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SOME STRONG AND WEAK LAWS OF LARGE NUMBERS IN $D(0,1)$. (U)

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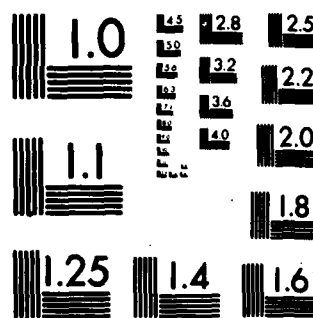
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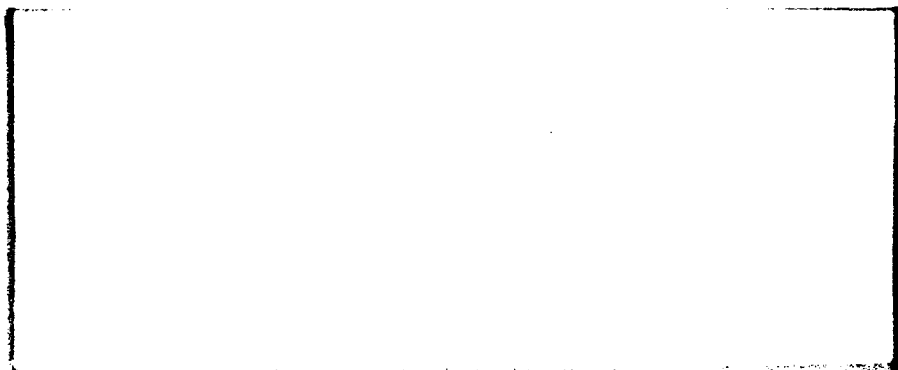
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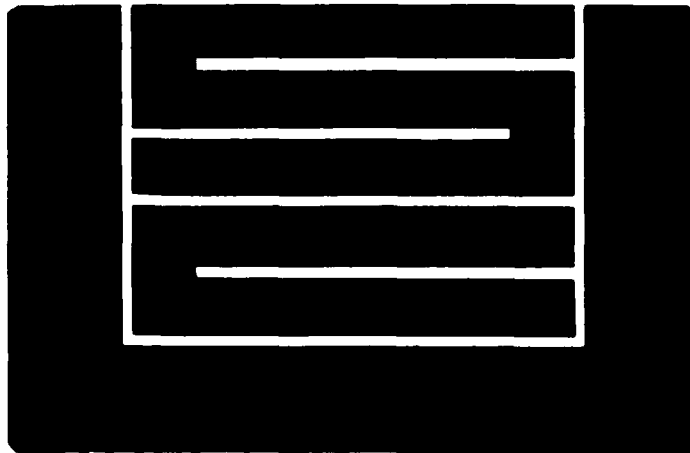
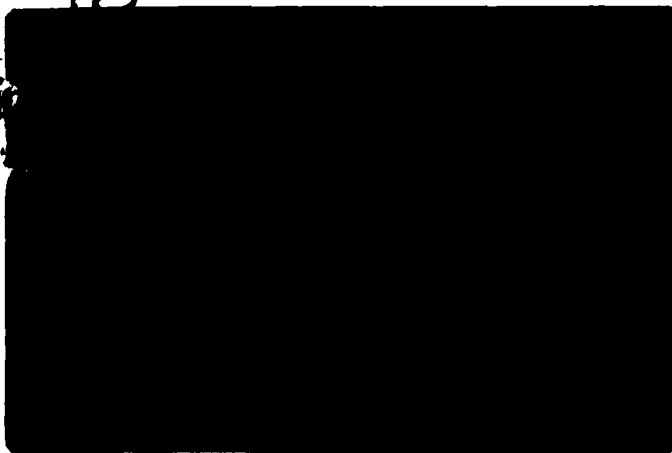
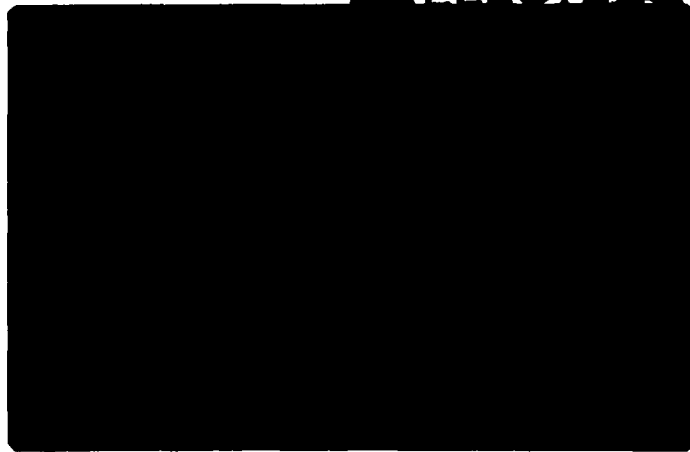
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**SOME STRONG AND WEAK LAWS
OF LARGE NUMBERS IN $D[0,1]$.**

by

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and

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Statistics Technical Report No. 57
60B12-3

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is preserved when random functions are centered, and this result is applied to improve some known strong laws for weighted sums in $D(0,1)$. A weak law of large numbers is proved using a new pointwise condition on the first moments and some weak laws for weighted sums are improved upon by weakening the hypotheses. A study is made of relationships among several conditions on ~~(X)~~ which appear as hypotheses in laws of large numbers.

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ABSTRACT

Strong laws of large numbers for a sequence (X_n) of random functions in $D[0,1]$ are derived using new pointwise conditions on the first absolute moments, which improve on known results. In particular, convex tightness is not implied by the hypotheses of the theorems. It is shown that convex tightness is preserved when random functions are centered, and this result is applied to improve some known strong laws for weighted sums in $D[0,1]$. A weak law of large numbers is proved using a new pointwise condition on the first moments and some weak laws for weighted sums are improved upon by weakening the hypotheses. A study is made of relationships among several conditions on (X_n) which appear as hypotheses in laws of large numbers.

§1. Introduction.

1.1 Laws of large numbers for sequences (X_n) of random functions in $D[0,1]$ have been obtained using a number of conditions on (X_n) , such as convex tightness and conditions on the moments $E \|X_n\|^r$, and others ([4],[13],[14],[12]). For random elements in a Banach space E , convexity conditions on E can be assumed, but such conditions are not available in $D[0,1]$ which is not locally convex with the Skorokhod topology.

For real-valued random variables the classical formulations of the strong and weak laws of large numbers are available and satisfactory formulations of necessary as well as sufficient conditions have been obtained ([5],[8],[7]). Thus, the major thrust centers around finding conditions which will convert pointwise convergence (in some mode) into convergence in the Skorokhod metric or, which is stronger, uniform convergence.

Necessary and sufficient conditions for pointwise convergence to imply Skorokhod convergence are known in terms of the moduli $w'_x(\delta)$ and $w''_x(\delta)$ which are used in $D[0,1]$ (the notation is that of [1]). See [11], 2.6.1, p. 277, for additional details. In the case of random functions in $D[0,1]$, however, more useful conditions implying Skorokhod convergence are desirable, preferably in terms of the individual summands. Various integral conditions have been used ([13],[14],[12]), some of which are listed in §3 and investigated in §6.

1.2 For a sequence (X_n) of random variables, tightness is neither

necessary nor sufficient for the law of large numbers, strong or weak, to hold. However, the concept of tightness, together with conditions on the moments of the random elements, has proved natural and useful in providing sufficient conditions for laws of large numbers, strong and weak, in Banach and Fréchet spaces ([16],[12]).

In $D[0,1]$, tightness has likewise played a central role, but hitherto this concept has taken the form of convex tightness, in which the compact sets involved are also required to be convex ([13],[4]). However, it was shown in [3] that any compact convex set in the Skorokhod topology is also compact and convex in the uniform topology on $D[0,1]$. This fact limits the scope of applicability of convex tightness as a condition on a sequence (X_n) of random functions in $D[0,1]$, since if (X_n) is convex tight, then all random functions X_n must necessarily take their values, with probability one, in a subspace of the Banach space $D[0,1]$, which is separable with respect to the uniform topology ([3],§1).

In the classical strong law of R. Ranga Rao for identically distributed summands ([9], or [6], p. 254 ff.) convex tightness is not required. In §4.1, strong laws of large numbers for non-identically-distributed random functions which do not require convex tightness are obtained, using a condition generalizing the basic lemma used by Ranga Rao in the proof of his result. In §4.2 previous results of [4] and [13] are strengthened. In §5.1 a new weak law is presented, and in §5.2 are found some improvements on weak laws in [13]. Finally, in §6 comparisons of the various integral conditions on the random functions is presented.

§2. Preliminaries.

2.1 For the definition of the space $D[0,1]$, as well as for the definition and properties of the Skorokhod topology, we refer to Chapter 3 of [1]. The Skorokhod metric is denoted throughout by d , and $\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$, for $x \in D[0,1]$.

2.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and make $D[0,1]$ into a measurable space by providing it with the σ -algebra generated by the Borel sets of the Skorokhod topology. A measurable map $X: \Omega \rightarrow D[0,1]$ is called a random element or random function. In particular, X is a random function in $D[0,1]$ if and only if $X(t)$ is a random variable for each $t \in [0,1]$. The expectation EX of a random function X can be defined pointwise by $(EX)(t) = E\{X(t)\}$ provided that it turns out that $EX \in D[0,1]$. A sufficient condition for this is that $E\|X\| < \infty$.

2.3 In general, when speaking of a partition P of $[0,1]$, a finite set of points $(t_0, t_1, \dots, t_m)$ is meant with $0 = t_0 < t_1 < \dots < t_{m-1} < t_m = 1$; or equivalently, intervals $I_1, I_2, \dots, I_m, I_i = [t_{i-1}, t_i]$, $i=1, \dots, m-1$, and $I_m = [t_{m-1}, t_m]$.

The norm $\|P\|$ of a partition is the length of the longest sub-interval: $\|P\| = \max_{i=0, \dots, m} \{t_{i+1} - t_i\}$. Given $\delta > 0$, a partition P is said to be δ -worse if $\min_{i=1, \dots, m} \{t_i - t_{i-1}\} > \delta$.

2.4 The indicator function I_A of a set $A \subset \Omega$, $A \in \mathcal{F}$, is defined by

$$I_A(\omega) = 0 \quad \text{if } \omega \notin A$$

and

$$I_A(\omega) = 1 \quad \text{if } \omega \in A.$$

Also, "a.s." stands for "almost surely" or "with probability one".

2.5 For a given partition $0 = t_0 < t_1 < \dots < t_m = 1$, define the operator T_m on $D[0,1]$ by

$$T_m x = \sum_{i=0}^{m-1} x(t_i) I_{[t_i, t_{i+1})} + x(1) I_{\{1\}}.$$

If a partition is not specified, then define the operator T_m by

$$T_m x = \sum_{i=0}^{2^m-1} x\left(\frac{i}{2^m}\right) I_{\left[\frac{i}{2^m}, \frac{i+1}{2^m}\right)} + x(1) I_{\{1\}}.$$

The operator T_m is a projection of $D[0,1]$ onto a finite-dimensional subspace which is additive but not continuous.

2.6 By a Toeplitz matrix we mean an array (a_{nk}) of real numbers satisfying (i) $\lim_{n \rightarrow \infty} a_{nk} = 0$ for each $k=1,2,\dots$, and (ii)

$$\sum_{k=1}^{\infty} |a_{nk}| \leq 1, \text{ for each } n = 1, 2, \dots.$$

2.7 Two lemmas from [4] are listed.

LEMMA 2.1: If K is a compact subset of $D[0,1]$, then

$$\lim_{m \rightarrow \infty} \sup_{x \in K} d(x, T_m x) = 0.$$

LEMMA 2.2: If $x, y, u, v \in D[0,1]$, then

$$d(x + u, y + v) \leq d(x, y) + \|u\| + \|v\|.$$

§3. Conditions on random functions in $D[0,1]$.

Let (X_n) be a sequence of random functions in $D[0,1]$. For quick reference various conditions on (X_n) are collected in this section.

3.1 (X_n) is said to be tight if, to every $\epsilon > 0$, there is $K \subset D[0,1]$, compact, such that $P[X_n \notin K] < \epsilon$, for every n .

3.2 (X_n) is said to be convex tight if, to every $\epsilon > 0$ there is $K \subset D[0,1]$, compact and convex, such that $P[X_n \notin K] < \epsilon$, for every n .

3.3 (X_n) is said to satisfy condition (T) if, to every $\epsilon > 0$, there is $K \subset D[0,1]$, compact, such that $E \|X_n I_{[X_n \notin K]}\| < \epsilon$, for every n .

3.4 (X_n) is said to satisfy condition (CT) if, to every $\epsilon > 0$, there is $K \subset D[0,1]$, compact and convex, such that $E \|X_n I_{[X_n \notin K]}\| < \epsilon$, for every n .

3.5 (X_n) is said to satisfy condition (MT) if, to every $\epsilon > 0$, there is a partition P of $[0,1]$ such that

$$E \left[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| \right] < \epsilon,$$

for every n .

3.6 (X_n) is said to satisfy condition (mT) if, to every $\epsilon > 0$, there is a partition P of $[0,1]$ such that

$$\max_{i=1, \dots, m} E \left[\sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| \right] < \epsilon,$$

for every n .

3.7 (X_n) is said to satisfy condition (RR) if, to every $\epsilon > 0$, there is a partition P of $[0, 1]$ such that

$$\max_{i=1, \dots, m} \sup_{t \in I_i} E |X_n(t) - X_n(t_{i-1})| < \epsilon,$$

for every n .

3.8 (X_n) is said to be uniformly integrable (UI) if, to every $\epsilon > 0$, there is $\delta > 0$ such that $E \|X_n\| I_{[X_n \notin B(\delta)]} < \epsilon$ for every n , where $B(\delta) = \{x: \|x\| \leq \delta\}$.

3.9 (X_n) is said to be stochastically bounded (SB) if, to every $\epsilon > 0$, there is $\delta > 0$ such that $P[\|X_n\| > \delta] < \epsilon$, for every n .

3.10 (X_n) is said to satisfy conditions (M)_r, $r \geq 0$, if (X_n) has uniformly bounded r^{th} moments, i.e., if there is a constant C such that $E \|X_n\|^r \leq C$, for every n . We abbreviate $(M)_r$, $r > 1$, to $(M)_{r>1}$.

3.11 For condition $(M)_1$, i.e., for uniformly bounded first moments, we write simply (M) .

3.12 For each condition listed on (X_n) , the corresponding condition for a single random function X in $D[0, 1]$ is obtained by taking (X_n) to be identically distributed.

§4. Strong Laws of Large Numbers.

4.1 The generalization of the classical strong law to $D[0,1]$ is the following theorem which was proved by R. Ranga Rao ([9], or [6], Chapter 7).

THEOREM 4.1: Let (X_n) be a sequence of independent, identically distributed random functions in $D[0,1]$ satisfying $E \|X_1\| < \infty$. Then $\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n X_k - EX_1 \right\| = 0$, with probability one.

THEOREM 4.2: Let (X_n) be a sequence of independent random functions in $D[0,1]$ satisfying condition (mT) and $\sum_{k=1}^{\infty} k^{-r} E \|X_k\|^r < \infty$ for some r , $1 \leq r \leq 2$. Then

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0, \text{ with probability one.}$$

PROOF: An easy calculation shows that

$$\begin{aligned} & \max_i E \left[\sup_{t \in I_i} |X_n(t) - EX_n(t) - (X_n(t_{i-1}) - EX_n(t_{i-1}))| \right] \\ & \leq 2 \max_i E \left[\sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| \right] \end{aligned}$$

and hence the sequence (X_n) can, without loss of generality, be assumed to satisfy $EX_n = 0$, for each n .

Let $\epsilon > 0$ be given and choose a partition P of $[0,1]$ such that

$$\sup_n \max_{i=1, \dots, m} E \left[\sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| \right] \leq \epsilon. \quad (1)$$

Write
$$\left\| \frac{1}{n} \sum_{k=1}^n X_k \right\| \leq \left\| \frac{1}{n} \sum_{k=1}^n (X_k - T_m X_k) \right\| + \left\| \frac{1}{n} \sum_{k=1}^n T_m X_k \right\|,$$

where T_m is defined in §2.5. Now

$$\lim_n \left\| \frac{1}{n} \sum_{k=1}^n T_m X_k \right\| = \lim_n \max_{i=0, \dots, m-1} \left| \frac{1}{n} \sum_{k=1}^n X(t_i) \right| = 0,$$

a.s., by Chung's strong law of large numbers.

$$\begin{aligned} \text{Next, } & \left\| \frac{1}{n} \sum_{k=1}^n (X_k - T_m X_k) \right\| \\ &= \max_{i=1, \dots, m} \sup_{t \in I_i} \left| \frac{1}{n} \sum_{k=1}^n (X_k(t) - X_k(t_{i-1})) \right| \\ &\leq \max_{i=1, \dots, m} \frac{1}{n} \sum_{k=1}^n \sup_{t \in I_i} |X_k(t) - X_k(t_{i-1})| \\ &= \max_{i=1, \dots, m} \frac{1}{n} \sum_{k=1}^n Y_k^i, \quad \text{where} \end{aligned}$$

$$Y_k^i = \sup_{t \in I_i} |X_k(t) - X_k(t_{i-1})|, \quad i = 1, \dots, m; \quad k = 1, 2, \dots$$

Now for each i , $(Y_k^i - E(Y_k^i))$ is a sequence of independent random variables with zero means and $E|Y_k^i - E(Y_k^i)|^r \leq 4^r E|X_k|^r$.

Thus, the strong law of large numbers yields, for each $i = 1, \dots, m$,

$$\lim_n \frac{1}{n} \sum_{k=1}^n (Y_k^i - EY_k^i) = 0, \quad \text{a.s.}$$

$$\text{Hence, } \overline{\lim}_n \left\| \frac{1}{n} \sum_{k=1}^n (X_k - T_m X_k) \right\|$$

$$\leq \lim_n \left[\max_{i=1, \dots, m} \frac{1}{n} \sum_{k=1}^n (Y_k^i - EY_k^i) + \max_{i=1, \dots, m} \frac{1}{n} \sum_{k=1}^n EY_k^i \right]$$

$$\leq 0 + \overline{\lim}_n \frac{1}{n} \sum_{k=1}^n E \left[\sup_{t \in I_i} |X_k(t) - X_k(t_{i-1})| \right]$$

$\leq \epsilon$ by (1),

a.s.

The proof is completed by letting $\epsilon \rightarrow 0$ and excluding a countable union of null sets. Q.E.D.

REMARK: The conditions of Theorem 4.2, in the case $1 < r \leq 2$, do not imply that (X_n) is tight. If $r = 1$, then tightness is implied.

THEOREM 4.3. Let (X_n) be a sequence of independent random functions in $D[0,1]$ satisfying condition (RR) and the following condition:

To every $\epsilon > 0$, there is a compact set $K \subset D[0,1]$ such that

$$1^\circ. \quad E \|X_n I_{[X_n \notin K]}\| < \epsilon, \quad \text{for every } n,$$

$$2^\circ. \quad \sum_{n=1}^{\infty} n^{-r} E \left(\left| \|X_n I_{[X_n \notin K]}\| - E \|X_n I_{[X_n \notin K]}\| \right|^r \right) < \infty,$$

for some $1 \leq r \leq 2$.

Then $\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0$, with probability one.

PROOF: Note that 1° implies that $E \|X_n\| < \infty$, which in turn implies that EX_n exists for each n . Let $\epsilon > 0$ be given and let K be a compact set such that both 1° and 2° hold. Put $X_k' = X_k I_{[X_k \in K]}$ and $X_k'' = X_k I_{[X_k \notin K]}$.

Since K is compact there is $\delta > 0$ such that, for any $x \in K$,

$$|x(t) - x(s)| \leq |x(u-0) - x(s)| + \epsilon, \quad (1)$$

whenever $0 \leq s \leq t < u < s + \delta$. (See [6], proof of Theorem 8.1, p. 257).

By (RR) choose a partition P such that

$$\sup_n \max_{i=1, \dots, m} \sup_{t \in I_i} E |X_n(t) - X_n(t_{i-1})| < \epsilon, \quad (2)$$

and, by adding points if necessary, arrange for

$$\|P\| < \delta. \quad (3)$$

Write
$$\left| \frac{1}{n} \sum_{k=1}^n (X_k(t) - EX_k(t)) \right| \leq$$

$$\left| \frac{1}{n} \sum_{k=1}^n (X_k'(t) - EX_k'(t)) \right| + \frac{1}{n} \sum_{k=1}^n \|X_k''\| + \frac{1}{n} \sum_{k=1}^n E \|X_k''\|. \quad (4)$$

Using (1) and (3) we have, for $t_{i-1} \leq t < t_i$, $i = 1, \dots, m$,

$$\begin{aligned} \left| \frac{1}{n} \sum_{k=1}^n (X_k'(t) - EX_k'(t)) \right| &\leq \left| \frac{1}{n} \sum_{k=1}^n (X_k'(t_{i-1}) - EX_k'(t_{i-1})) \right| \\ &+ \sup_{t \in I_i} \left| \frac{1}{n} \sum_{k=1}^n (X_k'(t) - X_k'(t_{i-1})) \right| \\ &+ \sup_{t \in I_i} \left| \frac{1}{n} \sum_{k=1}^n (EX_k'(t) - EX_k'(t_{i-1})) \right| \\ &\leq \left| \frac{1}{n} \sum_{k=1}^n (X_k'(t_{i-1}) - EX_k'(t_{i-1})) \right| \\ &+ \frac{1}{n} \sum_{k=1}^n |X_k'(t_{i-1}-0) - X_k'(t_{i-1})| + \epsilon \\ &+ \frac{1}{n} \sum_{k=1}^n |EX_k'(t_{i-1}-0) - EX_k'(t_{i-1})| + \epsilon. \end{aligned} \quad (5)$$

By the Strong Law of Large Numbers,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (X'_k(t_i) - EX'_k(t_i)) = 0 \quad \text{a.s.} \quad (6)$$

for each $i = 0, 1, \dots, m$.

$$\begin{aligned} \text{By (2),} \quad \max_{i=1, \dots, m} \frac{1}{n} \sum_{k=1}^n |EX'_k(t_{i-0}) - EX'_k(t_{i-1})| \\ \leq \max_{i=1, \dots, m} \frac{1}{n} \sum_{k=1}^n E|X'_k(t_{i-0}) - X'_k(t_{i-1})| < \epsilon. \end{aligned} \quad (7)$$

for every n . Now $\frac{1}{n} \sum_{k=1}^n |X'_k(t_{i-0}) - X'_k(t_{i-1})|$

$$\leq \frac{1}{n} \sum_{k=1}^n [|X'_k(t_{i-0}) - X'_k(t_{i-1})| - E|X'_k(t_{i-0}) - X'_k(t_{i-1})|] + \epsilon,$$

using (7) and thus, since the random variables are bounded, by the Strong Law of Large Numbers,

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n |X'_k(t_{i-0}) - X'_k(t_{i-1})| < \epsilon, \quad \text{a.s.} \quad (8)$$

for each $i = 1, \dots, m$.

From (6), (7), (8) and (5), we get

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X'_k - EX'_k) \right\| = 0, \quad \text{a.s.} \quad (9)$$

$$\text{Now} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\|X''_k\| - E \|X''_k\|) < 2\epsilon, \quad \text{a.s.} \quad (10)$$

by the Strong Law of Large Numbers, using hypothesis 2°. Using (10) and hypothesis 1° we then have

$$\begin{aligned}
& \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\|X_k''\| + E \|X_k''\|) \\
&= \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (\|X_k''\| - E \|X_k''\|) + \frac{2}{n} \sum_{k=1}^n E \|X_k''\| \\
&< 2\epsilon, \qquad \qquad \qquad \text{a.s.} \qquad \qquad (11)
\end{aligned}$$

Using (11) and (9) we get, via (4),

$$\overline{\lim}_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| < 4\epsilon, \qquad \text{a.s.}$$

Taking a sequence (ϵ_n) of positive numbers converging to 0, we get, taking a union of null sets,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0, \qquad \text{a.s.} \qquad \text{Q.E.D.}$$

REMARK: Condition 2° is somewhat complicated, but it would seem to render Theorem 4.3 independent of Theorem 4.2. By itself, 1° is of course condition (T), and (T) together with (RR) implies (mT) (§6, Lemma 6.11). Thus, if 2° is replaced by, say,

$$\sum_{n=1}^{\infty} n^{-r} E \|X_n I_{[X_n \notin K]}\|^r < \infty, \text{ this, for } 1 < r \leq 2, \text{ would imply}$$

$\sum_{n=1}^{\infty} n^{-r} E \|X_n\|^r < \infty$ and the hypotheses of Theorem 4.3 would imply those of Theorem 4.2. If $r = 1$, Theorem 4.3 may still be independent of Theorem 4.2 if 2° is replaced by $\sum_{n=1}^{\infty} n^{-1} E \|X_n I_{[X_n \notin K]}\| < \infty$

(here 1° follows and need not be stated).

COROLLARY 4.4: Let (X_n) be a sequence of independent random

functions in $D[0,1]$ satisfying (RR) and such that there is a compact set K with $P[X_n \in K] = 1$, for every n . Then

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0, \text{ with probability one.}$$

REMARK: Note that the convex hull of K need not be conditionally compact. In fact, it is shown in §6.4 that (RR) does not imply convex tightness.

4.2 Denote the topology on $D[0,1]$ generated by the Skorokhod metric by

T_s and that generated by the metric given by the supremum norm

$\|x\| = \sup_{0 \leq t \leq 1} |x(t)|$ by T_u . When $D[0,1]$ is provided with the supremum norm, it is a Banach space.

Denote by K the collection of all subsets K of $D[0,1]$ which have the property that their convex hulls $\text{co}(K)$ are conditionally compact re T_s . We shall need the following result from [3].

THEOREM 4.5: If $K \subset D[0,1]$ then $K \in K$ if and only if K is conditionally compact in T_u .

LEMMA 4.6: Let (X_n) be a sequence of convex tight random functions in $D[0,1]$. If $E \|X_n\|^r \leq C < \infty$ for all n , and some $r > 1$, then the sequence $(X_n - \mu_n)$, where $\mu_n = EX_n$, is convex tight.

PROOF: To every $m \in \mathbb{N}$ there is a (Skorokhod) compact, convex set K_m such that $P[X_n \notin K_m] < C^{-\frac{1}{r-1}} m^{-\frac{r}{r-1}}$, for every n . By Theorem 4.5, K_m is compact re T_u . Hence, $\overline{\text{co} K_m}$ is compact re T_u , and we assume without loss of generality that K_m is convex (and compact) re T_u , and also that $0 \in K_m$.

Since $D[0,1]$ with the supremum norm is a complete metric space, K_m is totally bounded (no separability is needed here).

Let $N(x, \epsilon) = \{y: \|x-y\| < \epsilon\}$, $x \in D[0,1]$, $\epsilon > 0$.

Let $\{N(x_i, \frac{1}{m})\}_{i=1}^{i_m}$ be a finite cover of K_m .

$$\begin{aligned} \text{Now, } \|\mu_n - EX_n I_{[X_n \in K_m]}\| &\leq E \|X_n I_{[X_n \notin K_m]}\| \\ &\leq (E \|X_n\|^r)^{\frac{1}{r}} (E I_{[X_n \notin K_m]})^{\frac{r-1}{r}} < \frac{1}{m}, \text{ for every } n. \end{aligned}$$

Since K_m is convex and $0 \in K_m$, $E[X_n I_{[X_n \in K_m]}] \in K_m$, for every n , and hence $d(\mu_n, K_m) < \frac{1}{m}$, for all n . Write $K^{(m)} = \bigcup_{i=1}^{i_m} N(x_i, \frac{2}{m})$.

By the triangle inequality $\mu_n \in K^{(m)}$ for every n . Since this holds for every m ,

$$\mu_n \in K_0 = \bigcap_{m=1}^{\infty} K^{(m)}, \text{ for every } n.$$

Since K_0 is obviously totally bounded, it is conditionally compact re $T_u^{(*)}$. The closed convex hull $K = \overline{\text{co}(K_0)}$ is compact re T_u , and consequently conditionally compact also re T_s .

Now let $\epsilon > 0$ be given and choose K_ϵ , compact and convex re T_s , such that $P[X_n \in K_\epsilon] > 1 - \epsilon$, for all n . Then $P[X_n - \mu_n \in K_\epsilon - K] \geq P[X_n \in K_\epsilon \text{ and } \mu_n \in K] = P[X_n \in K_\epsilon] > 1 - \epsilon$, for all n . Since $K_\epsilon \in K$ and $K \in K$, $K_\epsilon - K \in K$ by Theorem 9.8 of [2]. Thus $(X_n - \mu_n)$ is convex tight. Q.E.D.

It follows from Lemma 4.6 that the conclusion of Theorem 1

(*) See also [6], Lemma 3.1, p. 29.

of [4] can be strengthened to almost sure uniform convergence.

We now state this result in its strengthened form.

THEOREM 4.7: If (X_n) is a sequence of independent convex tight random functions in $D[0,1]$ satisfying $\sup_n E \|X_n\|^r < \infty$, with $r > 1$, then $\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0$, with probability one.

This result, however, is implied by Theorem 4 of [13], for which we now provide an alternate proof. Two preliminary results are needed.

LEMMA 4.8 ([12], p. 123): Let (X_n) be a sequence of real-valued random variables such that $\sup_n E|X_n|^r < \infty$, for some $r > 1$. Then there exists a random variable X such that

- (i) $P[|X_n| \geq a] \leq P[|X| \geq a]$, for all n and $a \geq 0$;
- (ii) $E\{|X|^{1+\frac{1}{s}}\} < \infty$ for $0 < \frac{1}{s} < r - 1$.

The following theorem is due to Rohatgi [10].

THEOREM 4.9 ([12], p. 68): Let (X_n) be a sequence of (real-valued) random variables, with $EX_n = 0$ for every n , and let (a_{nk}) be a Toeplitz sequence. If $\max_k |a_{nk}| = O(n^{-s})$ for some $s > 0$, and there is a random variable X satisfying

- (i) $P[|X_n| \geq a] \leq P[|X| \geq a]$ for all n and $a \geq 0$; and
- (ii) $E|X|^{1+\frac{1}{s}} < \infty$;

then

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} a_{nk} X_k = 0, \text{ with probability one.}$$

THEOREM 4.10: Let (X_n) be a sequence of independent random functions in $D[0,1]$ satisfying the following condition:

(CT) $_{r>1}$: To every $\epsilon > 0$ there is K , compact and convex, such that $E \|X_n I_{[X_n \notin K]}\|^r < \epsilon$, for every n , where $r > 1$.

Let $\{a_{nk}\}$ be an array of weights satisfying the additional condition that $\max_{k=1, \dots, n} |a_{nk}| = O(n^{-s})$, where $0 < \frac{1}{s} < r - 1$.

Then

$$\lim_{n \rightarrow \infty} d\left(\sum_{k=1}^n a_{nk} X_k, \sum_{k=1}^n a_{nk} EX_k\right) = 0, \text{ with probability one.}$$

PROOF: Let $\epsilon > 0$ be given and select K , compact and convex by (CT) $_{r>1}$ such that

$$E \|X_n I_{[X_n \notin K]}\|^r < \epsilon, \text{ for every } n. \quad (1)$$

Note that this implies the existence of EX_n , for each n . Without loss of generality K can be taken to be balanced and symmetric (write $K_1 = \bigcup_{|\alpha| \leq 1} \alpha K$ and replace K by $K_1 - K_1$: cf [2], Theorem 9.8, p.28); this in turn implies that K is absolutely convex. This we assume.

Write $X_k' = X_k I_{[X_k \in K]}$, $X_k'' = X_k I_{[X_k \notin K]}$. We have, using Lemma 2.2:

$$\begin{aligned} d\left(\sum_{k=1}^n a_{nk} X_k, \sum_{k=1}^n a_{nk} EX_k\right) &\leq d\left(\sum_{k=1}^n a_{nk} X_k', \sum_{k=1}^n a_{nk} T_m(X_k')\right) \\ &\quad + d\left(\sum_{k=1}^n a_{nk} T_m(X_k'), \sum_{k=1}^n a_{nk} T_m(EX_k')\right) \end{aligned}$$

$$\begin{aligned}
& + d\left(\sum_{k=1}^n a_{nk} T_m(EX'_k), \sum_{k=1}^n a_{nk} EX'_k\right) \\
& + \left\| \sum_{k=1}^n a_{nk} X''_k \right\| + \left\| \sum_{k=1}^n a_{nk} EX''_k \right\| \\
& = (I) + (II) + (III) + (IV) + (V).
\end{aligned} \tag{2}$$

Now $\sum_{k=1}^n a_{nk} X'_k \in K$, for all n , since K is absolutely convex. Thus, using Lemma 2.2,

$$(I) \leq \sup_{x \in K} d(x, T_m(x)) < \epsilon, \text{ for all sufficiently large } m. \tag{3}$$

$$\text{Now } (II) \leq \left\| T_m \sum_{k=1}^n a_{nk} (X'_k - EX'_k) \right\|$$

$$= \max_{i=0, \dots, 2^m-1} \left| \sum_{k=1}^n [X'_k(t_i) - EX'_k(t_i)] \right|,$$

using $d(x, y) \leq \|x - y\|$ and the additivity of T_m . Since K is compact, the random variables $X'_k(t_i) - EX'_k(t_i)$ are uniformly bounded and so an application of Theorem 3.6 yields

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{nk} (X'_k(t_i) - EX'_k(t_i)) = 0, \text{ a.s., for each } i = 0, 1, \dots, 2^m-1.$$

Hence

$$\lim_{n \rightarrow \infty} \max_{i=0, 1, \dots, 2^m-1} \sum_{k=1}^n a_{nk} (X'_k(t_i) - EX'_k(t_i)) = 0, \tag{4}$$

a.s.

Since K is convex, $EX'_k \in K$, for all k , and so, just as for (I),

$$(III) = d\left(T_m\left(\sum_{k=1}^n EX'_k\right), \sum_{k=1}^n EX'_k\right) \leq \sup_{x \in K} d(x, T_m(x)) < \epsilon, \tag{5}$$

for all sufficiently large m .

$$\begin{aligned} \text{Now (IV) + (V)} &\leq \sum_{k=1}^n |a_{nk}| (\|X_k''\| - E \|X_k''\|) \\ &+ 2 \sum_{k=1}^n E \|X_k''\|. \end{aligned} \quad (6)$$

$$\text{Since } E \left| \|X_k''\| - E \|X_k''\| \right|^r \leq 2^r E \|X_k''\|^r \leq 2^r \varepsilon < \infty,$$

for every k , where $r > 1$, by Lemma 4.8 there is a random variable X such that (i) $P[|X_n| \geq a] \leq P[|X| \geq a]$, all n , all $a \geq 0$; and (ii) $E|X|^{1+\frac{1}{s}} < \infty$, where $0 < \frac{1}{s} < r - 1$. Since $\max_k |a_{nk}| = O(n^{-s})$, Theorem 4.9 yields

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}| (\|X_k''\| - E \|X_k''\|) = 0, \quad \text{a.s.} \quad (7)$$

From (1) we have

$$2 \sum_{k=1}^{\infty} |a_{nk}| E \|X_k''\| \leq 2\varepsilon. \quad (8)$$

Using (3), (4), (5), (6), (7) and (8) in (2), we get

$$\lim_{n \rightarrow \infty} d \left(\sum_{k=1}^n a_{nk} X_k, \sum_{k=1}^n a_{nk} EX_k \right) < \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon, \quad \text{a.s.}$$

Taking a sequence (ε_n) of positive numbers converging to zero and taking a union of null sets, we get finally

$$\lim_{n \rightarrow \infty} d \left(\sum_{k=1}^n a_{nk} X_k, \sum_{k=1}^n a_{nk} EX_k \right) = 0, \quad \text{with probability one,}$$

Q.E.D.

That Theorem 4.10 is equivalent to Theorem 4 of [13] is seen as follows. On the one hand, $(CT)_{r>1}$ implies both (CT) and $(M)_{r>1}$, and (CT) implies convex tightness (Lemma 6.8). On the

other hand, convex tightness and $(M)_{r>1}$, imply $(CT)_{r>1}$. Indeed, let $1 < r' < r$. We have

$$\begin{aligned} E \|X_n\|^{r'} I_{[X_n \notin K]} &\leq [E(\|X_n\|^{r'})^{\frac{r}{r'}}]^{\frac{r'}{r}} [P[X_n \notin K]]^{\frac{r-r'}{r}} \\ &= (E \|X_n\|^r)^{\frac{r'}{r}} [P[X_n \notin K]]^{\frac{r-r'}{r}}. \end{aligned}$$

Now $E \|X_n\|^r$ are uniformly bounded by $(M)_{r>1}$, and $P[X_n \notin K]$ can be made uniformly arbitrarily small by convex tightness and the choice of K . Thus, to any $\varepsilon > 0$ there is K , compact and convex, such that $E \|X_n\|^{r'} I_{[X_n \notin K]} < \varepsilon$ for all n , which is condition $(CT)_{r'>1}$. Q.E.D.

Although the conclusion of Theorem 4.10 is stated in terms of convergence in the Skorokhod metric, this can be strengthened to uniform convergence by an application of Lemma 4.6.

§5. Weak Laws of Large Numbers.

5.1 Weak laws of large numbers for random elements in function spaces (or more general Banach or Fréchet spaces) come in two types. First, there are those based on sufficient conditions (usually some type of weak uncorrelation) which imply the convergence in probability of the weighted sums. Second, there are those which provide conditions sufficient to turn pointwise convergence in probability into convergence in probability in the metric of the function space; whereupon necessary and sufficient conditions for the classical weak laws can be invoked to provide the pointwise convergence. In this section we present one result

of the first type, and three of the second type which improve on existing results (for existing results see [12], [15]).

5.2 The following theorem uses a condition on second moments and an uncorrelation condition which looks rather like the condition (MT) which was used to obtain strong laws in the previous section.

DEFINITION: A sequence (X_n) of random functions in $D[0,1]$ is said to be pointwise uncorrelated if

$$\text{Cov}(X_k(t), X_\ell(t)) = E[(X_k(t) - EX_k(t))(X_\ell(t) - EX_\ell(t))] = 0$$

for each $t \in [0,1]$ when $k \neq \ell$.

THEOREM 5.1: Let (X_n) be a sequence of mean zero, pointwise uncorrelated random functions in $D[0,1]$ satisfying

$$1^\circ. \quad \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k=1}^n E \|X_n\|^2 = 0;$$

2°. To every $\epsilon > 0$ there is a partition P of $[0,1]$ such that

$$\sup_{k, \ell} E \left[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_k(t)X_\ell(t) - X_k(t_{i-1})X_\ell(t_{i-1})| \right] \leq \epsilon.$$

Then $\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n X_k \right\| = 0$, in probability.

PROOF: Let $\epsilon > 0$ be given, and choose, by 2°, a partition P such that

$$\sup_{k, \ell} E \left[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_k(t)X_\ell(t) - X_k(t_{i-1})X_\ell(t_{i-1})| \right] \leq \frac{\epsilon^3}{2}.$$

$$\begin{aligned}
\text{Then } P\left[\left\|\sum_{k=1}^n X_k\right\| \geq n\epsilon\right] &\leq \frac{1}{n^2\epsilon^2} E\left\|\sum_{k=1}^n X_k\right\|^2 \\
&= \frac{1}{n^2\epsilon^2} E\left[\sup_{0 \leq t \leq 1} \left|\sum_{k=1}^n X_k(t)\right|^2\right] \\
&= \frac{1}{n^2\epsilon^2} E\left[\sup_{0 \leq t \leq 1} \left(\sum_{k,\ell=1}^n (X_k(t)X_\ell(t) \right. \right. \\
&\quad \left. \left. - X_k(t_{i-1})X_\ell(t_{i-1})) + \sum_{k,\ell=1}^n X_k(t_{i-1})X_\ell(t_{i-1})\right)\right] \\
&\leq \frac{1}{n^2\epsilon^2} E\left[\max_i \sup_{t \in I_i} \sum_{k,\ell=1}^n (X_k(t)X_\ell(t) - X_k(t_{i-1})X_\ell(t_{i-1}))\right] \\
&\quad + \frac{1}{n^2\epsilon^2} E\left[\max_i \sum_{k,\ell=1}^n X_k(t_{i-1})X_\ell(t_{i-1})\right] \quad (*)
\end{aligned}$$

For the second term in (*)

$$\begin{aligned}
&\frac{1}{n^2\epsilon^2} E\left[\max_i \sum_{k,\ell=1}^n X_k(t_{i-1})X_\ell(t_{i-1})\right] \\
&\leq \frac{1}{n^2\epsilon^2} \sum_{i=1}^m E\left[\sum_{k,\ell=1}^n X_k(t_{i-1})X_\ell(t_{i-1})\right] \\
&= \frac{1}{n^2\epsilon^2} \sum_{i=1}^m \sum_{k=1}^n E(X_k(t_{i-1})^2) \leq \frac{m}{n^2\epsilon^2} \sum_{k=1}^n E\|X_k\|^2
\end{aligned}$$

which can be made less than $\frac{\epsilon}{4}$ for all $n \geq n_0$ from 1°. For the first term in (*)

$$\begin{aligned}
&\frac{1}{n^2\epsilon^2} E\left[\max_i \sup_{t \in I_i} \sum_{k,\ell=1}^n (X_k(t)X_\ell(t) - X_k(t_{i-1})X_\ell(t_{i-1}))\right] \\
&\leq \frac{1}{n^2\epsilon^2} E\left[\max_i \sup_{t \in I_i} \sum_{\substack{k,\ell \\ k \neq \ell}}^n (X_k(t)X_\ell(t) - X_k(t_{i-1})X_\ell(t_{i-1}))\right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n^2 \epsilon^2} E \left[\max_i \sup_{t \in I_i} \sum_{k=1}^n (X_k^2(t) - X_k^2(t_{i-1})) \right] \\
& \leq \frac{n(n-1)}{n^2 \epsilon^2} \sup_{k, \ell} E \left[\max_i \sup_{t \in I_i} |X_k(t) X_\ell(t) - X_k(t_{i-1}) X_\ell(t_{i-1})| \right] \\
& \quad + \frac{2}{n^2 \epsilon^2} \sum_{k=1}^n E \|X_k\|^2 \\
& < \frac{\epsilon}{4} + \frac{2\epsilon}{4} = \frac{3\epsilon}{4}
\end{aligned}$$

for all $n \geq n_0$. Hence, for $n \geq n_0$.

$$P \left[\left\| \sum_{k=1}^n X_k \right\| \geq n\epsilon \right] < \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon. \quad \text{Q.E.D.}$$

5.2 The next result provides sufficient conditions for the equivalence of pointwise and uniform convergence in probability, and improves on Theorem 1 of [13].

THEOREM 5.2: Let (X_n) be a sequence of random functions in $D[0,1]$ having property (MT) and such that $E \|X_n\| < \infty$, for each n . Let (a_{nk}) be a double array of real numbers satisfying $\sum_{k=1}^{\infty} |a_{nk}| \leq 1$, for each n . Then $\lim_{n \rightarrow \infty} \sum_{k=1}^n a_{nk} (X_k(t) - EX_k(t)) = 0$, in probability, for each $t \in [0,1]$, if and only if

$$\lim_{n \rightarrow \infty} \left\| \sum_{k=1}^n a_{nk} (X_k - EX_k) \right\| = 0, \text{ in probability.}$$

PROOF: Since (X_n) has property (MT), so does $(X_n - EX_n)$ ([14], proof of Theorem 1). Thus w.l.o.g., we assume $EX_n = 0$, for every n .

Let $\eta > 0$ and $\epsilon > 0$ be given and choose, by (MT), a partition P such that

$$\sup_n E[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_k(t) - X_k(t_{i-1})|] \leq \frac{\eta \epsilon}{4}.$$

Then
$$P[\| \sum_{k=1}^n a_{nk} X_k \| > \epsilon]$$

$$\leq P[\| \sum_{k=1}^n a_{nk} (X_k - T_m X_k) \| > \frac{\epsilon}{2}] + P[\| \sum_{k=1}^n a_{nk} T_m X_k \| > \frac{\epsilon}{2}].$$

Now
$$\begin{aligned} P[\| \sum_{k=1}^n a_{nk} (X_k - T_m X_k) \| > \frac{\epsilon}{2}] \\ &\leq \frac{2}{\epsilon} E \| \sum_{k=1}^n a_{nk} (X_k - T_m X_k) \| \\ &\leq \frac{2}{\epsilon} \sum_{k=1}^n |a_{nk}| E \| X_k - T_m X_k \| \\ &= \frac{2}{\epsilon} \sum_{k=1}^n |a_{nk}| E[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_k(t) - X_k(t_{i-1})|] \\ &\leq \frac{2}{\epsilon} \sum_{k=1}^n |a_{nk}| \frac{\eta \epsilon}{4} \leq \frac{\eta}{2}. \end{aligned}$$

$$P[\| \sum_{k=1}^n a_{nk} T_m X_k \| > \frac{\epsilon}{2}]$$

$$= P[\max_{i=1, \dots, m} | \sum_{k=1}^n a_{nk} X_k(t_i) | > \frac{\epsilon}{2}] < \frac{\eta}{2},$$

for all sufficiently large n .

Thus, to every $\epsilon > 0$ and $\eta > 0$, $P[\| \sum_{k=1}^n a_{nk} X_k \| > \epsilon] < \eta$,

for all sufficiently large n .

Q.E.D.

Theorem 3 of [4], strengthened to yield uniform convergence, will now be obtained as a corollary.

COROLLARY 5.3: Let (X_n) be a sequence of convex tight random functions in D satisfying $(M)_{r>1}$. Then $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n [X_k(t) - EX_k(t)] = 0$, in probability, for each $t \in [0,1]$, if and only if

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0, \text{ in probability.}$$

PROOF: We have $(M)_{r>1} \Rightarrow (UI)$ and convex tightness and $(UI) \Rightarrow (MT)$ (§6, Theorem 6.7), and $a_{nk} = \frac{1}{n}$, $k=1, \dots, n$; $a_{nk} = 0$, $k > n$, satisfies the condition of Theorem 5.2. Q.E.D.

COROLLARY 5.4: Let (X_n) be a sequence of random functions in $D[0,1]$ having property (MT) and such that $E \|X_n\| < \infty$, for each n . If

$$1^\circ \text{ cov}(X_k(t), X_\ell(t)) = 0 \text{ for each } k \neq \ell, \text{ for each } t \in [0,1];$$

and

$$2^\circ \sum_{k=1}^n \text{var}(X_k(t)) = o(n^2), \text{ for each } t \in [0,1],$$

then,

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{k=1}^n (X_k - EX_k) \right\| = 0, \text{ in probability.}$$

§6. A Comparison of Various Conditions on Random Functions in $D[0,1]$.

6.1 In this section we investigate relationships among the conditions on a sequence (X_n) of random functions in $D[0,1]$, which were defined and collected together in §3. The most striking result

is Theorem 6.7, which asserts the equivalence of convex tightness, (CT) and (MT) for uniformly integrable sequences. Also, (CT) is shown to imply convex tightness; (CT) also implies (MT) but uniform integrability (UI) appears necessary for the converse. Examples are given to show that many implications cannot be reversed; however, some open questions remain.

6.2 LEMMA 6.1: If (X_n) satisfies (T) then (X_n) is stochastically bounded.

PROOF: Let $\epsilon > 0$ be given and let K be compact such that $E \|X_n I_{[X_n \notin K]}\| < \epsilon$, for each n . Let $c = \sup_{x \in K} \|x\|$ and let $\delta > \max\{2, 2c\}$. Then

$$P[\|X_n\| > \delta] \leq P[\|X_n I_{[X_n \in K]}\| > \delta/2] + P[\|X_n I_{[X_n \notin K]}\| > \delta/2]$$

$$= 0 + P[\|X_n I_{[X_n \notin K]}\| > \delta/2]$$

$$\leq \frac{2}{\delta} E \|X_n I_{[X_n \notin K]}\| < \frac{2}{\delta} \epsilon < \epsilon,$$

Q.E.D.

LEMMA 6.2: (CT) implies (MT).

PROOF: Let $\epsilon > 0$ be given. By (CT) choose K , compact and convex, such that $E \|X_n I_{[X_n \notin K]}\| < \frac{\epsilon}{4}$, for every n .

Since K is compact and convex there is a partition P of $[0, 1]$ and $\delta > 0$, using Theorem 3.6 of [3], such that

$$\sup_{x \in K} \max_{i=1, \dots, m} \sup_{t, s \in I_i} |x(s) - x(t)| < \frac{\epsilon}{2}. \quad (*)$$

Then

$$\begin{aligned} & E\left[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| \right] \\ & \leq E\left[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| I_{[X_n \in K]} \right] \\ & \quad + E\left[\max_{i=1, \dots, m} \sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| I_{[X_n \notin K]} \right] \\ & \leq E\left(\frac{\epsilon}{2}\right) + E\left[2 \|X_n\| I_{[X_n \notin K]} \right] \quad (\text{using } (*)) \\ & < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for every } n, \quad \text{Q.E.D.} \end{aligned}$$

LEMMA 6.3: Let (X_n) be a sequence of random functions in $D[0,1]$. If (X_n) satisfies (MT) and is stochastically bounded, then (X_n) is convex tight.

PROOF: Given $\epsilon > 0$, find by (MT) a partition P such that

$$\sup_n E\left[\max_i \sup_{t_i \leq t < t_{i+1}} |X_n(t) - X_n(t_i)| \right] \leq \epsilon.$$

This implies that

$$\sup_n P\left[\max_i \sup_{t_i \leq t < t_{i+1}} |X_n(t) - X_n(t_i)| > a \right] < \frac{\epsilon}{a}, \quad (*)$$

for any $a > 0$.

Let (ϵ_k) and (η_k) be sequences of positive numbers such that $\lim \eta_k = 0$ and $\sum_{k=1}^{\infty} \epsilon_k = \epsilon$.

Let δ_k be chosen by stochastic boundedness so that

$$\sup_n P[\|X_n\| \geq \delta_k] \leq \epsilon_k.$$

Let $B(\delta) = \{x \in D[0,1] : \|x\| \geq \delta\}$ and define the sets

$$A_k(\epsilon) = \{x \in B(\delta_k) : \max_i \sup_{t_i \leq t < t_{i+1}} |x(t) - x(t_i)| \leq \eta_k\},$$

where the partition P_k which is used is chosen so that

$$\sup_n E[\max_i \sup_{t_i \leq t < t_{i+1}} |X_n(t) - X_n(t_i)|] \leq \eta_k \epsilon_k,$$

which is possible by (MT).

Then, using (*), we have

$$\sup_n P[X_n \notin A_k(\epsilon)] < \epsilon_k. \quad (**)$$

Now put $A(\epsilon) = \bigcup_k A_k(\epsilon)$. Let $\eta > 0$ be given and find k_0 , such that $\eta_{k_0} < \frac{1}{2} \eta$.

For $x \in A(\epsilon)$ we have,

$$\max_i \sup_{t_i \leq t < t_{i+1}} |x(t) - x(t_i)| \leq \eta_{k_0} < \frac{1}{2} \eta,$$

because $x \in A_{k_0}(\epsilon)$. It follows that

$$\max_i \sup_{t_i < t < t_{i+1}} |x(t) - x(t-0)| < \eta, \text{ and so}$$

$S_\eta(A(\epsilon)) = \{t \in [0,1] : \sup_{x \in A(\epsilon)} |x(t) - x(t-0)| > \eta\}$ is finite

(at most m_{k_0} , the number of points of the partition P) for each $n > 0$.

If the set $A(\epsilon)$ is conditionally compact, it will follow by Theorem 6 of [4] that the convex hull $\text{co}(A(\epsilon))$ is conditionally compact.

To prove that $A(\epsilon)$ is conditionally compact, we use Theorem 14.3 of [1]. The set $A(\epsilon)$ is bounded since $x \in A(\epsilon)$ implies $\|x\| \leq \delta_1$. Now let $\alpha > 0$ be given and let k be such that $\eta_k \leq \frac{\alpha}{2}$. Then there is a partition P , by definition of $A_k(\epsilon)$, such that for each $x \in A_k(\epsilon)$ (a fortiori each $x \in A(\epsilon)$), we have

$$\max_1 \sup_{t_i \leq t, s < t_{i+1}} |x(t) - x(s)| < \alpha.$$

If $\delta = \min_i \{t_{i+1} - t_i\}$ for P , then $\sup_{x \in A(\epsilon)} w'_x(\delta) < \alpha$, and since

$\alpha > 0$ is arbitrary, $\lim_{\delta \rightarrow 0} \sup_{x \in A(\epsilon)} w'_x(\delta) = 0$, so that by Theorem 14.3 of [1], $A(\epsilon)$ is conditionally compact in $D[0,1]$.

Thus, $\text{co}(A(\epsilon))$ is conditionally compact in $D[0,1]$. Now,

$$P[X_n \notin \text{co}(A(\epsilon))] \leq P[X_n \notin A(\epsilon)] = P[X_n \in \cup_k A_k^C(\epsilon)] \leq \sum_k P[X_n \in A_k^C(\epsilon)]$$

$$= \sum_k P[X_n \notin A_k(\epsilon)] \leq \sum_k \sup_n P[X_n \notin A_k(\epsilon)] < \sum_k \epsilon_k = \epsilon, \text{ by } (**).$$

Since ϵ was arbitrary, (X_n) is convex tight,

Q.E.D.

LEMMA 6.4: If (X_n) is convex tight and uniformly integrable, then (X_n) satisfies (CT).

PROOF: Given $\epsilon > 0$, find by uniform integrability δ such that

$$E \|X_n I_{[X_n \notin B(\delta)]}\| < \epsilon, \text{ for each } n.$$

By convex tightness, find K , compact and convex, such that

$$P[X_n \notin K] < \frac{\epsilon}{\delta}, \text{ for every } n.$$

Then

$$\begin{aligned} E \|X_n I_{[X_n \notin K]}\| &= E \|X_n I_{[X_n \notin K; X_n \notin B(\delta)]}\| \\ &\quad + E \|X_n I_{[X_n \in K; X_n \in B(\delta)]}\| \\ &< \epsilon + \delta E I_{[X_n \notin K; X_n \in B(\delta)]} \\ &\leq \epsilon + \delta E I_{[X_n \notin K]} \\ &= \epsilon + \delta P[X_n \notin K] < \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Thus (X_n) satisfies (CT).

Q.E.D.

However, by Example 6.3 below, uniform integrability cannot be replaced by stochastic boundedness in Lemma 6.4.

LEMMA 6.5: If (X_n) is tight and uniformly integrable, then (X_n) satisfies (T).

PROOF: Exactly the same as that of Lemma 6.4.

LEMMA 6.6: Uniform integrability implies stochastic boundedness.

PROOF: Let $B(\delta) = \{x: \|x\| \leq \delta\}$. Let $\epsilon > 0$ be given and choose, by (UI), δ such that $E \|X_n I_{[X_n \notin B(\delta)]}\| \leq \epsilon$, for each n .

Then, for $\delta \geq 1$, for every n ,

$$P[\|X_n\| \geq \delta] = P[\|X_n I_{[X_n \notin B(\delta)]}\| \geq \delta]$$

$$\leq \frac{1}{\delta} E[\|X_n I_{[X_n \notin B(\delta)]}\|]$$

$$\leq E \|X_n I_{[X_n \notin B(\delta)]}\| \leq \epsilon,$$

Q.E.D.

THEOREM 6.7: If (X_n) is uniformly integrable, then the following are equivalent:

1° (X_n) is convex tight;

2° (X_n) satisfies (CT);

3° (X_n) satisfies (MT).

PROOF: 1° $\Rightarrow$ 2° follows from Lemma 6.4; 2° $\Rightarrow$ 3° from Lemma 6.2; 3° $\Rightarrow$ 1° from Lemma 6.6 and Lemma 6.3. Q.E.D.

LEMMA 6.8: If (X_n) satisfies (CT) then (X_n) is convex tight.

PROOF: Lemma 6.1 yields: (CT) implies stochastic boundedness. Now Lemma 6.2 and Lemma 6.3 yield the result, Q.E.D

6.3 Easy examples show that (MT) does not imply tightness nor does tightness imply (MT).

We write (UI) for uniform integrability and (SB) for stochastic boundedness.

EXAMPLE 6.1: (UI) does not imply tightness. The counter-example is the sequence (X_n) of deterministic random functions

$$X_n = I_{[2^{-n-1}, 2^{-n}]}, \quad n = 1, 2, \text{etc.}$$

Thus, by Lemma 6.6, (SB) does not imply tightness either.

EXAMPLE 6.2: (SB) $\not\Rightarrow$ (UI). Let (X_n) be the sequence of random functions defined by

$$X_n(t) = \begin{cases} n, & \text{with probability } \frac{1}{n}, \\ 0, & \text{with probability } 1 - \frac{1}{n}, \end{cases} \text{ for } 0 \leq t \leq 1.$$

EXAMPLE 6.3: Convex tightness and (SB) do not imply (MT).

Let $X_n = X$ for every n , where X is defined as follows. Let

$$x_n = 2^{2n} I_{\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)}, \quad n = 2, 3, \dots, \quad x_1 = 4 I_{\left[\frac{1}{2}, 1\right]}.$$

$P[X = x_n] = 2^{-n}$, $n = 1, 2, \text{etc.}$ Then X is convex tight and stochastically bounded. But for any partition P of $[0, 1]$,

$$\begin{aligned} E\left\{ \max_{i=1, \dots, m} \sup_{t \in I_i} |X(t) - X(t_{i-1})| \right\} &\geq E\left\{ \sup_{0 \leq t < t_1} |X(t) - X(0)| \right\} \\ &= E\left\{ \sup_{0 \leq t < t_1} |X(t)| \right\} = \sum_{n=n_0}^{\infty} 2^{-n} \cdot 2^{2n} = +\infty \end{aligned}$$

where n_0 is such that $2^{-n_0+1} < t_1$. Thus (MT) fails, Q.E.D.

Thus, by Lemma 6.2, convex tightness and (SB) do not imply (CT).

6.4 We have $(CT) \Rightarrow (MT) \Rightarrow (mT) \Rightarrow (RR)$. The first implication follows from Lemma 6.2 and the last two are obvious.

EXAMPLE 6.4: (mT) $\not\Rightarrow$ (MT). Define a random function X as follows. Let $\Omega = [0, 1]$ with Lebesgue measure. Let $X(\omega) = I_{[\omega, 1]}$.

Let $\varepsilon > 0$ be given and choose a partition P of $[0, 1]$ such that

$$\max_{i=1, \dots, m} \{t_i - t_{i-1}\} < \varepsilon. \quad \text{Then}$$

$$E[\sup_{t \in I_i} |X(t) - X(t_{i-1})|] = t_i - t_{i-1} < \epsilon,$$

for each i , so that (mT) is satisfied. However,

$$E[\max_{i=1, \dots, m} \sup_{t \in I_i} |X(t) - X(t_{i-1})|] = 1,$$

for any partition P , so that (MT) fails.

Q.E.D.

Since tightness implies (SB), by Lemma 6.3, (MT) and tightness imply convex tightness. That (mT) and tightness do not imply convex tightness is shown by Example 6.4.

EXAMPLE 6.5: (T) $\not\Rightarrow$ (MT). Let X be the random function of Example 6.4. (MT) fails. Let

$$K_\epsilon = \{x = I_{[\omega, 1]} : 0 < \frac{\epsilon}{2} \leq \omega \leq 1 - \frac{\epsilon}{2} < 1\}$$

where $0 < \epsilon < 1$. Then K_ϵ is conditionally compact in D , and

$$E \|X I_{[K]}\| = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \text{ Since } \epsilon \text{ is arbitrary, (T) is satisfied.}$$

Q.E.D.

Clearly, also (MT) $\not\Rightarrow$ (T), so that no implication holds between the two conditions.

LEMMA 6.9: For a random function X in D with $E \|X\| < \infty$, condition (mT) is satisfied.

PROOF: For $0 \leq \alpha < \beta \leq 1$, let $\rho(\alpha, \beta) = E[\sup_{\alpha \leq t, s < \beta} |X(t) - X(s)|]$.

Let $\tau_1 = 1$ if $\rho(0, 1) \leq \epsilon$; otherwise, let $\tau_1 = \inf\{t: \rho(0, t) > \epsilon\}$.

$$\text{Since } \lim_{n \rightarrow \infty} \rho(0, \frac{1}{n}) = \lim_{n \rightarrow \infty} E[\sup_{0 \leq s, t < \frac{1}{n}} |X(t) - X(s)|]$$

$$= E[\lim_{n \rightarrow \infty} \sup_{0 \leq t, s < \frac{1}{n}} |X(t) - X(s)|] = 0,$$

by the dominated convergence theorem, using $E \|X\| < \infty$, we have

$$\tau_1 > 0.$$

In general, let $\tau_j = 1$ if $\rho(\tau_{j-1}, 1) \leq \epsilon$, and let $\tau_j = \inf\{t: t > \tau_{j-1} \text{ and } \rho(\tau_{j-1}, t) > \epsilon\}$, otherwise. Again $\tau_j < \tau_{j+1}$.

Now suppose that $\tau_j < 1$, for every $j = 1, 2, \dots$. Since (τ_n) is monotonically increasing it converges to some $t_0 \in (0, 1]$, and since $X \in D$,

$$\lim_{n \rightarrow \infty} \sup_{\tau_n \leq t, s < \tau_{n+1}} |X(s) - X(t)| = |X(t_0) - X(t_0)| = 0.$$

Thus, $\epsilon < \lim_{n \rightarrow \infty} E[\sup_{\tau_n \leq t, s < \tau_{n+1}} |X(t) - X(s)|] = 0$, a contradiction.

This proves the lemma.

Q.E.D.

EXAMPLE 6.6: (RR) and (M) $\nRightarrow$ tightness. Let

$$x_{in} = I_{[\frac{2i-1}{2^n}, \frac{2i}{2^n})}, \quad i = 1, 2, \dots, 2^{n-1} - 1$$

$$x_{2^{n-1}, n} = I_{[\frac{2^{n-1}}{2^n}, 1]}$$

Define the random function X_n by

$$P[X_n = x_{in}] = \frac{1}{2^{n-1}}, \quad \text{for } i = 1, 2, \dots, 2^{n-1}.$$

We show that (mT) is satisfied. Let P be a partition and

let $\|P\|$ be its norm.

Given an interval $[t_{i-1}, t_i]$ of P , for fixed n there can be at most $[2^n \|P\|] + 1$ values of j such that the interval where x_{jn} is = 1 has a non-empty intersection with $[t_{i-1}, t_i]$. ($[\cdot]$ is the greatest integer function). Thus,

$$\begin{aligned} P\{\omega: \sup_{t_{i-1} \leq t < t_i} |X_n(t) - X_n(t_{i-1})| = 1\} &\leq ([2^n \|P\|] + 1) \frac{1}{2^n} \\ &\leq (2^n \|P\| + 1) \frac{1}{2^n} = \|P\| + \frac{1}{2^n}, \text{ and so} \end{aligned}$$

$$\begin{aligned} E\left[\sup_{t_{i-1} \leq t < t_i} |X_n(t) - X_n(t_{i-1})|\right] &= P\left\{\sup_{t_{i-1} \leq t < t_i} |X_n(t) - X_n(t_{i-1})| = 1\right\} \\ &\leq \|P\| + \frac{1}{2^n}, \text{ and this holds for each } i = 1, \dots, 2^{n-1}. \end{aligned}$$

Let $\varepsilon > 0$ be given. Choose $\|P\| < \frac{\varepsilon}{2}$ and n_0 such that $\frac{1}{2^{n_0}} < \frac{\varepsilon}{2}$. Then

$$\sup_{n=n_0, n_0+1, \dots} \max_i E\left[\sup_{t_{i-1} \leq t < t_i} |X_n(t) - X_n(t_{i-1})|\right] \leq \varepsilon.$$

Now refine P so that this holds for $n=1, \dots, n_0-1$. Also

$$\sup_n \max_i E\left[\sup_{t_{i-1} \leq t < t_i} |X_n(t) - X_n(t_{i-1})|\right] \leq \varepsilon.$$

But this is (mT).

But the sequence (X_n) is obviously not tight; this can be seen by Theorem 15.2 of Billingsley; in fact

$$\lim_{n \rightarrow \infty} P[w_{X_n}'(\delta) > \epsilon] = 1,$$

if $0 < \epsilon < 1$, for any $\delta > 0$.

Since (X_n) satisfies (mT) but is not tight, it satisfies (RR) and is not tight. (M) holds since every X_n is bounded by one.

Q.E.D.

LEMMA 6.10: If (X_n) satisfies (RR) and (T), then it satisfies (mT).

PROOF: Let $\epsilon > 0$ be given, and by (T), choose K , compact, such that $E \|X_n^I_{[X_n \in K]}\| < \epsilon$, for all n , and let P be a partition of $[0,1]$ such that (RR) holds. Choose $\delta > 0$ such that $0 \leq s \leq t < n < s + \delta \leq 1$ implies

$$|x(t) - x(s)| \leq |x(u-0) - x(s)| + \epsilon, \text{ for all } x \in K.$$

By adding points to P if necessary, arrange for $\max_{i=1, \dots, m} \{t_i - t_{i-1}\} < \delta$.

Write $X_n' = X_n^I_{[X_n \in K]}$ and $X_n'' = X_n^I_{[X_n \notin K]}$. Then,

$$\begin{aligned} & \max_{i=1, \dots, m} E \left[\sup_{t \in I_i} |X_n(t) - X_n(t_{i-1})| \right] \\ & \leq \max_{i=1, \dots, m} E \left[\sup_{t \in I_i} |X_n'(t) - X_n'(t_{i-1})| \right] \\ & \quad + \max_{i=1, \dots, m} E \left[\sup_{t \in I_i} |X_n''(t) - X_n''(t_i)| \right] \\ & \leq \max_{i=1, \dots, m} E[|X_n'(t_{i-0}) - X_n'(t_{i-1})|] + \epsilon + 2E \|X_n''\| \end{aligned}$$

$$\leq \max_{i=1, \dots, m} \sup_{t \in I_i} E[|X_n'(t) - X_n(t_{i-1})|] + \epsilon + 2\epsilon$$

$< \epsilon + \epsilon + 2\epsilon = 4\epsilon$, and thus (mT) is satisfied.

Q.E.D.

6.4 We have the following diagram:

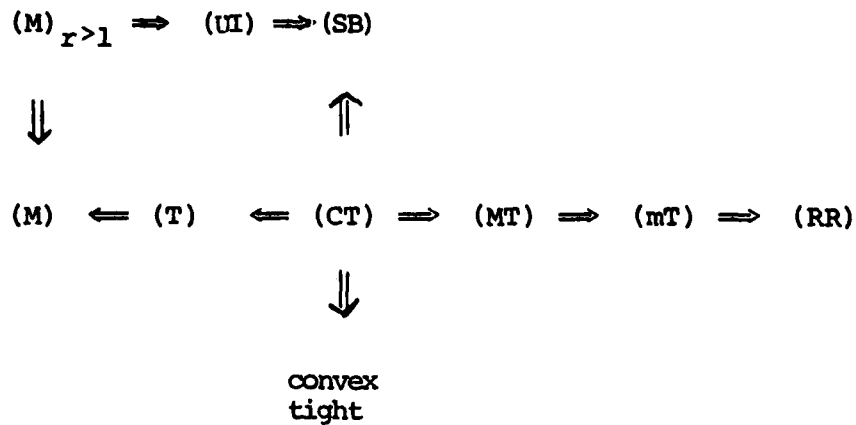


Figure 1.

6.5 Some Open Questions

- 1° Does (RR) imply (mT) ?
- 2° Does (RR) together with (M) imply (mT) ?

REFERENCES

- [1] Billingsley, P. (1968). Convergence of Probability Measures, Wiley, New York.
- [2] Daffer, P. Z. (1978). Laws of large numbers for $D[0,1]$ and related topics. Ph.D. Thesis, University of South Carolina.
- [3] Daffer, P. Z. (1979). On compact convex subsets of $D[0,1]$. To appear: Rocky Mountain Journal of Mathematics.
- [4] Daffer, P. Z., and Taylor, R. L. (1979). Laws of large numbers for $D[0,1]$. Annals of Probability, 7, 85-95.
- [5] Gnedenko, B. V., and Kolmogoroff, A. N. (1954). Limit Distributions for Sums of Independent Random Variables. (Translated by K. L. Chung). Addison-Wesley, Cambridge, Mass.
- [6] Parthasarathy, K. R. (1967). Probability Measures on Metric Spaces. Academic Press, New York and London.
- [7] Petrov, V. V. (1969). On the strong law of large numbers. Theory of Probability and its Applications, 14, 183-192.
- [8] Prokhorov, Yu. V. (1959). Some remarks on the strong law of large numbers. Theory of Probability and its Applications, 4, 204-208.
- [9] Rao, R. Ranga (1963). The law of large numbers for $D[0,1]$ -valued random variables. Theory of Probability and its Applications 8, 70-74.
- [10] Rohatgi, V. K. (1971). Convergence of weighted sums of independent random variables. Proc. Camb. Phil. Soc. 69, 305-307.
- [11] Skorokhod, A. V. (1956). Limit theorems for stochastic processes. Theory of Probability and its Applications, 1, 261-290.
- [12] Taylor, R. L. (1978). Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces. Lecture Notes in Mathematics, 672. Springer-Verlag, Berlin, Heidelberg, New York.
- [13] Taylor, R. L. and Daffer, P. Z. (1979). Convergence of weighted sums of random elements in $D[0,1]$. Journal of Multivariate Analysis 9. 1-12.

- [14] Taylor, R. L. and Daffer, P. Z. (1979). On the weak law of large numbers for $D[0,1]$. (Submitted for publication).
- [15] Taylor, R. L. and Padgett, W. J. (1973). Laws of Large Numbers for Normed Linear Spaces and Certain Fréchet Spaces. Lecture Notes in Mathematics 360. Springer-Verlag, Berlin, Heidelberg, New York.
- [16] Taylor, R. L. and Wei, Duan (1979). Laws of large numbers for tight random elements in normed linear spaces. Annals of Probability 7, 150-155.

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20. (cont.)

is preserved when random functions are centered, and this result is applied to improve some known strong laws for weighted sums in $D[0,1]$. A weak law of large numbers is proved using a new pointwise condition on the first moments and some weak laws for weighted sums are improved upon by weakening the hypotheses. A study is made of relationships among several conditions on (X_n) which appear as hypotheses in laws of large numbers.

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