylor's celebrated assertion[6] that the stability of resistive modes can be determined by computing the energy (i.e. the eigenvalue $\tilde{\mu}$) of nonaxisymmetric states is equivalent to the assertion that δW^* without constraint is the relevant energy principle for resistive modes. While this is an intriguing proposition, we shall see that there is no need to invoke it, because the tilting mode ($m = 1$) becomes unstable for parameters with which no mode rational surface exists.

III. SOLUTION BY SERIES EXPANSION

Proceeding to obtain solutions of (1), we invoke the representation[12]

$$\underset{\sim}{B} = \hat{z} x \nabla \psi + 1/\mu \ \nabla x \hat{z} x \nabla \psi \ , \tag{4}$$

which satisfies (1) if ψ is a solution of the scalar Helmholtz equation $(\nabla^2 + \mu^2)\psi = 0$. The boundary conditions $B_r = 0$ at $r = a$ and $B_z = 0$ at $z = 0$, $z = L$, become

$$\frac{im}{r}\psi + \frac{1}{\mu}\frac{\partial^2\psi}{\partial r \partial z} = 0 \qquad \text{at } r = a \tag{5a}$$

$$\frac{1}{\mu}\frac{\partial^2\psi}{\partial z^2} + \mu\psi = 0 \qquad \text{at } z = 0,\ z = L. \tag{5b}$$

Proceeding by the method of separation of variables, we find that solutions of the differential equation with condition (5b) are of the form

$$\psi = \sum_{n=1}^{\infty} A_n J_m(k'r) \sin kz \ e^{im\theta} + A_o r^m e^{im\theta} e^{i\mu z}, \tag{6}$$

where $k' = (\mu^2 - k^2)^{\frac{1}{2}}$ and $k = n\pi/L$. [The Bessel function is understood to mean $I_m(\sqrt{k^2 - \mu^2}\, r)$ for $k > \mu$.] The last term in (6) has B_z identically equal to zero, and therefore trivially satisfies (5b). Now, imposing (5a) we have the condition

$$-\frac{im}{a}\sum_{n=1}^{\infty} A_n J_m(k'a)\sin kz - \frac{1}{\mu}\sum_{n=1}^{\infty} A_n kk' J_m'(k'a)\cos kz$$

$$- 2imA_o a^{m-1} e^{i\mu z} = 0 \tag{7}$$

for all z between 0 and L. We have also started directly from the three components of (1) and obtained (7), showing that the representation (4) contains all solutions of (1) satisfying the boundary conditions which we impose.

For $m = 0$ we find that only one coefficient A_n is nonzero and $k'a = j_{1\ell}$, the ℓ'th zero of J_1. Thus we have

$$\mu = (n^2\pi^2/L^2 + j_{1\ell}^2/a^2)^{\frac{1}{2}} \tag{8a}$$

$$\psi = (\mu/k')\, J_o\,(k'r)\,\sin\,(n\pi\, z/L), \tag{8b}$$

$$B_r = -k\; J_1(k'r)\,\cos(n\pi\, z/L), \tag{8c}$$

$$B_\theta = \mu\; J_1(k'r)\,\sin(n\pi\, z/L), \tag{8d}$$

$$B_z = k'\; J_o(k'r)\,\sin(n\pi\, z/L). \tag{8e}$$

The term in (6) proportional to A_o does not contribute to $\underset{\sim}{\Gamma}$ for $m = 0$. The axisymmetric equilibrium with the smallest eigenvalue μ (hence the lowest energy) has $\ell = n = 1$. The flux surfaces for this state are shown in Fig. 1. The poloidal flux is

$$rA_\theta = rJ_1(j_{11}r/a)\,\sin(\pi\, z/L), \tag{9}$$

and the safety factor $q \equiv \oint (\underset{\sim}{B}\cdot\nabla\theta/\underset{\sim}{B}\cdot\nabla r)dr$ equals

$$q = \frac{\mu r A_\theta}{\pi k}\int_{r_1}^{r_2} \frac{dr}{r\left[r^2 J_1(j_{11}r/a)^2 - r^2 A_\theta^2\right]^{1/2}}, \tag{10a}$$

$$= \frac{\mu L}{\pi^2} \phi \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho \left[\rho^2 J_1 (j_{11}\rho)^2 - \phi^2 \right]^{1/2}} \tag{10b}$$

where r_1 and r_2 are the two points where the flux surface crosses the midplane $z = L/2$, $\rho = r/a$, and $\phi = rA_\theta/a$ is the normalized flux. From (10b) it is easily seen that q factors into a product of a term depending on the normalized flux and another term dependent on only the elongation L/a. That is, $q(rA_\theta)$ is self similar for every L/a. Furthermore, we have shown that q decreases monotonically with distance from the magnetic axis and that

$$q_s = \mu/4k = (1 + j_{1,1}{}^2 L^2/\pi^2 a^2)/4, \tag{11a}$$

$$q_a = 4q_s/j_{0,1} \simeq q_s/0.6, \tag{11b}$$

where q_s is evaluated at the separatrix $rA_\theta = 0$ and q_a is evaluated at the magnetic axis, and $j_{0,1}$ is the first zero of J_o. The q profile monotonically increases with L/a (because the poloidal connection length increases) and the q = 1 surface first appears at the magnetic axis for L/a = 1.793. Notice that our model has somewhat more shear than the spherical model of Ref. 2.

For $m \neq 0$, Eq. (7) cannot be satisfied by a single n, because J_m and J'_m cannot both be zero for the same argument. Similarly, a finite set of nonzero A_n cannot satisfy (7) because any such finite set of trigonometric functions are linearly independent. Finally, we observe that the infinite set of functions $\sin(n\pi z/L)$, $\cos(n\pi z/L)$ are not linearly independent. An obvious way to approach (7) is to expand $\cos kz$ and $e^{i\mu z}$ in a sin series, i.e. extending all functions to be odd on the interval $-L < z < L$. These expansions have Fourier coefficients that decrease as n^{-1}. We use finite elements rather than sin functions as basis functions to avoid these convergence problems. That is, we truncate the series (7) at $n = N$, producing a set of equations with $2N + 2$ real unknowns (since A_0, A_1, ..., A_N are complex). We then evaluate (7) at $2N + 2$ points in the interval $0 < z < L$. The resulting $(2N+2) \times (2N+2)$ matrix is a complicated function of the eigenvalue μ. We compute numerically the determinant $d(\mu)$ as a function of μ. This determinant exhibits nonphysical roots $d(\mu) \sim |\mu - n\pi/L|^{2+m}$. This is because, as $\mu \to n\pi/L$, the $e^{i\mu z}$ solution of (6) is redundant [leading to two pairs of identical columns in the matrix] and because $J_m(k'a) \sim k'J_m'(k'a) \sim (\mu - n\pi/L)^{m/2}$ [there are two columns proportional to this factor.] The other roots of $d(\mu)$ are double roots, due to the symmetry $\theta \to \theta + \pi/2m$ [which exchanges real and imaginary parts of $A_0, A_1, \ldots, A_N$.] The determinant $d(\mu)$ does not reach zero for finite N. For $N = 10$, $d(\mu)$ exhibits fairly sharp dips down to 10^{-2} compared to neighboring values. For $N = 15$, the dips are more pronounced, around 10^{-4}. This shows that quite a few Fourier harmonics are necessary to describe the solution accurately. The complexity of these eigenfunctions, in particular the fact that they are not separable, is related to the fact that (5) represents an unconventional set of boundary conditions for an elliptic equation.

In Fig. 2 we show the first two physical roots for $m = 1$ and the first root (8a) for $m = 0$, as a function of the elongation L/a. There are crossings at $L/a = 1.67$ and $L/a = 4.14$. Since no mode rational surface $q = n$ exists for $L/a < 1.793$ (the maximum q at $L/a = 1.67$ is 0.944) this implies that all $m = 1$ modes are stable for $0 < L/a < 1.67$ and that an $m = 1$ instability exists for $1.67 < L/a < 1.793$. The second crossing, which occurs when the $q = 2$ surface is present (q has the range $1.29 < q < 2.14$) indicates but does not prove the existence of another $m = 1$ instability. There does not seem to be a further crossing of the two $m = 1$ states, and therefore we believe that the first $m = 1$ mode is always dominant in terms of growth rate and saturation amplitude. Because this mode has similar structure to that of Rosenbluth and Bussac[2] in spherical geometry, and because it becomes unstable at a similar elongation and when no mode rational surface exists, we refer to this as the tilting mode.

The lowest eigenvalues for the $m = 2$, 3, and 4 modes, as well as for $m = 0$ are shown in Fig. 3. Since there is no crossing with the lowest $m = 0$ eigenvalue, it follows that there are no $m = 2$, 3 or 4 instabilities. [Note that, as $L/a \to 0$, the second term of (7) becomes dominant, indicating that the eigenvalue μ should become independent of m in this limit. The results of Fig. 2 and 3 show that this is indeed the case.] Furthermore, for $m \to \infty$ and $A_o = 0$, (7) is satisfied with a single nonzero A_n, giving $\mu = (n^2\pi^2/L^2 + j_{m\ell}^2/a^2)^{\frac{1}{2}}$. The lowest energy state has $n = \ell = 1$. The $m = 4$ results of Fig. 3 agree with this value within a few percent and we therefore conclude that the axisymmetric equilibrium (8) is unstable only to $m = 1$ modes.

IV. LINEARIZED CODE

Here we present an alternate approach to that of Sec. III. First, notice that the linearized equation of motion of ideal magnetohydrodynamic for force free equilibria

$$\rho\, \partial^2 \underset{\sim}{\xi}/\partial t^2 = \delta \underset{\sim}{j} \times \underset{\sim}{B} + \underset{\sim}{j} \times \delta \underset{\sim}{B} \tag{12}$$

takes the form

$$(\rho/B^2)\ (\partial^2/\partial t^2)\delta\underset{\sim}{A} \ = \ (\mu\, \delta \underset{\sim}{B} - \delta \underset{\sim}{j})_\perp \tag{13}$$

for equilibria satisfying (1), where $\delta\underset{\sim}{A} = \underset{\sim}{\xi} \times \underset{\sim}{B}$ and $\perp$ signifies the components perpendicular to the equilibrium field $\underset{\sim}{B}$. Thus we may consider $\delta\underset{\sim}{A}$ and not $\underset{\sim}{\xi}$ our basic dependent variable; clearly $\underset{\sim}{B}\cdot\delta\underset{\sim}{A} = 0$ for all time if $\underset{\sim}{B}\cdot\delta\underset{\sim}{A}$ and $(\partial/\partial t)\underset{\sim}{B}\cdot\delta\underset{\sim}{A}$ are both zero at $t = 0$. This set of equations, of course, has a variational principle with δW^*, with normalization $T = \int d^3 x\, \rho |\delta\underset{\sim}{A}|^2/B^2$ and with constraint $\underset{\sim}{B}\cdot\delta\underset{\sim}{A} = 0$. The modified set of equations

$$(\rho/B^2)(\partial^2/\partial t^2)\ \delta\underset{\sim}{A} = \mu\delta\underset{\sim}{B} - \delta\underset{\sim}{j} \tag{14}$$

has the same variational principle but without the constraint. As discussed in Sec. II, the stability criterion obtained by integrating (13) or (14) must be identical if no mode rational surface exists in the plasma. Furthermore (14) has the advantage of producing nonaxisymmetric solutions to (14) at marginal stability. [Taylor's assertion,[6] mentioned earlier, is therefore equivalent to the statement that tearing modes are stable in an axisymmetric system satisfying (1) if and only if (14) is stable, regardless of the existence of mode rational surfaces.]

Our code integrates (14) in time until one unstable mode if any dominates all others. Runs are terminated when the growth rate, based upon the kinetic energy, varies by less than a percent. Results from a series of such runs with normalization ρ/B^2 = const. are shown in Fig. 4, for $m = 1$. The marginal stability point at $L/a \simeq 1.65$ agrees well with that shown on Fig. 2. The perturbed magnetic field of this unstable mode near marginal stability ($L/a = 1.75$) is shown in Fig. 5. Notice that the real part of δB_r (i.e. the $\cos m\theta$ component) looks like a pure sine wave as a function of z, but with a period slightly less than L, whereas the imaginary part of δB_r (i.e. the $-\sin m$ component) seems to be a combination of $\sin(\pi z/L)$ and $\cos(2\pi z/L)$. The other components of $\delta \underset{\sim}{B}$ also show a complex structure of Fourier harmonics in z, including aperiodic behavior. See especially the imaginary part of δB_θ, which seems to have a period 1.16 L. This is to be expected since the $m = 1$ solutions to (7) contain many harmonics as well as the anharmonic component proportional to $e^{i\mu z}$.

ACKNOWLEDGMENTS

We would like to thank Z. G. An, A. Bondeson, H. H. Chen, A. Drobot, Y. C. Lee, C. S. Liu, G. Marklin, and N. K. Winsor for illuminating discussions.

This work was supported under U.S. Department of Energy Contract No. EX-76-A-34-1006.

APPENDIX

Our purpose here is to prove the two propositions of Sec. II. The first is that a gauge transformation can be found so that $\delta\underset{\sim}{A}$ is normal, i.e. $\hat{n} \times \delta\underset{\sim}{A} = 0$ on the boundary, if the volume is simply connected and if $\hat{n}\cdot\delta\underset{\sim}{B} = 0$ on the boundary. The second proposition is that the constraint $\underset{\sim}{B}\cdot\delta\underset{\sim}{A} = 0$ can be satisfied throughout the plasma if no mode rational surface exists in the plasma (as is the case in the neighborhood of marginal stability for the tilting mode).

To prove the first proposition, we ask whether there is a scalar function χ such that $\hat{t}\cdot\nabla\chi = -\hat{t}\cdot\delta\underset{\sim}{A} = 0$ on the boundary. Assuming $\chi = 0$ at some reference point $\underset{\sim}{x}_o$ on the boundary, we find

$$\chi(\underset{\sim}{x}) = -\int_{\underset{\sim}{x}_o}^{\underset{\sim}{x}} \delta\underset{\sim}{A} \cdot d\underset{\sim}{\ell}.$$

The function χ so defined is single valued because the integral over a closed path equals the magnetic flux through that part of the boundary encircled by the path.

To prove the second proposition, we must find another scalar function χ that satisfies $\underset{\sim}{B}\cdot\nabla\chi = -\underset{\sim}{B}\cdot\delta\underset{\sim}{A}$ (where $\hat{n} \times \delta\underset{\sim}{A} = 0$ is now assumed at the boundary). Writing $\underset{\sim}{B} = \nabla\psi \times \nabla\theta + g(\psi)\nabla\theta$ (ψ is now the poloidal flux rA_θ) we find

$$\frac{1}{J}\frac{\partial\chi}{\partial\varphi} + \frac{g(\psi)}{r^2}\frac{\partial\chi}{\partial\theta} = -\underset{\sim}{B}\cdot\delta\underset{\sim}{A} , \tag{A.1}$$

where φ is a coordinate along the flux surfaces ψ = const., and J is the Jacobian $(\nabla\psi \times \nabla\theta \cdot \nabla\varphi)^{-1}$. We can always assume $J = r^2K(\psi)$, where $K(\psi) = (1/2\pi) \oint d\ell/(r^2B_p)$ and B_p is the magnitude of the poloidal field, and that φ varies from 0 to 2π. Also, since $\delta \underset{\sim}{A} \sim e^{im\theta}$, we can assume $\chi = \chi(\psi,\varphi)e^{im\theta}$. (It is also possible to add an arbitrary function of ψ.) Substituting in (A.1), we find

$$\partial\chi/\partial\varphi + imq(\psi)\chi = -r^2 K(\psi)\underset{\sim}{B}\cdot\delta\underset{\sim}{A}, \qquad (A.2)$$

where $q(\psi) = g(\psi)K(\psi)$ is the safety factor $(1/2\pi)\oint(\underset{\sim}{B}\cdot\nabla\theta)d\varphi/(\underset{\sim}{B}\cdot\nabla\varphi)$. Expanding χ and the right hand side of (A.2) as

$$\sum_{n=-\infty}^{\infty} \chi_n(\psi)e^{-in\varphi}$$

and

$$\sum_{n=-\infty}^{\infty} f_n(\psi)e^{-in\varphi}$$

and substituting in (A.2), we find

$$\chi_n(\psi) = i\, f_n(\psi)/(n - mq(\psi)). \qquad (A.3)$$

If $\psi = \psi_o$ is a mode rational surface, i.e. has $n - mq(\psi_o) = 0$ for some n, we must have $f_n(\psi_o) = 0$. If no mode rational surface exists, on the other hand, there is no constraint on $\delta\underset{\sim}{A}$, i.e. on the coefficients $f_n(\psi)$. It is obvious that this second gauge transformation does not destroy the property guaranteed by the first. We wish to emphasize that these two properties are general and do not depend upon $\underset{\sim}{B}$ being a solution of (1).

REFERENCES

1. M. N. Bussac, H. P. Furth, M. Okabayashi, M. N. Rosenbluth, and A. M. M. Todd, in Plasma Physics and Controlled Nuclear Fusion Research (International Atomic Energy Agency, Vienna, 1979), v.3, p.251.
2. M. N. Rosenbluth and M. N. Bussac, Nucl. Fusion 19, 489 (1979).
3. G. C. Goldenbaum, J. H. Irby, Y. P. Chong, and G. W. Hart, Phys. Rev. Lett. 44, 393 (1980).
4. Z. G. An, A. Bondeson, H. Bruhns, H. H. Chen, Y. T. Chong, J. M. Finn, G. C. Goldenbaum, H. R. Griem, G. W. Hart, R. Hess, J. H. Irby, Y. C. Lee, C. S. Liu, W. M. Manheimer, G. Marklin, E. Ott, to appear in Plasma Physics and Controlled Nuclear Fusion Research (International Atomic Energy Agency, Vienna).
5. L. Woltjer, Proc. Nat. Acad. Sci. 44, 489 (1958).
6. J. B. Taylor, Phys. Rev. Lett. 33, 1139 (1974).
7. Z. G. An, A. Bondeson, H. H. Chen, Y. C. Lee, and C. S. Liu, private communication.
8. G. Marklin, private communication.
9. Z. G. An, private communication.
10. A. Reiman, Phys. Fluids 23, 230 (1980).
11. I. B. Bernstein, E. A. Frieman, M. A. Krushkal, and R. M. Kulsrud, Proc. Roy. Soc. London Ser. A244, 17 (1958).
12. P. M. Morse and H. Feshbach, Methods in Theoretical Physics, v.2 p.1766 (1953).

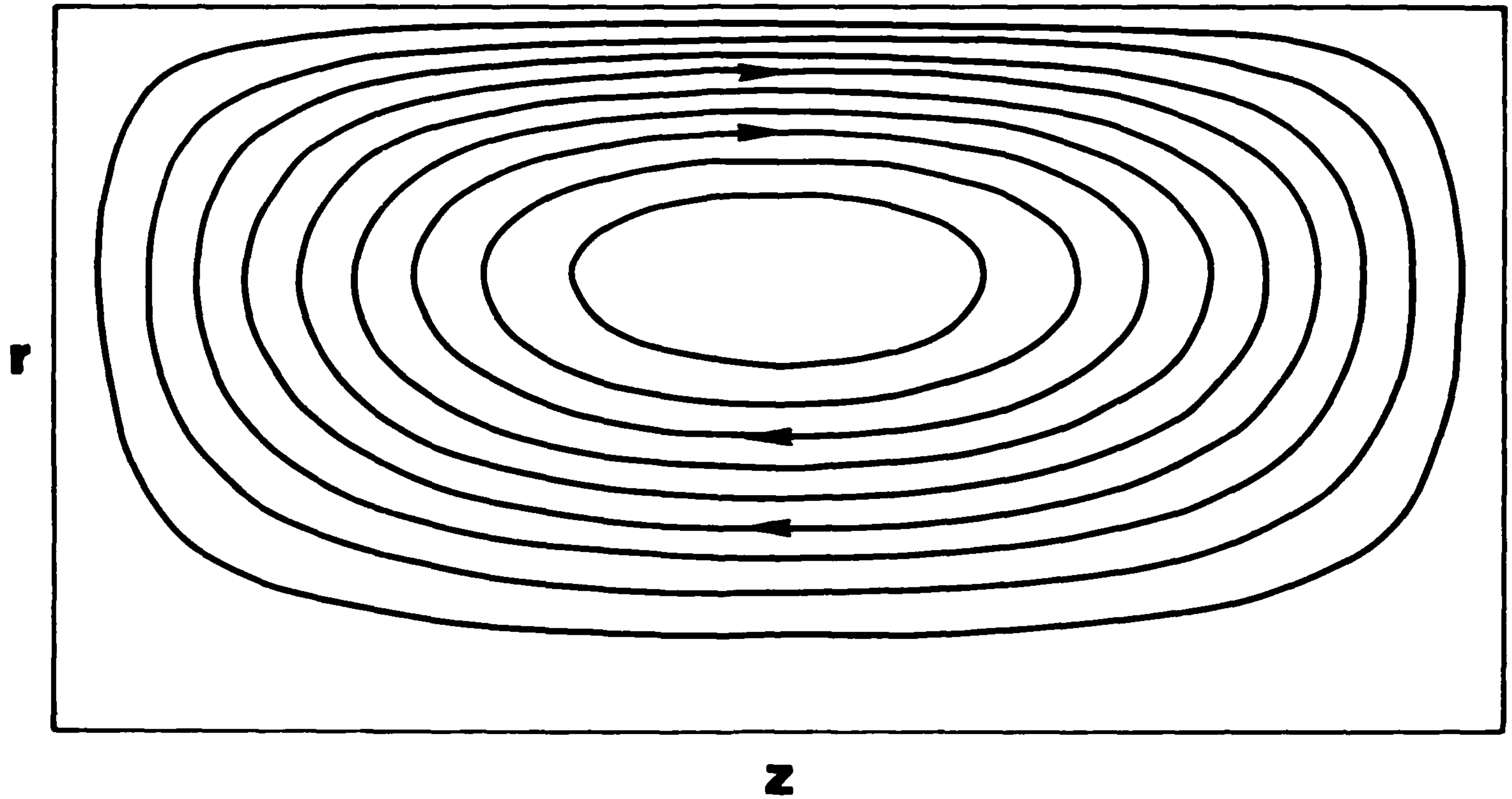


Fig. 1 — Flux surfaces rA_θ = const. for the axisymmetric equilibrium given by equation (8)

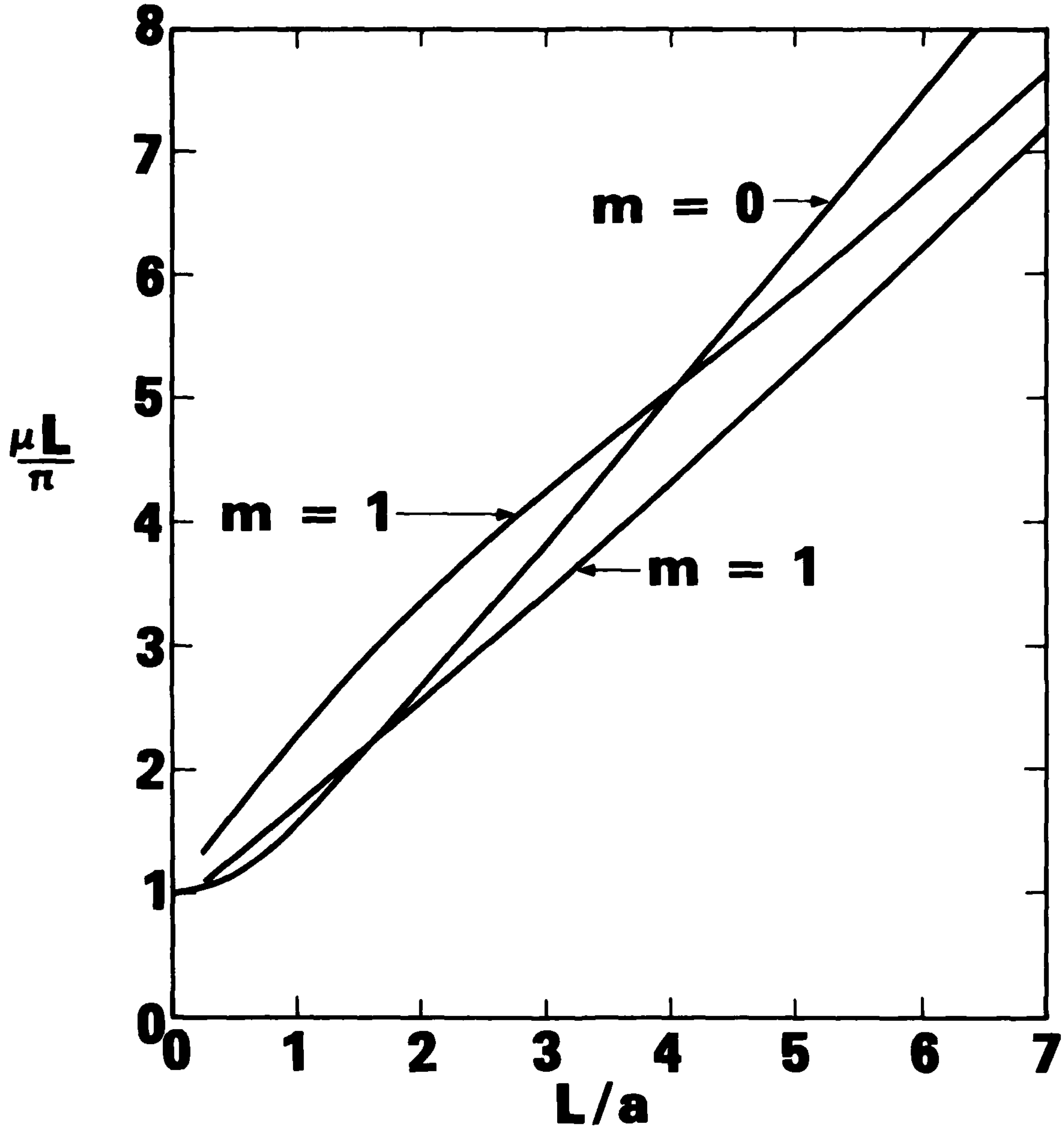


Fig. 2 — Normalized eigenvalue $\mu L/\pi$ as a function of elongation L/a for the lowest m = 0 equilibrium and for the two lowest m = 1 equilibria. Note the crossings at L/a = 1.67 and at L/a = 4.14

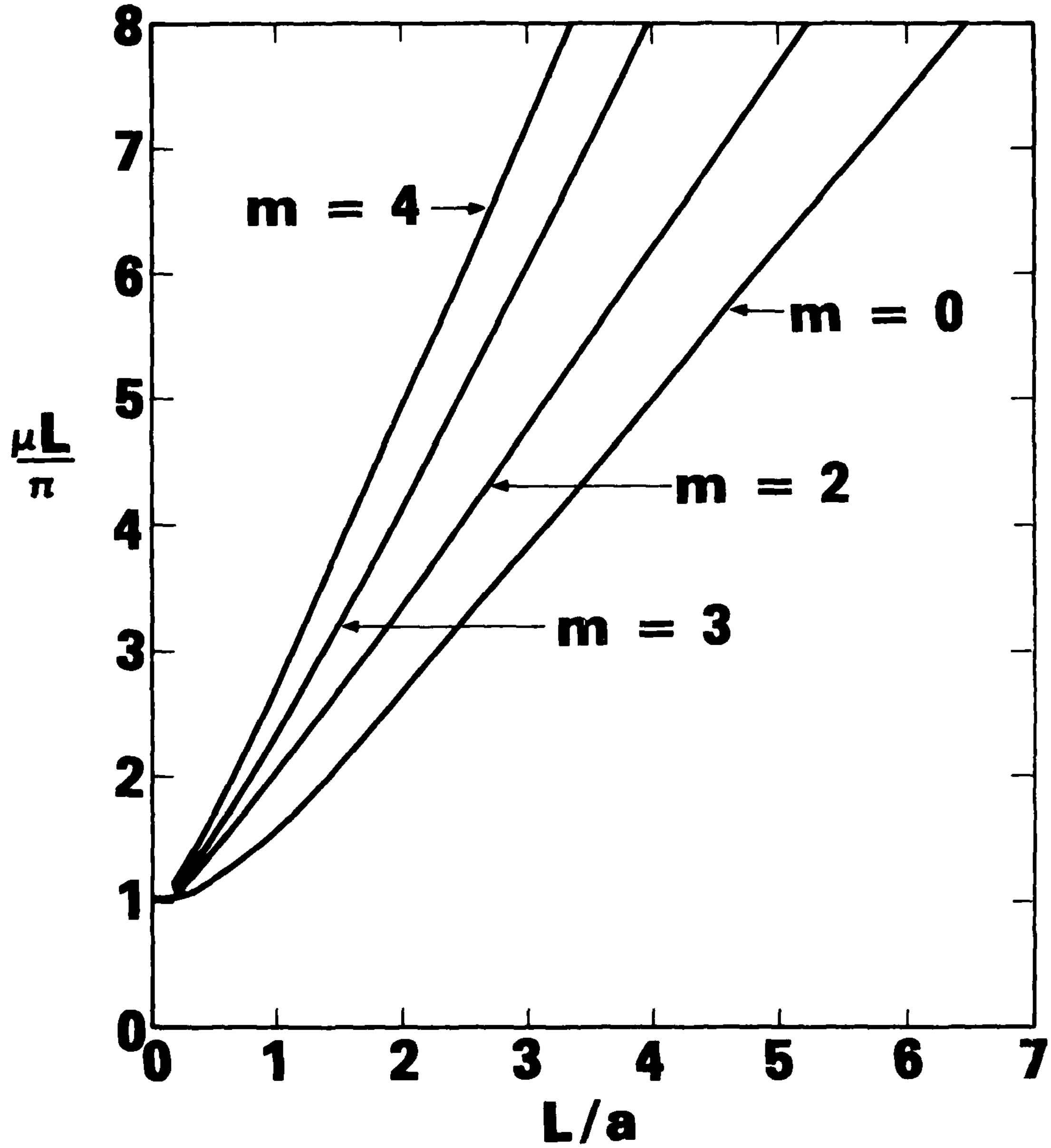


Fig. 3 — Normalized eigenvalue $\mu L/\pi$ as a function of L/a for m = 0,2,3, and 4 equilibria

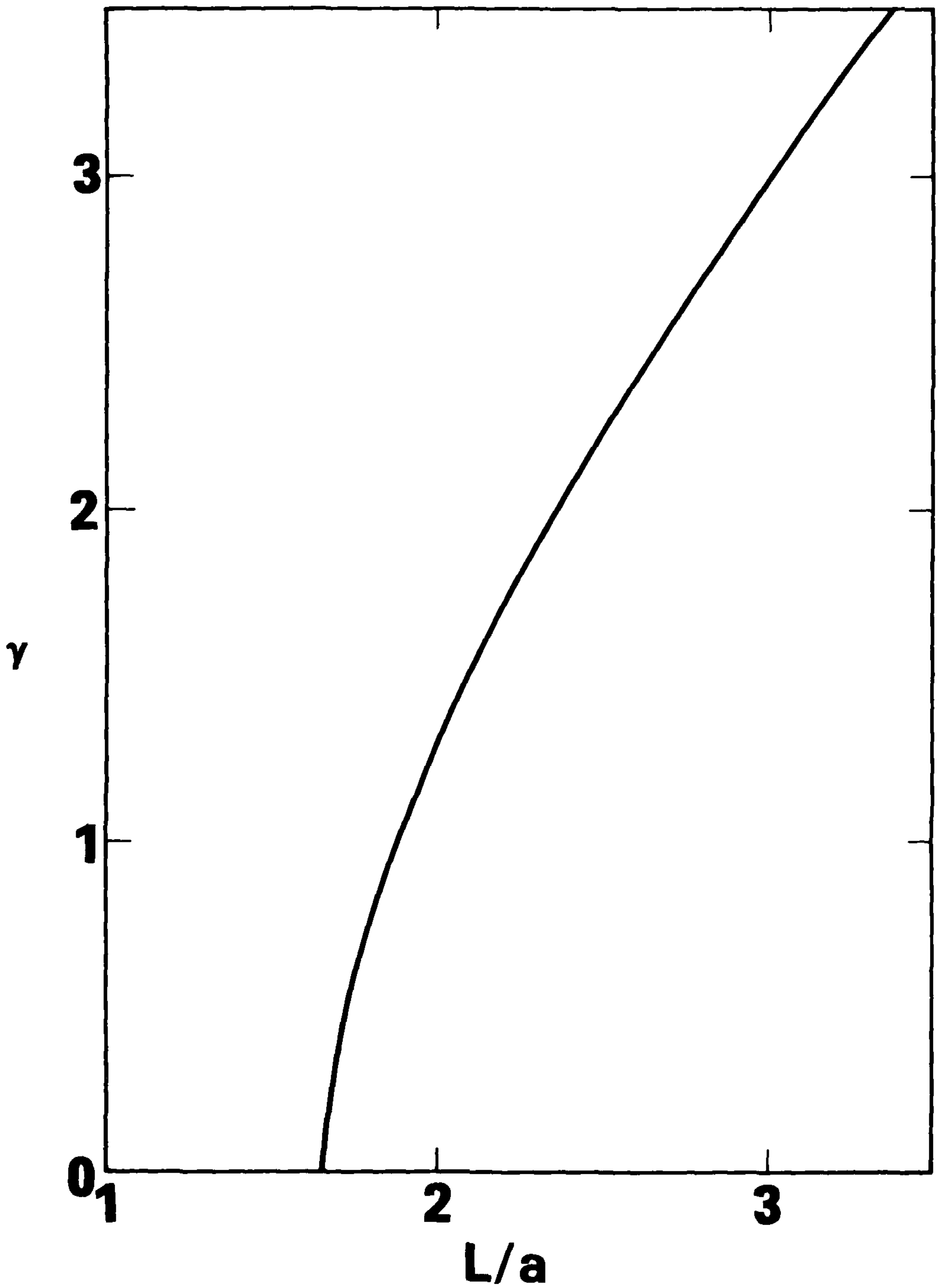
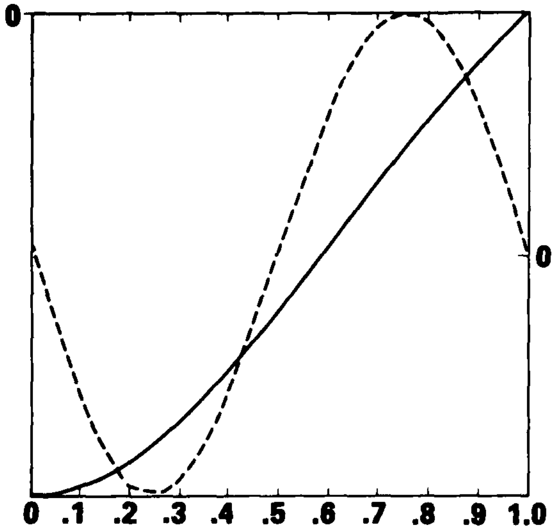
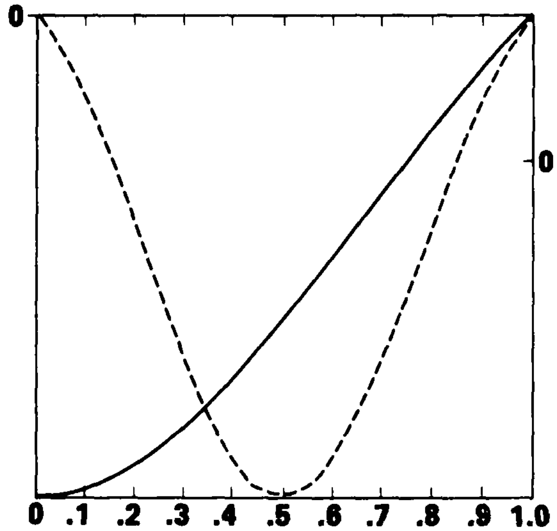


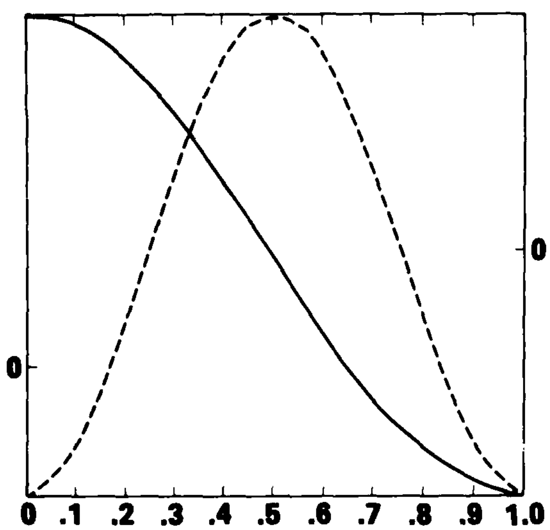
Fig. 4 — Normalized growth rate $\gamma = |\omega| L/V_A$ for the most unstable mode of the system (14) with m = 1, as a function of elongation L/a, showing marginal stability at L/a $\simeq$ 1.65


0
0
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0

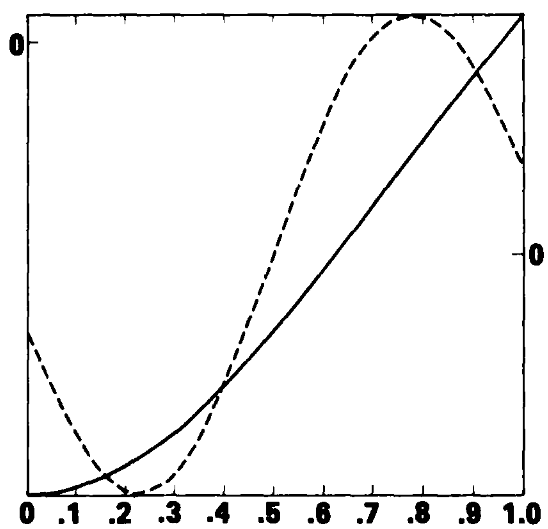
(a)


0
0
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0

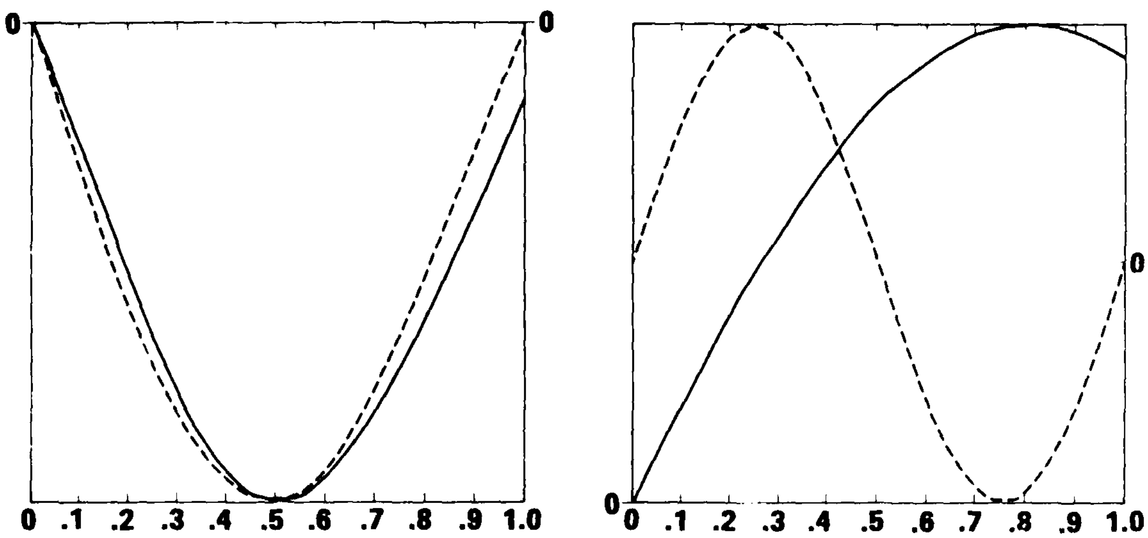
(b)


0
0
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0

(c)


0
0
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0

(d)



(e) (f)

Fig. 5 — Magnetic fields produced by the m = 1 tilting instability from (14) near marginal stability ($L/a = 1.75$). Respectively, these are the real and imaginary parts of δB_r (a,b), of δB_θ (c,d) and of δB_z (e,f). For the solid curve the horizontal axis represents r (at $z = L/2$) and the zero point is marked on the left hand vertical axis. For the dashed curve the horizontal axis represents z (at $r = a/2$) and the zero point is marked on the right hand vertical axis. The scale of $\delta \underset{\sim}{B}$ is arbitrary.

Distribution List

DOE
P.O. Box 62
Oak Ridge, Tenn. 37830

UC20 Basic List (116 copies)
UC20f (75 copies)
UC20g (62 copies)

NAVAL RESEARCH LABORATORY
Washington, D.C. 20375

Code 4700 (25 copies)
Code 4790 (150 copies)
D. Spicer, Code 4169

DEFENSE TECHNICAL INFORMATION CENTER
Cameron Station
5010 Duke Street
Alexandria, VA 22314 (12 copies)