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EXISTENCE OF OPTIMAL CONTROLS FOR
PARTIALLY OBSERVED DIFFUSIONS

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EXISTENCE OF OPTIMAL CONTROLS FOR PARTIALLY OBSERVED DIFFUSIONS

WENDELL H. FLEMING AND ETIENNE PARDOUX

1. Introduction. In this paper we are concerned with the existence of optimal controls for problems of the following kind. Let X_t denote the process which we wish to control, Y_t the observation process, and U_t the control process, $0 \leq t \leq T$, with T fixed. The state and observation processes are governed by stochastic differential equations

$$(1.1) \quad \begin{aligned} (a) \quad dX_t &= b(X_t, Y_t, U_t)dt + \sigma(X_t, Y_t)dW_t \\ (b) \quad dY_t &= h(X_t)dt + d\tilde{W}_t. \end{aligned}$$

X_t has values in N -dimensional $\mathbb{R}^N$, Y_t values in $\mathbb{R}^M$, and U_t values in $\mathcal{U} \subset \mathbb{R}^L$. X_0 has given distribution μ , and $Y_0 = 0$. In (1.1), W and $\tilde{W}$ are independent standard Wiener processes, with values in $\mathbb{R}^D, \mathbb{R}^M$ respectively. The matrix σ is thus $N \times D$.

The problem is to minimize a criterion of the form

$$(1.2) \quad J = E\left\{\int_0^T F(X_t, U_t)dt + G(X_T)\right\}.$$

It is customary to require that U_t be measurable with respect to the σ -algebra generated by the observations Y_s , $0 \leq s \leq t$. We call this the strict sense version of the problem (§6). For several years the question of proving a general theorem about existence

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of optimal controls in the strict sense has been open. We do not obtain such a result here. In fact, our results together with a counterexample of Varadhan (§6) strongly suggest that, if there is indeed a general existence theorem for strict sense optimal controls, then standard methods are not adequate to prove it. There is a similar difficulty in proving existence of optimal controls with complete observations with singular noise coefficient σ , if the term "complete observations" is taken in the strict sense that U_t depends on the past of the Wiener process driving the system.

Instead of allowing only strict-sense controls, we obtain existence of a minimum in a wider class of controls. Roughly speaking, this wider class is obtained as follows. Let

$$(1.3) \quad Z_t = \exp\left[\int_0^t h(X_s) \cdot dY_s - \frac{1}{2} \int_0^t |h(X_s)|^2 ds\right].$$

Then W_t, Y_t are independent standard Wiener processes under a new probability measure $\overset{\circ}{P}$ related to the original probability measure $\overset{\circ}{P}$ by $\frac{d\overset{\circ}{P}}{dP} = Z_T^{-1}$. In the wider sense formulation we wish to require merely that U_t be independent of future increments $Y_r - Y_t$ for $t \leq r$ and independent of the W process, with respect to $\overset{\circ}{P}$. In §2 we give a precise formulation of this idea, in which the control is defined as the joint probability distribution measure π of the processes Y, U .

Our method depends on introducing a "pathwise" version $\Lambda_t^{Y,U}$ of an unnormalized conditional distribution measure for X_t given past values of the observation process Y and the control process U (§3). An important fact is the continuous dependence

of Λ_t on Y, U and the initial distribution μ (Lemma 3.2). In §4 we introduce a "separated" control problem, equivalent to the original problem formulated in §2. The "state" in the separated problem is the measure Λ_t . In §4 we study the dynamics of Λ_t , using a method of forward and backward partial differential equations. Similar ideas were used in [13], for the nonlinear filter problem. The forward equation (5.4) is linear parabolic (possibly degenerate) with coefficients depending parametrically on observations Y_t and controls U_t . Under suitable regularity assumptions, Λ_t has a density $q(t, x)$, related in a simple way to a solution $p(t, x)$ of the forward equation via (5.6). Without the regularity assumptions, one uses instead a weak sense version (5.4') of the forward equation. We also show that Λ_t satisfies the Zakai equation (5.8) of nonlinear filtering. The method of backward and forward equations was applied to the nonlinear filter problem in [13], working directly with the Zakai equation and its adjoint and allowing correlations between the Wiener processes $W, \tilde{W}$ driving the state and observation equations. However, technical difficulties are encountered in adapting that method to the control problem.

In [8] another "separated" control problem was considered. In that formulation the "state" for the separated problem corresponds to the (normalized) conditional distribution measure of X_t given past observations. Some of the results in [8] are proved under assumptions not satisfied when X_t is a controlled, partially observed solution to (1.1a). Hence, the results of [8] are complementary to those in the present paper.

A first existence theorem asserting that there is a control minimizing J is proved in §4, when $F = 0$ in (1.2). When $F \neq 0$

different methods are needed. In §7 we assume that the coefficient σ in (1.1a) is $N \times N$ nonsingular, and use methods of the L^2 -theory of parabolic partial differential equations.

In [4] Christopeit proved an existence theorem for optimal stochastic controls under partial observations. In that work, the observation process is a deterministic function of (part of) the past trajectory of the state process, and the optimal control is sought in a class of feedback controls. Both our results and our methods of proof differ significantly from his.

2. Formulation of the problem. We make the following assumptions about the functions appearing in (1.1).

(A₁) σ is a bounded, continuous $N \times D$ matrix-valued function on $\mathbb{R}^{N+M}$. Moreover, $\sigma(\cdot, y)$ is Lipschitz on $\mathbb{R}^N$ with Lipschitz constant not depending on $y \in \mathbb{R}^M$.

(A₂) $b(x, y, u) = b^0(x, y) + b^1(x, y)u$, where b^0, b^1 are bounded, continuous functions on $\mathbb{R}^{N+M}$. Moreover, $b^\ell(\cdot, y)$ is Lipschitz on $\mathbb{R}^N$ with Lipschitz constant not depending on $y \in \mathbb{R}^M$, for $\ell = 0, 1$.

Note that in (A₂), b^0 has values in $\mathbb{R}^N$, while b^1 has $N \times L$ matrices as values.

We write $C_b(\mathbb{R}^N)$ for the space of bounded continuous real-valued functions on $\mathbb{R}^N$, and $C_0(\mathbb{R}^N)$ for the space of continuous functions with compact support. We write $C_b^k(\mathbb{R}^N)$, $C_0^k(\mathbb{R}^N)$ for the spaces of functions whose partial derivatives of orders $\leq k$ are in $C_b(\mathbb{R}^N)$, $C_0(\mathbb{R}^N)$ respectively. Similarly we write $C_b^k(\mathbb{R}^N; \mathbb{R}^M)$, $C_0^k(\mathbb{R}^N; \mathbb{R}^M)$ if the functions are $\mathbb{R}^M$ -valued.

(A₃) $h \in C_b^2(\mathbb{R}^N; \mathbb{R}^M)$.

In §7 we shall assume that σ is nonsingular $N \times N$. One could also let b, σ, h depend on t , with minor changes in the results, and proofs. This would only be a generalization in §7, since in §'s 2-6 t can be adjoined as an additional x component.

(A₄) $\mathcal{Z}$ is a convex, compact subset of $\mathbb{R}^L$.

Choose any $T > 0$ which will be fixed throughout the paper. We formulate the control problem on the "canonical" sample space

$$\Omega = \Omega_0 \times \Omega_1 \times \Omega_2 \times \Omega_3,$$

where $\Omega_0, \Omega_1, \Omega_2$ are $C([0, T]; \mathbb{R}^m)$ with $m = D, N, M$ respectively and

$$\Omega_3 = L^2([0, T]; \mathcal{U}).$$

The elements $\omega = (W, X, Y, U)$ of Ω satisfy

$$\omega(t) = (W_t(\omega), X_t(\omega), Y_t(\omega), U_t(\omega)), \quad 0 \leq t \leq T.$$

We give $\Omega_0, \Omega_1, \Omega_2$ the usual norm topology; and Ω_3 the weak topology, which is metrizable and separable since $\mathcal{U}$ is compact [2, p. 238]. Let

$$\Omega^1 = \Omega_0 \times \Omega_1, \quad \Omega^2 = \Omega_2 \times \Omega_3,$$

whose respective elements are pairs $(W, X), (Y, U)$. Let

$$\mathcal{F}_t(W) = \sigma\{W_s, 0 \leq s \leq t\}, \text{ with } \mathcal{F}_t(X), \mathcal{F}_t(Y) \text{ defined similarly.}$$

Let

$$\mathcal{F}_t(U) = \sigma\{V_s, 0 \leq s \leq t\}, \quad V_t = \int_0^t U_s ds.$$

The elements of these σ -algebras are subsets of $\Omega_0, \dots, \Omega_3$ respectively. However, we can also regard them as σ -algebras of subsets of Ω, Ω^1 , or Ω^2 , with the obvious identifications. For example $A \in \mathcal{F}_t(X)$ can be identified with $\Omega_0 \times A \times \Omega_2 \times \Omega_3$.

We shall also use the σ -algebras

$$\mathcal{G}_t^1 = \mathcal{F}_t(W) \times \mathcal{F}_t(X)$$

$$\mathcal{G}_t^2 = \mathcal{F}_t(Y) \times \mathcal{F}_t(U)$$

$$\mathcal{A}_t = \mathcal{G}_t^1 \times \mathcal{G}_t^2 = \mathcal{F}_t(W) \times \dots \times \mathcal{F}_t(U).$$

We note that $\mathcal{F}_T(U)$ is the Borel σ -algebra of Ω_3 , and thus $\mathcal{G}_T^2$ is the Borel σ -algebra of Ω^2 .

Remark. Intuitively, by using the indefinite integral V_t instead of U_t in defining $\mathcal{F}_t(U)$, we need not be concerned with changes in U_t on subsets of $[0, T]$ of Lebesgue measure 0. An alternative to our formulation would be to consider quadruples (W, X, Y, V) instead of (W, X, Y, U) , using the uniform norm on V . By (A_2) the control enters linearly in b . Hence, one can write, in the integrated form of (1.1a),

$$\int_0^t b^1(X_s, Y_s) U_s ds = \int_0^t b^1(X_s, Y_s) dV_s,$$

the right side being a Riemann-Stieltjes integral. This device was used in [9], but we use here U_t instead.

Distribution of (W, X) conditioned on (Y, U) . Let $Y = Y_\cdot$, $U = U_\cdot$ be given sample paths for the observation and control processes; thus $(Y, U) \in \Omega^2$. Consider equation (1.1a) with initial data $W_0 = 0$, $X_0 = x$. Assumptions (A_1) , (A_2) imply the Ito conditions. There is a solution to (1.1a) which is pathwise unique, and hence also unique

in probability law. Let $\bar{P}_x^{Y,U}$ denote the distribution measure of (W,X) given (Y,U) . Then $\bar{P}_x^{Y,U}$ lies in the space of probability measures on $\mathcal{G}_T^1$. By convergence of a sequence of probability measures $\bar{P}_n$ to $\bar{P}$ we mean weak convergence, namely

$$\int_{\Omega} g(W,X) d\bar{P}_n \rightarrow \int_{\Omega} g(W,X) d\bar{P} \quad \text{for all } g \in C_b(\Omega^1).$$

Lemma 2.1. $\bar{P}_x^{Y,U}$ depends continuously on x, Y, U .

This lemma is essentially known (cf. Stroock-Varadhan [14]). However, for completeness we outline a proof in the Appendix.

Following the motivation described in §1, the formal definition of admissible control is as follows.

Definition. An admissible control π is a probability measure on $(\Omega^2, \mathcal{G}_T^2)$ such that Y is a π , $\{\mathcal{G}_T^2\}$ Wiener process.

The projection $(Y,U) \rightarrow Y$ maps π onto Wiener measure. The definition of admissible control requires, in addition, that $\int_0^t U_s ds$ be independent of $Y_r - Y_t$ for $t \leq r \leq T$.

Let $\mathfrak{A}$ denote the set of all admissible controls π . Given a distribution μ for X_0 , each $\pi \in \mathfrak{A}$ determines a joint distribution measure P_π of (W,X,Y,Z) as follows. Define $\bar{P}^{Y,U} = \bar{P}_\mu^{Y,U}$ by

$$\bar{P}^{Y,U}(A) = \int_{\mathbb{R}^N} P_x^{Y,U}(A) d\mu(x), \quad A \in \mathcal{G}_T^1.$$

We then define $\bar{P}_\pi^\circ$ on $\mathcal{G}_T$ by

$$(2.1) \quad \overset{\circ}{P}_{\pi}(dW, dX, dY, dU) = \bar{P}^{Y,U}(dW, dX) \pi(dY, dU).$$

The projection of $\overset{\circ}{P}_{\pi}$ under $(W, X, Y, U) \rightarrow (Y, U)$ is π . The family of probability measures $\bar{P}^{Y,U}$ gives a regular conditional distribution for (W, X) . If $g(W, X)$ is $\mathcal{G}_t^1$ -measurable, then $\bar{E}^{Y,U} g(W, X)$ is $\mathcal{G}_t^2$ -measurable, where we write $\bar{E}^{Y,U}, \overset{\circ}{E}_{\pi}$ for expectations with respect to $\bar{P}^{Y,U}, \overset{\circ}{P}_{\pi}$. We then have for any $\mathcal{G}_t$ -measurable ψ with $\overset{\circ}{E}_{\pi} |\psi| < \infty$

$$(2.2) \quad \overset{\circ}{E}_{\pi}(\psi | \mathcal{G}_t^2) = \bar{E}^{Y,U}(\psi), \quad \pi - \text{a.s.}$$

We define P_{π} by

$$(2.3) \quad \frac{dP_{\pi}}{d\overset{\circ}{P}_{\pi}} = Z_T,$$

with Z_T as in (1.3). Since $h(x)$ is bounded, $P_{\pi}(\Omega) = \overset{\circ}{E}_{\pi}(Z_T) = 1$.

For each (Y, U) , W is a $\bar{P}^{Y,U}$ -standard Wiener process, and X satisfies the stochastic differential equation (1.1a) $\bar{P}^{Y,U}$ -a.s. With respect to $\overset{\circ}{P}_{\pi}$, W and Y are independent standard Wiener processes.

Lemma 2.2. Let $\tilde{W}_t = Y_t - \int_0^t h(X_s) ds$. Then $\tilde{W}, W$ are independent standard Wiener processes under P_{π} and the stochastic differential equations (1.1a), (1.1b) hold P_{π} - a.s.

Proof. Since the pair $\begin{pmatrix} W \\ Y \end{pmatrix}$ is a $\overset{\circ}{P}_{\pi}$ -standard Wiener process, of dimension $N + M$, the Cameron-Martin-Girsanov formula and (2.3) imply that $\begin{pmatrix} W \\ \tilde{W} \end{pmatrix}$ is a P_{π} -standard Wiener process. Since (1.1a)

holds $\overset{\circ}{P}_\pi$ - a.s. and $P_\pi << \overset{\circ}{P}_\pi$, (1.1a) holds P_π - a.s.; while (1.1b) holds by definition of $\tilde{W}$.

We have defined as admissible control a probability measure π belonging to the class $\mathfrak{A}$. Convergence of sequences π_n of admissible controls is taken in the sense of weak convergence of probability measures. $\mathfrak{A}$ is a metric space under (for instance) the Prokhorov metric [2]. Moreover, $\mathfrak{A}$ is a convex set.

Lemma 2.3. $\mathfrak{A}$ is compact under weak sequential convergence.

Proof. Since every measure $\pi \in \mathfrak{A}$ projects onto Wiener measure under $(Y, U) \rightarrow Y$ and the second component U lies in the compact (weak topology) space $L^2([0, T]; \mathscr{U})$, tightness of $\mathfrak{A}$ follows by standard arguments. Hence [2, p. 37] it remains only to show that $\mathfrak{A}$ is closed. Suppose that $\pi_n \rightarrow \pi$, $\pi_n \in \mathfrak{A}$. We must show that Y is a π , $\{\mathscr{G}_t^2\}$ Wiener process. Since π_n projects onto Wiener measure for each n , so does π . We need only verify that $Y_r - Y_t$ is independent of $\mathscr{G}_t^2$ for $t \leq r$. For this it suffices that for any $\mathscr{G}_t^2$ -measurable $\phi \in C_b(\Omega^2)$ and $f \in C_b(\mathbb{R}^M)$

$$\int_{\Omega^2} \phi f(Y_r - Y_t) d\pi = \int_{\Omega^2} \phi d\pi \int_{\Omega^2} f(Y_r - Y_t) d\pi.$$

But this holds for each π_n , and we pass to the limit. This proves Lemma 2.3.

In §6 we shall consider the subclass $\mathfrak{A}^S$ of strict-sense controls.

3. The unnormalized conditional distribution. We wish to introduce an unnormalized conditional distribution of X_t given controls and observations up to t . Let us take a version of the $\mathbb{P}_\pi$ -martingale Z such that Z_t is $\mathcal{G}_t$ -measurable for $0 \leq t \leq T$. Consider any $f \in C_b(\mathbb{R}^N)$. By (2.2) with $\psi = f(X_t)Z_t$,

$$(3.1) \quad E_\pi(f(X_t)Z_t | \mathcal{G}_t^2) = \bar{E}^{Y,U}(f(X_t)Z_t), \quad 0 \leq t \leq T, \quad \pi - \text{a.s.}$$

Let us rewrite (3.1) in such a way that it is defined for all Y, U , not just π -a.s., and depends continuously on (Y, U) . See Lemma 3.2 below. Since $h \in C_b^2(\mathbb{R}^N; \mathbb{R}^M)$, we can integrate

$\int_0^t h(X_s) \cdot dY_s$ by parts:

$$\int_0^t h(X_s) \cdot dY_s = h(X_t) \cdot Y_t - \int_0^t Y_s \cdot L_s h(X_s) ds - \int_0^t Y_s \cdot \nabla h(X_s) \sigma(X_s, Y_s) dW_s,$$

where $Y_s \cdot \nabla h$ is the gradient in x of $Y_s \cdot h$ and

$$(3.2) \quad L_s = \frac{1}{2} \sum_{i,j=1}^N a_{ij}(x, Y_s) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x, Y_s, U_s) \frac{\partial}{\partial x_i},$$

with $a = \sigma \sigma'$. For fixed Y, U , $\frac{\partial}{\partial s} + L_s$ is the backward operator corresponding to (1.1a). Let

$$(3.3) \quad e(s, x) = \frac{1}{2} (a Y_s \cdot \nabla h, Y_s \cdot \nabla h) - Y_s \cdot L_s h - \frac{1}{2} |h|^2,$$

where in (3.3) $(a\xi, \xi) = |\xi \sigma|^2$ denotes the dot product in $\mathbb{R}^N$ of $a\xi$ with ξ , and $\cdot$ denotes the dot product in $\mathbb{R}^M$. From (1.3)

$$Z_t = \overset{Y}{Z}_t \exp Y_t \cdot h(X_t) \exp \int_0^t e(s, X_s) ds$$

where

$$\overset{Y}{Z}_t = \exp \left[- \int_0^t Y_s \cdot \nabla h(X_s) \sigma(X_s, Y_s) dW_s - \frac{1}{2} \int_0^t (a(X_s, Y_s) Y_s \cdot \nabla h(X_s), Y_s \cdot \nabla h(X_s)) ds \right].$$

For fixed (Y, U) let us define another probability measure $\overset{Y}{P}^{Y, U}$ on $(\Omega^1, \mathcal{G}_T^1)$ by

$$(3.4) \quad \frac{d\overset{Y}{P}^{Y, U}}{dP^{Y, U}} = \overset{Y}{Z}_T.$$

This corresponds to a change in drift coefficient in equation (1.1a) from b to $\overset{Y}{b} = b - aY_s \cdot \nabla h$, and changes L_s in (3.2) to the operator

$$(3.5) \quad \overset{Y}{L}_s = L_s - (aY_s \cdot \nabla h, \nabla).$$

From (3.1) we then have

$$\overset{\circ}{E}_\pi(f(X_t) Z_t | \mathcal{G}_t^2) = \overset{Y}{E}^{Y, U}(f(X_t) \exp(Y_t \cdot h(X_t)) \exp \int_0^t e(s, X_s) ds),$$

where the right side is now defined for all $(Y, U) \in \Omega^2$, not merely π -a.s. For fixed (Y, U) the right side is a bounded linear functional on $C_b(\mathbb{R}^N)$. Hence, for every $(Y, U) \in \Omega^2$ and $0 \leq t \leq T$ there exists a measure $\Lambda_t^{Y, U}$ on the Borel σ -algebra $\mathcal{B}(\mathbb{R}^N)$ such that

$$(3.6) \quad \langle f, \Lambda_t^{Y, U} \rangle = \overset{Y}{E}^{Y, U}(f(X_t) \exp(Y_t \cdot h(X_t)) \exp \int_0^t e(s, X_s) ds),$$

for all $f \in C_b(\mathbb{R}^N)$ where for any measure ν with $\nu(\mathbb{R}^N) < \infty$

$$\langle f, \nu \rangle = \int_{\mathbb{R}^N} f(x) d\nu(x).$$

Definition. $\Lambda_t^{Y,U}$ is the unnormalized conditional distribution measure.

The unnormalized conditional distribution measure satisfies, for all $f \in C_b(\mathbb{R}^N)$,

$$(3.7) \quad \langle f, \Lambda_t^{Y,U} \rangle = \overset{\circ}{E}_\pi(f(X_t) Z_t | \mathcal{G}_t^2).$$

As is well-known, the (normalized) conditional distribution of X_t satisfies, for all $f \in C_b(\mathbb{R}^N)$,

$$E_\pi(f(X_t) | \mathcal{G}_t^2) = \frac{\langle f, \Lambda_t^{Y,U} \rangle}{\langle 1, \Lambda_t^{Y,U} \rangle},$$

where E_π denotes expectation with respect to the measure P_π defined by (2.3). For fixed t , let

$$(3.8) \quad v_0^{Y,U}(x) = \overset{Y,U}{E}_x(f(X_t) \exp(Y_t \cdot h(X_t)) \exp \int_0^t e(s, X_s) ds),$$

where $\overset{Y,U}{E}_x$ denotes expectation with respect to the probability measure $\overset{Y,U}{P}_x$ in (3.4) for initial state $X_0 = x$. For initial distribution μ for X_0 ,

$$\overset{Y,U}{E}(g(X)) = \int_{\mathbb{R}^N} \overset{Y,U}{E}_x(g(X)) d\mu(x).$$

Therefore, by (3.6), (3.8)

$$(3.9) \quad \langle f, \Lambda_t^{Y,U} \rangle = \langle v_0^{Y,U}, \mu \rangle$$

for all $(Y,U) \in \Omega^2$ and $f \in C_b(\mathbb{R}^N)$.

In §5 we shall see that (3.9) has a natural interpretation in terms of solutions to forward and backward partial differential equations.

Remark. We shall later wish to consider $\Lambda_t^{Y,U}$ corresponding to any $\mu \geq 0$ with $\mu(\mathbb{R}^N) < \infty$, not merely for probability measures μ on $\mathcal{D}(\mathbb{R}^N)$. Given Y,U , and μ , the right side of (3.9) is a bounded, nonnegative linear functional of f , by (3.8). This gives an alternate way to define the measure $\Lambda_t^{Y,U}$ without the restriction $\mu(\mathbb{R}^N) = 1$, in such a way that (3.9) holds.

Lemma 3.1. (a) $v_0^{Y,U}(x)$ is a continuous function of (x,Y,U)

(b) Given $f \in C_0(\mathbb{R}^N)$ and $a > 0$, there exist

$c, k > 0$ (depending on f, a and bounds for $|b|, |\sigma|, |\nabla h|$), such that $|v_0^{Y,U}(x)| \leq c \exp(-kx)$ for all $x \in \mathbb{R}^N$ and Y,U such that $\|Y\| \leq a$.

Proof. By Lemma A.1 (Appendix) $p_x^{Y,U}$ depends continuously on x, Y, U . Let $x_n \rightarrow x, Y_n \rightarrow Y, U_n \rightarrow U$, and let (for fixed t)

$$\psi_n(X) = f(X_t) \exp(Y_{nt} \cdot h(X_t)) \exp \int_0^t e_n(s, X_s) ds$$

$$\psi(X) = f(X_t) \exp(Y_t \cdot h(X_t)) \exp \int_0^t e(s, X_s) ds,$$

where e_n is defined by (3.3) with Y, U replaced by Y_n, U_n . For any compact $\Gamma \subset C([0, T]; \mathbb{R}^N)$, $\int_0^t e_n ds \rightarrow \int_0^t e ds$ as $n \rightarrow \infty$, uniformly on Γ . This is proved by the same reasoning used in the proof of Lemma A.1. Then $\psi_n \rightarrow \psi$ uniformly on Γ . From this we conclude that $v_0^{Y_n, U_n}(x_n) \rightarrow v_0^{Y, U}(x)$ as $n \rightarrow \infty$, which proves (a).

To prove (b), U_s is bounded by (A_4) . For $\|Y\| \leq a$, $Y_t \cdot h(X_t)$ and $e(s, X_s)$ are bounded. Hence, for some c_1 ,

$$|v_0^{Y, U}(x)| \leq c_1 P_x^{Y, U}(X_t \in \text{spt } f).$$

However, $P_x^{Y, U}$ - a.s.

$$dX_t = \overset{v}{b}(t, X_t)dt + \sigma d\overset{v}{W}_t,$$

$$\overset{v}{b} = b - aY_s \cdot \nabla h, \quad \overset{v}{W}_t = W_t + \int_0^t Y_s \cdot \nabla h(X_s) \sigma(X_s, Y_s) ds$$

and $\overset{v}{W}_t$ is a $P_x^{Y, U}$ -Wiener process. For $\|Y\| \leq a$, $\overset{v}{b}$ is bounded; and σ is bounded by (A_1) . By standard estimates

$$|X_t - x| \leq c_2 \|z\| + c_2' t, \quad z_t = \int_0^t \sigma d\overset{v}{W}_s,$$

$$P_x^{Y, U}(X_t \in \text{spt } f) \leq P_x^{Y, U}(\|z\| > k_1|x| - k_2)$$

for some $k_1 > 0$ and k_2 ($\| \cdot \|$ is as usual the sup norm). Using the fact that σ is bounded and an exponential martingale inequality,

$$p_x^{Y,U}(|\zeta| > k_1|x| - k_2) \leq c_3 \exp(-k|x|)$$

for some $c_3, k > 0$. See for example [14, p. 87]. This proves (b).

For $r > 0$ let

$$\mathcal{M}_r = \{\mu \geq 0 \text{ on } \mathcal{D}(\mathbb{R}^N) : ||\mu|| \leq r\},$$

where $||\mu|| = \mu(\mathbb{R}^N)$. We use vague convergence for sequences of measures: $\nu_n \rightarrow \nu$ means that $\langle f, \nu_n \rangle \rightarrow \langle f, \nu \rangle$ for all $f \in C_0(\mathbb{R}^N)$. The next lemma asserts continuous dependence of $\Lambda_t^{Y,U}$ on μ, Y, U for fixed t , provided we restrict μ to $\mathcal{M}_r$ (see the Remark preceding Lemma 3.1).

Lemma 3.2. $\Lambda_t^{Y,U}$ is a continuous function of μ, Y, U , on $\mathcal{M}_r \times \Omega^2$.

Proof. Let $\mu_n \rightarrow \mu$, $(Y_n, U_n) \rightarrow (Y, U)$. Given $f \in C_0(\mathbb{R}^N)$ let

$$\nu_n(x) = v_0^{Y_n, U_n}(x), \quad \nu(x) = v_0^{Y, U}(x).$$

By (3.9) it suffices to show that

$$(*) \quad \lim_{n \rightarrow \infty} \langle \nu_n, \mu_n \rangle = \langle \nu, \mu \rangle.$$

By Lemma 3.1(a), $x_n \rightarrow x$ implies $\nu_n(x_n) \rightarrow \nu(x)$. Hence $\nu_n \rightarrow \nu$ uniformly on compact subsets of $\mathbb{R}^N$. Since $Y_n \rightarrow Y$, $||Y_n|| \leq a$ for some a . By Lemma 3.1(b), $\nu_n(x) \rightarrow 0$ as $|x| \rightarrow \infty$, uniformly with respect to n . Since $||\mu_n|| \leq r$, this implies (*) and hence Lemma 3.2.

4. The "separated" control problem. As in (1.2) let

$$(4.1) \quad J(\pi) = E_{\pi} \left\{ \int_0^T F(X_t, U_t) dt + G(X_T) \right\},$$

with E_{π} the expectation with respect to the probability measure P_{π} in (2.3). The minimum problem is: given a distribution measure μ for X_0 , find a control $\pi^* \in \mathfrak{A}$ such that $J(\pi^*) \leq J(\pi)$ for all $\pi \in \mathfrak{A}$.

(A₅) F, G are continuous, and $F \geq 0, G \geq 0$. There exists $\pi \in \mathfrak{A}$ such that $J(\pi) < \infty$.

We sometimes impose the stronger condition:

(A'₅) F, G are continuous, $F \geq 0, G \geq 0$, and for some positive $C, m, \ell > m$

$$|F(x, u)| \leq C(1 + |x|^m), \quad |G(x)| \leq C(1 + |x|^m), \quad \int |x|^{\ell} d\mu < \infty.$$

Since X_t satisfies the stochastic differential equation (1.1a) with bounded coefficients b, σ , $J(\pi) < \infty$ for all $\pi \in \mathfrak{A}$ provided that (A'₅) holds. See [10, p. 48].

From (2.3) and the fact that X_t, U_t are $\mathscr{A}_t$ -measurable

$$J(\pi) = E_{\pi} \left\{ \int_0^T Z_t F(X_t, U_t) dt + Z_T G(X_T) \right\}.$$

Upon taking conditional expectations and using (3.7)

$$(4.2) \quad J(\pi) = E_{\pi} \left\{ \int_0^{T_0} E_{\pi} (Z_t F(X_t, U_t) | \mathscr{A}_t^2) dt + E_{\pi} (Z_T G(X_T) | \mathscr{A}_T^2) \right\},$$

$$J(\pi) = \int_{\Omega^2} \left\{ \int_0^T \langle F(\cdot, U_t), \Lambda_t^{Y, U} \rangle dt + \langle G, \Lambda_T^{Y, U} \rangle \right\} d\pi(Y, U).$$

In the separated problem, we regard the unnormalized conditional distribution measure $\Lambda_t = \Lambda_t^{Y,U}$ as the "state", and (4.2) as the criterion to be minimized. Initially, $\Lambda_0 = \mu$. The dynamics of the measure-valued process Λ_t will be described in §5.

In our formulation, the separated control problem is completely equivalent to the problem originally formulated in §2. An optimal control π^* for either problem is also optimal for the other.

In the case $F = 0$ we can now prove the existence of an optimal π^* . In §7 we shall prove another existence theorem, with $F \neq 0$, using methods of partial differential equations. One cannot, in general, reduce $F \neq 0$ to $F = 0$ by adding a new state variable since linearity would then no longer hold in (A_2) , §2.

Theorem 4.1. Let $F = 0$. There exists $\pi^* \in \mathfrak{A}$ such that
 $J(\pi^*) \leq J(\pi)$ for all $\pi \in \mathfrak{A}$.

Proof. By Lemma 2.3, $\mathfrak{A}$ is compact. It suffices to show that

$$J(\pi) = \int_{\Omega^2} \langle G, \Lambda_T^{Y,U} \rangle d\pi(Y,U)$$

is lower semicontinuous on $\mathfrak{A}$. For $\rho \in C_0(\mathbb{R}^N)$, $H \in C_b(\mathbb{R}^1)$, $0 \leq \rho \leq 1$, $H \geq 0$, let

$$\tilde{J}(\pi) = \int_{\Omega^2} H[\langle \rho G, \Lambda_T^{Y,U} \rangle] d\pi(Y,U).$$

By Lemma 3.2, with μ fixed, the integrand is a bounded continuous function on Ω^2 . Hence $\tilde{J}$ is continuous on $\mathfrak{A}$. Let $\rho = \rho_n$, $H = H_n$ be increasing sequences such that $\rho_n(x) \rightarrow 1$, $H_n(z) \rightarrow z$ as $z \rightarrow \infty$. Then $J(\pi)$ is the limit of the corresponding increasing sequence $\tilde{J}_n(\pi)$, which implies that $J(\pi)$ is lower semicontinuous on $\mathfrak{A}$.

5. Dynamics of Λ_t . We begin by imposing rather stringent regularity conditions on the coefficients in (1.1), and by assuming that the initial distribution μ has a density $p_0 \in C_0^\infty(\mathbb{R}^N)$. Then Λ_t turns out to have a density $q(t,x)$ which obeys the Zakai stochastic partial differential equation, as in case of nonlinear filtering. However, it is more convenient to consider instead $p = q \exp(-Y_t \cdot h)$, which obeys the partial differential equation (5.4). Later in the section we drop the regularity assumptions, and obtain the same equation in a weak form.

The regular case. We fix $(Y,U) \in \Omega^2$, and for the present assume that U is continuous on $[0,T]$. We also assume for the present that, for fixed Y, σ, b^0, b^1, h are of class $C_b^\infty(\mathbb{R}^N)$. Given $t > 0$ and $f \in C_0(\mathbb{R}^N)$ consider the following "backward" partial differential equation

$$(5.1) \quad \frac{dv}{ds} + \overset{Y}{L}_s v + e(s)v = 0, \quad 0 \leq s \leq t, \\ v(t) = f \exp(Y_t \cdot h),$$

where we have written $v(s), e(s)$ for $v(s, \cdot), e(s, \cdot)$ and $\overset{Y}{L}_s$ is defined by (3.5). The Cauchy problem (5.1) has the (unique) probabilistic solution

$$(5.2) \quad v(s,x) = \overset{Y,U}{E}_{sx} [f(X_t) \exp(Y_t \cdot h(X_t)) \exp \int_s^t e(\theta, X_\theta) d\theta],$$

where $\overset{Y,U}{P}_{sx}$ is the distribution measure of $(\overset{Y}{W}_t, X_t)$ satisfying

$$dX_t = \dot{b}dt + \sigma d\dot{W}_t, \quad s \leq t \leq T, \quad \text{with } X_s = x \quad (\text{in particular, } \dot{P}_{ox}^{Y,U} = \dot{P}_x^{Y,U}). \quad \text{By (3.8)}$$

$$(5.3) \quad v(0) = v_0^{Y,U}.$$

Under our regularity conditions, $v(s) \in C^\infty(\mathbb{R}^N)$ for $0 \leq s \leq t$.

This follows from the smooth dependence on the initial state x of path-wise solutions $\bar{X}$ to $d\bar{X}_t = \dot{b}dt + \sigma d\bar{W}_t$, $\bar{X}_s = x$, with $\bar{W}_t$ a fixed Wiener process on some $(\bar{\Omega}, \{\bar{\mathcal{F}}_t\}, \bar{P})$. By essentially the same proof as [10, p. 74] dv/ds is continuous and (5.1) holds. Moreover, each partial derivative of any order of v in the variables $x_1, \dots, x_n$ tends to 0 exponentially as $|x| \rightarrow \infty$. For instance by replacing X by $\bar{X}$ and $\dot{E}_{sx}^{Y,U}$ by $\bar{E} = E_{\bar{P}}$ in (5.2), and differentiating with respect to x_i , we get an estimate

$$|v_{x_i}(s, x)| \leq C \max_{s \leq \tau \leq t} \bar{E}(\chi_f |\xi_i(\tau)|)$$

with $\xi_i = \partial \bar{X} / \partial x_i$ and χ_f the indicator function of the event $X \in \text{spt } f$. By [10, p. 61], $\bar{E}|\xi_i(\tau)|^p$ is bounded (independent of τ and x) for each $p > 0$. By taking $p = 2$ and using Cauchy-Schwartz we get

$$|v_{x_i}(s, x)| \leq C_1 [\bar{P}(\bar{X}_t \in \text{spt } f)]^{1/2}.$$

Since $P(\bar{X}_t \in B) = \dot{P}_{sx}^{Y,U}(X_t \in B)$, the proof of Lemma 3.1b then shows that $v_{x_i}(s, x) \rightarrow 0$ exponentially as $|x| \rightarrow \infty$. Similarly, higher order derivatives of v tend to 0 exponentially as $|x| \rightarrow \infty$, using the fact that partial derivatives of $\bar{X}$ of all orders with respect to $x_1, \dots, x_n$ have bounded expectations [10, p. 61].

Let us also consider the following initial value problem for the equation adjoint to (5.1):

$$(5.4) \quad \frac{dp}{dt} = L_t^* p + e(t)p, \quad t \geq 0,$$

$$p(0) = p_0,$$

where $p_0 \in C_0^\infty(\mathbb{R}^N)$. The time reversal $s = T - t$ changes (5.4) into a problem of the same form as (5.1), but with $\tilde{L}_s$ replaced by another degenerate parabolic operator L_s' and $e(s)$ by another $e'(s)$. Therefore, (5.4) has a unique solution with $p(t) \in C^\infty(\mathbb{R}^N)$ and with all partial derivatives of any order in $x_1, \dots, x_N$ tending to 0 exponentially as $|x| \rightarrow \infty$.

Let us write $(\cdot, \cdot)$ for scalar product in $L^2(\mathbb{R}^N)$. Integrations by parts imply $(v(t), p(t)) = \text{constant}$. In particular,

$$(v(t), p(t)) = (v_0^{Y,U}, p_0).$$

If p_0 is the density of μ , then we have from (3.9) since $v(t) = f \exp(Y_t \cdot h)$

$$(5.5) \quad \int_{\mathbb{R}^N} p(t, x) \exp(Y_t \cdot h(x)) f(x) dx = \langle f, \Lambda_t^{Y,U} \rangle.$$

Let

$$(5.6) \quad q(t, x) = p(t, x) \exp(Y_t \cdot h(x)).$$

(of course, $q = q^{Y,U}$ depends on the observation and control trajectories.) Then (5.5) implies that $q(t)$ is the density of the unnormalized conditional distribution measure $\Lambda_t = \Lambda_t^{Y,U}$, under the above regularity assumptions. The partial differential

equation (5.4) determines the dynamics of $p(t)$, hence also of $q(t)$.

Equation (5.4) is a linear partial differential equation in which the processes Y, U enter parametrically. In contrast, the Zakai equation for $q(t)$, see (5.8) below, is a stochastic partial differential equation driven by the Y process. The technique of replacing the Zakai equation by (5.4) is analogous to the technique of Doss [7] and Sussmann [15] for reducing certain finite dimensional Ito-sense stochastic differential equations to ordinary differential equations depending parametrically on a Wiener process. The same idea has been used in nonlinear filtering by Liptser-Shiryaev [12], Clark [5], and others. See Davis [6].

The general case. Let us return to the assumptions $(A_1)-(A_3)$ on σ, b^0, b^1, h . We consider fixed $(Y, U) \in \Omega^2$, and any distribution μ for X_0 . Let us rewrite (5.4) in a weak form. Define the measure $\tilde{\Lambda}_t$ by

$$(5.7) \quad \langle g, \tilde{\Lambda}_t \rangle = \langle g \exp(-Y_t \cdot h), \Lambda_t \rangle, \quad g \in C_b(\mathbb{R}^N).$$

In the regular case, $\tilde{\Lambda}_t$ has density $p(t)$. By multiplying (5.4) by $g \in C_0^\infty(\mathbb{R}^N)$ and integrating by parts, we get

$$(5.4') \quad \frac{d}{dt} \langle g, \tilde{\Lambda}_t \rangle = \langle L_t^v g, \tilde{\Lambda}_t \rangle + \langle e(t)g, \tilde{\Lambda}_t \rangle, \quad g \in C_0^\infty(\mathbb{R}^N).$$

This is the weak form of (5.4). The initial data are now $\tilde{\Lambda}_0 = \mu$.

Theorem 5.1. Equation (5.4') holds, for any $(Y, U) \in \Omega^2$, any $g \in C_b^2(\mathbb{R}^N)$ and initial distribution μ for X_0 .

Proof. For $g \in C_0^\infty(\mathbb{R}^N)$ (5.4') holds in the regular case. For fixed Y , take $\sigma_n, b_n^0, b_n^1, h_n$ of class $C_b^\infty(\mathbb{R}^N)$, uniformly bounded and tending uniformly to σ, b^0, b^1, h as $n \rightarrow \infty$ with partial derivatives $\sigma_{x_i}, b_{x_i}^0, b_{x_i}^1, h_{x_i x_j}$ uniformly bounded. Moreover, let U_n tend to U almost everywhere on $[0, T]$, $U_{nt} \in \mathcal{U}$, and μ_n tend weakly to μ , where U_n is continuous on $[0, T]$ and μ_n has density $p_{n0} \in C_0^\infty(\mathbb{R}^N)$. Let $\tilde{p}_{nx} = \tilde{p}_{nx}^{Y, U_n}$, where the subscript n means that σ, b^ℓ, h are replaced by σ_n, b_n^ℓ, h_n , $\ell = 0, 1$. Lemma A.1 implies that $\tilde{p}_{nx} \rightarrow \tilde{p}_x^{Y, U}$ if $x_n \rightarrow x$ as $n \rightarrow \infty$. Let $f \in C_0(\mathbb{R}^N)$. The same proof as for Lemma 3.1(a) implies that

$$v_{0n}(x) = E_{x_n}(f(X_t) \exp(Y_t \cdot h(X_t)) \exp \int_0^t c_n(s, X_s) ds)$$

tends uniformly on any compact set to $v_0^{Y, U}(x)$. Moreover, by Lemma 3.1(b), $v_{0n}(x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly with respect to n . Let Λ_{nt} be the corresponding unnormalized conditional distribution, with $\Lambda_{n0} = \mu_n$. By (3.9)

$$\langle f, \Lambda_{nt} \rangle = \langle v_{0n}, \mu_n \rangle, \quad \langle f, \Lambda_t \rangle = \langle v_0, \mu \rangle,$$

where $v_0 = v_0^{Y, U}$. Then $\langle v_{0n}, \mu_n \rangle \rightarrow \langle v_0, \mu \rangle$. Since this is true for every $f \in C_0(\mathbb{R}^N)$, $\Lambda_{nt} \rightarrow \Lambda_t$ vaguely as $n \rightarrow \infty$ with $\|\Lambda_{nt}\|$ bounded. From (5.7), $\tilde{\Lambda}_{nt} \rightarrow \tilde{\Lambda}_t$ vaguely with $\|\tilde{\Lambda}_{nt}\|$ bounded. We rewrite (5.4') in the regular case in integrated form:

$$\langle g, \tilde{\Lambda}_{tn} \rangle = \langle g, \mu_n \rangle + \int_0^t \langle \tilde{L}_{sn} g, \tilde{\Lambda}_{sn} \rangle ds + \int_0^t \langle e_n(s) g, \tilde{\Lambda}_{sn} \rangle ds.$$

For each $g \in C_0^\infty(\mathbb{R}^N)$, $\overset{Y}{L}_{sn}g$, $e_n(s)$ are uniformly bounded and tend to $\overset{Y}{L}_sg, e(s)$ uniformly on $\mathbb{R}^N$, for almost all $s \in [0, T]$.

By passing to the limit we get (5.4'), when $g \in C_0^\infty(\mathbb{R}^N)$. Finally, we approximate $g \in C_b^2(\mathbb{R}^N)$ by $g_n \in C_0^\infty(\mathbb{R}^N)$ such that $g_n, \overset{Y}{L}_sg_n$ are uniformly bounded and tend to $g, \overset{Y}{L}_sg$ as $n \rightarrow \infty$, uniformly on compact subsets of $\mathbb{R}^N$. By passing to the limit in (5.4') we get Theorem 5.1.

We do not have a uniqueness result for equation (5.4'), in contrast with the nondegenerate case to be considered in §7. Moreover, in §7 we will be able to use results from the theory of parabolic PDE concerning the continuous dependence of solutions on the coefficients to get a stronger existence theorem for an optimal stochastic control.

The Zakai equation. The unnormalized conditional distribution Λ_t satisfies the following (Zakai) equation, written in a weak form. Recall that Y is a $\overset{\circ}{P}_\pi, \{\mathcal{G}_t^2\}$ -brownian motion for every admissible control π .

Theorem 5.2. For every $f \in C_b^2(\mathbb{R}^N)$

$$(5.8) \quad d\langle f, \Lambda_t \rangle = \langle L_t f, \Lambda_t \rangle dt + \langle hf, \Lambda_t \rangle \cdot dY_t.$$

Proof. Let $\psi(t, x) = f(x) \exp(Y_t \cdot h(x))$. Then

$$\langle f, \Lambda_t \rangle = \langle \psi(t), \tilde{\Lambda}_t \rangle,$$

where as before we set $\psi(t) = \psi(t, \cdot)$. For fixed x , the Ito differential rule implies that

$$d\psi = \frac{1}{2} \psi |h|^2 dt + \psi h \cdot dY.$$

Given $t > 0$, we partition $[0, t]$ into m subintervals $[t_{j-1}, t_j]$ of length $m^{-1}t$. Then

$$\begin{aligned} \langle f, \Lambda_t \rangle - \langle f, \Lambda_0 \rangle &= \sum_{j=1}^m \langle \psi(t_j), \tilde{\Lambda}_{t_j} - \tilde{\Lambda}_{t_{j-1}} \rangle \\ &+ \sum_{j=1}^m \langle \psi(t_j) - \psi(t_{j-1}), \tilde{\Lambda}_{t_{j-1}} \rangle \\ &= \sum_{j=1}^m \int_{t_{j-1}}^{t_j} [\langle L_s \psi(t_j) + e(s) \psi(t_j), \tilde{\Lambda}_s \rangle] ds \\ &+ \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \frac{1}{2} \langle \psi(s) |h|^2, \tilde{\Lambda}_{t_{j-1}} \rangle ds \\ &+ \sum_{j=1}^m \int_{t_{j-1}}^{t_j} \langle \psi(s) h, \tilde{\Lambda}_{t_{j-1}} \rangle \cdot dY_s. \end{aligned}$$

(To justify the exchange of stochastic and Lebesgue integrals see the Note below.) For fixed Y, U ,

$$\begin{aligned} |\langle \psi(s) h, \tilde{\Lambda}_s \rangle - \langle \psi(s) h, \tilde{\Lambda}_{t_{j-1}} \rangle| &= \\ &= \left| \int_{t_{j-1}}^s [\langle L_\theta(\psi(s) h), \tilde{\Lambda}_\theta \rangle + \langle e(\theta) \psi(s) h, \tilde{\Lambda}_\theta \rangle] d\theta \right| \\ &\leq c(s - t_{j-1}) \leq c m^{-1} T, \end{aligned}$$

where c depends on Y, U . Hence as $m \rightarrow \infty$ the last term tends in $\mathbb{P}_\pi$ -probability to $\int_0^t \langle \psi(s)h, \tilde{\Lambda}_s \rangle \cdot dY_s$. By using a similar estimate for the integrand in the middle term, and elementary estimates for the first term, we get

$$\begin{aligned} \langle f, \Lambda_t \rangle - \langle f, \Lambda_0 \rangle &= \int_0^t \langle L_s^\vee \psi(s) + e(s)\psi(s) + \frac{1}{2} \psi(s)|h|^2, \tilde{\Lambda}_s \rangle ds \\ &\quad + \int_0^t \langle \psi(s)h, \tilde{\Lambda}_s \rangle \cdot dY_s. \end{aligned}$$

A straightforward calculation, using (3.2), (3.3), (3.5) gives

$$\exp(Y_s \cdot h) L_s f = L_s^\vee \psi(s) + e(s)\psi(s) + \frac{1}{2} \psi(s)|h|^2.$$

Moreover, from (5.7)

$$\begin{aligned} \langle \exp(Y_s \cdot h) L_s f, \tilde{\Lambda}_s \rangle &= \langle L_s f, \Lambda_s \rangle \\ \langle \psi(s)h, \tilde{\Lambda}_s \rangle &= \langle \exp(Y_s \cdot h) h f, \tilde{\Lambda}_s \rangle = \langle h f, \Lambda_s \rangle. \end{aligned}$$

Therefore

$$\langle f, \Lambda_t \rangle - \langle f, \Lambda_0 \rangle = \int_0^t \langle L_s f, \Lambda_s \rangle ds + \int_0^t \langle h f, \Lambda_s \rangle \cdot dY_s.$$

This is the integrated form of (5.8), and proves Theorem 5.2.

Note. In the proof we have used

$$(5.9) \quad \int_{t_{j-1}}^{t_j} \langle \psi(s)h, \tilde{\Lambda} \rangle \cdot dY_s = \langle \zeta, \tilde{\Lambda} \rangle$$

where for brevity we now write $\tilde{\Lambda}_{t_{j-1}} = \tilde{\Lambda}$ and where (pointwise on $\mathbb{R}^N$)

$$\zeta = \int_{t_{j-1}}^{t_j} \psi(s)h \cdot dY_s = \psi(t_j) - \psi(t_{j-1}) - \frac{1}{2} \int_{t_{j-1}}^{t_j} \psi(s) |h|^2 dx.$$

The functions $\zeta, \psi(s)h$ are bounded and uniformly continuous on $\mathbb{R}^N$. The bounds and moduli of continuity depend on f and $||Y||$, but not on s . For $n = 1, 2, \dots$, partition $B_n = \{|x| \leq n\}$ into Borel sets $A_1^n, \dots, A_{m_n}^n$ of diameter $< n^{-1}$ and choose $x_i^n \in A_i^n$. Then

$$(5.10) \quad \int_{t_{j-1}}^{t_j} \left[\sum_i \psi(s, x_i^n) h(x_i^n) \tilde{\Lambda}(A_i^n) \right] \cdot dY_s = \sum_i \zeta(x_i^n) \tilde{\Lambda}(A_i^n).$$

For each (Y, U) , the right side tends to $\langle \zeta, \tilde{\Lambda} \rangle$ as $n \rightarrow \infty$. The sum in brackets on the left side tends to $\langle \psi(s)h, \tilde{\Lambda} \rangle$ uniformly with respect to s . Hence the stochastic integral converges in probability to the left side of (5.9) as $n \rightarrow \infty$ [10, p. 11, IV]. This proves (5.9).

Theorem 5.3. For $K = 1, 2, \dots, m \geq 0$

$$(5.11) \quad \mathbb{E}_\pi \left(\langle (1+|x|^2)^{\frac{m}{2}}, \Lambda_t \rangle^K \right) \leq C \langle (1+|x|^2)^{\frac{m}{2}}, \mu \rangle^K,$$

where C depends on K, m , and t (but not on $\pi \in \mathfrak{A}$).

Proof. For $0 < \alpha < 1$, let

$$f_\alpha(x) = (1 + |x|^2)^{\frac{m}{2}} \exp[-\alpha(1 + |x|^2)^{1/2}].$$

Any easy calculation shows that $f_\alpha \in C_b^2(\mathbb{R}^N)$ and $|L_s f_\alpha| \leq C_1 f_\alpha$ for suitable C_1 depending on m . The Zakai equation (5.8) and Ito differential rule imply

$$\begin{aligned} d\langle f_\alpha, \Lambda_t \rangle^K &= [K\langle f_\alpha, \Lambda_t \rangle^{K-1} \langle L_t f_\alpha, \Lambda_t \rangle \\ &\quad + K(K-1)\langle f_\alpha, \Lambda_t \rangle^{K-2} |\langle h f_\alpha, \Lambda_t \rangle|^2] dt \\ &\quad + K\langle h f_\alpha, \Lambda_t \rangle^{K-1} \cdot dY_t. \end{aligned}$$

For $a > 0$ let $\tau_a = \inf\{t: ||\Lambda_t|| \geq a\}$. From (3.6) with $f = 1$, $||\Lambda_t|| = \langle 1, \Lambda_t \rangle$ is continuous in t and $\{\mathcal{G}_t^2\}$ -adapted. Hence τ_a is a stopping time. Let χ_a be the indicator function of the set $\{s \leq \tau_a\}$. Then

$$\begin{aligned} \overset{\circ}{E}_\pi \langle f_\alpha, \Lambda_{t \wedge \tau_a} \rangle^K &= \langle f_\alpha, \mu \rangle^K \\ &\quad + K \overset{\circ}{E}_\pi \int_0^t \chi_a \langle f_\alpha, \Lambda_s \rangle^{K-1} \langle L_s f_\alpha, \Lambda_s \rangle ds \\ &\quad + K(K-1) \overset{\circ}{E}_\pi \int_0^t \chi_a \langle f_\alpha, \Lambda_s \rangle^{K-2} |\langle h f_\alpha, \Lambda_s \rangle|^2 ds. \end{aligned}$$

We have since $f_\alpha > 0$ and $|L_s f_\alpha| \leq C_1 f_\alpha$,

$$|\langle L_s f_\alpha, \Lambda_s \rangle| \leq C_1 \langle f_\alpha, \Lambda_s \rangle,$$

$$|\langle h f_\alpha, \Lambda_s \rangle| \leq \|h\| \langle f_\alpha, \Lambda_s \rangle,$$

$$\overset{\circ}{E}_\pi \langle f_\alpha, \Lambda_{t \wedge \tau_a} \rangle^K \leq \langle f_\alpha, \mu \rangle^K + (KC_1 + K(K-1)\|h\|^2) \overset{\circ}{E}_\pi \int_0^t \chi_a \langle f_\alpha, \Lambda_s \rangle^K ds.$$

However, $\chi_a \langle f_\alpha, \Lambda_s \rangle \leq \langle f_\alpha, \Lambda_{s \wedge \tau_a} \rangle$. Gronwall's inequality then implies

$$\overset{\circ}{E}_\pi \langle f_\alpha, \Lambda_{t \wedge \tau_a} \rangle^K \leq C \langle f_\alpha, \mu \rangle^K,$$

$$C = \exp[(KC_1 + K(K-1)\|h\|^2)t].$$

We let $a \rightarrow \infty$ and then $\alpha \rightarrow 0$ to obtain (5.11).

6. Strict-sense admissible controls. We recall the notations of §2.

Definition. We say that $\pi \in \mathfrak{A}$ is a strict-sense admissible control if there exists $\underline{u}: \Omega_2 \rightarrow \Omega_3$ such that $\underline{u}$ is $(\mathcal{F}_T(Y), \mathcal{F}_T(U))$ measurable for $0 \leq t \leq T$, and for every $\mathcal{G}_T^2$ -measurable $\psi \geq 0$

$$\int_{\Omega^2} \psi(Y, U) d\pi = \int_{\Omega_2} \psi(Y, \underline{u}(Y)) dw,$$

where w is Wiener measure on $(\Omega_2, \mathcal{F}_T(Y))$.

For any $\pi \in \mathfrak{A}$

$$\pi(dY, dU) = \pi^Y(dU)w(dY),$$

where π^Y is a regular conditional distribution for π . Strict-sense admissible controls are those such that $\pi^Y = \delta_{\underline{u}(Y)}$, w -a.s., where δ_u = Dirac measure on $(\Omega_3, \mathcal{F}_T(U))$ concentrated at u . By admitting in §2 controls $\pi \in \mathfrak{A}$ which are not strict-sense, we are in effect allowing the choice of U_t to depend on auxiliary randomizations. Let

$$\mathfrak{A}^S = \{\text{strict-sense admissible } \pi\}.$$

Corresponding to $\pi \in \mathfrak{A}^S$ there is a causal functional γ such that $U_t = \gamma(t, Y)$ Lebesgue $\times \pi$ -almost everywhere [16]. Causal is in the sense that $Y_s = Y'_s$ for $0 \leq s \leq t$ implies $\gamma(t, Y) = \gamma(t, Y')$ for $Y, Y' \in C([0, T]; \mathbb{R}^M)$. (We do not use this result in this paper.)

It can be shown that $\mathfrak{A}^S$ is dense in $\mathfrak{A}$. We shall not prove this here. However, we shall show that the infimum of $J(\pi)$ on $\mathfrak{A}^S$

is the same as on $\mathfrak{A}$ (Theorem 6.1). For this purpose we consider approximations by piecewise constant controls.

For $m = 1, 2, \dots$ let us partition $[0, T]$ into m equal subintervals $[t_{j-1}, t_j]$, $t_j = j\Delta$, $\Delta = m^{-1}T$. Let

$$\Omega_{3m} = \{U \in \Omega_3: U_t = \text{constant on } [t_{j-1}, t_j], j = 1, \dots, m\}.$$

On Ω_{3m} weak and strong convergence of a sequence are both equivalent to pointwise convergence on each subinterval $[t_{j-1}, t_j]$. Define

$\phi_m: \Omega_3 \rightarrow \Omega_{3m}$ by $\phi_m(U) = U_m$, where

$$U_{mt} = \begin{cases} 0, & 0 \leq t \leq \Delta \\ \Delta^{-1} \int_{t_{j-1}}^{t_j} U_s ds, & t_j \leq t \leq t_{j+1}. \end{cases}$$

As $m \rightarrow \infty$, $\phi_m(U) \rightarrow U$ in L^2 -norm, for every $U \in \Omega_3$. Let $\Omega_m^2 = \Omega_2 \times \Omega_{3m}$, and

$$\mathfrak{A}_m = \{\pi \in \mathfrak{A}: \pi(\Omega_m^2) = 1\}.$$

If $\pi \in \mathfrak{A}_m$, $t \in [t_j, t_{j+1})$, then U_t is independent of the increments $Y_r - Y_s$ for $t_j \leq s \leq r$ under π .

We call $\psi(Y, U)$ strongly continuous on $\Omega^2 = \Omega_2 \times \Omega_3$ if ψ is continuous when Ω_3 has the L^2 -norm topology rather than the weak topology. We also denote by ϕ_m the mapping from $\Omega^2 \rightarrow \Omega_m^2$, such that $(Y, U) \rightarrow (Y, \phi_m(U))$.

Lemma 6.1. Let ψ be bounded and strongly continuous on Ω^2 . Let $\pi_m = \phi_m \pi$. Then

$$\lim_{m \rightarrow \infty} \int_{\Omega_m^2} \psi(Y, U) d\pi_m = \int_{\Omega^2} \psi(Y, U) d\pi.$$

Proof. By definition

$$\int_{\Omega_m^2} \psi(Y, U) d\pi_m = \int_{\Omega^2} \psi(Y, \phi_m U) d\pi.$$

Since $\phi_m(U) \rightarrow U$ strongly, the lemma follows from the dominated convergence theorem.

In particular, we may take in Lemma 6.1 any ψ bounded and continuous on Ω^2 , where Ω_3 has the weak topology. Thus:

Corollary 6.1. As $m \rightarrow \infty$, $\phi_m \pi \rightarrow \pi$, for every $\pi \in \mathfrak{A}$.

$$\text{Let } \mathfrak{A}_m^S = \mathfrak{A}_m \cap \mathfrak{A}^S.$$

Lemma 6.2. Let ψ be bounded on Ω^2 and continuous on any compact subset of Ω_m^2 . Then

$$\inf_{\mathfrak{A}_m} \int_{\Omega^2} \psi d\pi = \inf_{\mathfrak{A}_m^S} \int_{\Omega^2} \psi d\pi.$$

We leave the proof of this lemma, which depends on standard but tedious arguments, to the Appendix.

In addition to $(A_1)-(A_4)$ in §2 we assume (A_5') in §4. We use the "separated" formula (4.2) for $J(\pi)$.

Theorem 6.1. $\inf_{\mathfrak{A}} J(\pi) = \inf_{\mathfrak{A}^S} J(\pi).$

Proof. Since $\mathfrak{A}^S \subset \mathfrak{A}$, we have $\leq$. Let $\rho \in C_0(\mathbb{R}^N)$ with $0 \leq \rho \leq 1$, $H \in C_b(\mathbb{R}^1)$, and

$$\psi(Y, U) = \int_0^T H[\langle \rho F(\cdot, U_t), \Lambda_t^{Y, U} \rangle] dt + H[\langle \rho G, \Lambda_T^{Y, U} \rangle],$$

$$\tilde{J}(\pi) = \int_{\Omega^2} \psi d\pi.$$

By Lemma 3.2, ψ satisfies the hypotheses of both Lemmas 6.1 and 6.2. Hence, for every $\varepsilon > 0$, and $\pi \in \mathfrak{A}$ there exist m and $\pi_1 \in \mathfrak{A}_m^S$ such that

$$J(\pi_1) < \tilde{J}(\pi) + \varepsilon.$$

Therefore,

$$\inf_{\mathfrak{A}^S} \tilde{J}(\pi) = \inf_{\mathfrak{A}} \tilde{J}(\pi).$$

Now take ρ_n such that $\rho_n(x) = 1$ for $|x| \leq n$, $\rho_n(z) = \min(z, n)$, and the corresponding $\tilde{J}_n(\pi)$. To complete the proof it suffices to show that $\tilde{J}_n(\pi) \rightarrow \tilde{J}(\pi)$ uniformly on $\mathfrak{A}$ as $n \rightarrow \infty$. For brevity, we write $\Lambda_t = \Lambda_t^{Y, U}$. We have from (A'_5), §4,

$$(*) \quad 0 \leq \int_{\Omega} [\langle F(\cdot, U_t), \Lambda_t \rangle - H_n(\langle F(\cdot, U_t), \Lambda_t \rangle)]$$

$$\leq C \left[\int_{\Omega} \langle (1 + \rho_n)(1 + |x|^m), \Lambda_t \rangle d\pi + \int_{\mathbb{R}^n} \langle 1 + |x|^m, \Lambda_t \rangle d\pi \right]$$

where $B_n = \{ \langle (1+|x|^m), \Lambda_t \rangle > C^{-1}n \}$. Let $\ell > m$ as in (A_5^1) and $p = m^{-1}\ell$. From Hölder's inequality

$$\begin{aligned} \langle (1-\rho_n)(1+|x|^m), \Lambda_t \rangle &\leq \left(\int_{|x| \geq n} 1 d\Lambda_t \right)^{1/p'} \left(\int_{\mathbb{R}^n} (1+|x|^m)^p d\Lambda_t \right)^{1/p} \\ &\leq c_1 n^{-\ell/p'} \left(\int_{\mathbb{R}^N} (1+|x|^\ell) d\Lambda_t \right)^{1/p+1/p'}. \end{aligned}$$

Since $p^{-1} + (p')^{-1} = 1$ and $(p')^{-1}\ell = \ell - m$,

$$\mathring{E}_n \langle (1-\rho_n)(1+|x|^m), \Lambda_t \rangle \leq c_1 n^{-(\ell-m)} \mathring{E}_n \langle 1+|x|^\ell, \Lambda_t \rangle.$$

By Theorem 5.3 the expectation on the right side is finite. By Cauchy-Schwartz

$$\int_{B_n} \langle 1+|x|^m, \Lambda_t \rangle d\pi \leq \pi(B_n)^{1/2} [\mathring{E}_n \langle 1+|x|^m, \Lambda_t \rangle^2]^{1/2}.$$

Moreover,

$$\pi(B_n) \leq C n^{-1} \mathring{E}_n \langle 1+|x|^m, \Lambda_t \rangle.$$

By using again Theorem 5.3, the right side of (*) is bounded above by $c_2 n^{-\beta}$, where $\beta = \min(\frac{1}{2}, \ell-m)$. A similar estimate holds if $F(\cdot, \Pi_t)$ is replaced by $G(\cdot)$. We then have, for all $n \in \mathbb{N}$,

$$0 \leq J(\pi) - \tilde{J}_n(\pi) \leq c_2(T+1)n^{-\beta},$$

as required. This proves Theorem 6.1.

Extreme points of $\mathfrak{A}$. Under the hypotheses of the existence theorem 4.1 or of Theorem 7.2 below, $J(\pi)$ is linear and lower semicontinuous on the compact, convex set $\mathfrak{A}$. Hence, $J(\pi)$ has a minimum at some extreme point of $\mathfrak{A}$. Let

$$\mathfrak{A}^e = \{\text{extreme points of } \mathfrak{A}\}.$$

It can be shown that $\mathfrak{A}^s \subset \mathfrak{A}^e$. However, the following counterexample, due essentially to Varadhan, shows that $\mathfrak{A}^s \neq \mathfrak{A}^e$.

An example of Cirelson [3] provides a bounded causal drift coefficient $\alpha(t, \eta)$, such that the stochastic differential equation

$$d\eta_t = \alpha(t, \eta)dt + dY,$$

with Y a Wiener process, $\eta_0 = Y_0 = 0$ has no strong solution. However, the Carmon-Martin-Girsanov formula gives a weak solution, uniquely determining the joint distribution measure π' of (Y, η) on $C([0, T]; \mathbb{R}^2)$. Let $\mathcal{U} = [-1, 1]$, and $U_t = \phi^{-1}(\eta_t)$ where $\phi(u) = (1-u^2)^{-1}u$, $-1 < u < 1$. Let $\phi(Y, U) = (Y, \eta)$, $\eta_t = \phi(U_t)$. Then

$\pi = \phi^{-1}\pi'$ is in $\mathfrak{A}$, but not in $\mathfrak{A}^S$ since no strong solution exists. In fact, $\pi \in \mathfrak{A}^e$. To see this, suppose that $\pi = \lambda\pi_1 + (1-\lambda)\pi_2$, $0 < \lambda < 1$, $\pi_i \in \mathfrak{A}$ for $i = 1, 2$. Let $\pi'_i = \phi\pi_i$. Then, for $i = 1, 2$

$$\pi'_i(\eta_t = \int_0^t \alpha(s, \eta) ds + Y_t, \quad 0 \leq t \leq T) = 1.$$

Hence, $\pi'_i = \pi'$, $\pi_i = \pi$, $i = 1, 2$, which implies $\pi \in \mathfrak{A}^e$.

The following characterization [17] of $\mathfrak{A}^e$ was pointed out to the authors by J-M. Bismut: $\pi \in \mathfrak{A}^e$ if and only if every bounded π , $\{\mathcal{G}_t^2\}$ -martingale M_t has the form

$$M_t = c + \int_0^t f_s dY_s$$

with c a constant and f_t some integrable predictable process.

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7. The nondegenerate case. Let us now assume, instead of (A_1) in §2:

(A'_1) σ is a bounded, continuous $N \times N$ matrix-valued function on $\mathbb{R}^{N+M}$ with bounded inverse. Moreover, $\partial\sigma/\partial x_i \in C_b(\mathbb{R}^{N+M})$ for $i = 1, \dots, N$.

We also assume:

(A_6) The distribution μ of X_0 has a density $p_0 \in L^2(\mathbb{R}^N)$.

Let us show that, for fixed $(Y, U) \in \Omega^2$, the forward equation (5.4) is still correct, if suitably interpreted in the L^2 theory of parabolic partial differential equations.

Consider the Sobolev space

$$H^1 = \{v \in L^2(\mathbb{R}^N) : \frac{\partial v}{\partial x_i} \in L^2(\mathbb{R}^N), \quad i = 1, \dots, N\},$$

and $H^{-1} = (H^1)'$. Let $\hat{L}_t$ be the bounded linear operator from H^1 to H^{-1} , such that for all $p, v \in H^1$

$$\begin{aligned} \langle \hat{L}_t p, v \rangle = & - \frac{1}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij} \frac{\partial p}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^N \int_{\mathbb{R}^N} \hat{a}_i p \frac{\partial v}{\partial x_i} dx \\ & + \frac{1}{2} Y_s \cdot \sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij} \frac{\partial h}{\partial x_j} \frac{\partial p}{\partial x_i} v dx, \end{aligned}$$

where $\langle, \rangle$ denotes here pairing of H^1 and H^{-1} and

$$\hat{a}_i(s, x) = b_i(x, Y_s, U_s) - \frac{1}{2} \sum_{j=1}^N \frac{\partial a_{ij}}{\partial x_j}(x, Y_s) - \frac{1}{2} Y_s \cdot \sum_{j=1}^N a_{ij}(x, Y_s) \frac{\partial h}{\partial x_j}.$$

In the "regular case" integrations by parts show that equation (5.4) is equivalent to

$$(7.1) \quad \frac{dp}{dt} = \hat{L}_t p + \hat{e}p, \quad t \geq 0$$

$$p(0) = p_0, \quad \text{where}$$

$$\hat{e}(s, x) = \frac{1}{2} (a Y_s \cdot \nabla h, Y_s \cdot \nabla h) - \hat{b} \cdot (Y_s \cdot \nabla h) - \frac{1}{2} |h|^2,$$

$$\hat{b}_i(s, x) = b_i(x, Y_s, U_s) - \frac{1}{2} \sum_{j=1}^N \frac{\partial a_{ij}}{\partial x_j}(x, Y_s).$$

The initial value problem has, for fixed Y, U , a unique solution [1]

$$p \in L^2([0, T]; H^1) \cap C([0, T]; L^2(\mathbb{R}^N)).$$

Theorem 7.1. $q(t) = p(t) \exp(Y_t \cdot h)$ is the density of the unnormalized conditional distribution Λ_t .

Proof. From (5.6), this is true in the regular case. Following the proof of Theorem 5.1, we make approximations $\sigma_n, b_n^0, b_n^1, h_n$, such that $\sigma_n, \partial \sigma_n / \partial x_i, b_n^0, b_n^1, h_n$ are uniformly bounded and tend uniformly to $\sigma, \partial \sigma / \partial x_i, \dots, h$ as $n \rightarrow \infty$ with $a_n = \sigma_n^* \sigma_n \geq \alpha I (\alpha > 0)$ for all n . U_n is continuous and tends to U strongly in $L^2([0, T]; \mathcal{Z})$, while μ_n has density $p_{n0} \in C_0^\infty(\mathbb{R}^N)$ tending to p_0 strongly in $L^2(\mathbb{R}^N)$. The density $p_n(t)$ of the

corresponding $\tilde{\Lambda}_{nt}$ satisfies

$$\begin{aligned}\frac{dp_n}{dt} &= \hat{L}_{nt}p_n + \hat{e}_n p_n \\ p_n(0) &= p_{no},\end{aligned}$$

where $\hat{L}_{nt}, \hat{e}_n$ are obtained by replacing $\sigma, \dots, U$ above by $\sigma_n, \dots, U_n$.

Rewrite $A_n p_n = \hat{L}_n p_n + \hat{e}_n p_n$, and $Ap = \hat{L}p + \hat{e}p$. Then:

$$\begin{cases} \frac{d}{dt} (p - p_n) = A_n (p - p_n) + g_n \\ p(0) - p_n(0) = p_0 - p_{no} \end{cases}$$

where $g_n = (A - A_n)p$.

It follows from the above hypotheses that there exists c , independent of n , such that for all $v \in H^1$:

$$\langle -A_n v, v \rangle + c |v|_{L^2(\mathbb{R}^N)}^2 \geq \frac{\alpha}{2} \|v\|_{H^1}^2.$$

Consequently, by standard PDE arguments, see [1], there exist c' and c'' such that

$$\sup_{0 \leq t \leq T} |p(t) - p_n(t)|_{L^2(\mathbb{R}^N)}^2 \leq c' |p_0 - p_{no}|_{L^2(\mathbb{R}^N)}^2 + c'' \|g_n\|_{L^2(0,T;H^{-1})}^2.$$

One easily checks that $g_n \rightarrow 0$ in $L^2(0,T;H^{-1})$.

Finally, $p_n(t) \rightarrow p(t)$ in $L^2(\mathbb{R}^N)$. Then

$$\lim_{n \rightarrow \infty} \langle f, \Lambda_{nt} \rangle = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f \exp(Y_t \cdot h) p_n dx = \int_{\mathbb{R}^N} f \exp(Y_t \cdot h) p dx$$

for any $f \in C_0(\mathbb{R}^N)$. However, the proof of Theorem 5.1 showed that $\langle f, \Lambda_{nt} \rangle \rightarrow \langle f, \Lambda_t \rangle$. Thus, $q = p \exp(Y_t \cdot h)$ is the density of Λ_t , which is Theorem 7.1.

Let us write $p = p^{Y,U}$ to emphasize the dependence on Y, U of the solution to the initial value problem (7.1). From (4.2) and Theorem 7.1 we can rewrite the criterion to be minimized as

$$(7.2) \quad J(\pi) = \int_{\Omega^2} \left[\int_0^T \int_{\mathbb{R}^N} F(x, U_t) p^{Y,U}(t, x) \exp(Y_t \cdot h(x)) dx dt \right. \\ \left. + \int_{\mathbb{R}^N} G(x) p^{Y,U}(T, x) \exp(Y_T \cdot h(x)) dx \right] d\pi(Y, U).$$

Let us suppose:

(A''₅) Condition (A₅) in §4 holds, and $F(x, \cdot)$ is convex on $\mathcal{U}$ for all $x \in \mathbb{R}^N$.

Theorem 7.2. There exists $\pi^* \in \mathfrak{A}$ such that $J(\pi^*) \leq J(\pi)$ for all $\pi \in \mathfrak{A}$.

Let us first prove two lemmas.

Lemma 7.1. For every $\rho \in C_0(\mathbb{R}^N)$, $\rho \geq 0$, and $(Y, U) \in \Omega^2$, the function $\psi(V)$ defined by

$$\psi(V) = \int_0^T \int_{\mathbb{R}^N} \rho(x) F(x, V_t) p^{Y,U}(t, x) \exp[Y_t \cdot h(x)] dx dt$$

is lower-semicontinuous on Ω_3 .

Proof: Since $\psi(V)$ is convex from (A_5^1) , it suffices to show that it is continuous on $L^2(0,T;\mathcal{U})$ endowed with the strong topology.

Let $V^n \rightarrow V$ in $L^2(0,T;\mathcal{U})$ strongly. Let $V^{n'}$ be a subsequence such that $V_t^{n'}$ converges for almost all t . Then $\psi(V^{n'}) \rightarrow \psi(V)$. Consequently, any convergent subsequence of $\{\psi(V^n)\}$ has $\psi(V)$ as its limit. But ψ is uniformly bounded on $L^2(0,T;\mathcal{U})$. It follows that $\psi(V^n) \rightarrow \psi(V)$.

Lemma 7.2. Let $(Y_n, U_n) \rightarrow (Y, U)$ in Ω^2 . Denote $p^n = p^{Y_n, U_n}$, $p = p^{Y, U}$. Then for every D bounded open subset of $\mathbb{R}^N$ with smooth boundary,

(a) $p^n(T) \rightarrow p(T)$ in $L^2(D)$ weakly.

(b) $p^n \rightarrow p$ in $L^2((0,T) \times D)$ strongly.

Proof. Equation (7.1) can be rewritten in the form:

$$(7.3) \quad \begin{cases} \frac{d}{dt} + A_0 p + U_t A_1 p = 0 \\ p(0) = 0 \end{cases}$$

where for all $p, v \in H^1$,

$$\begin{aligned} \langle A_0 p, v \rangle &= \frac{1}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij} \frac{\partial p}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^N \alpha_i p \frac{\partial v}{\partial x_i} dx - \\ &- \sum_{i=1}^N \int_{\mathbb{R}^N} \gamma_i \frac{\partial p}{\partial x_i} v dx + \int_{\mathbb{R}^N} \delta p v dx. \end{aligned}$$

$$\langle A_1 p, v \rangle = - \sum_{i=1}^N \int_{\mathbb{R}^N} \beta_i p \frac{\partial v}{\partial x_i} dx + \int_{\mathbb{R}^N} \theta p v dx$$

with

$$\alpha_i(t, x) = -b_i^0(x, Y_t) + \frac{1}{2} \sum_{j=1}^N \frac{\partial a_{ij}}{\partial x_j}(x, Y_t) + \frac{1}{2} Y_t \cdot \sum_{j=1}^N a_{ij}(x, Y_t) \frac{\partial h(x)}{\partial x_j}$$

$$\beta_i(t, x) = b_i^1(x, Y_t)$$

$$\gamma_i(t, x) = \frac{1}{2} Y_t \cdot \sum_{j=1}^N a_{ij}(x, Y_t) \frac{\partial h}{\partial x_j}(x)$$

$$\begin{aligned} \delta(t, x) &= -\frac{1}{2} (a Y_t \cdot \nabla h, Y_t \cdot \nabla h) + Y_t \cdot \sum_{i=1}^N \frac{\partial h}{\partial x_i}(x) [b_i^0(x, Y_t) - \\ &- \frac{1}{2} \sum_{j=1}^N \frac{\partial a_{ij}}{\partial x_j}(x, Y_t)] - \frac{1}{2} |h|^2(x) \end{aligned}$$

$$\theta(t, x) = Y_t \cdot \sum_{i=1}^N b_i^1(x, Y_t) \frac{\partial h}{\partial x_i}(x)$$

all these coefficients being continuous and bounded functions of (t, Y_t) .

It follows from standard arguments, after multiplication of (7.3) by p and making use of (A'_2) , that there exists a unique constant K (depending only on $\sup_t |Y_t|$) such that:

$$(7.4) \quad |p(t)|_{L^2(\mathbb{R}^N)}^2 + \frac{\alpha}{2} \int_0^t |\nabla p(s)|_{(L^2(\mathbb{R}^N))^N}^2 ds \leq |p_0|^2 + K \int_0^t |p(s)|_{L^2(\mathbb{R}^N)}^2 ds.$$

Let now $(Y_n, U_n) \rightarrow (Y, U)$ in Ω^2 . Then $\sup_t Y_t^n$ is uniformly bounded, and it follows from (7.4) and (7.3) that $(p^n, \frac{dp^n}{dt}, p^n(T))$ remains in a bounded subset of $L^2(0, T; H^1) \times L^2(0, T; H^{-1}) \times L^2(\mathbb{R}^N)$.

We can then extract a subsequence, still denoted p^n , such that:

$(p^n, \frac{dp^n}{dt}, p^n(T)) \rightarrow (\bar{p}, \bar{q}, \bar{r})$ weakly, and it is easy to check that $\bar{q} = \frac{d\bar{p}}{dt}$, $\bar{r} = \bar{p}(T)$. Then $\bar{p} \in L^2(0, T; H^1)$. If we still denote by p^n and $\bar{p}$ the restriction of p^n and $\bar{p}$ to $[0, T] \times D$, we have:

- (i) $p^n \rightarrow \bar{p}$ in $L^2(0, T; H^1(D))$ weakly
- (ii) $\frac{dp^n}{dt} \rightarrow \frac{d\bar{p}}{dt}$ in $L^2(0, T; H^{-1}(D))$ weakly
- (iii) $p^n(T) \rightarrow \bar{p}(T)$ in $L^2(D)$ weakly

where D is open, bounded and with smooth boundary.

Since D is bounded, the injection from $H^1(D)$ into $L^2(D)$ is compact, and it follows from (i) and (ii) by a compactness Lemma [11] that

- (iv) $p^n \rightarrow \bar{p}$ in $L^2([0, T] \times D)$ strongly.

It remains to show that $\bar{p} = p$. Choose any $\phi \in C_0^\infty(\mathbb{R}^1)$, and $v \in C_0^\infty(D)$. Multiply (7.3)ⁿ by ϕv , and integrate by parts:

$$\begin{aligned}
\phi(T)(p^n(T), v) &+ \int_0^T \phi(t) \langle p^n, (A_0^n)^* v \rangle dt + \\
&+ \int_0^T U_t^n \phi(t) \langle p^n, (A_1^n)^* v \rangle dt = \phi(0)(p_0, v) \\
&+ \int_0^T \frac{d\phi}{dt}(p^n, v) dt
\end{aligned}$$

where $(\cdot, \cdot)$ denotes the inner product in $L^2(\mathbb{R}^N)$.

Now, $(A_0^n)^* v \rightarrow A_0^* v$ in $L^2(0, T; H^{-1}(D))$ strongly and $(A_1^n)^* v \rightarrow A_1^* v$ in $L^2([0, T] \times D)$ strongly. It follows from (i), (iii) and (iv) that we can take the limit in the above equality. Since D, v and ϕ are arbitrary, $\bar{p} = p$, the unique solution of (7.3).

Proof of Theorem 7.2. As in Theorem 4.1, it suffices to show that $J(\pi)$ is lower semi-continuous on $\mathfrak{A}$, and this will be true if for all $\rho \in C_0(\mathbb{R}^N)$, $\rho \geq 0$, and $H \in C_b(\mathbb{R}^1)$ monotone, the following functional is lower semi-continuous on $\mathfrak{A}$:

$$\begin{aligned}
\tilde{J}(\pi) = \int_{\Omega^2} H \left(\int_0^T \int_{\mathbb{R}^N} \rho(x) F(x, U_t) \exp(Y_t \cdot h(x)) p^{Y, U}(t, x) dx dt + \right. \\
\left. + \int_{\mathbb{R}^N} \rho(x) G(x) \exp(Y_T \cdot h(x)) p^{Y, U}(T, x) dx \right) d\pi(Y, U).
\end{aligned}$$

A sufficient condition for $\tilde{J}$ to be l.s.c. (lower semi-continuous) on $\mathfrak{A}$ is that the integrand be l.s.c. on Ω^2 .

Since H is continuous and monotone, it suffices to show that the following functional is l.s.c. on Ω^2 :

$$\begin{aligned} \textcircled{II}(Y,U) &= \int_0^T \int_{\mathbb{R}^N} \rho(x) F(x, U_t) \exp(Y_t \cdot h(x)) p^{Y,U}(t,x) dx dt + \\ &+ \int_{\mathbb{R}^N} \rho(x) G(x) \exp(Y_T \cdot h(x)) p^{Y,U}(T,x) dx. \end{aligned}$$

Let now (Y^n, U^n) be a sequence such that $(Y^n, U^n) \rightarrow (Y, U)$ in Ω^2 , and consider (with the notations of Lemma 7.2):

$$\begin{aligned} \textcircled{II}(\bar{Y}^n, U^n) - \textcircled{II}(\bar{Y}, U) &= \int_0^T \int_{\mathbb{R}^N} \rho [F(U^n) - F(U)] \exp(Y_t \cdot h) p dx dt + \\ &+ \int_0^T \int_{\mathbb{R}^N} \rho F(U^n) [\exp(Y_t^n \cdot h) p^n - \exp(Y_t \cdot h) p] dx dt + \\ &+ \int_{\mathbb{R}^N} \rho G [\exp(Y_T^n \cdot h) p^n(T) - \exp(Y_T \cdot h) p(T)] dx. \end{aligned}$$

When $n \rightarrow \infty$, it follows from Lemma 7.1 that $\liminf$ of the first term in the right hand side is ≥ 0 . The two other terms tend to zero from Lemma 7.2. Then $\textcircled{II}(Y^n, U^n) \geq \textcircled{II}(Y, U)$.

APPENDIX

In this Appendix we prove two results used in the paper. The first result concerns the continuous dependence on the coefficients and initial state of solutions to martingale problems associated with stochastic differential equations of the form

$$(A.1) \quad dX_t = (\beta^0(t, X_t) + \beta^1(t, X_t)U_t)dt + \gamma(t, X_t)dW_t, \quad 0 < t < T$$

$$X_0 = x.$$

Let us write for brevity $X'_t = (W_t, X_t)$, and consider the "canonical" sample space $\Omega^1, \{\mathcal{G}_t^1\}$ in the notation of §2. For $f \in C_0^2(\mathbb{R}^{D+N})$ let

$$(A.2) \quad M_f(t) = f(X'_t) - f(X'_0) - \int_0^t L'_s f(X'_s) ds,$$

$$(A.3) \quad L'_t f = \frac{1}{2} \Delta_w f + \frac{1}{2} \sum_{i,j=1}^N \alpha_{ij}(t,x) f_{x_i x_j} + \sum_{i=1}^N \sum_{k=1}^D \gamma_{ik}(t,x) f_{x_i} w_k + (\beta^0(t,x) + \beta^1(t,x)U_t) \cdot \nabla_x f$$

where $\Delta_w f(w,x)$ is the Laplacean with respect to w , ∇_x the gradient in x , and $\alpha = \gamma\gamma'$. The martingale problem is to find a probability measure P_x on $\{\mathcal{G}_t^1\}$ such that $P_x(X'_0 = (0,x)) = 1$ and $M_f(t)$ is a $P_x, \{\mathcal{G}_t^1\}$ martingale for every $f \in C_0^2(\mathbb{R}^{D+N})$. See [14, Ch. 6].

Let us call a function β of class $\mathcal{L}_K$ if β is Borel measurable on $[0,T] \times \mathbb{R}^N$, $|\beta(t,x)| \leq K$, and $\beta(t,\cdot)$ is Lipschitz with constant K . If β^0, β^1, γ are of class $\mathcal{L}_K$ and $U \in L^2([0,T]; \mathcal{U})$, then the Ito conditions hold in (A.1). This implies existence and

uniqueness pathwise of solutions to (A.1), and consequently existence and uniqueness of the solution P_x to the martingale problem.

We write P_{nx} for the solution to the martingale problem if β^ℓ, γ, U are replaced by $\beta_n^\ell, \gamma_n, U_n$, $n = 1, 2, \dots$, $i = 0, 1$.

Lemma A.1. Assume that β_n^ℓ, γ_n are of class $\mathcal{L}_k$ and tend to β^i, γ as $n \rightarrow \infty$, uniformly on compact subsets of $[0, T] \times \mathbb{R}^N$, $i = 0, 1$. Moreover, assume that $U_n \rightarrow U$ weakly in $L^2([0, 1]; \mathcal{U})$, $x_n \rightarrow x$ as $n \rightarrow \infty$. Then $P_{nx_n} \rightarrow P_x$.

Proof. The sequence $P_n = P_{nx_n}$ of probability measures is tight [11, Ch. 6.1]. Hence, any subsequence has a further subsequence tending to a limit P_0 . It suffices to show that P_0 is a solution to the martingale problem. Uniqueness then implies $P_0 = P_x$. Clearly, $P_0(X'_0 = (0, x)) = 1$. Let us write $M_{nf, l'_{nt}}$ in (A.2), (A.3) when β^ℓ, γ, U are replaced by $\beta_n^\ell, \gamma_n, U_n$. Let us show that for fixed $f \in C_0^2(\mathbb{R}^{D+N})$ and compact $\Gamma \subset \Omega^1$,

$$(A.4) \quad \lim_{n \rightarrow \infty} \int_0^t L'_{ns} f(X'_s) ds = \int_0^t L'_s f(X'_s) ds$$

uniformly on $[0, T] \times \Gamma$. Since $M_{nf}(t)$ is a $P_n, \{\mathcal{G}_t^1\}$ martingale and $P_n \rightarrow P_0$ (n in a subsequence) (A.4) will imply that $M_f(t)$ is a $P_0, \{\mathcal{G}_t^1\}$ martingale. Now

$$L'_{ns} f - L'_s f = (\beta_n^1 - \beta^1) U_{ns} \cdot \nabla_x f + \beta^1 (U_{ns} - U_s) \nabla_x f + \theta_n(s, x),$$

where $\theta_n \rightarrow 0$ uniformly on compact subsets of $[0, T] \times \mathbb{R}^N$. Since U_{ns} is bounded (see (A₄), §2) and $\beta_n^1 \rightarrow \beta^1$ uniformly on

compact sets,

$$\lim_{n \rightarrow \infty} \int_0^t [\alpha_n^1(s, X_s) - \beta^1(s, X_s)] U_{ns} \cdot \nabla_x f(X_s^1) ds = 0$$

$$\lim_{n \rightarrow \infty} \int_0^t \alpha_n(s, X_s) ds = 0$$

uniformly for $0 \leq t \leq T$, $X_s^1 \in \Gamma$. To obtain (A.4) it remains to show that

$$(A.5) \quad \lim_{n \rightarrow \infty} \int_0^t \alpha^1(s, X_s) (U_{ns} - U_s) \cdot \nabla f(X_s^1) ds = 0$$

uniformly on $[0, T] \times \Gamma$. Now $\beta^1(s, \cdot)$ and ∇f are bounded and Lipschitz, with some constant K . Moreover, functions $X_s^1 \in \Gamma$ are uniformly bounded and equicontinuous. Therefore, given $\varepsilon > 0$ the integral in (A.1) can be approximated to within ε , uniformly with respect to $X_s^1 \in \Gamma$ and $n = 1, 2, \dots$, by a finite sum

$$(A.6) \quad \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \alpha^1(s, x_i^1) (U_{ns} - U_s) \cdot \nabla f(x_i^1) ds$$

where $0 = t_0 < t_1 < \dots < t_m = T$ and $x_1^1, \dots, x_m^1$ are suitably chosen. (The t_i, x_i^1 depend on ε and Γ .) Since $U_n \rightarrow U$ weakly, (A.6) tends to 0 as $n \rightarrow \infty$. This proves Lemma A.1.

Note. In this paper we appeal to Lemma A.1 three times. In Lemma 2.1 we take

$$\gamma_n(t, x) = \sigma(x, Y_{nt}), \quad \gamma(t, x) = \sigma(t, Y_t)$$

$$\beta_n^\ell(t, x) = b^\ell(x, Y_{nt}), \quad \beta^\ell(t, x) = b^\ell(x, Y_t)$$

where $\|Y_n - Y\| \rightarrow 0$ as $n \rightarrow \infty$ (sup norm). In that case, $P_X = P_X^{Y, U}$. In the proof of Lemma 3.1 we use instead of b^ℓ the modified drift coefficient $b^{\vee \ell}$ corresponding to the change of probability measures (3.4). Then $P_X = P_X^{Y, U}$ and L_S is replaced by $L_S^\vee$ in (3.5). Finally, we use Lemma A.1 in the proof of Theorem 5.1 as indicated there.

In §6 we postponed the proof of Lemma 6.2.

Proof of Lemma 6.2. Since $\mathfrak{A}_m^S \subset \mathfrak{A}_m$ it suffices to show that, for every $\pi \in \mathfrak{A}_m$ and $\varepsilon > 0$, there exists $\pi_1 \in \mathfrak{A}_m^S$ such that

$$(\#) \quad \int_{\Omega^2} \psi d\pi_1 < \int_{\Omega^2} \psi d\pi + \varepsilon.$$

Let us fix Δ and consider different $T = m\Delta$, $m = 1, 2, \dots$. We prove (#) by induction on m . For $m = 1$, each admissible control π , with U_t constant on $[0, \Delta)$ π -almost surely, corresponds to a product measure: $\pi = w \times \alpha$ where w is Wiener measure on $\Omega_{21} = C([0, \Delta]; \mathbb{R}^M)$ and α is a probability measure on $\mathcal{U}$. Let u^* minimize $\int_{\Omega_{21}} \psi(Y, u) dw(Y)$ on $\mathcal{U}$. The control π_1 such that $U_t = u^*$, $0 \leq t < \Delta$ with probability 1 satisfies

$$\int_{\Omega_1^2} \psi d\pi_1 = \int_{\Omega_{21}^2} \psi(Y, u^*) dw(Y),$$

where $\Omega_1^2 = \Omega_{21} \times \Omega_3$. We then have

$$\int_{\Omega_1^2} \psi d\pi = \int_{\mathcal{U}} \int_{\Omega_{21}} \psi(Y, u) dw(Y) d\alpha(u) \geq \int_{\Omega_1^2} \psi d\pi_1$$

as required.

Now suppose that (#) has been proved when m is replaced by $m - 1$ (i.e., T by $T - \Delta$). Let $(\bar{Y}, \bar{U})$ denote the restriction of (Y, U) to $[0, T - \Delta]$, and $\bar{\Omega}^2$ the space of such $(\bar{Y}, \bar{U})$. Let $\bar{\pi}$ be the measure on $\bar{\Omega}^2$ induced from π by restriction. We write similarly $\bar{\Omega}_m^2, \bar{\Omega}_2, \bar{w}$ when T is replaced by $T - \Delta$. Let $U_t = U_m$ on $[T - \Delta, T]$, where $U_m \in \mathcal{U}$. Let $Y_{mt} = Y_t - Y_{t-T+\Delta}$ on $[T - \Delta, T]$, and w_m Wiener measure on $C_m = C([T - \Delta, T]; \mathbb{R}^M)$. We can identify Y with $(\bar{Y}, Y_m)$ and piecewise constant U with $(\bar{U}, U_m)$. Let

$$\begin{aligned} \gamma(\bar{Y}, \bar{U}, u) &= \int_{C_m} \psi(\bar{Y}, Y_m, \bar{U}, u) dw_m(Y_m), \\ \zeta(\bar{Y}, \bar{U}) &= \min_{\mathcal{U}} \gamma(\bar{Y}, \bar{U}, u). \end{aligned}$$

Since ψ is bounded and continuous on any compact subset of Ω_m^2 , ζ is bounded and continuous on any compact subset of $\bar{\Omega}_m^2$.

Consider any $\pi \in \mathfrak{A}_m$, with corresponding $\bar{\pi}$ determined by restriction. By induction there exists $\bar{\pi}_1$ strict-sense admissible, such that U_t is constant on $[t_{j-1}, t_j]$, $j = 1, \dots, m-1$, $\bar{\pi}_1$ -almost surely, and

$$\int_{\bar{\Omega}^2} \zeta d\bar{\pi}_1 < \int_{\bar{\Omega}^2} \zeta d\bar{\pi} + \frac{\varepsilon}{3}.$$

we define $\phi: \bar{\Omega}^2 \rightarrow \mathcal{U}$ as follows. Let $K \subset \bar{\Omega}^2$ be compact with

$$\bar{\pi}(\bar{\Omega}^2 - K) + \bar{\pi}_1(\bar{\Omega}^2 - K) < \varepsilon(3||\psi||)^{-1},$$

where $||\psi|| = \sup$ norm. Choose a partition $K = K_1 \cup \dots \cup K_n$ with each $K_i \in \mathcal{C}_{T-\Delta}^2$ and $(\bar{Y}_i, \bar{U}_i) \in K_i$ such that

$$\zeta(\bar{Y}_i, \bar{U}_i) < \zeta(\bar{Y}, \bar{U}) + \frac{\varepsilon}{6}, \quad \gamma(\bar{Y}, \bar{U}, u) \leq \gamma(\bar{Y}_i, \bar{U}_i, u) + \frac{\varepsilon}{6}$$

for all $(\bar{Y}, \bar{U}) \in K_i$, $u \in \mathcal{U}$. Let $u_i \in \mathcal{U}$ minimize $\gamma(\bar{Y}_i, \bar{U}_i, u)$ on $\mathcal{U}$, $i = 1, \dots, n$. Let $u_0 \in \mathcal{U}$ be arbitrary; and take

$$\phi(\bar{Y}, \bar{U}) = \begin{cases} u_i, & (\bar{Y}, \bar{U}) \in K_i \\ u_0, & (\bar{Y}, \bar{U}) \in \bar{\Omega}^2 - K. \end{cases}$$

The control $\pi_1 \in \mathfrak{A}_m^S$ is defined by taking $U_m = \phi(\bar{Y}, \bar{U})$ π_1 -almost surely, and $\bar{\pi}_1$ the restriction of π_1 . Then

$$\begin{aligned} \int_{\bar{\Omega}^2} \psi d\pi_1 &= \int_{\bar{\Omega}^2} \gamma(\bar{Y}, \bar{U}, \phi(\bar{Y}, \bar{U})) d\pi_1 \\ &\leq \sum_{i=1}^n \int_{K_i} \zeta(\bar{Y}_i, \bar{U}_i) d\bar{\pi}_1 + \int_{\bar{\Omega}^2 - K} \gamma(\bar{Y}, \bar{U}, u_0) d\bar{\pi}_1 + \frac{\varepsilon}{6} \\ &\leq \int_{\bar{\Omega}^2} \zeta d\bar{\pi}_1 + \frac{2\varepsilon}{3} \leq \int_{\bar{\Omega}^2} \zeta d\bar{\pi} + \varepsilon. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\Omega^2} \psi d\pi &= \int_{\overline{\Omega}^2} \int_{\mathcal{U}} \gamma(\overline{Y}, \overline{U}, u) d\pi^{\overline{Y}, \overline{U}}(u) d\overline{\pi}(\overline{Y}, \overline{U}) \\
&\geq \int_{\overline{\Omega}^2} \zeta(\overline{Y}, \overline{U}) d\overline{\pi}(\overline{Y}, \overline{U}).
\end{aligned}$$

This gives (#), and hence Lemma 6.2.

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