AD-A091 954 LEE (J S) ASSOCIATES INC ARLINGTON VA F/G 12/1
MULTI-SENSOR DETECTION STUDY.(U)
SEP 80 L E MILLER N60921-80-C-0107
UNCLASSIFIED JTR-80-02 NL

1 OF 2
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MULTI-SENSOR DETECTION STUDY

JTR-80-02

September 30, 1980

Prepared for:

Signal Processing Branch, Code U22
Naval Surface Weapons Center
White Oak Laboratory
Silver Spring, Maryland 20910

Contract N60921-80-C-0107

Prepared by:

J. S. LEE ASSOCIATES, INC.
2001 Jefferson Davis Highway
Suite 1100, Crystal Plaza 1
Arlington, Virginia 22202
(703) 979-2230

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER

2. GOVT ACCESSION NO. AD-A091 954

3. RECIPIENT'S CATALOG NUMBER (9) Final rept. Apr-Sep 80

4. TITLE (and Subtitle)

(6) Multi-Sensor Detection Study.

5. TYPE OF REPORT & PERIOD COVERED

Final, April-September, 1980

6. PERFORMING ORG. REPORT NUMBER

(14) JTR-80-02

7. AUTHOR(s)

(10) Leonard E. Miller

8. CONTRACT OR GRANT NUMBER(s)

(15) N60921-80-C-0107

9. PERFORMING ORGANIZATION NAME AND ADDRESS

J. S. LEE ASSOCIATES, INC.
2001 Jefferson Davis Highway, Suite 1100
Arlington, Virginia 22202

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

NIF; 0; 0; 0U34QA

11. CONTROLLING OFFICE NAME AND ADDRESS

Code U22, Naval Surface Weapons Center
White Oak Laboratory
Silver Spring, Maryland 20910

12. REPORT DATE

(11) Sep[redacted]80 (30)

13. NUMBER OF PAGES

108 (12) 112

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Detection, multisensor, covariance, correlation, multivariate analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The methods of complex Gaussian multivariate analysis are applied to the problem of detecting signals in noise using data from multiple sensors whose relative locations are arbitrary. Theory is used to predict detection performance in terms of minimum detectable signal (MDS). A computer program implementing the multivariate detection approach is described, demonstrating that computational requirements are modest and verifying theory by simulation. Significant improvements over single-sensor processing (reduction in MDS by a factor equal to the number of sensors) is shown for certain cases.

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MULTISENSOR DETECTION STUDY

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MULTISENSOR DETECTION STUDY

1.0 INTRODUCTION

The intent of the work which is reported herein is to explore the possibility that improvements can be made in the initial detection of signals in noise using multiple sensors, arbitrarily located, by basing the detection strategy upon a multivariate statistical approach.

1.1 Background

This question of detection is part of the larger task depicted by Figure 1-1, in which the data from $\underline{m}$ sensors are somehow to be combined and operated upon to produce "decisions and numbers". The "decisions" we shall be concerned with here consist of detection of a source of interest in the medium being considered (e.g., underwater) along with whatever parameter estimates ("numbers"), are required to carry out detection. We restrict our attention to a single source, whose waveform we denote by $s(t;\theta)$ to indicate variation in time and dependence upon certain parameters θ. Whether this source of interest is present or not, the medium is such that there exists at each of m sensors a noise waveform $n_i(t,\eta_i)$, $i=1, 2,\ldots,m$; the noise parameters $\{\eta_i\}$ are in general different in value at each sensor. In this work, the signal and noise parameters are considered to be unknown *a priori* and therefore must be estimated. Also both random and deterministic signals are considered.

By "sensor" we shall refer in this work to whatever appropriate transducer and conditioning may be required to acquire data, plus additional processing such as sampling and analog-to-digital conversion. In some instances a discrete Fourier transform (DFT) is performed to obtain what is taken to be "sensor data". The physical locations of the sensors are assumed to be

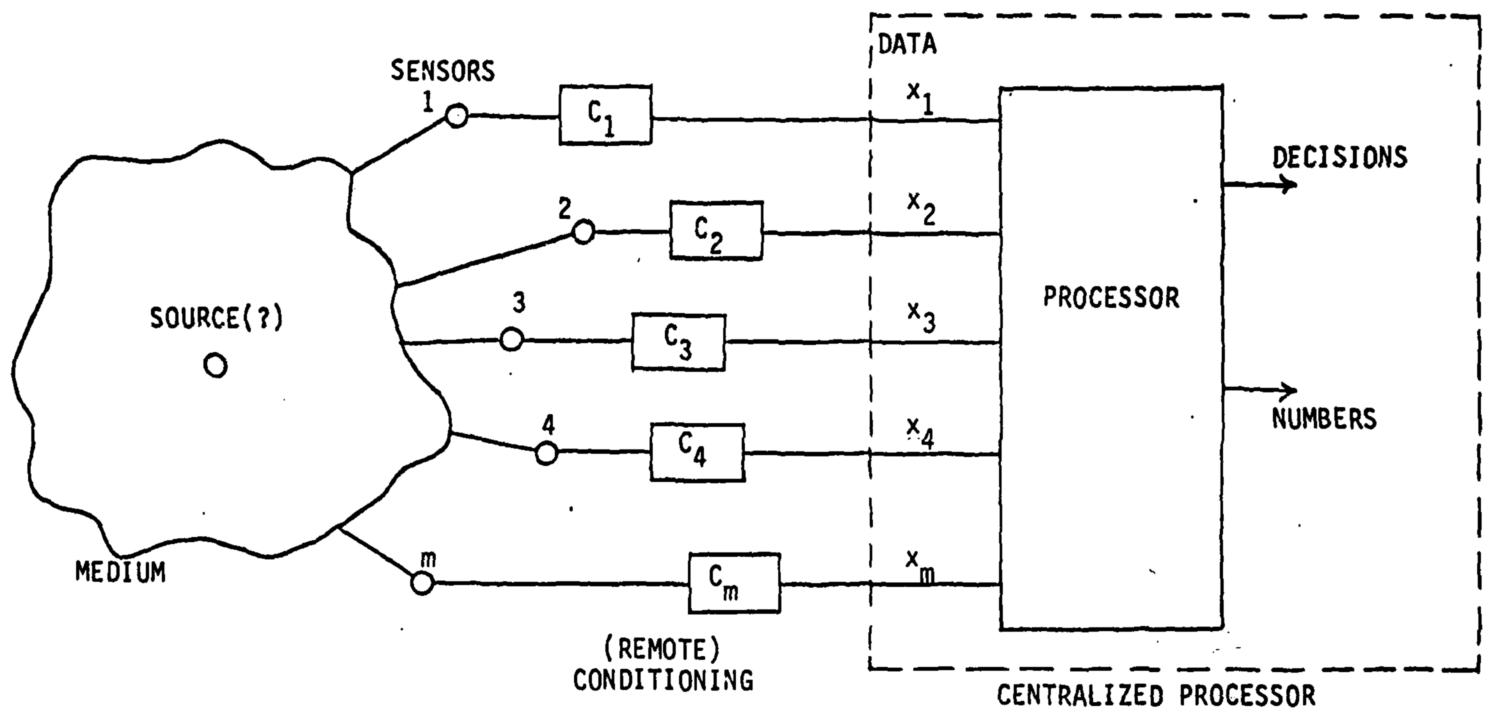


FIGURE 1-1 MULTISENSOR SYSTEM MODEL

different, so that observation in both space and time is performed by collection of sensors, whose outputs are assumed to be available to a centralized processor, not necessarily in "real time". Our primary concern is with the structure of this processor; therefore, our modelling effort begins with the "data" $x_i(t)$ from the m channels, as called out in Figure 1-1.

1.1.1 Concept of Data Vectors, Matrix

At a given sampling time t_k the data from the sensors may be considered as an $(m \times 1)$ vector $\underline{x}_k$. Over an observation period if $\underline{n}$ such vector samples are collected, the totality of the data may be represented by an $(m \times n)$ matrix X as illustrated in Figure 1-2. The columns of the matrix are the vector data samples:

$$X = \{\underline{x}_1, \underline{x}_2, \ldots \underline{x}_k, \ldots, \underline{x}_n\}. \tag{1-1}$$

The elements of the data matrix are

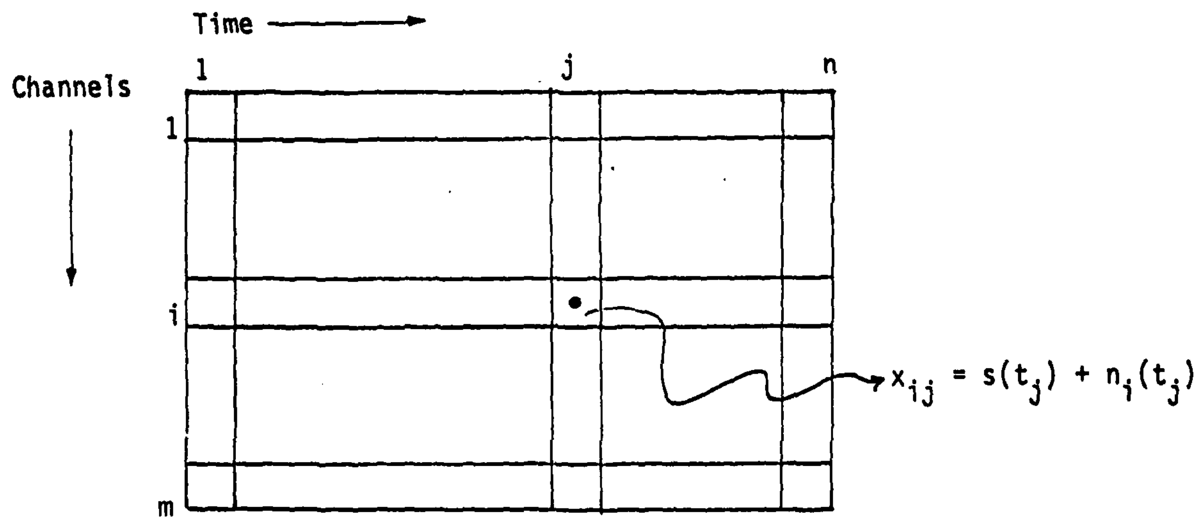
$$x_{ik} = x_i(t_k), i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, n. \tag{1-2}$$

As discussed in Section 1.3, for narrowband signals and noise, the $\{x_{ik}\}$ can be considered complex numbers describing the waveform components referenced to the center frequency. For broadband signals and noise, the $\{x_{ik}\}$ are more appropriately taken to be the complex Fourier coefficients of the input waveforms.

1.1.2 Objectives of the Study

An earlier work [1] explored the general subject of maximum likelihood (ML) signal detection and estimation when the data vector and matrix formulations are used. It was shown that in general, ML processors of vector data are analogous to their familiar scalar counterparts, but are not simple

DATA MATRIX



$\{X_{ij}\}$: mn Data Samples of, say,

m Jointly Gaussian Random Processes.

FIGURE 1-2 DATA MATRIX

"building block" or modular extensions of scalar processors. For example, the analogy to noise power (variance) on a single sensor is, for multiple sensors, the determinant of the matrix of covariance between pairs of sensors. This quantity is a nonlinear combination of the single channel variances and also of correlations; thus something more than just putting together m scalar processors is required to process m sensors properly.

In this study, the theme of ML detection using data from multiple sensors is pursued further, with the following objectives:

(a) Describe the theoretical performance of the ML detector of multisensor (multivariate) signals in noise, using conventional figures of merit.

(b) Develop an computer program which implements the ML detector and which can be used in experiments, also giving an indication of the complexity of the ML multivariate detector.

1.2 Methodology

In accordance with its objectives, this study employs the following analytical procedures.

1.2.1 Theoretical Base

In order to determine the maximum likelihood detection strategy using the sensor data these procedures are followed:

(a) Model the joint probability density function (pdf) for the data X at the sensor outputs under the hypotheses H_0: noise only, and H_1: signal plus noise.

(b) Take the pdf's $p_1(X;\theta,\eta|H_1)$ and $p_0(X;\eta|H_0)$ to be likelihood functions with θ and η representing signal and noise parameters, respectively. Maximize these likelihood functions by deriving maximum likelihood (ML) estimates $\hat{\theta}$ and $\hat{\eta}$ for these parameters, which are NOT ASSUMED TO BE KNOWN <u>a priori</u>.

(c) Use the ML detection criterion

$$\Lambda(X) = p_1(x;\hat{\theta},\hat{n}\ H_1)/p_0(x;\hat{n}|H_0) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \tag{1-3}$$

to find the statistic or function of the data z(X) which tests the hypothesis H_0. That is, the above equation reduces to

$$z(X) \underset{H_0}{\overset{H_1}{\gtrless}} z_0 \tag{1-4}$$

where z_0 is a threshold.

(d) Determine the distribution of the test statistic z(X) so that the probability integrals

$$Q(z_0;n|H_0) = \Pr\{z>z_0|H_0\} = \Pr\{H_0 \text{ rejected } |H_0\} \tag{1-5}$$

and

$$Q(z_0;\theta,n|H_1) = \Pr\{z>z_0|H_1\} = \Pr\{H_0 \text{ rejected } |H_1\} \tag{1-6}$$

may be computed.

1.2.2 Application

The ML detection strategy is applied to the sensor detection problem by carrying out the next steps:

(a) Compute the thresholds z_{oF} which satisfy

$$Q(z_{oF};n|H_0) = P_{FA} \tag{1-7}$$

for various values of the false alarm probability P_{FA} and using those thresholds, compute the detection probability

$$P_D = Q(z_{oF};\theta,n|H_1) \tag{1-8}$$

as a function of the actual values of the parameters (thus obtaining receiver operating characteristics or ROC).

(b) Relate the statistical detection parameters to the system variables (number of sensors, number of samples) as well as to conventional input data parameters (signal-to-noise ratios, signal and noise inter-sensor correlations, signal phases, etc.) in order to show the dependencies and to permit comparisons with other systems.

(c) Develop a computer program which accepts data in the form studied and which performs the statistical procedures involved in the detection strategies; exercise the program with simulated data to the extent necessary to verify that it works as predicted by theory.

1.2.3 Assessment

The performance of the multivariate detection strategy is to be assessed with respect to the following considerations:

(a) Minimum detectable signal.

(b) Complexity and storage requirements of implementation.

(c) Compatibility of configuration with other signal processing tasks.

1.3 Models and Notation

A glossary of mathematical symbols and notation employed is given at the end of the report. The concept of denoting the n samples of waveforms at m sensors by a data matrix X has been introduced previously. Although throughout the report the data is considered as narrowband (complex) time domain sampled data, as described next, the model used can also be extended to broadband data as developed by discrete Fourier transforms, as shown in Section 1.3.2. Complex data are treated to cover the subject thoroughly; real (baseband) data is somewhat simpler than the complex, and can be derived from it.

1.3.1 Narrowband Signals and Noise

The notation which is used is based upon the assumption that the vector $\underline{x}(t)$ of the m waveforms, when referenced to a given frequency and phase, can be represented by the narrowband (Rician) decomposition

$$\underline{x}(t) = u(t)\cos(\omega t + \phi) - \underline{v}(t)\sin(\omega t + \phi), \tag{1-9}$$

in which $\underline{u}(t)$ and $\underline{v}(t)$ are the in-phase and quadrature components of $\underline{x}(t)$ with respect to $\cos(\omega t + \phi)$. We may just as well represent $\underline{x}(t)$ as the (lowpass) complex vector waveform

$$\underline{x}(t) = \underline{u}(t) + j\underline{v}(t), \quad j = \sqrt{-1}, \tag{1-10}$$

and the matrix of samples is

$$X = ||x_{ik}|| = ||u_{ik} + jv_{ik}|| = U + jV. \tag{1-11}$$

The columns of these matrices retain the interpretation

$$(x_{ik})_{k=k_0} = \underline{x}_{k_0} = \underline{x}(t_{k_0}), \tag{1-12}$$

while the rows are the observations of the output of a single sensor over time:

$$(x_{ik})_{i=i_0} = \{x_{i_0}(t_k),\ k = 1, 2, \ldots, n\}. \tag{1-13}$$

An example of how this data may be collected in practice is diagrammed in Figure 1-3a.

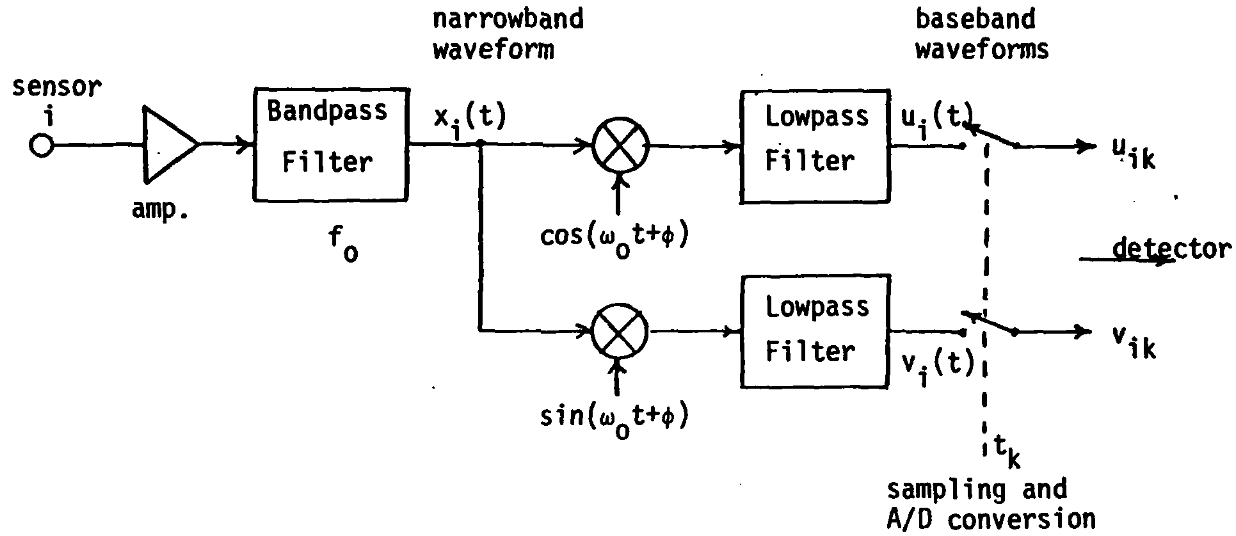
1.3.2 Multivariate Gaussian Model

Now if the waveforms are from stationary, jointly Gaussian random processes with $(m \times m)$ covariance matrix $\Sigma = ||\sigma_{ir}||$ and mean vector $\underline{\mu} = \underline{a} + j\underline{b}$, then the pdf for a single vector data sample is

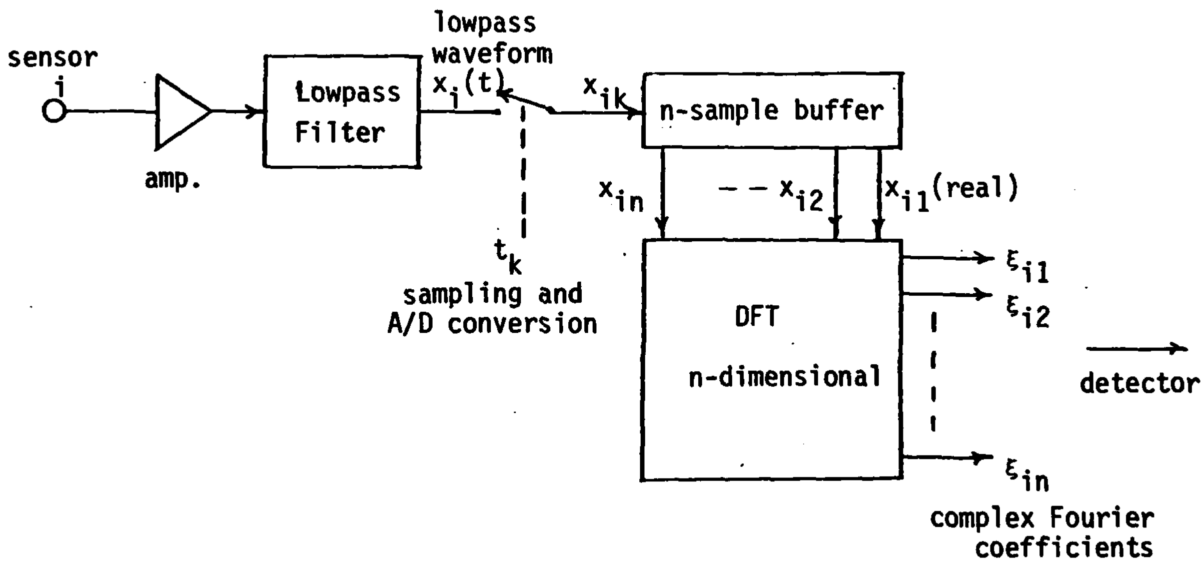
$$p(\underline{x}_k; \underline{\mu}, \Sigma) = [(2\pi)^m|\Sigma|]^{-1} \exp\left\{-\frac{1}{2}[(\underline{u}_k-\underline{a})'\Sigma^{-1}(\underline{u}_k-\underline{a}) + (\underline{v}_k-\underline{b})'\Sigma^{-1}(\underline{v}_k-\underline{b})]\right\}. \tag{1-14}$$

By noticing that the scalar quantities in the exponent may be understood as traces of matrices we can write

$$\begin{aligned}(\underline{u}_k-a)'\Sigma^{-1}(\underline{u}_k-\underline{a}) &= \mathrm{tr}[(\underline{u}_k-\underline{a})'\Sigma^{-1}(\underline{u}_k-\underline{a})] \text{ (scalar)} \\ &= \mathrm{tr}[\Sigma^{-1}(\underline{u}_k-\underline{a})(\underline{u}_k-\underline{a})']\end{aligned} \tag{1-15}$$



(a) NARROWBAND QUADRATURE PROCESSING
Data: $x_{ik} = u_{ik} + jv_{ik}$; i = sensor index,
k = time index.



(b) DISCRETE FOURIER TRANSFORM PROCESSING
Data: $\xi_{ik} = \xi_{ikR} + j\xi_{ikI}$; i = sensor index,

FIGURE 1-3. SCHEMATIC REPRESENTATION OF POSSIBLE SOURCES OF DATA AS MODELLED (not necessarily in real time).

since tr(PQ) = tr(QP). Therefore the pdf of $\underline{x}_k$ can alternately be written

$$p(\underline{x}_k;\underline{\mu},\Sigma) = [(2\pi)^m|\Sigma|]^{-1} \operatorname{etr}\left\{-\frac{1}{2}\Sigma^{-1}(\underline{x}_k-\underline{\mu})(\underline{x}_k-\underline{\mu})^*\right\}. \tag{1-16}$$

This form takes advantage of the facts that

$$\operatorname{tr}[PQ] = \operatorname{tr}[P\cdot \operatorname{Re}\{Q\}],\ P = P'(\text{real}),\ Q = Q^*; \tag{1-17}$$

$$\operatorname{tr}[PQ_1] + \operatorname{tr}[PQ_2] = \operatorname{tr}[P(Q_1 + Q_2)]; \tag{1-18}$$

and that $(\underline{u}_k-\underline{a})(\underline{u}_k-\underline{a})' + (\underline{v}_k-\underline{b})(\underline{v}_k-\underline{b})' = \operatorname{Re}\{\underline{x}_k\underline{x}_k^*\}$. (1-19)

For independent data vector samples, the joint pdf for all the observed data is given by

$$p(X) = \prod_{k=1}^{n} p(\underline{x}_k) = [(2\pi)^m|\Sigma|]^{-n} \exp\left\{-\frac{1}{2}\sum_{k=1}^{n}[(\underline{u}_k-\underline{a})'\Sigma^{-1}(\underline{u}_k-\underline{a}) + (\underline{v}_k-\underline{b})'\Sigma^{-1}(\underline{v}_k-\underline{b})]\right\}$$

$$= [(2\pi)^m|\Sigma|]^{-n} \operatorname{etr}\left\{-\frac{1}{2}\Sigma^{-1}\sum_{k=1}^{n}(\underline{x}_k-\underline{\mu})(\underline{x}_k-\underline{\mu})^*\right\}. \tag{1-20}$$

Also, by defining a matrix (m × n) whose columns all are equal to the mean vector,

$$M \equiv (\underline{\mu}, \underline{\mu}, \underline{\mu}, \ldots, \underline{\mu}), \tag{1-21}$$

then we may utilize the observation that

$$\sum_k(\underline{x}_k-\underline{\mu})(\underline{x}_k-\underline{\mu})^* = (X-M)(X-M)^* \tag{1-22}$$

to write, finally,

$$p(X) = [(2\pi)^m|\Sigma|]^{-n} \operatorname{etr}\left\{-\frac{1}{2}\Sigma^{-1}(X-M)(X-M)^*\right\}. \tag{1-23}$$

The identification of the mean vector and covariance matrix with signal and noise parameters will be made in the text in various ways. However throughout the report we shall assume that Gaussian form is suitable for representing the data.

1.3.3 Extension to Broadband Signals and Noise

In many applications the narrowband representation just discussed is not suitable. However, the same pdf model (1-23) can be used if the interpretation of the data matrix X is changed from time samples to Fourier coefficients, as illustrated in Figure 1-3b.

Consider that the time sampling takes place as previously described and that the n (real) samples from each sensor are transformed by a discrete Fourier transform (DFT). A new (complex) data matrix $F = ||\xi_{ir}||$ is then created. Its element at the i:th sensor and r:th frequency bin is given by [7]

$$\xi_{ir} = \frac{1}{n} \sum_{k=1}^{n} x_{ik}\, w^{r(k-1)}, \quad w = \exp\left\{-j2\pi/n\right\}. \qquad (1\text{-}24)$$

Assuming that the time samples have zero means, the covariance between elements is

$$E\left\{\xi_{ir}\bar{\xi}_{ps}\right\} = \frac{1}{n^2} \sum_{k=1}^{n}\sum_{\ell=1}^{n} E\left\{x_{ik}x_{p\ell}\right\} w^{r(k-1)-s(\ell-1)}. \qquad (1\text{-}25)$$

If the time sample vectors $\underline{x}_k$ are independent, then

$$E\left\{\xi_{ir}\bar{\xi}_{ps}\right\} = \frac{1}{n^2} \sum_{k=1}^{n} E\left\{x_{ik}x_{pk}\right\} w^{(r-s)(k-1)}$$

$$= \frac{1}{n^2}\, \sigma_{ip} \sum_{K=1}^{n} w^{(r-s)(k-1)} = \frac{\sigma_{ip}}{n}\, \delta_{rs}. \qquad (1\text{-}26)$$

That is, if the $\underline{x}_k$ are independent, then so also are the frequency sample vectors $\underline{\xi}_r$ which form the columns of F, and it is also evident from (1-26) that

$$E\left\{\underline{\xi}_r\underline{\xi}_r^*\right\} = \frac{1}{n}\sum = \frac{1}{n}\, E\left\{\underline{x}_k\underline{x}_k'\right\},\ \underline{x}_k \text{ independent.*} \qquad (1\text{-}27)$$

For $\underline{x}(t)$ real, the element obey the symmetry relationships

$$\xi_{ir} = \overline{\xi}_{i,n-r},\ r = 1,2,\ldots,n-1 \qquad (1\text{-}28)$$

so that in total for each sensor the n time samples produce n spectral samples ($\frac{n}{2}$ + 1 real, $\frac{n}{2}$ - 1 imaginary). If a subset of $n_F < \frac{n}{2} - 1$ frequencies is selected for processing ($r=r_0 > 0$ to $r_0 + n_F - 1 < \frac{n}{2}$), then there are n_F complex vectors in the data base.

1.3.4 Distribution of DFT Components

The real and imaginary components $\underline{\xi}_{rR}$ and $\underline{\xi}_{rI}$ of these vectors are independent, with covariance matrix

$$E\left\{\underline{\xi}_{rR}\underline{\xi}_{rR}'\right\} = E\left\{\underline{\xi}_{rI}\underline{\xi}_{rI}'\right\} = \frac{1}{2n}\Sigma \qquad (1\text{-}29)$$

Therefore the n_F vectors have the joint pdf

$$p(F;n_F) = [(2\pi)^m\, |\tfrac{1}{2n}\Sigma|]^{-n_F} \exp\left\{-n\sum_{r=1}^{n_F} \underline{\xi}_r^*\ \Sigma^{-1}\ \underline{\xi}_r\right\}. \qquad (1\text{-}30)$$

If the time samples contain a deterministic signal, then (1-33) is modified to become

$$p(F;n_F) = [(2\pi)^m\, |\tfrac{1}{2n}\Sigma|]^{-n_F} \exp\left\{-n\sum_{r=1}^{n_F}(\underline{\xi}_r - \underline{\psi}_r)^*\ \Sigma^{-1}(\underline{\xi}_r - \underline{\psi}_r)\right\}, \qquad (1\text{-}31)$$

where $\underline{\psi}_r$ is used to write the DFT component vectors of the signal.

It is evident, then that solution of the narrowband detection problem for the time-sampling model of the data X will furnish also the solution of the broadband detection problems for the DFT model of the data

*The independent sampling assumption is not difficult to justify in the case of one sensor since it is a matter of bandwidth. For multiple sensors, one can expect cross correlations to occur between different sensors at different times for directional sources. By ignoring this, in effect we are requiring that either the sensor field is small or that the data are aligned in time (i.e., the array is steered in the direction of the source).

F when it is assumed that $\underline{\psi}_r = \underline{\psi}$. This assumption corresponds to a signal whose spectrum is "flat" over the bandwidth spanned by the n_F frequency bins. For this reason, only the narrowband, time domain case is treated in detail.

2.0 MULTIVARIATE MAXIMUM LIKELIHOOD DETECTION STATISTICS

In this chapter the likelihood functions corresponding to hypotheses concerning signal and noise parameters are formed and used to derive statistics or functions of the data for testing the hypotheses (detection). The true values of the parameters are treated as unknown and are estimated so as to maximize the likelihood functions under the given hypothesis.

2.1 Unknown Deterministic Signals

If the signal portion $\{\underline{s}_k\}$ of the data vectors may be considered to be constant over the interval in which the samples are observed, then we may use the model

$$E\{\underline{x}_k\} \equiv \underline{\mu} = \underline{s},\ \underline{s}_k = \underline{s},\ k = 1, 2, \ldots, n. \qquad (2\text{-}1)$$

This model corresponds to a signal whose amplitude and phase with respect to the reference frequency and phase are "slowly varying" or constant (though different in value at each sensor), an idealistic extension of the narrowband assumption

$$\text{Bandwidth (signal)} \ll \text{center frequency} \qquad (2\text{-}2)$$

The covariance matrix Σ of the data in this case corresponds to that of the noise component.

If instead of time samples the data are Fourier coefficents, the assumption that the mean vector is a constant (over frequency) corresponds to a signal whose spectrum is flat over the n frequency bins.

Detection of the signal under these assumptions then is equivalent to choosing between the hypothesis

$$H_0:\ \underline{\mu} = \underline{0}. \qquad (2\text{-}3a)$$

and its alternative H_1: $\underline{\mu} \neq \underline{0}$. (2-3b)

2.1.1 Multivariate Processing

The pdf of the data given the presence of a signal is

$$p(X;\underline{\mu},\Sigma|H_1) = [(2\pi)^m|\Sigma|]^{-n}\ \mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}(X-M)(X-M)^*\right\}. \qquad (2\text{-}4)$$

Taking this function for the likelihood function corresponding to H_1 and maximizing it with respect to the unknown parameters $\underline{\mu}$ and Σ results in [1-3]

$$\hat{\underline{\mu}} = \frac{1}{n} \sum_{k=1}^{n} \underline{x}_k \tag{2-5}$$

and

$$\hat{\Sigma} = \frac{1}{2n} \sum_{k=1}^{n} (\underline{x}_k - \hat{\underline{\mu}})(\underline{x}_k - \hat{\underline{\mu}})^* \triangleq \frac{1}{2n} A. \tag{2-6}$$

That is, the ML estimates for the mean vector $\underline{\mu}$ and the covariance matrix Σ are the sample mean vector $\hat{\underline{\mu}}$ and the sample covariance $\hat{\Sigma}$. If analogous to the matrix M previously defined, we define the $(m \times n)$ matrix

$$X_0 \triangleq (\hat{\underline{\mu}}, \hat{\underline{\mu}}, \ldots, \hat{\underline{\mu}}), \tag{2-7}$$

then we can write

$$\begin{aligned} \hat{\Sigma} &= \frac{1}{2n} (X - X_0)(X - X_0)^* = \frac{1}{2n} (XX^* - X_0 X_0^*) \\ &= \frac{1}{2n} (X - M)(X - M)^* - \frac{1}{2} (\hat{\underline{\mu}} - \underline{\mu})(\hat{\underline{\mu}} - \underline{\mu})^*. \end{aligned} \tag{2-8}$$

We are not here concerned with the properties of these estimates, except that they are statistically independent as will be discussed. Substituting the estimates in the pdf (2-4) yields the H_1 likelihood function

$$L_1(X) = \max_{\underline{\mu}, \Sigma} p(X; \underline{\mu}, \Sigma | H_1) = [(2\pi)^m |\hat{\Sigma}|]^{-n} e^{-mn}. \tag{2-9}$$

Under H_0 the mean is assumed zero and the covariance ML estimate is

$$\hat{\Sigma}_0 = \frac{1}{2n} \sum_{k=1}^{n} \underline{x}_k \underline{x}_k^* = \frac{1}{2n} XX^* = \frac{1}{2n} A_0 \tag{2-10}$$

with the resulting likelihood function

$$L_0(X) = \max_{\Sigma} p(x; \Sigma | H_0) = [(2\pi)^m |\hat{\Sigma}_0|]^{-n} e^{-mn}. \tag{2-11}$$

The maximum likelihood detection in this case would be implemented by the test of the statistic

$$\Lambda(X) = \frac{|A_0|}{|A|} = \frac{|XX^*|}{|XX^* - X_0 X_0^*|}, \tag{2-12}$$

the ratio of the determinants of the estimated covariance matrices. This expression may be reduced to another form as follows:

$$\Lambda(X) = |A|^{-1}\ |A_0| = |A^{-1}|\,|A + n\,\hat{\underline{\mu}}\,\hat{\underline{\mu}}^*| = |I + nA^{-1}\ \hat{\underline{\mu}}\,\hat{\underline{\mu}}^*|. \qquad (2\text{-}13)$$

Middleton [4] gives the identity

$$|I + \gamma G| = \exp\left\{-\sum_{r=1}^{\infty} \frac{(-\gamma)^r}{r}\ \mathrm{tr}\ G^r\right\}; \qquad (2\text{-}14)$$

here, since $\mathrm{tr}\left[(A^{-1}\hat{\underline{\mu}}\hat{\underline{\mu}}^*)^r\right] = \mathrm{tr}\left[A^{-1}\hat{\underline{\mu}}(\hat{\underline{\mu}}^*A^{-1}\hat{\underline{\mu}})^{r-1}\hat{\underline{\mu}}^*\right]$

$$= (\hat{\underline{\mu}}^*A^{-1}\hat{\underline{\mu}})^r, \quad (\hat{\underline{\mu}}^*A^{-1}\hat{\underline{\mu}} \text{ is scalar})$$

we have $\Lambda(X) = \exp\left\{-\sum_{r=1}^{\infty} (-n\,\hat{\underline{\mu}}^*A^{-1}\underline{\mu})^r/r\right\}$

$$= \exp\left\{\ln\left[1 + n\,\hat{\underline{\mu}}^*A^{-1}\hat{\underline{\mu}}\right]\right\} = 1 + n\,\hat{\underline{\mu}}^*A^{-1}\hat{\underline{\mu}}. \qquad (2\text{-}15)$$

The ML detector performs the test

$$z(X) = n\hat{\underline{\mu}}^*A^{-1}\hat{\underline{\mu}} \ \underset{H_0}{\overset{H_1}{\gtrless}}\ z_0. \qquad (2\text{-}16)$$

Computational forms are to be discussed in another section; our next concern is to determine the distribution of this test statistic so that probabilities of false alarm and detection may be calculated.

2.1.2 Distribution of the Multivariate Test Statistic

If the data were real instead of complex, then the test statistic (2-16) could be identified with the "Hotelling T^2 statistic" [2, 5], so named because for m=1 it reduces to the square of the familiar Student's t-statistic (used for testing the mean of a normal population of unknown variance). It is shown by Anderson [2] that the T^2 statistic is distributed as Fisher's noncentral F-statistic with m and n-m degrees of freedom:

$$z(X) \text{ distributed as } \frac{m}{n-m} F_{m,n-m}(\lambda), \; X \text{ real}, \tag{2-17}$$

where the noncentrality parameter λ is

$$\lambda = n\underline{a}' \, \Sigma^{-1}\underline{a}, \tag{2-18}$$

to use $\underline{a} = \mathrm{Re}\{\underline{c}\}$ to represent real data (Σ is the true noise covariance matrix).

The question now is, what is the corresponding distribution for complex data? In Appendix A it is shown that in this case

$$z(X) \text{ distributed as } \frac{m}{n-m} F_{2m,2(n-m)}(\lambda) \, , \; X \text{ complex}, \tag{2-19}$$

where the noncentrality parameter now is

$$\lambda = n\underline{a}' \, \Sigma^{-1} \, \underline{a} + nb' \, \Sigma^{-1} \, \underline{b} = n\underline{\mu}^* \Sigma^{-1} \, \underline{\mu}. \tag{2-20}$$

2.1.3 Combined Single-Sensor Processing

The stated objectives of this study include determining what improvements in detection performance may be obtained by using multivariate processing of multiple sensor data, rather than some form of modular or "built up" approach based upon single-sensor processing. The fact that the ML detector has been shown to be indeed a multivariate processor is reason enough to expect improvements. However, it is natural to wonder just how much improvement can be expected, since multivariate processing at least seems to be a complicated procedure.

In order to provide a measure for detection improvements, we now consider ways of combining single-sensor detection processing (performance figures of merit are discussed in Chapter 4). The test statistic for a single-sensor is found by substituting m=1 into the multivariate expression, (2-12) to get at the ith sensor

$$\Lambda_i(X_i) = \hat{\sigma}^2_{0,i}/\hat{\sigma}^2_{1,i} = \frac{a_{ii} + n|\hat{\mu}_i|^2}{a_{ii}}$$

$$= 1 + n|\hat{\mu}_i|^2/a_{ii}, \tag{2-21}$$

where

$$a_{ii} = \sum_{k=1}^{n} |x_{ik} - \hat{\mu}_i|^2 \equiv 2n\hat{\sigma}^2_{1,i} \tag{2-22a}$$

and

$$\hat{\mu}_i = \frac{1}{n} \sum_{k=1}^{n} x_{ik} \tag{2-22b}$$

are the sample variance and mean. From (2-19) we know that

$$z_i(X_i) = n|\hat{\mu}_i|^2/a_{ii} \text{ distributed as } \frac{1}{n-1} F_{2,2n-2}(\lambda_i) \tag{2-23}$$

and the noncentrality parameter λ_i is related to the true mean and variance at sensor i by

$$\lambda_i = n|\mu_i|^2/\sigma_i^2. \tag{2-24}$$

One method for combining these functions of the data at single sensors is to sum or average them, as illustrated in Figure 2-1 when the combining operation is addition. That, is we can create a single test statistic from the m sensors by calculating

$$\Lambda_c = (X) = \frac{1}{m} \sum_{i=1}^{m} \Lambda_i(X_i)$$

$$= 1 + \frac{n}{m} \sum_{i=1}^{m} |\hat{\mu}_i|^2/a_{ii}. \tag{2-25}$$

Another method which follows naturally from the multivariate case is to combine the single sensor test statistic by forming their product. In effect, this is the multivariate solution for the special case of independence between sensor noises.

Unfortunately, the distribution for either of these combinations has not been found, so their detection performances cannot be calculated.

A method which can be evaluated is the following: perform individual detections for each sensor. If the decision at sensor i is "SIGNAL," assign the value $y_i = 1$; if the decision is "NO SIGNAL" at the single sensor i, assign the value $y_i = 0$. Then add up the $\{y_i\}$ and decide for the collection of sensors whether a signal is present using the rule

$$z_d \triangleq \sum_{i=1}^{m} y_i \underset{\text{NO SIGNAL}}{\overset{\text{SIGNAL}}{\gtrless}} k, \quad 0 < k \leq m. \tag{2-26}$$

This type of "majority vote" technique may be evaluated fairly readily. The threshold (identical) at each sensor is chosen by requiring that

$$\Pr\{z_d \geq k|H_0\} = \sum_{r=k}^{m} \binom{m}{r} p_F^r (1-p_F)^{m-r} = P_{FA}(\text{given}) \tag{2-27a}$$

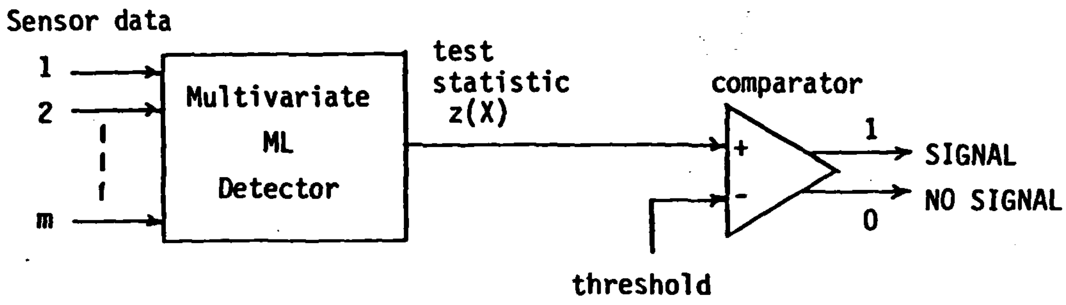
and solving for p_F as a function of P_{FA} and k. Then the required individual sensor detection parameters (all equal to λ_d) are found by requiring that

$$\Pr\{z_d \geq k|H_1\} = \sum_{r=k}^{m} \binom{m}{r} p_D^r (1-p_D)^{m-r} = P_D(\text{given}) \tag{2-27b}$$

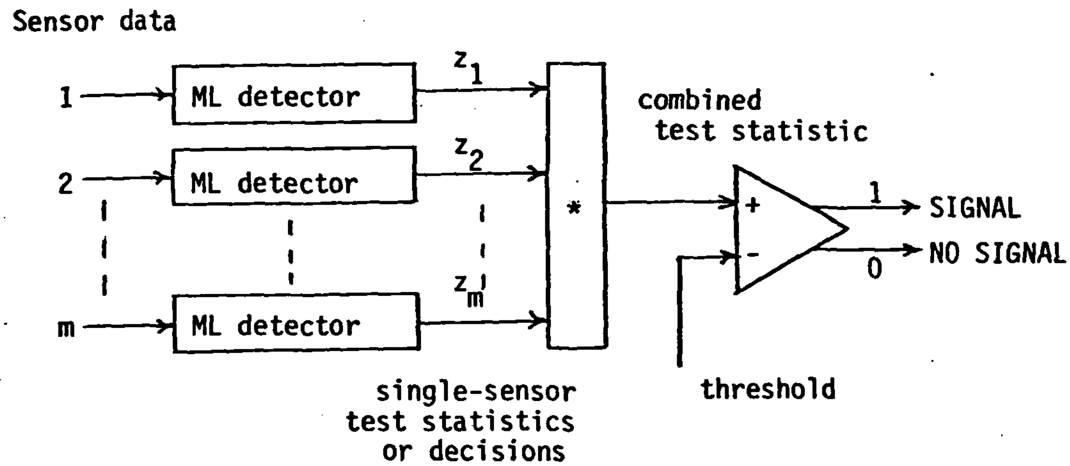
the solution for p_D can be related to λ_D.

Without regard to the distribution of the data (except assuming sensor data are independent) we may calculate p_F and p_D for given P_{FA} and P_D. For $P_{FA} = .01$ and $P_D = .9$, the results are

m	p_f	p_D	decision rule
2	.1	.9487	2 out of 2 (k = 2)
3	.0589	.8042	2 out of 3 (k = 2)
4	.0420	.6795	2 out of 4 (k = 2)
5	.1056	.7534	3 out of 5 (k = 3)
10	.1504	.6458	5 out of 10 (k = 5)
20	.2498	.6145	10 out of 20 (k = 10)



VS.



* = some combining operation

FIGURE 2-1. MULTIVARIATE VS. COMBINED SINGLE-SENSOR PROCESSING.

The value of p_F determines the threshold and one then computes the λ required to achieve a single sensor detection probability of p_D. How this is done is shown in Section 3.1; each sensor decision is based on the F-statistic distribution (2-19) with $m = 1$.

2.2 Random Signals

If the signal portion $\{\underline{s}_k\}$ of the data vectors may be considered to be independent samples of an m-dimensional, zero-mean random process, then we may use the model

$$E\{\underline{x}_k\} \equiv \underline{\mu} = \underline{0},\ k = 1, 2, \ldots, n. \tag{2-28}$$

The covariance matrix Σ of the data in this case corresponds to that of the noise component plus that of the signal component, when present.

Detection of the signal under these assumptions then is equivalent to choosing between the hypothesis

$$H_0\colon \Sigma = \Sigma_{noise} \equiv \Sigma_0 \tag{2-29a}$$

and its alternative $H_1\colon \ = \Sigma_{noise} + \Sigma_{signal} \equiv \Sigma_1$. (2-29b)

2.2.1 Multivariate Processing

The pdf of the data given the presence of a signal is

$$p(X;\Sigma|H_1) = [(2\pi)^m|\Sigma_1|]^{-n} \operatorname{etr}\{-\frac{1}{2}\Sigma_1^{-1} XX^*\}, \tag{2-30}$$

where $\Sigma_1 \equiv \Sigma_{noise} + \Sigma_{signal}$. Maximizing with respect to the unknown matrix Σ_1 results in the estimate

$$\hat{\Sigma}_1 = \frac{1}{2n} \sum_{k=1}^{n} \underline{x}_k \underline{x}_k^* = \frac{1}{2n} A_0, \tag{2-31}$$

and substituting the estimate in the pdf yields the likelihood function

$$L_1(X) = [(2\pi)^m\ |\hat{\Sigma}_1|]^{-n}\ e^{-mn}. \tag{2-32}$$

The H_0 likelihood function was found previously (equation (2-11)). Immediately we find that the likelihood ratio is identically one for testing these hypotheses because both Σ_0 and Σ_1 are estimated by the same quantity,

$A_0/2n$. Put simply, we cannot tell the difference between signal plus noise and noise only, under these assumptions. Therefore, the hypotheses need to be modified by additional assumptions.

Let the noise at each sensor be assumed independent of noise at different sensors, that is, let us take Σ_0 to be diagonal:

$$H_0': \quad \Sigma = \mathrm{dia}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2) \equiv \Sigma_0. \tag{2-33}$$

Under this hypothesis, the pdf of the data is

$$p(X; \Sigma_0|H_0') = \left[(2\pi)^m \prod_{i=1}^{m} \sigma_i^2\right]^{-n} \exp\left\{-\frac{1}{2} \sum_{i=1}^{m}\sum_{k=1}^{n} |x_{ik}|^2/\sigma_i^2\right\}. \tag{2-34}$$

Maximizing this pdf with respect to the noise powers $\{\sigma_i^2\}$ yields the estimates

$$\hat{\sigma}_i^2 = \frac{1}{2n} \sum_{k=1}^{n} |x_{ik}|^2 \equiv \frac{1}{2n} (A_0)_{ii} \tag{2-35}$$

and the likelihood function

$$L_0(X) = \left[(2\pi)^m \prod_{i=1}^{m} \hat{\sigma}_i^2\right]^{-n} e^{-mn}. \tag{2-36}$$

The likelihood ratio test then becomes

$$\left[\Lambda(X)\right]^{1/n} = \frac{|\hat{\Sigma}_0|}{|\hat{\Sigma}_1|} = \frac{\prod (A_0)_{ii}}{|A_0|} \underset{H_0}{\overset{H_1}{\gtrless}} 1, \tag{2-37}$$

that is, if the product of the diagonals of the sample covariance matrix A_0 is greater than its determinant, then we decide that there is some correlation existing between data from different sensors because a signal is present.

For example, suppose that we are examining data from two sensors ($m=2$). Then the test is

$$\left[\Lambda(X)\right]^{1/n} = \frac{a_{11}a_{22}}{a_{11}a_{22}-|a_{12}|^2} = \frac{1}{1 - R(x)} \underset{H_0'}{\overset{H_1}{\gtrless}} 1 \tag{2-38a}$$

or

$$R(X) \equiv \frac{|a_{12}|^2}{a_{11}a_{22}} \underset{H_0'}{\overset{H_1}{\gtrless}} 0. \tag{2-38b}$$

Here R(X) is the estimated square of the cross-correlation coefficient between sensors 1 and 2:

$$R(X) \equiv \frac{\left|\sum_{k=1}^{n} x_{1k}\bar{x}_{2k}\right|^2}{\left(\sum_{k=1}^{n} |x_{1k}|^2\right)\left(\sum_{k=1}^{n} |x_{2k}|^2\right)} \equiv |\hat{\rho}_{12}|^2. \tag{2-38c}$$

If the data in this case were the Fourier coefficients at a single frequency, then R(X) would correspond to what is called the "magnitude squared coherence function" estimate at that frequency [8].

For m sensors we can interpret the test statistic given by (2-37) by further development:

$$\frac{\prod (A_0)_{ii}}{|A_0|} = |I + R|^{-1} = \exp\left\{\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \operatorname{tr}(R^r)\right\} \underset{H_0'}{\overset{H_1}{\gtrless}} 1, \tag{2-39}$$

In this relation we have used (2-14) and have defined the matrix of sample correlation coefficients by

$$R \triangleq \begin{bmatrix} 0 & \hat{\rho}_{12} & \hat{\rho}_{13} \cdots & \hat{\rho}_{1m} \\ \hat{\bar{\rho}}_{12} & 0 & \rho_{23} \cdots & \rho_{2m} \\ \vdots & \vdots & & \\ \hat{\bar{\rho}}_{1m} & \hat{\bar{\rho}}_{2m} & \hat{\bar{\rho}}_{3m} \cdots & 0 \end{bmatrix}, \quad \hat{\rho}_{ir} \equiv \frac{a_{ir}}{\sqrt{a_{ii}a_{rr}}}, \ i \neq r. \tag{2-40}$$

The trace of R is zero, and we can rewrite (2-39) as

$$\frac{1}{2}\operatorname{tr}R^2 - \frac{1}{3}\operatorname{tr}R^3 + \ldots \underset{H_0'}{\overset{H_1}{\gtrless}} 0. \tag{2-41}$$

Now, the traces of the powers of R are

$$\operatorname{tr}R^2 = \sum_i \sum_r \rho_{ir}\,\rho_{ri} = 2\sum_{i<r} |\rho_{ir}|^2$$

$$\mathrm{tr}R^3 = \sum_i \sum_r \sum_p \hat{\rho}_{ir}\, \hat{\rho}_{rp}\, \hat{\rho}_{pi}, \text{ etc.} \tag{2-42}$$

Since $|\hat{\rho}_{ir}|<1$, we can see that an approximation to the test (2-41) would be to use the first term, or the statistic

$$z(X) = \frac{1}{2}\,\mathrm{tr}R^2(X) = \sum_{i<r} |\hat{\rho}_{ir}|^2$$

$$= \frac{\sum_{i<r} \left|\sum_k x_{ik}\bar{x}_{rk}\right|^2}{\left(\sum_k |x_{ik}|^2\right)\left(\sum_k |x_{rk}|^2\right)}. \tag{2-43}$$

This quantity is simply the sum of the magnitude squared of all the measured oorrelation coefficients between sensors.

2.2.2 Distribution of the Multivariate Test Statistic

In order to set a threshold for the test, we need to determine its distribution. For two sensors this task is not too difficult; in Appendix D, the pdf for $z(X) = |\hat{\rho}_{12}|^2$ is shown to be

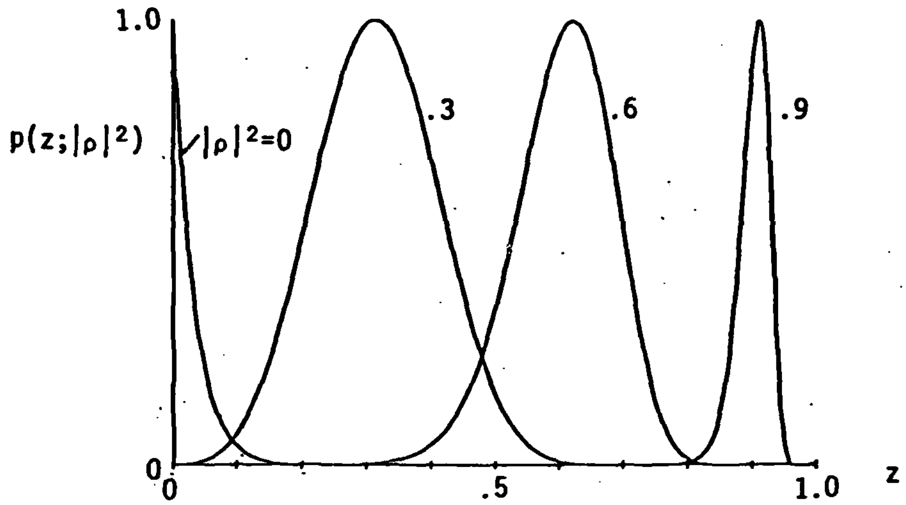
$$p(z) = (n-1)(1-\rho_{12}^2)^n\,(1-z)^{n-2}\;{}_2F_1(n,n;1;\rho_{12}^2 z), \tag{2-44}$$

where ρ_{12} is the true correlation coefficient and the ${}_2F_1(\quad)$ is the Gaussian hypergeometric function. This expression is in complete agreement with the pdf for the magnitude squared coherence function estimate reported in [8], which also gives the distribution function (see Figure 2-2)

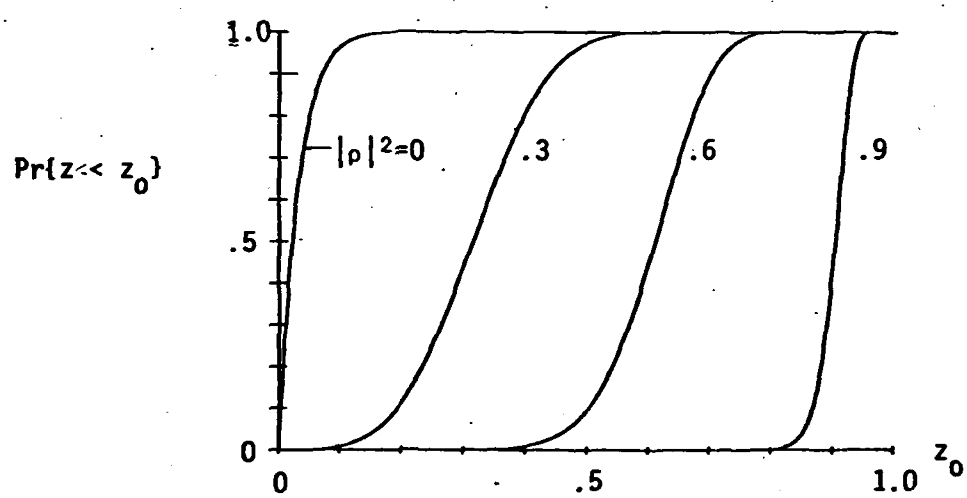
$$P_z(z_0) \equiv \Pr\left\{z \le z_0\right\}$$

$$= z_0 \left[\frac{1-\rho_{12}^2}{1-\rho_{12}^2 z_0}\right]^n \sum_{k=0}^{n-2} \left(\frac{1-z_0}{1-\rho_{12}^2 z_0}\right)^k {}_2F_1(-k,1-n;1;\rho_{12}^2 z_0). \tag{2-45}$$

The more general case is straightforward but extremely complicated [9, 10]; so much so, that the general practice among statisticians is to calculate moments of the test statistic (and these are tractable for H_0' only!).



(a) Probability density function (normalized by maximum value) for random signal test statistic, correlation varied (n=32).



(b) Cumulative distribution function for random signal test statistic, correlation varied (n=32).

FIGURE 2-2. DISTRIBUTION OF RANDOM SIGNAL TEST STATISTIC FOR TWO SENSORS (from [8])

Then the probability of rejecting H_0' (false alarm) is obtained by fitting a pdf to the moments. In Appendix B it is shown that the moments of $z(X) = |I + R|^{-1}$ are

$$E\left\{z^{\nu}\right\} = \prod_{k=2}^{m} \frac{B(n-\nu+1-k,\ k-1)}{B(n+1-k,\ k-1)}. \qquad (2\text{-}46)$$

By noting that

$$B(n-\nu+1-k,\ k-1) = \int_0^1 dx\ x^{n-k-\nu}(1-z)^{k-2}$$

$$= \int_0^1 dx\ x^{-\nu}\ p_{\beta}(x;n-k+1,\ k-1) \qquad (2\text{-}47)$$

we are able to say that the inverse of z is distributed as the product of m-1 independent beta-distributed variables. But again, this is not especially helpful for more than a few sensors. So the moment method is the reasonable way to proceed, and is considered further in the next chapter.

3.0 THEORETICAL CALCULATIONS

A large part of the work consists of obtaining numerical results to portray the theory developed and to enable assessment of the detection procedures. In this section the various computational methods used are explained and the results tabulated. Computational features of any software for implementing the detectors are considered separately in another chapter.

3.1 Probability Integral for the F-statistic

Both detection statistics being compared may be treated by considering the probability

$$\Pr\left\{F_{2n,2(n-m)}(\lambda)>\eta\right\} = \int_{\eta}^{\infty} dF\ p(F|n,m;\lambda).$$

$$\equiv Q(\eta|2n,2(n-m);\lambda) \tag{3-1}$$

This probability is given by [6]

$$Q(\eta|2n,2n-2m;\lambda) = e^{-\lambda/2}\sum_{k=0}^{\infty}\frac{(\lambda/2)^k}{k!}\, I_x(n-m,\ m+k) \tag{3-2a}$$

where $x \equiv \dfrac{n-m}{n-m+m\eta}$ (3-2b)

and Pearson's incomplete Beta function is given by

$$I_x(n-m,\ m+k) = 1 - I_{1-x}(m+k,\ n-m)$$

$$= \int_0^x d\xi\ \frac{\xi^{n-m-1}(1-\xi)^{m+k-1}}{B(n-m,\ m+k)} \tag{3-3}$$

A particularly simple computational form is

$$I_x(n-m,\ m+k) = x^{n-m}\sum_{r=0}^{m+k-1}\frac{(1-x)^r}{r!}\,(n-m)_r, \tag{3-4}$$

using $(n-m)_r \equiv (n-m)(n-m+1)\ldots(n-m+r-1)$. (3-5)

3.1.1 False Alarm Threshold

For no signal, λ=0 and it is obvious from equation (3-2) that the probability of false alarm is computed using

$$P_{FA} = Q(z_0|2n,\ 2n-2m,0)$$
$$= I_{x_0}(n-m,m), \quad x_0 \equiv \frac{n-m}{n-m+mz_0} \quad . \qquad (3\text{-}6)$$

Since we wish to calculate the threshold z_0 (or x_0) for given values of P_{FA}, it was convenient to perform the calculation of (3-6) using (3-4) on a calculator (hp 34C), iterating on x_0 until the desired P_{FA} was obtained. (Program P-1).

For the special case of m=1, which applies to a single sensor, (3-6) reduces to the simple form (see (3-5))

$$Q(z_0|2n,\ 2n-2) = x_0^{n-1} \ . \qquad (3\text{-}7)$$

Table 3-1 gives z_0 and x_0 for P_{FA} = .1, .01, and .001 respectively.

3.1.2 Detection Probability

Given the threshold corresponding to a chosen false alarm probability the probability of detection P_D is computed using (3-2) with $\eta = z_0$. That is, $P_D = Q(z_0|2n,\ 2n-2m,\lambda) = P_D(\lambda)$. This was done using the FORTRAN program P-2. The results are tabulated in Table 3-2.

The behavior of the probability of detection can be understood by considering Figures 3-1 and 3-2, which are plots of data selected from the tables.

P_{FA}	m	n = 10	20	50	100	200
10^{-1}	1	.774267	.885867	.954095	.977010	.988496
		2.6239	2.4479	2.3375	2.3296	2.3160
	2	.63164	.810237	.922922	.961279	.980595
		2.33272	2.10604	2.00436	1.97375	1.93932
	3	.50992	.743495	.89501	.947135	.973477
		2.24255	1.95499	1.83779	1.80471	1.78913
	4	.40058	.68141	.868716	.933764	.966738
		2.24457	1.87018	1.73793	1.70243	1.68591
	5	.30097	.62247	.84345	.920878	.960233
		2.32259	1.81951	1.67046	1.63248	1.61514
	10		.35793	.72559	.860288	.929548
			1.79384	1.51276	1.46161	1.44004
	20			.510666	.74769	.872118
				1.43734	1.34981	1.31970
10^{-2}	1	.599484	.784760	.910298	.954548	.977124
		6.0129	5.2112	4.8285	4.7140	4.6589
	2	.45595	.6982	.87207	.93482	.96711
		4.77289	3.89029	3.52073	3.41651	3.33284
	3	.3437	.62593	.83929	.91779	.958431
		4.45553	3.38653	2.99990	2.89622	2.84809
	4	.25001	.56125	.80925	.90209	.95041
		4.49976	3.12695	2.71069	2.60488	2.55670
	5	.17098	.50175	.78095	.88721	.94279
		4.84864	2.97907	2.52443	2.41545	2.36658
	10		.25395	.6541	.81954	.90792
			2.93778	2.11527	1.98177	1.92695
	20			.43655	.69929	.84515
				1.93603	1.72009	1.64900
10^{-3}	1	.464159	.695193	.868511	.932603	.965883
		10.389912	8.330548	7.418375	7.154455	7.029047
	2	.3349	.60661	.82662	.91051	.95455
		7.943864	5.836551	5.033897	4.815993	4.713792
	3	.23885	.53459	.79131	.89173	.944863
		7.435699	4.933357	4.131727	3.925773	3.831945
	4	.16285	.47145	.75936	.87459	.936
		7.710930	4.484463	3.644332	3.441430	3.350427
	5	.10253	.41445	.7296	.8585	.92765
		8.753243	4.238509	3.335526	3.131625	3.041718
	10		.18904	.59904	.78655	.890005
			4.289886	2.677350	2.442375	2.348195
	20			.38378	.66204	.82372
				2.408489	2.041931	1.926043

$$P_{FA} = Q(z_0 | 2m, 2n-2m) = I_{x_0}(n-m, m), \quad x_0 = (n-m)/(n-m + mz_0)$$

Table entries: x_0 (top), z_0

TABLE 3-1. FALSE ALARM THRESHOLDS FOR F-STATISTIC.

$n = 10$

λ(dB)	P_{FA} = .1							.01							.001						
	m = 1	2	3	4	5	10	20	1	2	3	4	5	10	20	1	2	3	4	5	10	20
-6	.1257	.1158	.1113	.1085	.1066			.0148	.0126	.0118	.0113	.0109			.0017	.0013	.0012	.0012	.0011		
-3	.1516	.1318	.1228	.1172	.1133			.0202	.0155	.0136	.0126	.0119			.0025	.0017	.0015	.0013	.0012		
0	.2036	.1649	.1466	.1352	.1270			.0325	.0220	.0178	.0155	.0140			.0045	.0027	.0020	.0017	.0015		
3	.3065	.2335	.1965	.1728	.1556			.0632	.0381	.0279	.0222	.0186			.0105	.0053	.0035	.0026	.0021		
6	.4947	.3725	.3018	.2533	.2169			.1453	.0821	.0548	.0396	.0301			.0315	.0140	.0080	.0052	.0036		
8	.6698	.5243	.4255	.3515	.2930			.2647	.1515	.0976	.0667	.0473			.0721	.0314	.0167	.0098	.0062		
10	.8459	.7151	.6014	.5020	.4150			.4664	.2878	.1866	.1237	.0830			.1710	.0771	.0394	.0215	.0122		
12	.9606	.8901	.8004	.6988	.5914			.7277	.5179	.3584	.2413	.1580			.3793	.1925	.1005	.0529	.0278		
14	.9965	.9809	.9456	.8843	.7940			.9301	.7925	.6238	.4548	.3072			.6915	.4346	.2521	.1365	.0693		
15				.9458	.8807				.8986	.7696	.5947	.4194			.8357	.5998	.3803	.2162	.1109		
16	.9999	.9992	.9950	.9807	.9434			.9949	.9632	.8804	.7380	.5536			.9345	.7641	.5406	.3310	.1757		
17										.9543	.8607	.6966				.8927	.7110	.4808	.2716		
18											.9433	.8262				.9658	.8560	.6499	.4021		
19												.9212					.9480	.8068	.5602		
20																		.9193	.7236		
21																			.8604		
22																			.9475		

$n = 20$

λ(dB)	P_{FA} = .1							.01							.001						
	m = 1	2	3	4	5	10	20	1	2	3	4	5	10	20	1	2	3	4	5	10	20
-6	.1275	.1178	.1135	.1110	.1093	.1048		.0155	.0132	.0123	.0118	.0115	.0107		.0018	.0015	.0013	.0012	.0012	.0011	
-3	.1551	.1360	.1275	.1223	.1188	.1097		.0217	.0168	.0149	.0138	.0131	.0114		.0028	.0020	.0017	.0015	.0014	.0012	
0	.2107	.1736	.1563	.1459	.1386	.1198		.0362	.0252	.0208	.0183	.0167	.0130		.0055	.0039	.0026	.0022	.0019	.0014	
3	.3205	.2517	.2175	.1960	.1808	.1409		.0732	.0467	.0357	.0295	.0255	.0165		.0139	.0075	.0052	.0040	.0033	.0018	
6	.5185	.4087	.3462	.3039	.2728	.1868		.1728	.1077	.0780	.0609	.0498	.0252		.0444	.0228	.0145	.0103	.0077	.0031	
8	.6974	.5745	.4932	.4331	.3863	.2459		.3154	.2045	.1479	.1133	.0902	.0386		.1056	.0552	.0344	.0234	.0169	.0052	
10	.8687	.7697	.6878	.6180	.5578	.3462		.5433	.3866	.2906	.2254	.1788	.0672		.2491	.1414	.0899	.0607	.0428	.0104	
12	.9706	.9267	.8763	.8226	.7677	.5063		.8044	.6573	.5374	.4395	.3598	.1318		.5194	.3434	.2361	.1659	.1185	.0251	
14	.9979	.9911	.9785	.9596	.9341	.7193		.9645	.9019	.8233	.7365	.6475	.2735		.8331	.6747	.5330	.4132	.3156	.0693	
15						.8246				.9237	.8659	.7957	.3888		.9347	.8325	.7136	.5922	.4782	.1177	
16		.9998	.9991	.9975	.9941	.9097		.9986	.9923	.9771	.9500	.9091	.5337		.9831	.9381	.8657	.7718	.6654	.1978	
17													.6928				.9570	.9069	.8344	.3207	
18													.8365						.9436	.4877	
19													.9357							.6772	
20																				.8442	
21																				.9485	
22																					

TABLE 3-2(a) PROBABILITY OF DETECTION FOR F-STATISTIC (n = 10,20) n = 50

λ(dB)	P_{FA} = .1, m = 1	2	3	4	5	10	20	.01, m = 1	2	3	4	5	10	20	.001, m = 1	2	3	4	5	10	20
-6	.1285	.1190	.1149	.1124	.1108	.1068	.1039	.0159	.0136	.0127	.0122	.0119	.0111	.0106	.0019	.0015	.0014	.0013	.0013	.0011	.0011
-3	.1572	.1385	.1301	.1252	.1219	.1138	.1079	.0227	.0177	.0157	.0146	.0139	.0123	.0112	.0031	.0022	.0019	.0017	.0016	.0013	.0012
0	.2149	.1787	.1620	.1520	.1452	.1282	.1160	.0387	.0274	.0228	.0203	.0186	.0149	.0125	.0063	.0039	.0030	.0026	.0023	.0017	.0013
3	.3285	.2624	.2297	.2094	.1952	.1593	.1331	.0797	.0526	.0411	.0347	.0305	.0211	.0155	.0164	.0093	.0067	.0052	.0044	.0026	.0017
6	.5319	.4292	.3713	.3326	.3044	.2286	.1707	.1905	.1252	.0947	.0768	.0650	.0383	.0229	.0545	.0302	.0204	.0152	.0121	.0058	.0028
8	.7125	.6017	.5297	.4773	.4368	.3184	.2200	.3467	.2396	.1835	.1483	.1241	.0674	.0343	.1306	.0756	.0512	.0376	.0291	.0122	.0048
10	.8803	.7968	.7300	.6745	.6273	.4664	.3066	.5871	.4471	.3593	.2980	.2525	.1343	.0596	.3035	.1940	.1372	.1026	.0798	.0313	.0098
12	.9751	.9421	.9067	.8707	.8352	.6763	.4537	.8414	.7283	.6357	.5586	.4936	.2856	.1195	.6021	.4490	.3481	.2765	.2236	.0922	.0250
14	.9984	.9942	.9875	.9782	.9667	.8844	.6696	.9766	.9402	.8964	.8487	.7992	.5658	.2620	.8920	.7915	.6969	.6109	.5344	.2725	.0758
15						.9514	.7864			.9650	.9413	.9129	.7323	.3847	.9656	.9153	.8555	.7911	.7256	.4370	.1359
16			.9997	.9992	.9984	.9857	.8870	.9994	.9971	.9925	.9851	.9744	.8731	.5437	.9933	.9776	.9531	.9206	.8815	.6382	.2395
17							.9552						.9590	.7185					.9674	.8264	.3998
18														.8689						.9462	.6067
19														.9596							.8089
20																					.9412

n = 100

λ(dB)	P_{FA} = .1, m = 1	2	3	4	5	10	20	.01, m = 1	2	3	4	5	10	20	.001, m = 1	2	3	4	5	10	20
-6	.1288	.1194	.1153	.1129	.1113	.1074	.1047	.0161	.0138	.0128	.0123	.0120	.0112	.0108	.0020	.0016	.0014	.0013	.0013	.0012	.0011
-3	.1579	.1393	.1310	.1262	.1229	.1150	.1095	.0231	.0180	.0160	.0149	.0142	.0126	.0115	.0032	.0023	.0019	.0017	.0016	.0014	.0012
0	.2163	.1804	.1639	.1540	.1473	.1308	.1194	.0395	.0281	.0235	.0210	.0193	.0156	.0133	.0066	.0041	.0032	.0027	.0024	.0018	.0014
3	.3312	.2659	.2337	.2137	.1999	.1650	.1406	.0819	.0547	.0431	.0365	.0323	.0228	.0172	.0174	.0100	.0072	.0057	.0048	.0030	.0020
6	.5363	.4358	.3795	.3419	.3146	.2417	.1877	.1966	.1313	.1007	.0827	.0706	.0433	.0275	.0582	.0331	.0229	.0173	.0139	.0070	.0037
8	.7173	.6103	.5413	.4912	.4527	.3408	.2500	.3572	.2516	.1960	.1610	.1367	.0789	.0444	.1397	.0834	.0581	.0437	.0345	.0157	.0070
10	.8839	.8049	.7426	.6912	.6478	.5015	.3592	.6011	.4668	.3824	.3231	.2783	.1613	.0835	.3226	.2135	.1559	.1202	.0960	.0425	.0165
12	.9764	.9463	.9148	.8834	.8528	.7191	.5378	.8523	.7493	.6653	.5954	.5362	.3430	.1783	.6284	.4845	.3884	.3187	.2662	.1284	.0473
14	.9986	.9950	.9895	.9822	.9734	.9140	.7704	.9797	.9495	.9140	.8758	.8365	.6504	.3919	.9074	.8230	.7435	.6705	.6045	.3662	.1529
15							.8753	.9959	.9868	.9733	.9561	.9358	.8095	.5545	.9725	.9338	.8884	.8396	.7895	.5584	.2697
16			.9998	.9995	.9990	.9921	.9488	.9995	.9979	.9949	.9902	.9836	.9248	.7315	.9951	.9844	.9682	.9471	.9218	.7604	.4459
17									.9998	.9995	.9988	.9977	.9813	.8801		.9980	.9950	.9899	.9826	.9113	.6621
18												.9998	.9976	.9657			.9996	.9991	.9981	.9812	.8551
19													.9998	.9948						.9982	.9637
20														.9996							.9958

TABLE 3-2(b) PROBABILITY OF DETECTION FOR F-STATISTIC (n = 50,100) n = 200

λ(dB)	P_{FA} = .1, m = 1	2	3	4	5	10	20	.01, 1	2	3	4	5	10	20	.001, 1	2	3	4	5	10	20
-6	.1290	.1196	.1155	.1131	.1115	.1077	.1051	.0162	.0138	.0129	.0124	.0121	.0113	.0108	.0020	.0016	.0014	.0013	.0013	.0012	.0011
-3	.1582	.1397	.1315	.1267	.1234	.1156	.1103	.0232	.0182	.0162	.0151	.0143	.0127	.0117	.0032	.0023	.0019	.0018	.0016	.0014	.0012
0	.2170	.1812	.1648	.1550	.1484	.1321	.1211	.0400	.0285	.0239	.0213	.0196	.0159	.0136	.0067	.0042	.0033	.0028	.0025	.0019	.0015
3	.3325	.2676	.2357	.2159	.2022	.1679	.1441	.0831	.0557	.0441	.0375	.0332	.0237	.0181	.0179	.0104	.0075	.0060	.0051	.0032	.0022
6	.5384	.4391	.3835	.3465	.3196	.2481	.1957	.1996	.1344	.1038	.0856	.0735	.0460	.0300	.0602	.0347	.0242	.0185	.0149	.0077	.0043
8	.7196	.6145	.5469	.4980	.4604	.3518	.2643	.3624	.2578	.2024	.1675	.1432	.0850	.0499	.1444	.0876	.0617	.0469	.0375	.0178	.0084
10	.8856	.8088	.7486	.6992	.6576	.5182	.3839	.6080	.4767	.3939	.3357	.2921	.1755	.0967	.3323	.2237	.1658	.1296	.1049	.0491	.0208
12	.9770	.9483	.9186	.8892	.8608	.7383	.5749	.8575	.7593	.6795	.6131	.5568	.3719	.2104	.6412	.5023	.4086	.3404	.2885	.1492	.0622
14	.9886	.9953	.9903	.9838	.9761	.9256	.8086	.9811	.9536	.9217	.8877	.8528	.6883	.4560	.9144	.8374	.7649	.6982	.6337	.4146	.2017
15	.9998	.9993	.9982	.9965	.9942	.9741	.9046	.9963	.9883	.9766	.9620	.9450	.8406	.6292	.9754	.9417	.9024	.8602	.8170	.6151	.3473
16			.9998	.9995	.9991	.9941	.9654	.9996	.9983	.9958	.9920	.9869	.9427	.8004	.9958	.9871	.9740	.9571	.9370	.8093	.5489
17						.9942	.9920			.9996	.9991	.9983	.9875	.9249	.9996	.9985	.9962	.9926	.9875	.9384	.7643
18							.9990						.9986	.9833			.9997	.9994	.9988	.9893	.9206
19														.9982						.9992	.9861
20																					.9990
21																					
22																					

TABLE 3-2(c) PROBABILITY OF DETECTION FOR F-STATISTIC
(n = 200)

Figure 3-1 shows P_D vs λ, the noncentrality parameter, for n=50 and various values of P_{FA} and m. This type of plot is also known as "receiver operating characteristics" (ROC). As $\lambda \to 0$ ($-\infty$ dB), $P_D(\lambda) \to P_{FA}$ and this trend can be observed on the left side of the figure. Also the approach of $P_D(\lambda) \to 1$ as $\lambda \to \infty$ is seen on the right.

What is interesting in this figure is that for all P_{FA},

$$P_D(\lambda;\ m_1) < P_D(\lambda;\ m_2) \quad \text{for } m_1 > m_2. \tag{3-8}$$

This behavior is attributable to the loss of degrees of freedom in estimating $\underline{\mu}$ and Σ; the larger m given n, the smaller the "left over" degrees of freedom 2(n-m). Thus, for example, if P_{FA} = .1 and a P_D of .5 is desired then a larger value of λ(6.9 dB) is required for m=2 than for m=1 (5.6 dB). At first glance, it might seem that there is a disadvantage in increasing m (more data is bad)! However, as will be discussed in Chapter 4, other things being equal, the increased m automatically insures a λ sufficiently larger to produce an improved P_D.

Figure 3-2 illustrates how for fixed m(=3), the probability of detection increases with greater n, the number of vector samples observed. This trait is entirely expected since both degrees of freedom and the amount of information (data) are proportional to n. The most informative aspect of the figure is the demonstration that as n increases, the curves converge to a "limit curve". This happens because

$$P_D(\lambda;n,m,z_0) \to Q(\chi^2|2m,\lambda),\ \chi^2 = 2mz_0 \tag{3-9}$$

as $n \to \infty$: that is, the noncentral F distribution approaches the noncentral chi-squared distribution [6]. Therefore, we may write

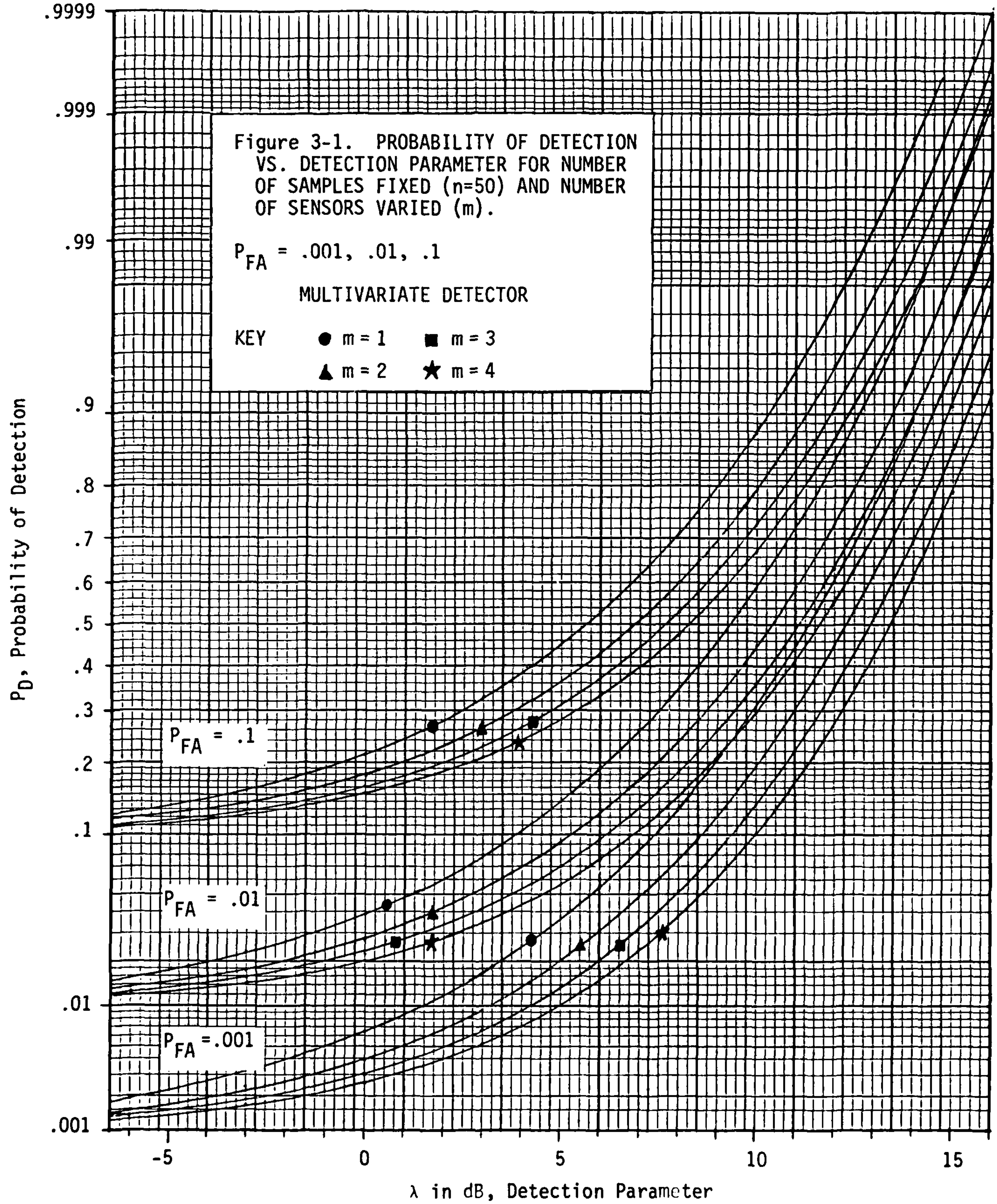


Figure 3-1. PROBABILITY OF DETECTION VS. DETECTION PARAMETER FOR NUMBER OF SAMPLES FIXED (n=50) AND NUMBER OF SENSORS VARIED (m).
P_{FA} = .001, .01, .1
MULTIVARIATE DETECTOR
KEY ● m = 1 ■ m = 3 ▲ m = 2 ★ m = 4
P_{FA} = .1
P_{FA} = .01
P_{FA} =.001
.9999
.999
.99
.9
.8
.7
.6
.5
.4
.3
.2
.1
.01
.001
P_D, Probability of Detection
-5
0
5
10
15
λ in dB, Detection Parameter

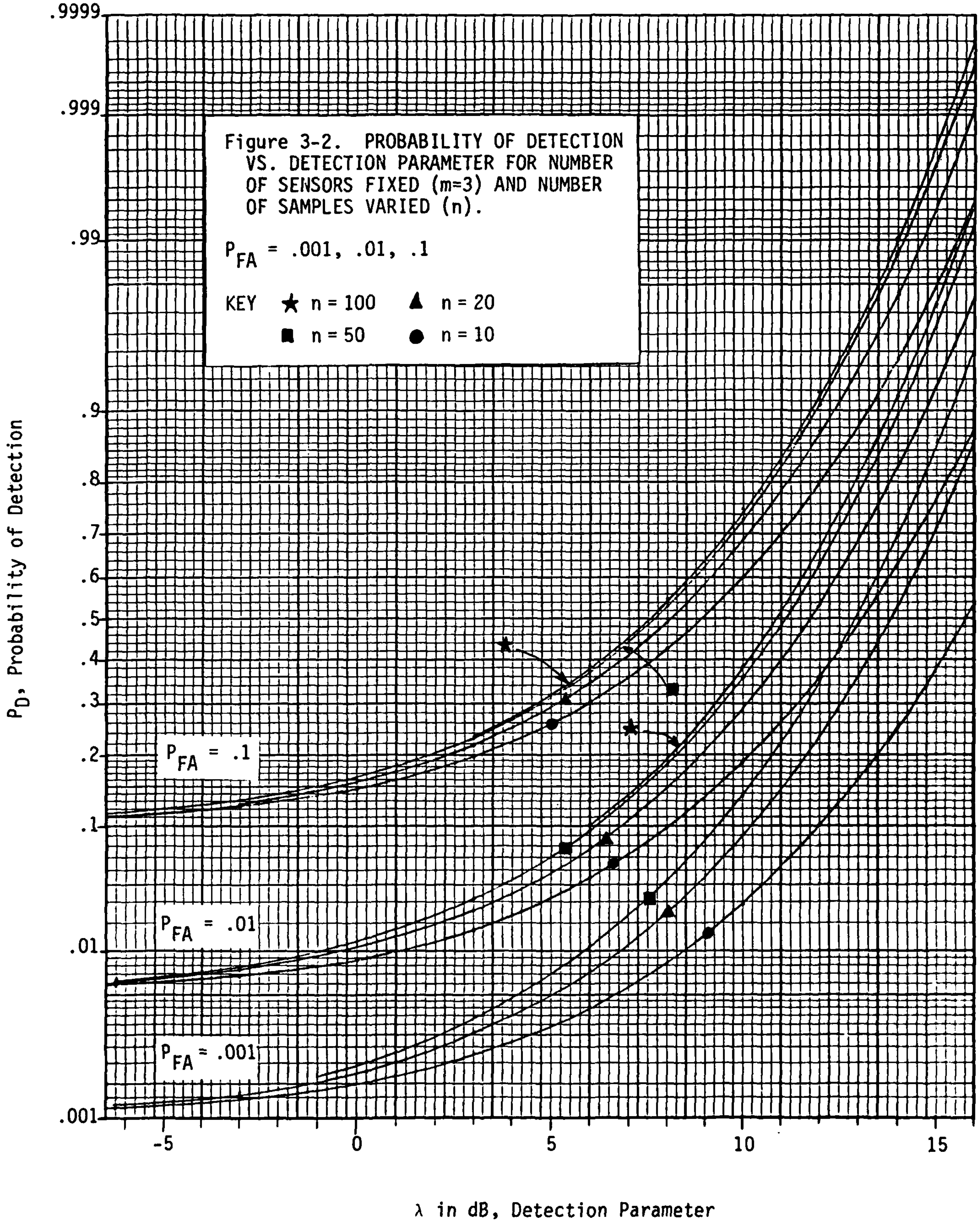


Figure 3-2. PROBABILITY OF DETECTION VS. DETECTION PARAMETER FOR NUMBER OF SENSORS FIXED (m=3) AND NUMBER OF SAMPLES VARIED (n).
P_{FA} = .001, .01, .1
KEY ★ n = 100 ▲ n = 20
■ n = 50 ● n = 10
P_{FA} = .1
P_{FA} = .01
P_{FA} = .001
.9999
.999
.99
.9
.8
.7
.6
.5
.4
.3
.2
.1
.01
.001
P_D, Probability of Detection
-5
0
5
10
15
λ in dB, Detection Parameter

$$Q(z_0|2m,\infty,\lambda) = e^{-\lambda/2} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} Q_{\chi^2}(2mz_0|2m+2k) \qquad (3\text{-}10)$$

$$= e^{-mz_0-\lambda/2} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} \sum_{r=0}^{m+k-1} \frac{(mz_0)^r}{r!} . \qquad (3\text{-}11)$$

3.2 Asymptotic Distribution of Random Signal Test Statistic

In Appendix B it is shown that the moments of the statistic given by (2-37) for testing the hypothesis H_0' (that the data is uncorrelated between sensors) can be written

$$E\{z^\nu\} = \prod_{k=1}^{m-1} \frac{\Gamma(n)\Gamma(n-\nu-k)}{\Gamma(n-k)\Gamma(n-\nu)} . \qquad (3\text{-}12)$$

Therefore the characteristic function for the new variable

$$y = 2\beta n \ \ell n\ z, \quad 0\leq\beta<1 \qquad (3\text{-}13)$$

is given by

$$\phi_y(t) = E\{e^{jty}\} = E\{z^{j2t\beta n}\}$$

$$= \prod_{k=1}^{m-1} \frac{\Gamma(n)\Gamma[\beta n(1-2jt)-k + n(1-\beta)]}{\Gamma(n-k)\Gamma[\beta n(1-2jt) + n(1-\beta)]} . \qquad (3\text{-}14)$$

We may use the asymptotic expansion [2, p. 204]

$$\ell n\Gamma(x + h) \sim \frac{1}{2} \ell n\ 2\pi + (x + h - \tfrac{1}{2})\ \ell n x - x$$

$$- \sum_{r=1}^{L-1} \frac{(-1)^r}{r(r+1)} x^{-r} B_{r+1}(h) + O(x^{-L}) \qquad (3\text{-}15)$$

which $x \to \infty$ and h is bounded and the $B_r(\cdot)$ are Bernoulli polynomials

[6, ch. 23]. Applying this expansion to (3-14) yields the expression

$$\phi_y(t) \sim (1-2it)^{-a/2} \exp\left\{\sum_{r=1}^{L} \omega_r[(1-2jt)^{-r} - 1]\right\} \tag{3-16a}$$

where $a = 2\sum_{k=1}^{m-1} k = m(m-1)$ (3-16b)

and

$$\omega_r = \frac{(-1)^r}{r(r+1)} (\beta n)^{-r} \sum_{k=1}^{m-1} \left\{B_{r+1}[n(1-\beta)] - B_{r+1}[n(1-\beta)-k]\right\}. \tag{3-16c}$$

Since $(1-2it)^{-\nu/2}$ is the characteristic function of a chi-squared variable with ν degrees of freedom, and $\omega_r \propto \frac{1}{n^r}$, the cumulative distribution function of y may be calculated using the asymptotic series

	Asymptotic Order
$\Pr\left\{y \leq u\right\} = P_a$	1
$+ \omega_1(P_{a+2} - P_a)$	n^{-1}
$+(\omega_2 + \frac{1}{2}\omega_1^2)P_{a+4} - \omega_1^2 P_{a+2} - (\omega_2 - \frac{1}{2}\omega_1^2)P_a$	n^{-2}
$+(\omega_3 + \omega_1\omega_2 + \frac{1}{6}\omega_1^3)P_{a+6} - \omega_1(\omega_2 + \frac{1}{2}\omega_1^2)P_{a+4}$ $- \omega_1(\omega_2 - \frac{1}{2}\omega_1^2)P_{a+2} - (\omega_3 - \omega_1\omega_2 + \frac{1}{6}\omega_1^3)P_a$	n^{-3}
$+ \ldots$	(3-17a)

where $P_\nu \triangleq P_r\left\{\chi_\nu^2 \leq u\right\}$. (3-17b)

This expression can be simplified considerably by choosing a value for the arbitrary coefficient β in (3-13) so as to make $\omega_1=0$. This value turns out to be

$$\beta = \frac{n-(m+1)/3}{n}. \tag{3-18}$$

With this value the asymptotic series becomes

$$
\begin{aligned}
\Pr\left\{y \le u\right\} = {} & P_a + \omega_2(P_{a+4} - P_a) + \omega_3(P_{a+6} - P_a) \\
& + \left[\omega_4(P_{a+8} - P_a) + \frac{1}{2}\,\omega_2^2(P_{a+8} - 2P_{a+4} + P_a)\right] \\
& + \left[\omega_5(P_{a+10} - P_a) + \omega_2\omega_3(P_{a+10} - P_{a+6} - P_{a+4} + P_a)\right] \\
& + \Big[\omega_6(P_{a+12} - P_a) + \omega_2\omega_4(P_{a+12} - P_{a+8} - P_{a+4} - P_a) \\
& + \frac{1}{2}\,\omega_3^2(P_{a+12} - 2P_{a+6} + P_a) \\
& + \frac{1}{6}\,\omega_2^3(P_{a+12} - 3P_{a+8} + 3P_{a+4} - P_a)\Big] + O(n^{-7}). \qquad (3\text{-}19)
\end{aligned}
$$

To calculate the ω_r's, we need the Bernoulli polynomials; these are listed conveniently in Table 23.1 of [6]. After the necessary algebra, we have

$$
\begin{aligned}
\omega_2 &= \frac{(m-2)(m-1)m(m+1)}{72[n-(m+1)/3]^2} \\
\omega_3 &= \frac{(m-2)(m-1)m(m+1)(2m-1)}{1620[n-(m+1)/3]^3} \\
\omega_4 &= \frac{(m-2)(m-1)m(m+1)(m^2-m-7)}{2160[n-(m+1)/3]^4} \qquad (3\text{-}20)
\end{aligned}
$$

Values of these coefficients for various numbers of sensors m and of samples n are given in Table 3-3.

We observe from these expressions that $\omega_r=0$ for m=2. This is true because [6]

$$B_{r+1}(1) - B_{r+1}(0) = B_{r+1}(0)[1-(-1)^r]$$

and

$$B_{2k+1} \equiv 0, \quad k > 0. \qquad (3\text{-}21)$$

Therefore for m=2, $\beta=(n-1)/n$ and a=2, with the results

$$y_2 = 2(n-1)\ln z_2 \text{ is } \chi_2^2. \qquad (3\text{-}22)$$

		n				
	m	10	20	50	100	200
ω_2	3	.004438	9.566(-4)	1.407(-4)	3.424(-5)	8.446(-6)
	4	.02400	.004959	7.134(-4)	1.724(-4)	4.237(-5)
	5	.07813	.01543	.002170	5.206(-4)	1.275(-4)
	10		.4123	.05124	.01185	.002854
	20			1.079	.2307	.05356
ω_3	3	1.138(-4)	1.139(-5)	6.426(-7)	7.712(-8)	9.447(-9)
	4	8.960(-4)	8.415(-5)	4.592(-6)	5.453(-7)	6.646(-8)
	5	.003906	3.429(-4)	1.808(-5)	2.125(-6)	2.577(-7)
	10		.02132	9.339(-4)	1.039(-4)	1.227(-5)
	20			.04349	.004299	4.810(-4)
ω_4	3	-1.969(-6)	-9.151(-8)	-1.981(-9)	-1.172(-10)	-7.133(-12)
	4	5.760(-5)	2.459(-6)	5.090(-8)	2.971(-9)	1.795(-10)
	5	5.290(-4)	2.064(-5)	4.082(-7)	2.349(-8)	1.410(-9)
	10		.004276	6.604(-5)	3.534(-6)	2.048(-7)
	20			.007255	3.316(-4)	1.788(-5)

Table 3-3 COEFFICIENTS FOR ASYMPTOTIC EXPANSION

For m=3, for example, the cumulative probability distribution function (3-19) becomes

$$\begin{aligned}
\Pr\left\{y \le u\right\} &= \Pr\left\{z^{2(n-4/3)} \le e^{u}\right\} \\
&= \Pr\left\{\chi_6^2 \le u\right\} + \frac{[n-4/3]^{-2}}{3}\left[\Pr\left\{\chi_{10}^2 \le u\right\} - \Pr\left\{\chi_6^2 \le u\right\}\right] \\
&\quad + \frac{2[n-4/3]^{-3}}{27}\left[\Pr\left\{\chi_{12}^2 \le u\right\} - \Pr\left\{\chi_6^2 \le u\right\}\right] \\
&\quad + [n-4/3]^{-4}\left[\frac{2}{45}\Pr\left\{\chi_{14}^2 \le u\right\} - \frac{1}{9}\Pr\left\{\chi_{10}^2 \le u\right\} + \frac{1}{15}\Pr\left\{\chi_6^2 \le u\right\}\right] \\
&\quad + 0(n^{-5}).
\end{aligned} \tag{3-23}$$

The required χ^2 probabilities can be found in tables such as *Table 26.8 of [6], or calculated using*

$$\Pr\left\{\chi_{2\nu}^2 \le u\right\} = 1 - \sum_{k=0}^{\nu-1} e^{-u/2} \frac{(u/2)^k}{K!} \tag{3-24}$$

as shown by program P-4 listed in the back of the report.

3.2.1 False Alarm Probability

We now wish to use the asymptotic distribution (3-19) to calculate the false alarm probability for the multisensor problem with random signals. The relation needed is

$$\begin{aligned}
P_{FA} &= \Pr\left\{z > z_0\right\} \\
&= 1 - \Pr\left\{z \le z_0\right\} \\
&= 1 - \Pr\left\{z^{2n\beta} \le z_0^{2n\beta}\right\} \\
&= 1 - \Pr\left\{y \le 2n\beta \ln z_0\right\};
\end{aligned} \tag{3-25}$$

that is, the threshold z_0 is found by converting the value of $u=u_0$ for which $\Pr\left\{y \le u_0\right\} = 1-P_{FA}$ by the operation

$$z_0 = \exp\left\{\frac{u_0}{2n\beta}\right\} . \tag{3-26}$$

For large numbers of samples it is evident from (3-16b) (3-18), and (3-20) that to an excellent approximation,

$$z_0 = \exp\left(\chi^2_{1-P_{FA}},m(m-1)\right)/\left(n - \frac{m+1}{3}\right) . \tag{3-27}$$

Also, since $z(x) = |I + R|^{-1} \approx \exp\left\{\frac{1}{2}\,\mathrm{tr}R^2\right\}$ as shown in Section 2.2.1, we may interpret (3-27) as a statement about the average values of the measured intersensor correlations. That is $z > z_0$ is equivalent to $\frac{1}{2}\,\mathrm{tr}R^2 > \ln z_0$, or

$$\frac{m(m-1)}{2}\,|\rho|^2_{\text{average}} > \left(\chi^2_{1-P_{FA}},\ m(m-1)\right)/\left(n - \frac{m+1}{3}\right) \tag{3-28}$$

Both z_0 and this $|\rho|_{\text{average}}$ defined by (3-28) are given for n=100, 200 and several values of m in Table 3-4. For example, for m=5 sensors and n=100, the ratio of the determinants of the estimated covariance matrix under H_0' and H_1 respectively has a 1% chance of exceeding the value 1.4672 when H_0' is true; this value would be exceeded by an average inter-sensor correlation coefficient of about 0.2 if H_1 were true.

P_{FA}	m	a= m(m-1)	$\chi^2_{1-P_{FA},a}$	n = 100 z_0	n = 100 $\lvert\rho\rvert_{average}$	n = 200 z_0	n = 200 $\lvert\rho\rvert_{average}$
P_{FA} = .1	2	2	4.60517	1.0476	.2157	1.0234	.1521
	3	6	10.6446	1.1139	.1896	1.0550	.1336
	4	12	18.5494	1.2076	.1773	1.0980	.1249
	5	20	28.4120	1.3363	.1703	1.1543	.1198
	10	90	107.565	3.0544	.1575	1.7296	.1103
	20	380	415.728	87.3735	.1534	8.6195	.1065
P_{FA} = .01	2	2	9.21034	1.0975	.3050	1.0474	.2151
	3	6	16.8119	1.1858	.2383	1.0883	.1680
	4	12	26.2170	1.3055	.2108	1.1413	.1484
	5	20	37.5662	1.4672	.1958	1.2089	.1377
	10	90	124.116	3.6270	.1692	1.8817	.1185
	20	380	447.1	122.4285	.1591	10.1409	.1104
.001	2	2	13.816	1.1498	.3736	1.0719	.2635
	3	6	22.458	1.2556	.2754	1.1197	.1941
	4	12	32.909	1.3975	.2362	1.1805	.1663
	5	20	45.315	1.5877	.2150	1.2572	.1513
	10	90	137.208	4.1550	.1779	2.0114	.1246
	20	380	470.9	158.1337	.1632	11.4719	.1133

TABLE 3-4. THRESHOLDS FOR RANDOM SIGNAL DETECTION.

3.3 Calculation of Multivariate Test Statistic

The statistics z(X) derived in Chapter two were derived, for analytical purposes, from forms originally expressed as ratios of the determinants of sample covariance matrices. Computationally it is simpler to calculate these determinants than to perform the matrix inversion called for by the analytical form. In this section, the particular method used for computing determinants is documented, and it is shown how to convert thresholds for the F-statistic to those for the ratio of determinants.

3.3.1 Computing the Determinants of the Sample Covariance Matrices

A well known iterative method for computing the determinant of a symmetric matrix is the "forward Doolittle, left side" scheme [15]. That basic approach is adapted here for the complex, Hermitian sample covariance matrices of the data. It is to successively reduce the dimension of the determinant by subtracting a multiple of the first row from the other rows so as to make the first column all zeros except for the "top" or "pivot" element. For example, for a 3×3 determinant, the algorithm works as follows:

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 0 & \alpha_{22} - \dfrac{\alpha_{22}}{\alpha_{11}}\alpha_{12} & \alpha_{23} - \dfrac{\alpha_{21}}{\alpha_{11}}\alpha_{13} \\ 0 & \alpha_{32} - \dfrac{\alpha_{31}}{\alpha_{11}}\alpha_{12} & \alpha_{33} - \dfrac{\alpha_{31}}{\alpha_{11}}\alpha_{13} \end{vmatrix}$$

$$= \alpha_{11} \begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} = \alpha_{11} \begin{vmatrix} \beta_{11} & \beta_{12} \\ 0 & \beta_{22} - \dfrac{\beta_{21}}{\beta_{11}}\beta_{12} \end{vmatrix}$$

$$= \alpha_{11}\beta_{11}\gamma_{11}. \tag{3-28}$$

An $m \times m$ determinant is found by an algorithm which may be written as follows:

Let $D = |A|$

Initially, $D = D_0 = 1$

For $i = 1$ to $m-1$

$D_i = D_{i-1} \cdot a_{ii}^{[i-1]}$ $\quad (A^{[0]} \equiv A)$

$p_i = a_{ii}^{[i-1]}$

For $k = i+1$ to m

For $\ell = i+1$ to m

$a_{k\ell}^{[i]} = a_{k\ell}^{[i-1]} - a_{k,i}^{[i-1]} \cdot a_{i\ell}^{[i-1]}/p_i$

next ℓ

next k

next i

Finally, $D = D_{m-1} \cdot a_{mm}^{[m-1]}$

3.3.2 Computer Implementation

The application of this algorithm to the complex-valued sample covariance matrices involved in the detection problems under consideration was tested by means of the simple simulation listed as program P-5. In this program, a Gaussian random number generator (ostensibly producing independent, zero-mean, unit variance numbers) was used to develop five-dimensional complex data vectors, $\underline{x}_k$.

The mean vector and H_0 sample covariance matrix were "built" or accumulated iteratively by the relations (n = number of samples = 10)

$$\hat{\mu}_i^{[k]} = \hat{\mu}_i^{[k-1]} + x_{ik}/n \ , \ i = 1,2,\ldots m(=5) \qquad (3\text{-}29)$$

$$\left.\begin{aligned}(\hat{\Sigma}_0^{[k]})_{i\ell} &\equiv \sigma_{0,i\ell}^{[k]} \\ &= \sigma_{0,i\ell}^{[k]} + x_{ik} \cdot \overline{x}_{\ell k}/2n\end{aligned}\right\} \quad i,\ell = 1,2,\ldots m. \tag{3-30}$$

A diagram of the simulation is shown in Figure 3-3. The estimated covariance matrix under the H_1 hypothesis is formed by subtracting one half the outer product of the mean vector from the H_0 hypothesis covariance matrix. Then the determinants of these matrices are computed by the algorithm given above, and the test statistic is taken to be (the real part of) their ratio.

Figure 3-4 shows the results of the calculations. The ratio for ten samples equals 1.813 for this particular set of random vectors. In the next section it is shown how to calculate threshold against which this number is to be tested.

3.3.3 Obtaining False Alarm Threshold for Ratio of Determinants

The statistic z(X) for which the false alarm detection probabilities were computed in Section 3.1 related to the ratio of covariance determinants by

$$\begin{aligned}\frac{|\hat{\Sigma}_0|}{|\hat{\Sigma}_1|} = R(X) &= 1 + n\underline{\hat{\mu}}^* \hat{\Sigma}_1^{-1} \underline{\hat{\mu}} \\ &= 1 + z(X) \\ &= 1 + \frac{m}{n-m} F_{2m,2(n-m)}.\end{aligned} \tag{3-31}$$

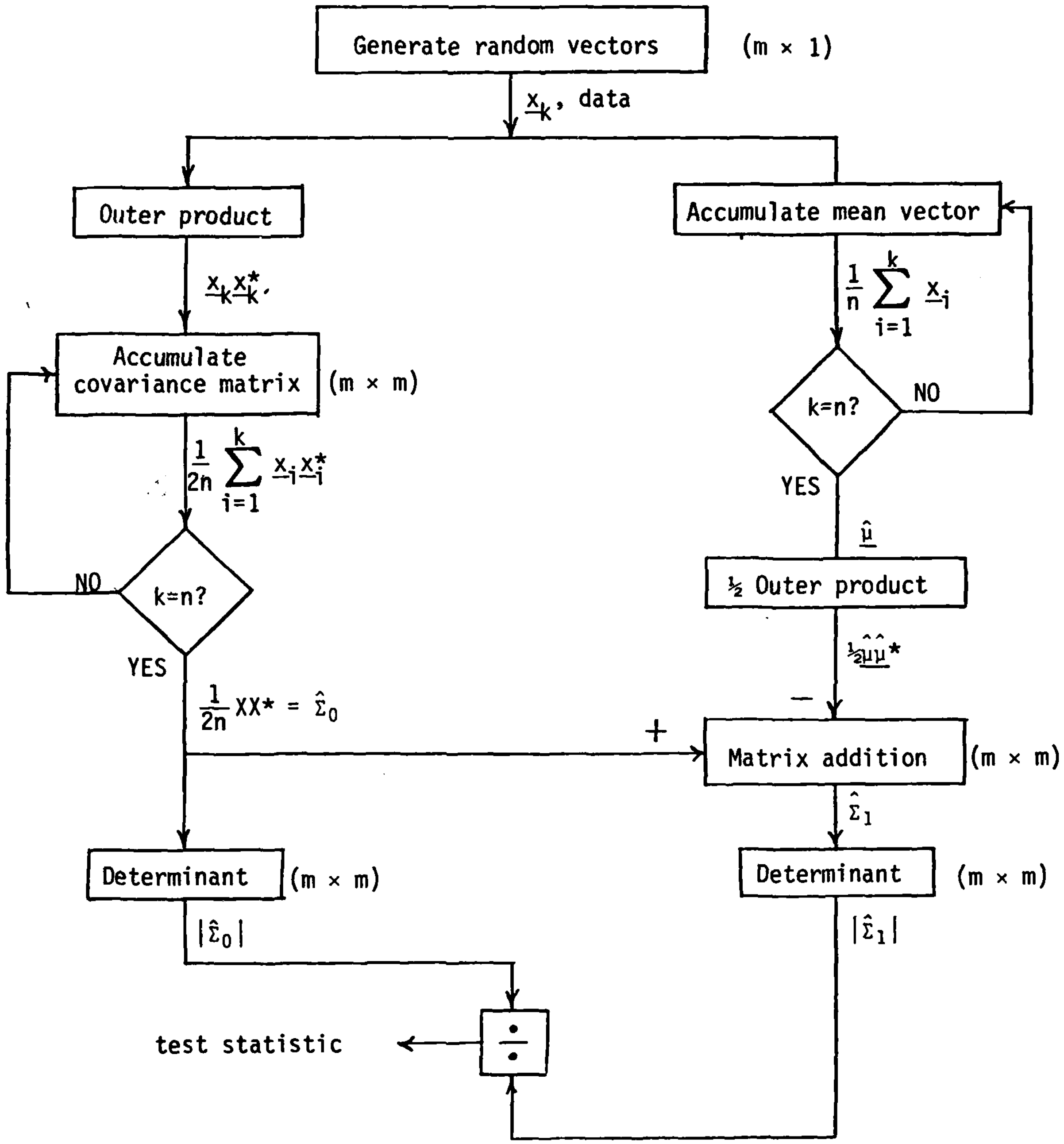


FIGURE 3-3. DIAGRAM OF MULTIVARIATE TEST STATISTIC CALCULATION.

MEAN VECTOR

(-0.163	0.351)	(0.347	-0.127)	(0.085	-0.316)	(-0.048	0.260)	(0.155	-0.323)

MATRIX XX*

(0.664	0.0)	(-0.241	-0.059)	(-0.293	0.182)	(0.276	0.097)	(-0.196	0.056)
-0.241	0.059	0.778	0.0	-0.014	-0.341	-0.039	-0.076	0.036	-0.048
-0.293	-0.182	-0.014	0.341	0.883	0.0	0.129	-0.117	0.002	-0.143
0.276	-0.097	-0.039	0.076	0.129	0.117	0.608	0.0	0.176	-0.076
-0.196	-0.056	0.036	0.048	0.002	0.143	0.176	0.076	0.974	0.0

MATRIX (X-XO)(X-XO)*

(0.590	0.0)	(-0.191	-0.110)	(-0.230	0.193)	(0.226	0.084)	(-0.126	0.055)
-0.191	0.110	0.710	0.0	-0.049	-0.391	-0.014	-0.034	-0.012	-0.094
-0.230	-0.193	-0.049	0.391	0.830	0.0	0.172	-0.114	-0.055	-0.132
0.226	-0.084	-0.014	0.034	0.172	0.114	0.573	0.0	0.222	-0.089
-0.126	-0.055	-0.012	0.094	-0.055	0.132	0.222	0.089	0.910	0.0

DETS AND RATIO

(0.068	0.0)	(0.037	0.0)	(1.813	0.0)
$\|\hat{\Sigma}_0\|$		$\|\hat{\Sigma}_1\|$		test statistic	

format: (real part, imaginary part)

FIGURE 3-4 SAMPLE OUTPUT OF PROGRAM P-5.

Therefore $R > R_0$ implies

$$F > \frac{n-m}{m} (R_0 - 1) = F_0, \tag{3-32a}$$

or

$$R > 1 + \frac{m}{n-m} F_0 = R_0. \tag{3-32b}$$

Now from Table 3-1 we find for n=10 and m=5 that the following values may be determined:

P_{FA}	F_0(from table)	R_0
.1	2.32	2.16
.01	4.85	3.43
.001	8.75	5.38

Thus the test value, being less than R_0 in each case results in the (correct) acceptance of H_0 at the 10%, 1%, and 0.1% levels.

4.0 APPLICATION OF THEORETICAL RESULTS

Having learned how to calculate the probability of detection P_D for the multivariate detection approach we are now in a position to apply these results to the detection problem. In this chapter, the objective is to find reasonable values for parameters which describe the detector performance and to show how the multivariate detector may be implemented.

4.1 Detection Performance Parameter, Deterministic Signals

It is convenient to extract a single parameter from the tables and curves of Chapter 3, one which represents the performance of detection (likelihood test) in achieving a useful P_D while rejecting false alarms. In receiver operating characteristics (ROC) for single sensor detectors, the noncentrality parameter λ which we have employed becomes signal-to-noise ratio (SNR). It is common to describe the detector's performance by specifying the SNR for which given P_D and P_{FA} are obtained. For example, we might choose the pair

$$(P_D, P_{FA}) = (.9, 10^{-2}) \tag{4-1}$$

and define the "minimum detectable signal" (MDS) as

$$\text{MDS} \equiv \text{SNR such that } P_D(\text{SNR}) = .9 \text{ for } P_{FA} = 10^{-2}. \tag{4-2}$$

and abbreviate this statement by the notation

$$\text{MDS}(.9) \equiv \text{SNR}(.9,.01)(\text{or SNR}(.9) \text{ with } P_{FA} = .01 \text{ understood}) \tag{4-3}$$

Since the multivariate parameter λ is analogous to SNR, we shall extract values of λ for which P_D = .5 and .9 from the ROC data already computed, then, in the subsequent sections, interpret this parameter in terms of minimum detectable signal by employing certain assumptions about signals and noise at the sensors.

As an example, from Figure 3-1 we see that for $P_{FA} = .1$, $n = 50$, and $m = 2$, a λ of 6.9 dB is required to produce $P_D = .5$ and a λ of 11.3 dB to give $P_D = .9$. This we write shorthand as

$$\left.\begin{array}{l}\lambda(.5) = 6.8 \text{ dB} \\ \lambda(.9) = 11.3 \text{ dB}\end{array}\right. \quad \text{for} \quad \begin{array}{l} P_{FA} = .1 \\ n=50,\ m=2. \end{array} \tag{4-4}$$

Table 4-1 gives $\lambda(.5)$ and $\lambda(.9)$ values extracted from the data generated in Chapter 3. The same information is plotted in Figures 4-1, 4-2, and 4-3. From these curves we see that $\lambda(.5)$ and $\lambda(.9)$ decrease to an asymptotic value with increasing n, and increase with m (the number of sensors) for a given n. This is in keeping with the discussion made of Figures 3-1 and 3-2 in the last chapter.

4.1.1 Structure of the Multivariate Detection Parameter

From Chapter 2 (or Appendix A) we recall that the multivariate noncentrality parameter is

$$\lambda = n\, \underline{\mu}^* \, \underline{\Sigma}^{-1} \, \underline{\mu} \tag{4-5}$$

where $\underline{\mu}$ is the mean of the complex data vectors $\{\underline{x}_k\}$ and Σ is the covariance matrix of their independent real and imaginary parts. What signal and noise models correspond to these quantities?

In Chapter 1, the complex data matrix elements were identified with the in-phase and quadrature components of the received sensor waveforms with respect to a center frequency:

$$x_{ik} = u_{ik} + jv_{ik}, \tag{4-6a}$$

P_{FA}	m	10	20	50	100	200
.1	1	6.1	5.8	5.6	5.6	5.6
	2	7.7	7.1	6.9	6.8	6.8
	3	8.9	8.1	7.7	7.6	7.5
	4	10.0	8.8	8.3	8.1	8.0
	5	11.1	9.5	8.8	8.5	8.4
	10	X	11.9	10.4	10.0	9.8
	20	X	X	12.5	11.7	11.3
.01	1	10.3	9.7	9.3	9.2	9.2
	2	11.9	10.9	10.4	10.2	10.2
	3	13.1	11.8	11.1	10.9	10.8
	4	14.4	12.5	11.6	11.4	11.3
	5	15.6	13.1	12.1	11.8	11.7
	10	X	15.8	13.6	13.1	12.9
	20	X	X	15.8	14.7	14.3
.001	1	12.8	11.9	11.4	11.3	11.2
	2	14.4	13.0	12.3	12.1	12.0
	3	15.7	13.9	12.9	12.7	12.6
	4	17.1	14.5	13.4	13.1	13.0
	5	18.6	15.1	13.9	13.5	13.3
	10	X	18.1	15.4	14.7	14.5
	20	X	X	17.5	16.2	15.8

Table 4-1(a) $\lambda(.5)$, REQUIRED VALUES OF DETECTION PARAMETER (dB) for $P_D = .5$

P_{FA}	m	10	20	50	100	200
.1	1	10.8	10.5	10.3	10.3	10.3
	2	12.1	11.6	11.3	11.2	11.1
	3	13.2	12.3	11.9	11.8	11.7
	4	14.2	12.9	12.4	12.2	12.1
	5	15.3	13.5	12.8	12.6	12.5
	10	X	15.9	14.2	13.8	13.7
	20	X	X	16.2	15.3	14.9
.01	1	13.6	12.9	12.6	12.5	12.4
	2	15.0	14.0	13.5	13.3	13.2
	3	16.2	14.8	14.0	13.8	13.7
	4	17.4	15.3	14.5	14.3	14.1
	5	18.7	15.9	14.9	14.6	14.5
	10	X	18.6	16.2	15.7	15.5
	20	X	X	18.3	17.2	16.7
.001	1	15.6	14.6	14.1	13.9	13.8
	2	17.1	15.6	14.9	14.6	14.5
	3	18.4	16.3	15.4	15.1	15.0
	4	19.8	16.9	15.8	15.5	15.3
	5	21.4	17.5	16.2	15.8	15.6
	10	X	20.4	17.6	16.9	16.6
	20	X	X	19.6	18.3	17.8

Table 4-1(b) $\lambda(.9)$, REQUIRED VALUES OF DETECTION PARAMETER (dB) for $P_D = .9$

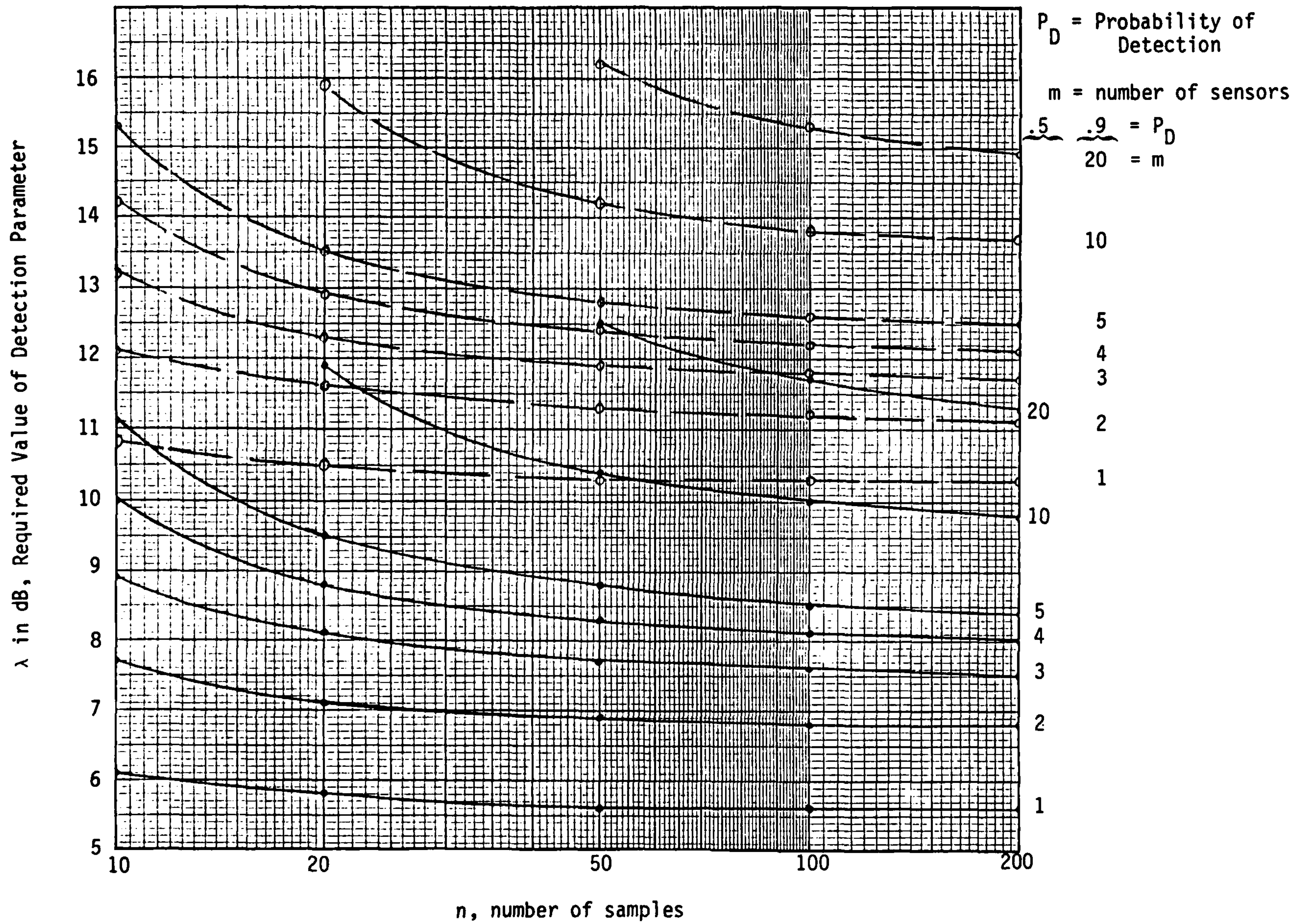


Figure 4-1. REQUIRED DETECTION PARAMETER VALUES vs. NUMBER OF SAMPLES, P_{FA} = .1

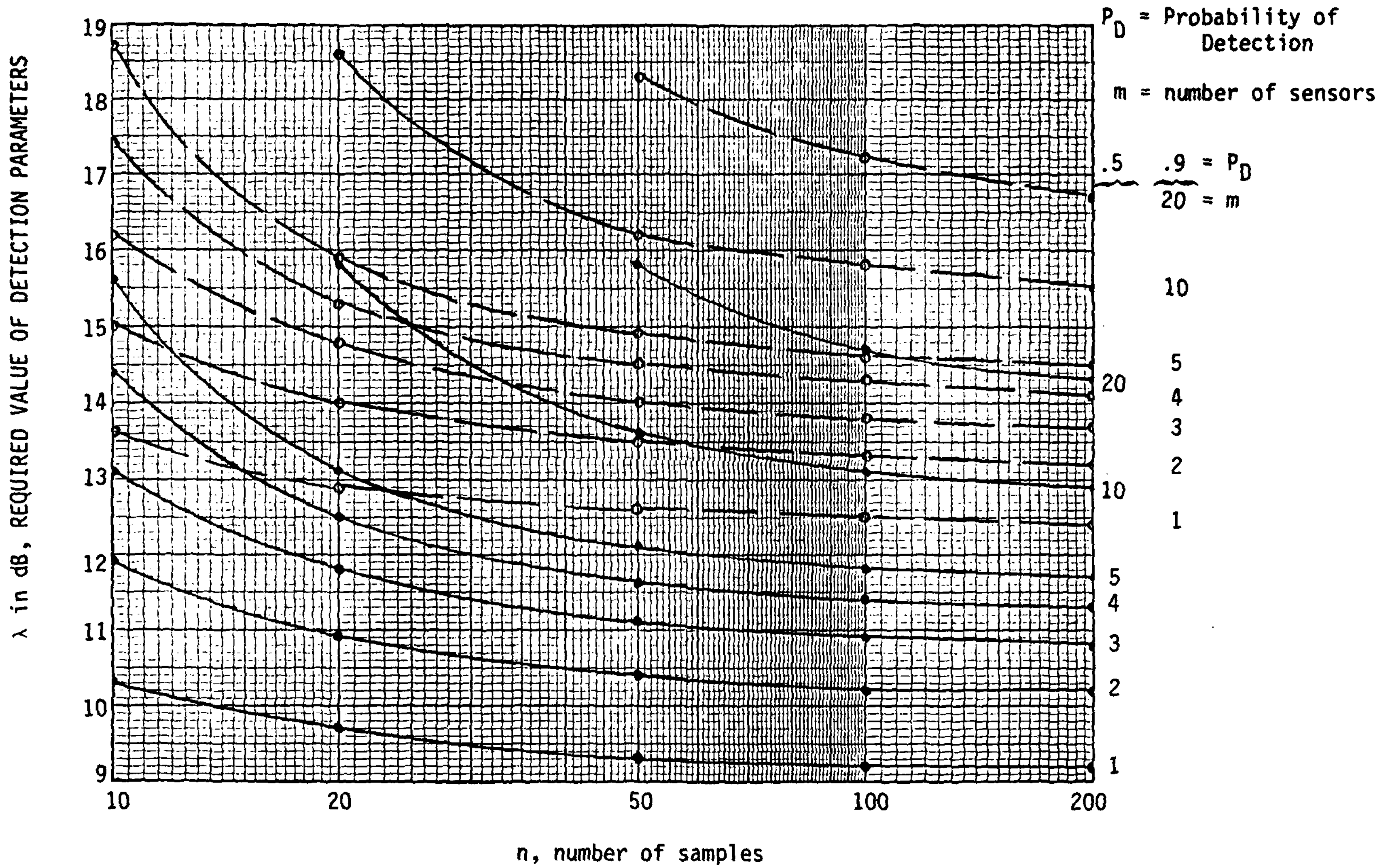


FIGURE 4-2. REQUIRED DETECTION PARAMETER VALUES vs. NUMBER OF SAMPLES, P_{FA} = .01

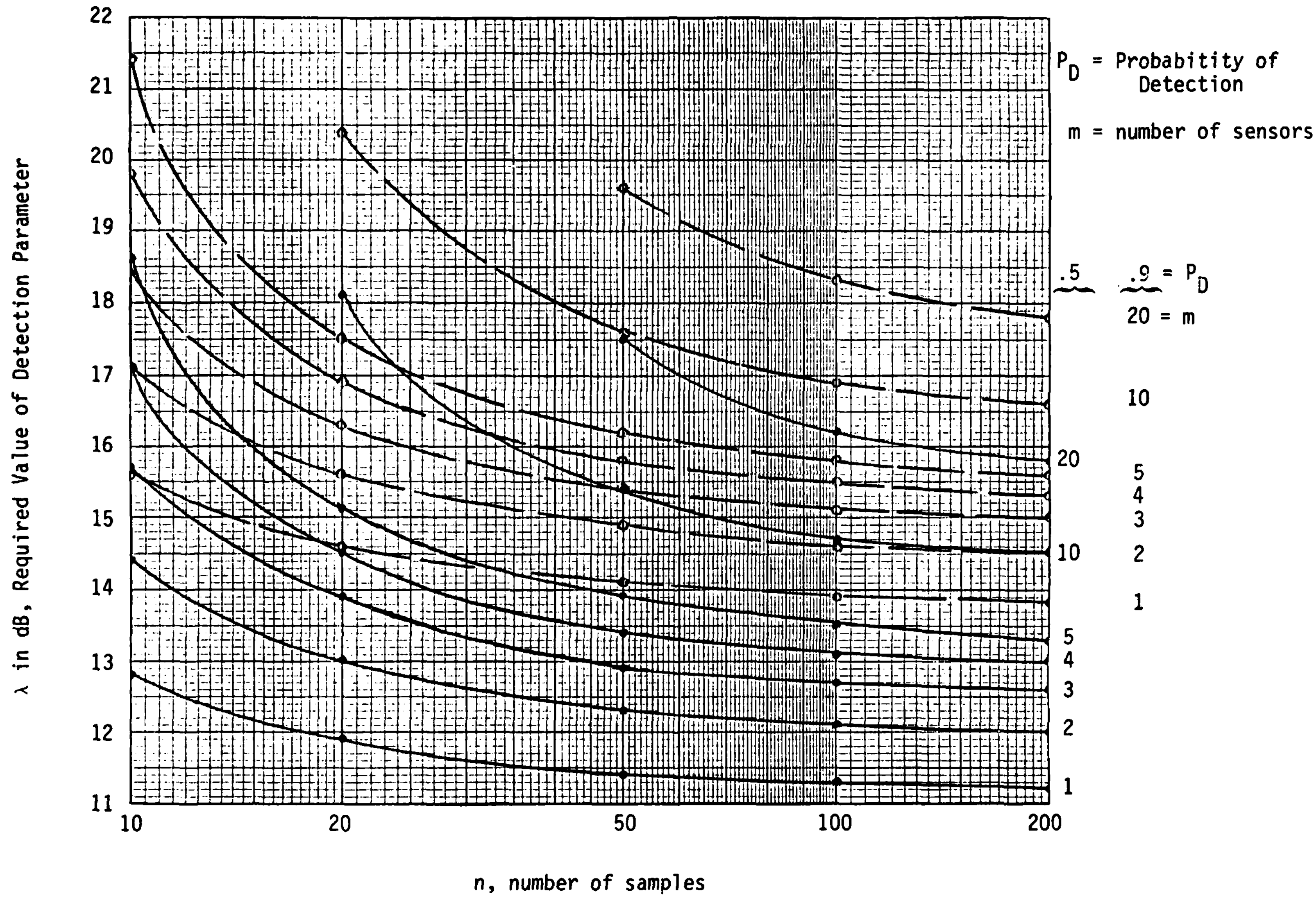


Figure 4-3. REQUIRED DETECTION PARAMETER VALUES vs. NUMBER OF SAMPLES P_{FA} = .001

or $x_i(t_k) = u_{ik} \cos\omega_c t_k - v_{ik} \sin\omega_c t_k$

$$= \sqrt{u_{ik}^2 + v_{ik}^2} \cos[\omega_c t_k + \tan^{-1} \frac{v_{ik}}{u_{ik}}]. \qquad (4\text{-}6b)$$

If we are considering only the possibility that a deterministic signal is present in the background noise, then the covariance matrix $\Sigma = ||\sigma_{ir}||$ is the covariance matrix of the zero-mean noise at the sensors:

$$\sigma_{ir} = E\{n_i n_r\}. \qquad (4\text{-}7)$$

The mean vector $\underline{\mu} = \underline{a} + j\underline{b}$ components then are due entirely to the signal, when present, and we assume that the signal components (in-phase and quadrature amplitudes or, alternatively, envelope and phase) are constant (or nearly so) while the n samples are observed. Let the signals at the sensor outputs have envelope values $\{S_i\}$ and phases $\{\theta_i\}$; then the elements of the mean vector can be identified as

$$\mu_i = a_i + jb_i = S_i \cos\theta_i + jS_i \sin\theta_i. \qquad (4\text{-}8)$$

Now, using σ^{ir} to denote elements of the inverse of Σ, we can express the multivariate noncentrality parameter by

$$\lambda = n\, \underline{\mu}^* \Sigma^{-1} \underline{\mu} = n \sum_{i=1}^{m} \sum_{r=1}^{m} (a_i - jb_i)\, \sigma^{ir} (a_r + jb_r)$$

$$= n \sum_{i,r} \sigma^{ir} [(a_i a_r + b_i b_r) + j(a_i b_r - a_r b_i)]$$

$$= n \sum_{i,r} \sigma^{ir} S_i S_r [\cos(\theta_i - \theta_r) - j \sin(\theta_i - \theta_r)]. \qquad (4\text{-}9)$$

Since Σ is a symmetric matrix, so also is Σ^{-1}, and the summation over the imaginary part is zero, leaving

$$\lambda_M \triangleq n \sum_{i=1}^{m} \sum_{r=1}^{m} \sigma^{ir} S_i S_r \cos(\theta_i - \theta_r) \tag{4-10}$$

The subscript "M" stands for "multivariate".

For the special case in which the noise is independent from sensor to sensor* (the covariance matrix is diagonal), (4-10) reduces to

$$\lambda_M = n \sum_{i=1}^{m} \sigma^{ii} S_i^2$$

$$= 2n \sum_{i=1}^{m} \frac{S_i^2}{2\sigma_i^2} = 2mn \times (\text{average SNR}). \tag{4-11}$$

With this result as motivation, we choose to define minimum detectable signal for the multivariate detector by

$$MDS_M(P_D) = \frac{1}{2nm} \lambda_M(P_D) = \frac{1}{2nm} \lambda(P_D;m,n). \tag{4-12}$$

4.1.2 Performance Predictions

Using the definitions for minimum detectable signal that have been given, we may predict the performance of detection based on the test statistics. For the multivariate detector, we use the numbers in Table 4-1 to calculate

$$MDS_M(P_D)(dB) = \lambda(P_D;\ m,\ n)(dB) - 10\ \log_{10}(2nm). \qquad (4\text{-}13)$$

Similarly, for the detector based on a single-sensor detector output the MDS is

$$MDS_1(P_D)(dB) = \lambda(P_D;\ 1,\ n)(dB) - 10\ \log_{10}(2n). \qquad (4\text{-}14)$$

For example, for a false alarm probability of .001 and a detection probability of .5, for m=10 sensors and n=50 samples, we have

$$MDS_M(.5) = 15.4\ dB - 10\ \log_{10}(1000) = -14.6\ dB \qquad (4\text{-}15)$$

and for m = 1,

$$MDS_1(.5) = 11.4\ dB - 10\ \log_{10}(100) = -8.6\ dB. \qquad (4\text{-}16)$$

Thus in this example, the multivariate detector has a processing gain of 6 dB over a single-sensor.

We may define a "multivariate processing gain" by

$$MPG \triangleq \frac{MDS_1(P_D)}{MDS_M(P_D)}\ . \qquad (4\text{-}17)$$

From (4-12) we may then write

$$MPG \approx m\,\frac{\lambda(P_D;\ 1,\ n)}{\lambda(P_D;\ m,\ n)}\ . \qquad (4\text{-}18)$$

Using the numbers for $\lambda_1(P_D)$ and $\lambda_M(P_D)$ given in Table 4-1 the MPG is calculated and plotted in Figures 4-4, 4-5, and 4-6 for the various P_{FA} values.

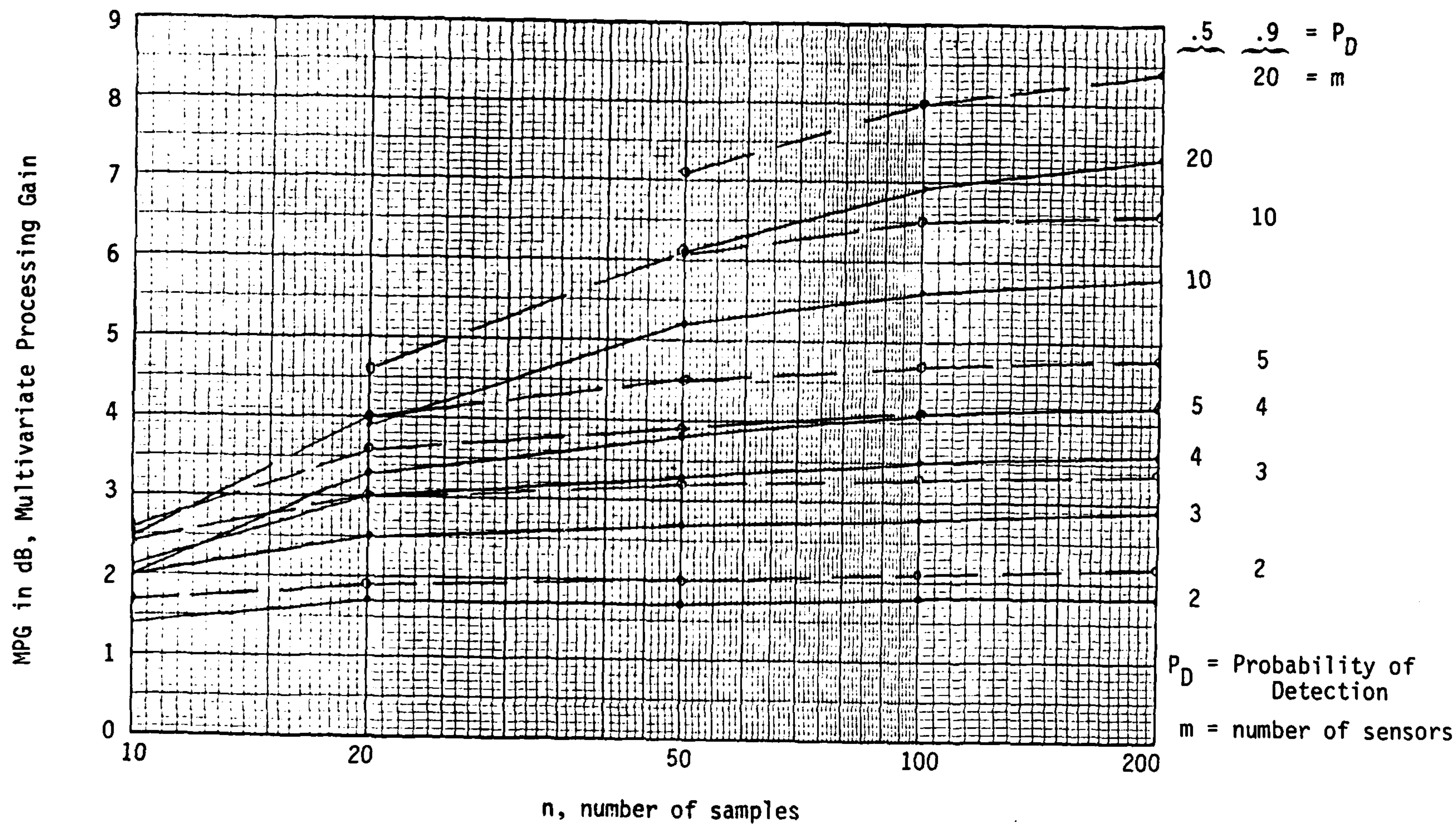


Figure 4-4. MULTIVARIATE PROCESSING GAIN vs. NUMBER OF SAMPLES, P_{FA} = .1

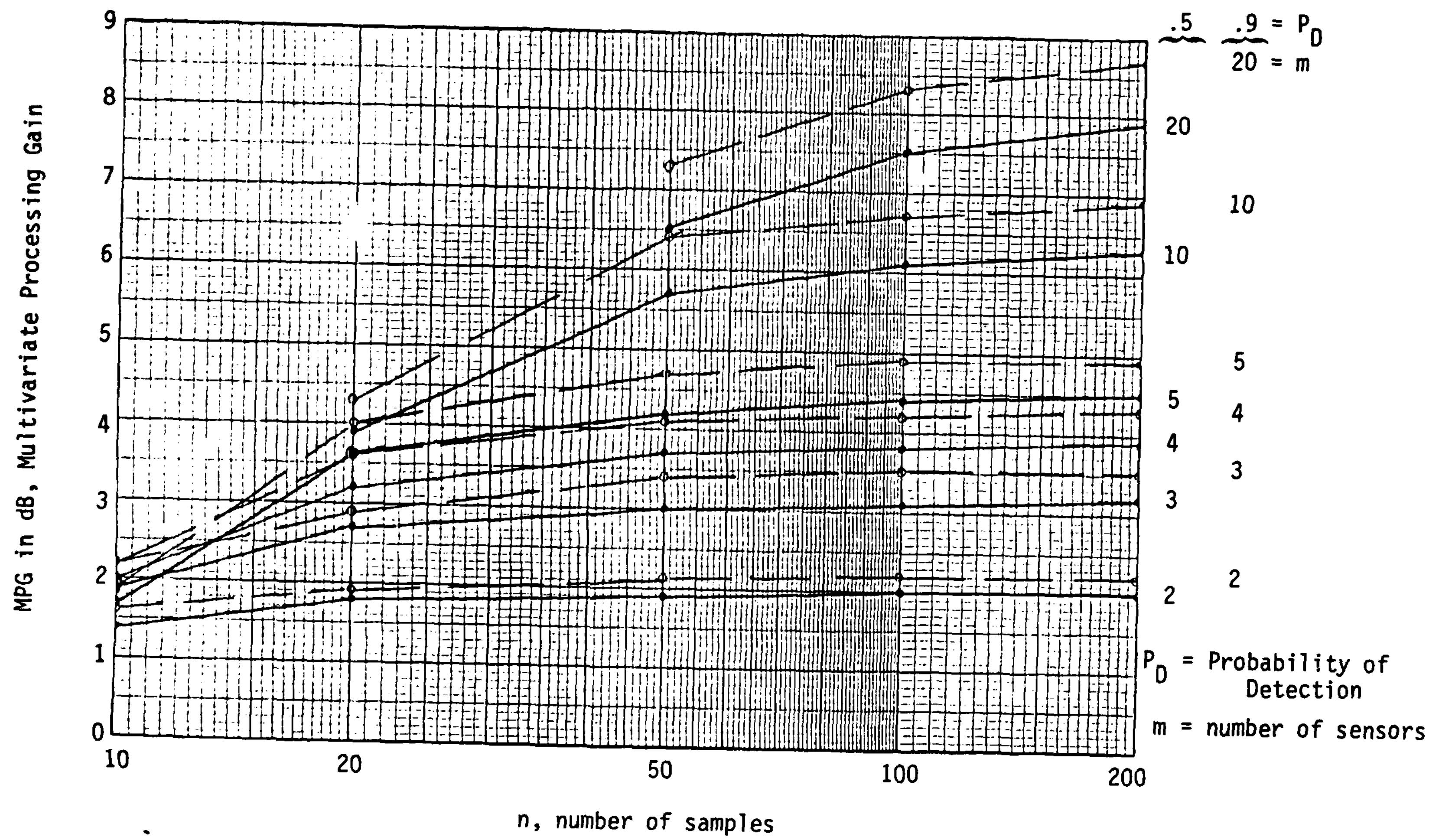


Figure 4-5. MULTIVARIATE PROCESSING GAIN vs. NUMBER OF SAMPLES, P_{FA} = .01

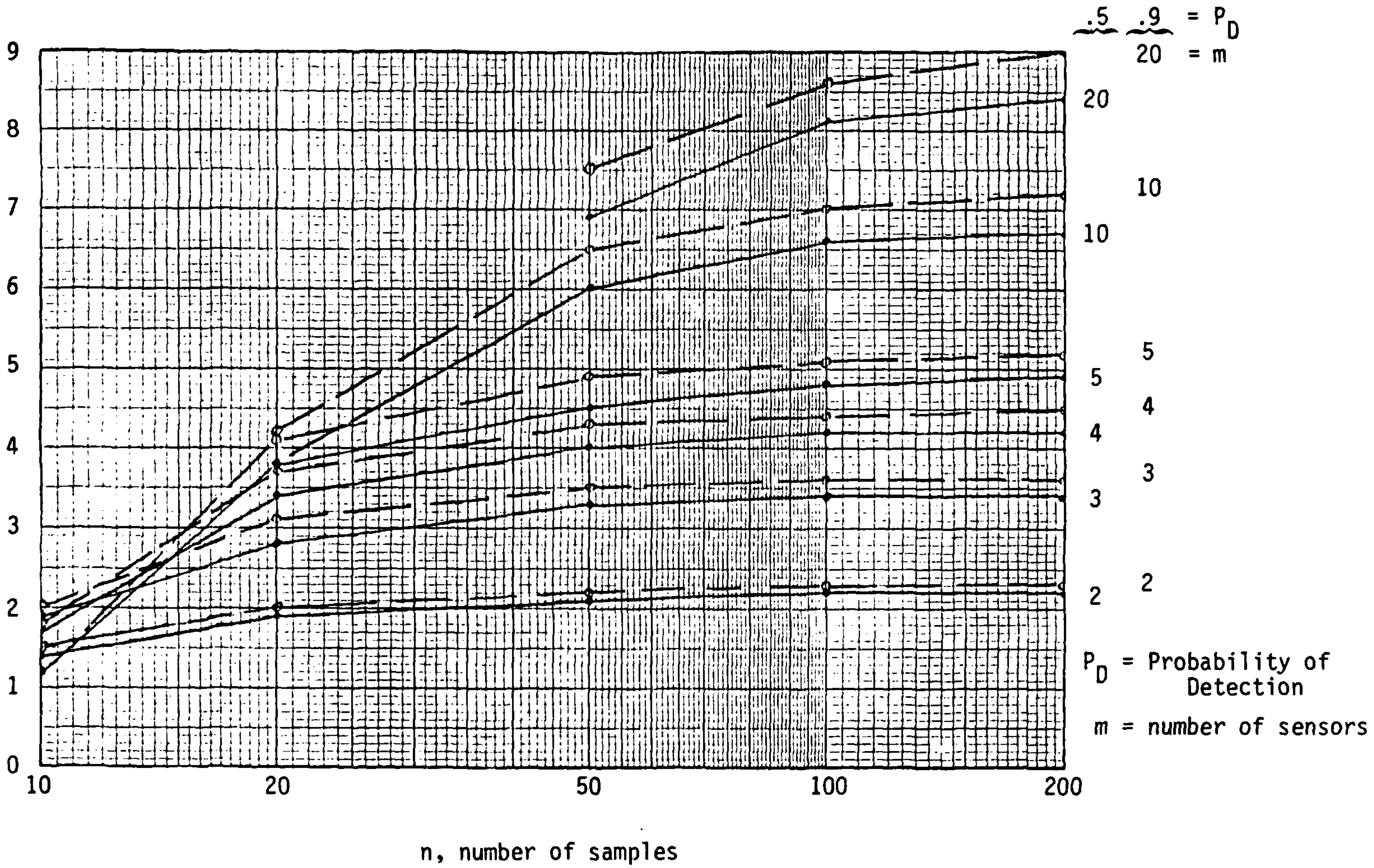


Figure 4-6. MULTIVARIATE PROCESSING GAIN vs. NUMBER OF SAMPLES P_{FA} = .001

It is evident from these figures that detection based upon the multivariate approach realizes appreciable gains over using a single sensor. Gains from 2 to 9 d B are shown. The curves rise to an asymptotic or limiting value as n, the number of observations, increases and this value is directly proportional to m, the number of sensors.

On each figure we see that the MPG is higher for $P_D = .9$ than for $P_D = .5$, with the amount of the increase a function of P_{FA}: the more stringent (smaller) the P_{FA} required, the smaller the increase. By comparing the figures we notice that MPG values tend to increase as P_{FA} decreases. This behavior is very interesting because it implies that the multvariate detector achieves gain in two dimensions (P_D, P_{FA}); the more difficult the detection criterion (either P_D or P_{FA}), the better the multivariate approach is compared to single-sensor processing. That is, this gain can be obtained for independent sensor noise terms. If the off-diagonal components of the inverse covariance matrix Σ^{-1} are nonzero, actually the λ_M is more or less than $2mn \cdot SNR$ (see (4-10)), and the performance of the multivariate approach can be slightly better or worse than has been shown, depending on the amount of correlation between sensor waveforms, and the signal phases (see Appendix C).

Comparison with Majority Decision Scheme. Analysis of the "majority decision" method described in Section 2.1.3 results in the following graphical comparison:

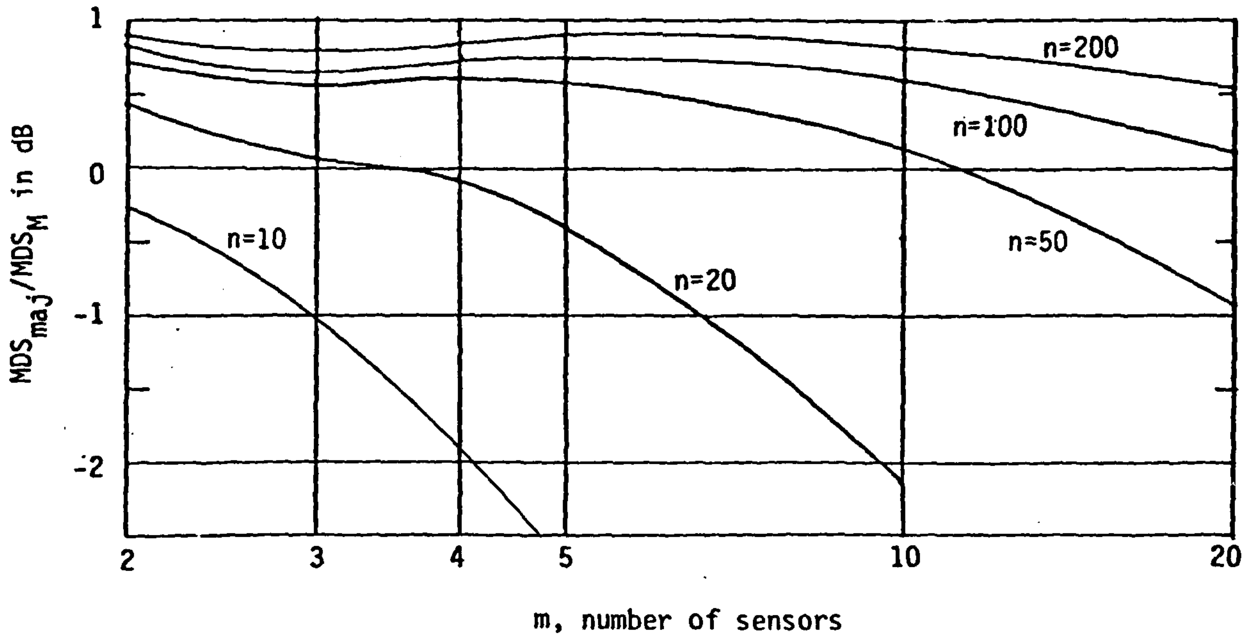


Figure 4-7 GAIN OF MULTIVARIATE DETECTOR OVER MAJORITY SCHEME

These results show the combination of single-sensor detector decisions (which assumes independent sensors) doing better than the multivariate for small numbers of samples n or for large number of sensors m. However, the comparison is not ideal since the multivariate processor is "working harder" than necessary: even though the sensor noises are independent (in this comparison), it is using up degrees of freedom (see Section 3.1.2) estimating correlations. It seems reasonable to expect that (a) a multivariate processor derived for the special case of independent sensors will uniformly do better that the combination scheme, and (b) when inter-sensor correlations are introduced, the performance of the majority decison scheme will deteriorate, while the multivariate will not. However, the distributions in either of these cases have not been found, so these expectations remain conjectures.

4.2 Computer Program for Performing Multivariate Detection of Deterministic Signals

In order to assess the implementation costs that may be associated with the multivariate detection processing, a computer program in FORTRAN was written and exercised. Limited simulations using the programs provided verification of theoretical detection performance predictions.

4.2.1 Program Requirements and Desirable Features

The basic requirement the program must satisfy has already been indicated in Section 3.3; the program must accept n vectors $\{\underline{x}_k\}$ which are $m \times 1$, accumulate mean vector and sample covariances, calculate determinants, and form test statistics which are ratios of determinants. The basic operation is specified by the equation

$$z(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n) = \frac{|2n\hat{\Sigma}_0|}{|2n\ \hat{\Sigma}_0 - n\ \hat{\underline{\mu}}\ \hat{\underline{\mu}}^*|} \underset{H_0}{\overset{H_1}{\gtrless}} z_0. \tag{4-19}$$

Although the detection problem has been formulated in this study as a "batch" decision rather than a sequential or an iterative one, certain inherent relationships in the functions to be computed allow accumulation of intermediate results. Therefore, only a minimum of input data has to be stored. Specifically, the sample mean vector can be accumulated using

$$\hat{\underline{\mu}} = \frac{1}{n}\sum_{k=1}^{n} \underline{x}_k = \frac{1}{n}\underline{x}_n + \frac{1}{n}\sum_{k=1}^{n-1} \underline{x}_k$$

$$= \hat{\underline{\mu}}^{[n-1]} + \underline{x}_n/n \tag{4-20}$$

where

$$\hat{\underline{\mu}}^{[i+1]} = \underline{\mu}^{[i]} + \underline{x}_{i+1}/n,\ \hat{\underline{\mu}}^{[0]} = 0.$$

Similiarly, the H_0 covariance can be accumulated using

$$2n\hat{\Sigma}_0 = A_0 = \sum_{k=1}^{n} \underline{x}_k\underline{x}_k^*$$

$$= \underline{x}_n\underline{x}_n^* + \sum_{k=1}^{n-1} \underline{x}_k\underline{x}_k^* = A_0^{[n-1]} + \underline{x}_n\underline{x}_n^* \tag{4-21}$$

where

$$A_0^{[i+1]} = A_0^{[i]} + \underline{x}_{i+1}\underline{x}_{i+1}^*; \; A_0^{[0]} = 0.$$

Certain economies are also possible due to the fact that the covariance matrices are positive definite and Hermitian.

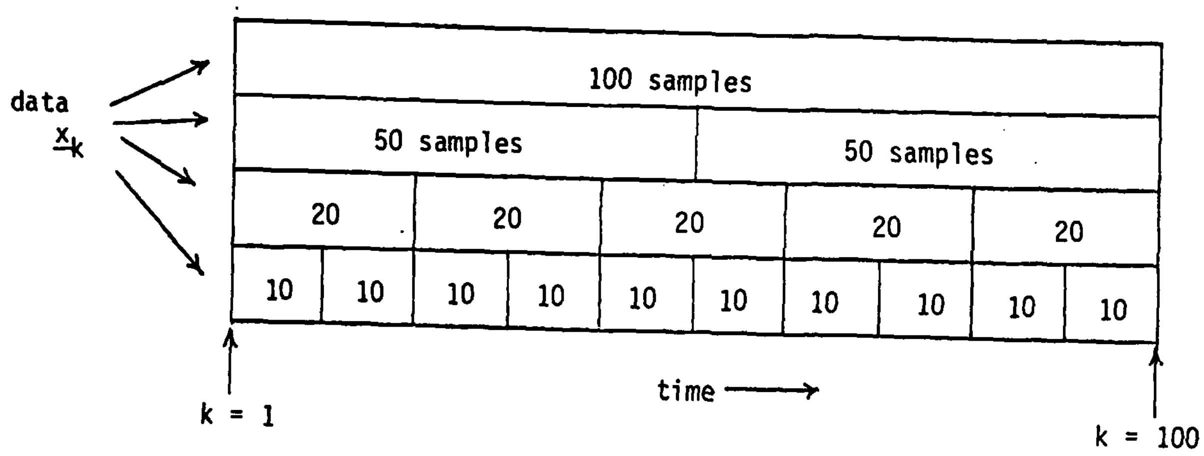
In practical detection systems it is desirable to have some control over "integration time" or, in this context, the number of samples. For weak, stationary signals one should like to make the number of samples high; for strong, intermittent signals a relatively short observation time would be selectable. Implementation of this second desirable feature is a simple matter of storing a table of thresholds.

How to accommodate the variable integration time feature is suggested by Figure 4-8. A 100-sample data base, for example, can be processed as a whole (maximum integration time) or as ten 10-sample data bases in succession (minimum integration time). In fact, both (plus in-between cases) can be done in parallel, and the results of each displayed simultaneously in some fashion.

Assuming simultaneous 10/20/50/100-sample detections, the basic requirements for the multivariate detector computer implementation may be summarized as in Figure 4-9.

4.2.2 Program Structure and Size

The basic requirements for computer implementation of the multivariate detector summarized in Figure 4-9 are met by the FORTRAN program listed as program P-6 in the back of this report. A flow diagram is presented in Figure 4-10. The symbology used is as follows:

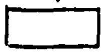


One 100-sample data base
= two 50-sample data bases
= five 20-sample data bases
= ten 10-sample data bases

FIGURE 4-8. CONCEPT OF MULTIPLE USAGE OF A DATA BASE.

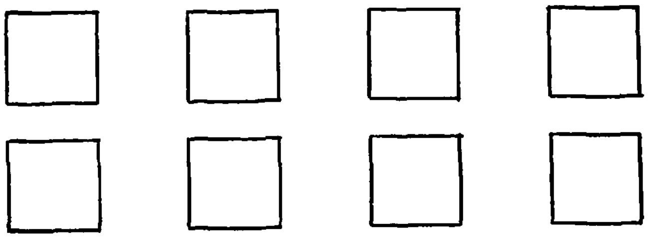
STORAGE: (not including that required for arithmetic, subroutines, overhead)

DATA: one $m \times 1$ input buffer for (complex) data vector



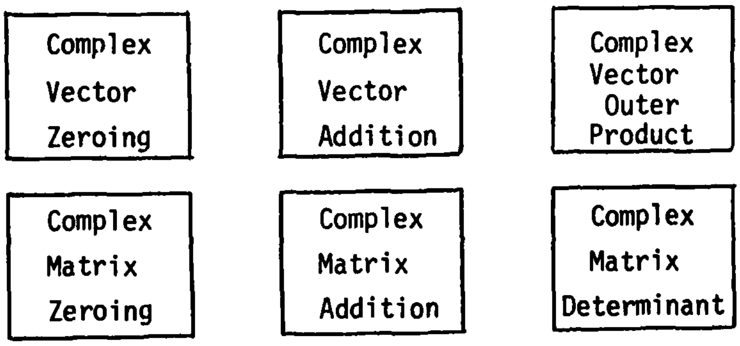
MEAN VECTORS: four $m \times 1$ (complex) arrays

COVARIANCES: eight $m \times m$ (complex) arrays



ARITHMETIC: complex conjugation, complex and scalar arithmetic

SUBROUTINES:



TABLES: False alarm thresholds vs. P_{FA} vs. number of samples

FIGURE 4-9. SUMMARY OF BASIC REQUIREMENTS FOR MULTIVARIATE DETECTOR COMPUTER IMPLEMENTATION.

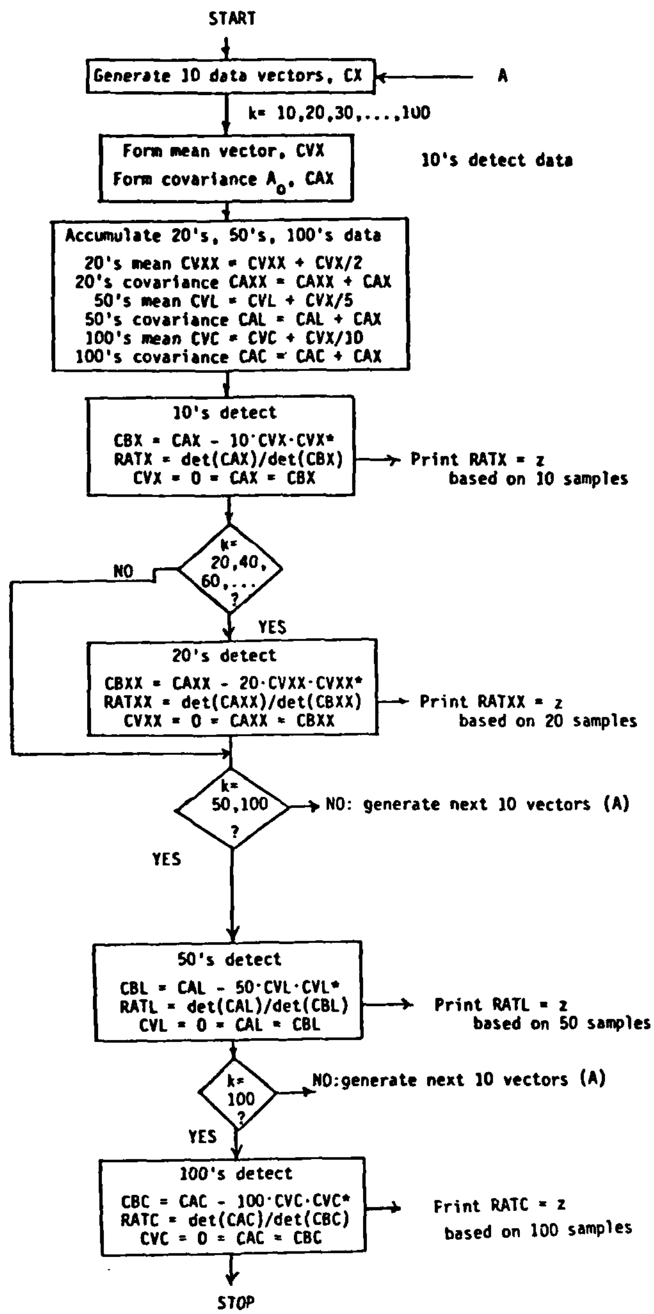


FIGURE 4-10 FLOW OF MULTIVARIATE DETECTOR PROGRAM.

Program Variable Name		Mathematical Equivalent
M(=5)		m, number of sensors
NSAMP		n, number of samples
CX(M)	(complex)	$\underline{x}_k$, data vector
CVX(M)	(complex)	$\hat{\mu}$, sample mean (10's base)
CVXX(M)	(complex)	$\hat{\mu}$, sample mean (50's base)
CVL(M)	(complex)	$\hat{\mu}$, sample mean (20's base)
CVC(M)	(complex)	$\hat{\mu}$, sample mean (100's base)
CAX(M,M)	(complex)	$2n\hat{\Sigma}_0$, H_0 covariance (10's base)
CAXX(M,M)	(complex)	$2n\hat{\Sigma}_0$, H_0 covariance (20's base)
CAL(M,M)	(complex)	$2n\hat{\Sigma}_0$, H_0 covariance (50's base)
CAC(M,M)	(complex)	$2n\hat{\Sigma}_0$, H_0 covariance (100's base)
CBX(M,M)	(complex)	$2n\hat{\Sigma}_1$, H_1 covariance (10's base)
CBXX(M,M)	(complex)	$2n\hat{\Sigma}_1$, H_1 covariance (20's base)
CBL(M,M)	(complex)	$2n\hat{\Sigma}_1$, H_1 covariance (50's base)
CBC(M,M)	(complex)	$2n\hat{\Sigma}_1$, H_1 covariance (100's base)
RATX, RATXX, RATL, RATC		z, computed test statistics
THRX, THRXX, THRL, THRC		z_0, stored thresholds

All vector and matrix operations (except for construction of CVX and CAX) are performed using subroutines. The main programs and subroutines, compiled as listed on an IBM 3033 FORTRAN system were sized for m=5 as follows:

Program or Subprogram	Source Lines	Size
MAIN (incl. data generation)	85	4442
CLRV (zero vector)	7	380
CLRM (zero matrix)	8	472
ADDV (add vectors)	7	588
ADDM (add matrices)	9	734
PRODV (vector product)	8	718
DETER (determinant)	15	910
		8244
complex arithmetic, random no. generator		500
		8744
I/O, FORTRAN, other system overhead		21208
		29952

The "size" is understood to be in units of 8-bit bytes, so that the program itself requires about 8k on an 8-bit machine, or 4k on a 16-bit machine.

As shown in the flow diagram, the program operation is straightforward: 10-sample mean vectors CVX and H_0 covariance matrices CAX are constructed from the input data (one input vector at a time). After incrementing the other (20-, 50-, and 100-sample) mean and covariance accumulations, the H_1 covariance CBX is formed, its determinant computed, and the test statistic z is computed as the ratio RATX.

In lieu of decision-making the program prints out z and the threshold z_0 corresponding to a one percent false alarm probability.

FORTRAN conditional (IF) statements are used to "count" the number of 10-sample blocks of data which have been "observed," and appropriately enable the 20-, 50-, and 100-sample detection computations at the proper "time". After each detection operation the accumulated means and covariances just used are reset.

4.2.3 Comparison of Computed Multivariate Results with Theory

To exercise the multivariate detection computer program and to collect simulated results for verification of the theory, a data base of 100 vectors with the following characteristics was generated and processed.

$$m=5, \; \Sigma = I, \; \underline{\mu} = S\underline{1} + j\underline{0}.$$

The signal components were chosen to be equal in phase (0^{o}) and amplitude at each sensor. On separate runs the amplitude was made 0, .2, .3, .5, and .9 to correspond to SNR's of $-\infty$, -17, -13.5, -9, and -3.9 dB respectively.

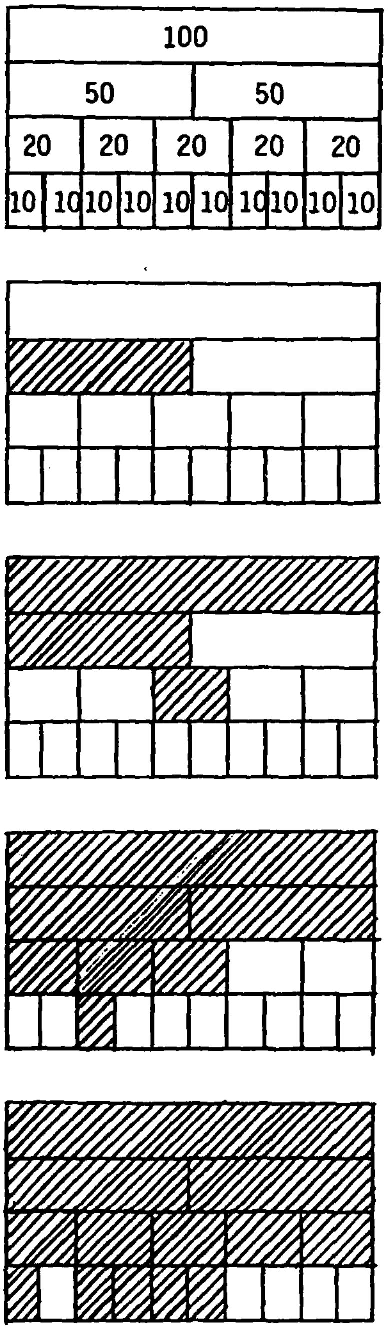
The results of these simulations and their theoretical interpretation may be displayed as shown in Figure 4-11. Detections for a particular block of data are indicated by shading the portion of a diagram (similar to Figure 4-9) which corresponds to that block in simulated time. Just to the right of these diagrams, a tally of detections is provided (1/5 = one detection in five trials, etc). The other experimental values presented are the arithmetical average values of the test statistics compute, normalized by the 1% false alarm threshold.

Thus, for example, on the run for which a SNR of -13.5 dB was simulated, the 20's detection yielded a detection at the third data block only, and the average value of the five statistics computed was 83.95% of the false alarm threshold.

Along with the experimental data in Figure 4-11 theoretical predictions are given for probability of detection and mean value of the test statistic relative to the threshold. The P_D values were determined by setting λ= 2nm SNR and using Table 3-2. The mean value of the test statistic was computed using [14]

$$E\left\{F_{\nu_1,\nu_2}(\lambda)\right\} = \frac{\nu_2(\nu_1 + \lambda)}{\nu_1(\nu_2 - 2)} \qquad (4\text{-}22)$$

EXPERIMENTAL RESULTS | THEORY



input SNR	scores and sample sets	tally	$\frac{z_{avg}}{z_o}$	P_D	$\frac{E\{z\}}{z_o}$
0 (S=0)	100	0/1	.9301	.01	.9344
	50 | 50	0/2	.8873	.01	.8697
	20 | 20 | 20 | 20 | 20	0/5	.6768	.01	.6810
	10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10	0/10	.3785	.01	.3847
-17 dB (S=.2)		0/1	.9851	.69	1.0288
		1/2	.9394	.25	.9584
		0/5	.7433	.05	.7526
		0/10	.4134	.02	.4275
-13.5 dB (S=.3)		1/1	1.0902	.99	1.1468
		1/2	1.0406	.73	1.0694
		1/5	.8395	.18	.8422
		0/10	.4604	.04	.4809
-9 dB (S=.5)		1/1	1.4653	.99+	1.5243
		2/2	1.3932	.99	1.4243
		3/5	1.1582	.65	1.1290
		1/10	.6135	.12	.6519
-3.9 dB (S=.9)		1/1	2.8478	.99+	2.8458
		2/2	2.6996	.99+	2.6667
		5/5	2.2995	.99+	2.1325
		5/10	1.1048	.55	1.2503

FIGURE 4-11. EXPERIMENTAL RESULTS VS. THEORY FOR MULTIVARIATE DETECTOR, P_{FA} = .01 and 5 sensors.

to write

$$E\{z\} = 1 + \frac{m}{n-m} \quad \frac{2(n-m)(2m + 2nm\ SNR)}{2m(2n - 2m - 2)}$$

$$= 1 + \frac{m}{n-m-1} \ (1 + nSNR). \qquad (4\text{-}23)$$

The z_0's were obtained from Table 3-1 as described in Section 3.3.3, resulting in

n	10	20	50	100
z_0 (m=5, P_{FA} = 1%)	5.8486	1.9930	1.2805	1.1271

. (4-24)

The experimental values are very close to theoretical predictions. For example, for SNR = -9 dB, the 50- and 100-sample detection calculations yielded "perfect scores"; theory predicted that detection was better than 99% probable. The 20-sample detection calculations scored 3/5 or 60% compared to the theoretical 65% probability, and the 10% score of the 10-sample detector was consistent with a predicted P_D of 12%.

Another way of viewing the results is given in Figure 4-12, in which the theoretical and experimental average detection statistics are compared as functions of SNR. The agreement is very good for this parameter also, considering the relatively small sample sizes.

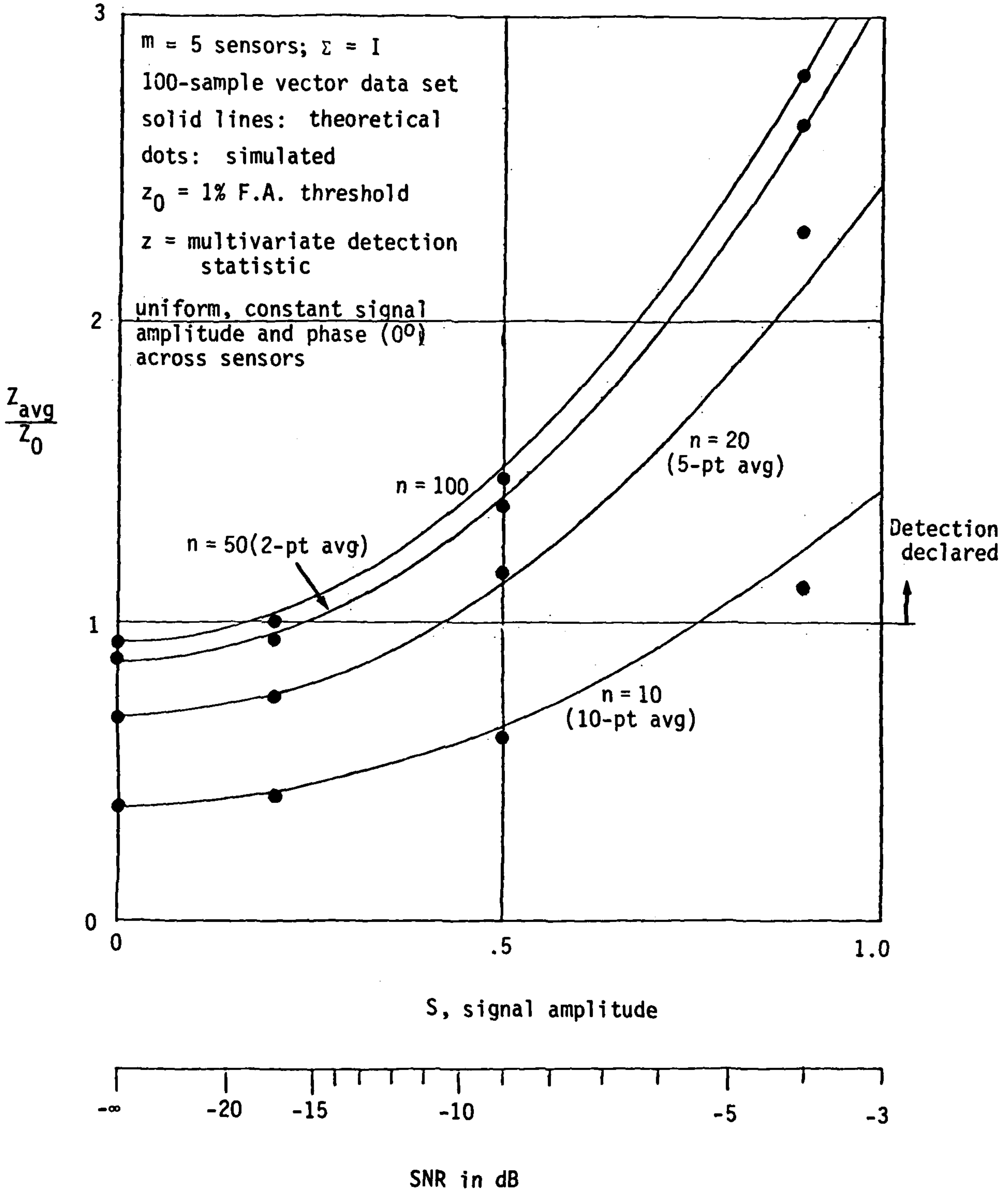


Figure 4-12 AVERAGE (NORMALIZED) DETECTION STATISTIC vs. SNR, P_{FA} = .01

4.3 Modification to Computer Program for Detecting Random Signals

The computer program for detecting deterministic signals may be easily modified in order to calculate test statistics for random signals. All that is required is the following subroutine:

```
      SUBROUTINE RHODET (CMAT, M, C1)
      IMPLICIT COMPLEX (C)
      C2 = (1., 0.)
      DO1  I = 1, M
    1 C2 = C2*CMAT(I,I)
      R1 = REAL (C2/C1)
      WRITE (6,2) R1
    2 FORMAT (1X, 'CORRELATION DET RATIO =', F9.4)
      RETURN
      END
```

In response to the statement, for example CALL RHODET (CAX, M, CNX) the subroutine calculates the ratio of the product of the diagonal elements of the covariance matrix CAX to its previously computed determinant, CNX, and prints it out.

5.0 ASSESSMENTS AND RECOMMENDATIONS

The study is concluded in this present phase by reviewing what has been done, assessing the results, and recommending further work.

5.1 Review

5.1.1 Summary of Work Performed

The mathematics of complex multivariate Gaussian statistics have been applied to a model of a multisensor detection problem. Testing hypotheses concerning the presence or absence of sources in the sensor medium have been shown to be equivalent to computing statistics or functions under the Gaussian noise model, corresponding to ratios of determinants of sample covariance matrices.

For the case of deterministic signals whose magnitude and phase remain constant during the period in which data are collected, the multivariate test statistic has been shown to be in the family of F-statistics. Using the properties of F-statistics distributions, theoretical predictions have been made of SNR's required to detect the signal while rejecting false alarms at a given level. The results confirm the expectation that multivariate processing of sensor data requires a significantly smaller SNR than does single-sensor processing. Also, a non-ideal comparison indicates that multivariate processing is better than a majority decision method for combining individual sensor decisions when large numbers of samples are used.

A computer program implementing the multivariate detection processing has been developed and tested, yielding simulation results compatible with theory.

For the case of random signals, the multivariate test statistic has been shown to have a distribution which approaches the chi-squared family asymptotically. Computation of the random signal test statistic may be accomplished by a very slight modification to the program for the detection of deterministic signals.

5.1.2 Summary of Multivariate Detection Performance

For deterministic (narrowband or broadband) signals the performance of the multivariate detector may be summarized as follows:

Minimum Detectable Signal in dB (P_D = .9, P_{FA} = .01, Σ = diagonal)

number of sensors	number of samples: 10	20	50	100	200
1	0.6	-3.1	-7.4	-10.5	-13.6
2	-1.0	-5.0	-9.5	-12.7	-15.8
3	-1.6	-6.0	-10.8	-14.0	-17.1
4	-1.6	-6.7	-11.5	-14.7	-17.9
5	-1.3	-7.1	-12.1	-15.4	-18.5
10	X	-7.4	-13.8	-17.2	-20.5
20	X	X	-14.7	-18.8	-22.3

Multivariate Processing Gain in dB over Single-Sensor Processing (P_D = .9, P_{FA} = .01, Σ = diagonal)

m	n: 10	20	50	100	200
2	1.6	1.9	2.1	2.2	2.2
3	2.2	2.9	3.4	3.5	3.5
4	2.2	3.6	4.1	4.2	4.3
5	1.9	4.0	4.7	4.9	4.9
10	X	4.3	6.4	6.7	6.9
20	X	X	7.3	8.3	8.7

5.2 Assessments

The multivariate detection approach is now assessed with regard to detection performance, implementation, and compatibility issues.

5.2.1 Detection Performance

Clearly the multivariate detection approach, achieving a SNR processing gain of around 2nm, is capable of performing very sensitive detections. It is the ML solution for the unknown signal case.

It is instructive to note that a beamformer utilizing m sensors and n samples, and processing the sum of m unknown signals in unknown noise by the ML method shown for single sensors, gets better results than the multivariate technique when steered at the target (equal signal phases). The gain of the beamformer over the multivariate is, for the parameters used in Section 5.1.2., given below:

Beamformer Gain over Multivariate in dB

	n = 10	20	50	100	200
m = 2	1.4	1.2	0.9	0.8	0.8
3	2.6	1.9	1.4	1.3	1.3
4	3.8	2.4	1.9	1.8	1.7
5	5.1	3.0	2.3	2.1	2.1
10	X	5.7	3.6	3.3	3.1
20	X	X	5.7	4.7	4.3

Thus it is seen that the multivariate detector is not far behind the beamformer in performance when their number of samples is large. Moreover, it must be noted that the multivariate detector can operate where good beamforming is not possible, such as sensor placements which are not well known or subject to random variation. This is because in effect the multivariate processor estimates the signal phases at each sensor.

Whether the multivariate processing gain is sufficiently great to be worth implementing is primarily a function of the effort required to collect the data. On the one hand, combined single-sensor processing or some "majority vote" of single-sensor decisions seems a simple procedure; one does not have to push m channels through the same I/O or FFT device in roughly the same time frame or buy a parallel processor. On the other hand, if the detection scenario tolerates a non-real time solution, such as time-sharing FFT and I/O devices, the only significant additional burden the multivariate approach imposes is the need for storing (buffering) the data segments from each sensor (rows in the X matrix) until the entire data matrix is collected. In this connection it should be noted again that the time alignment of sensor data is not critical to the performance of the multivariate detection approach, since for independent sensor noises, the sensor signal relative phases do not affect it.

The comments above apply to the detection of deterministic signals. Evaluation of the detection of random signals remains to be done.

5.2.2 Implementation

As demonstrated by the computer program for implementing the multivariate detector, the programming complexity and storage requirements are quite modest. This assessment, however, is based upon the availability of input data vectors (columns in the X matrix) in sequence. If the data collection situation differs from this assumption, additional storage for buffering will be required. Still, the advent of microprocessor-based, parallel processing at low cost would make an m-channel data bus going into a multivariate processor (another microprocessor) a strong possibility.

5.2.3 Compatibility

Since some detection scenarios there is a certain amount of competition for computing resources among the various signal processing tasks (detection, localization, tracking, etc.), it is appropriate to assess the compatibility of the multivariate detection approach with these other tasks. While it has not been shown in this work, in the previous study [1] it was demonstrated that the statistical quantities developed by the multivariate processor (means, variances, and covariances) are also sufficient statistics for estimation of certain parameters. Thus simultaneous detection and estimation can be carried out using some of the same data processing.

For example, the diagonal elements of the sample covariance matrix are estimates of noise power at each sensor and the off-diagonal elements are inter-sensor correlations. The mean vector contains estimates of the signals arriving at the sensors. Also, the single-channel test statistic is an estimate of SNR at one sensor, while the multivariate test statistic is a generalized SNR. Therefore, it appears that the multivariate processor is very compatible with signal processing tasks requiring the estimation of these quantities.

5.3 Recommendations

Further work recommended may be classified into two categories: continuations and extensions.

5.3.1 Continuations

Recommended further work in a continuation of the present work:

(a) Theory: Find receiver operating characteristics for random signals. Deal with DFT components explicitly for both random and deterministic signals. Obtain ROC for special cases of the ML detector such as independent sensor noises.

(b) Comparisons: Compare multivariate random signal detection performance against that of combined two-sensor correlation detectors.

(c) Algorithms: Refine the program for computing test statistics so that detection results for the various sample sizes (10/20/50/100) are available every ten samples by using a "sliding window" concept. Also, seek ways to iterate the computation of determinants so that accumulation and storage of covariance matrices is not required.

(d) Validation: Test the multivariate detection approach against simulated and actual multisensor data.

5.3.2 Extensions

Recommended further work extending the present work:

(a) Develop joint detection and estimation procedures which utilize the same block of multisensor data. For example, joint random signal (correlation) detection and localization via time delay estimation are especially compatible. Also, a generalized multivariate detector could be used to "educate" a beamformer and thereby obtain bearing estimates.

(b) Study in detail the impact on software and hardware requirements that implementation of multivariate processing would have, for a specific multisensor scenario such as airborne ASW surveillance using sonobuoys.

LIST OF PROGRAMS

PROGRAM P-1. INCOMPLETE BETA FUNCTION (F-STATISTIC FALSE ALARM PROBABILITY).

$$I_x(b,a) = x^b(1 + c_1 + c_2 + \ldots + c_{a-1})$$

where $c_{r+1} = (b + r)(1 - x)c_r/(r + 1)$, $c_0 = 1.$ $(a > 1)$

hp 34C keystroke codes:

LBL A	CHS	STO I	RCL 6	×
STO I	STO 4	RCL 4	ISG	GTO 0
x↔y	RCL 0	RCL 1	GTO 2	LBL 1
STO 0	1	STO 6	GTO 1	RCL 5
RTN	-	LBL 0	LBL 2	RCL 3
LBL B	EEX	×	RCL I	RCL 1
STO 3	3	STO+5	INT	y^x
1	÷	1	RCL 4	×
STO 5	1	STO+6	R↓	RTN
-	+	CLx	÷	

Routine A stores values of a and b.

To compute I, enter x, run routine B.

Example: $I_x(8,2) = .10000$ for $x = .63164$.

PROGRAM P-2. PROBABILITY OF DETECTION FOR F-STATISTIC.

```
//D2RFXACT JOB (0591,0000,,,,,Y,0),MILLER,CLASS=E
/*FETCH
/*NOTIFY
/*NOSETUP
//STEP1 EXEC FORTGCLG
//FORT.SYSIN DD *
      DIMENSION AM(20), PD(20),NN(10),MM(10),XX(7,5)
      READ(5,1) (NN(I),I=1,5),(MM(I),I=1,7)
1     FORMAT(12I5)
      READ(5,2) (AM(I),I=1,11)
2     FORMAT(11F5.1)
      READ(5,3) XX
3     FORMAT(7F11.6)
      WRITE(6,3) XX
      DO 32 NA=1,2
      NO=NN(NA)
      WRITE(6,4) NO
4     FORMAT(/1X,'N=',I3/2X,'M=      L=0    -6DB    -3DB     0DB     3DB     6
     CDB     8DB    10DB    12DB    14DB    16DB')
      DO 30 MC=1,3
      MA=MM(MC)
      MB=NO-MA
      X=XX(MC,NA)
```

(continued on next page)

```
      IF(X.EQ.0.) GOTO 30
      DO 25 I=1,11
      SUM1=0.
      C1=1.
      IF(I.GT.1) GOTO 5
      AMA=AM(I)
      GOTO 6
5     AMA=10.**(AM(I)/10.)
6     N=0
7     Y2=1.
      C2=1.
      JMAX=N+MA-1
      IF(JMAX.EQ.0) GOTO 9
      DO 8 J=1,JMAX
      C2=C2*(1.-X)*FLOAT(MB+J-1)/FLOAT(J)
      Y2=Y2+C2
8     CONTINUE
9     Y2=Y2*X**MB
      SUM1=SUM1+Y2*C1
      N=N+1
      C1=C1*AMA/2./FLOAT(N)
      IF(C1.LT.1E-5) GOTO 10
      GOTO 7
10    SUM1=SUM1*EXP(-AMA/2.)
      PD(I)=SUM1
25    CONTINUE
      WRITE(6,31) MA,(PD(I),I=1,11)
31    FORMAT(1X,I4,11F7.4)
30    CONTINUE
32    CONTINUE
      STOP
      END
/*
//GO.SYSIN DD *
  10   20   50  100  200    1    2    3    4    5   10   20
 0.0 -6.0 -3.   0.   3.    6.   8.  10.  12.  14.  16.
   .774267    .63164     .50992     .40058     .30097    0.         0.
   .885867    .810237    .743495    .68141     .62247     .33793    0.
   .954095    .922922    .89501     .868716    .84345     .72559     .510666
   .977010    .961279    .947135    .933764    .920878    .860288    .74769
   .988496    .980595    .973477    .966738    .960233    .929548    .872118
/*
```

PROGRAM P-3. ASYMPTOTIC DISTRIBUTION COEFFICIENTS.

```
//D2AXCOEF JOB (0591,0000,,,,,,Y,0),MILLER,CLASS=E
/*FETCH
/*NOTIFY
/*NOSETUP
//STEP1 EXEC FORTGCLG
//FORT.SYSIN DD *
      DIMENSION XM(6),XN(6),D2(6,6),D3(6,6),D4(6,6)
      READ(5,1) (XM(I),I=1,5),(XN(I),I=1,5)
1     FORMAT(10F7.2)
      DO 2 NM=1,5
      DO 3 NN=1,5
      IF(XM(NM).GE.XN(NN)) GOTO 3
      X=XN(NN)-(XM(NM)+1.)/3.
      D2(NM,NN)=(XM(NM)-2.)*(XM(NM)-1.)*XM(NM)*(XM(NM)+1.)/72./X**2
      D3(NM,NN)=D2(NM,NN)*(2.*XM(NM)-1.)/22.5/X
      D4(NM,NN)=D2(NM,NN)*(XM(NM)**2-XM(NM)-7.)/30./X**2
3     CONTINUE
2     CONTINUE
4     FORMAT(6E12.4)
      WRITE(6,4) D2
      WRITE(6,4) D3
      WRITE(6,4) D4
      STOP
      END
/*
//GO.SYSIN DD *
   3..    4.     5.     10.    20.    10.    20.    50.   100.   200.
/*
```

PROGRAM P-4 CHI-SQUARE CUMULATIVE DISTRIBUTION.

$\Pr\{\chi^2 \leq x\} = P(x|\nu) = 1 - Q(x|\nu)$, where ν is the number of degrees of freedom and

$$Q(x|\nu) = e^{-x/2} \sum_{n=0}^{\nu/2-1} (x/2)^n/n!$$

<u>hp 34C keystroke codes</u>:

<u>LBL A</u>	2	↑	ISG	e^x
CLR Σ	÷	RCL 7	GTO 1	STO 9
EEX	STO 7	x↔y	RCL 1	×
3	1	÷	RCL 7	RCL 9
÷	STO 8	RCL 8	2	×
STO I	<u>LBL 1</u>	×	÷	
x↔y	Σ+	STO 8	CHS	

Example: Q(447.1|380) = .0100; Q(4.60517|2) = .10000.

PROGRAM P-5 TRYOUT OF MATRIX AND VECTOR OPERATIONS.

```
//D2TRYOUT JOB (0591,0000,,,,,Y,0),MILLER,CLASS=E
/*FETCH
/*NOTIFY
/*NOSETUP
//STEP1 EXEC FORTGCLG
//FORT.SYSIN DD *
      IMPLICIT COMPLEX(C)
      DIMENSION CX(5),CC(5,5),CMU(5),CCO(5,5)
      CALL STARTR(65539)
      DO 1 K=1,5
      CMU(K)=CMPLX(0.,0.)
      DO 2 K1=1,5
      CC(K,K1)=CMPLX(0.,0.)
      CCO(K,K1)=CMPLX(0.,0.)
2     CONTINUE
1     CONTINUE
C GENERATE RANDOM VECTORS, MEAN VECTOR, AND XX* MATRIX
      DO 8 N=1,10
      DO 3 J=1,5
      A=RANDN(1.)
      B=RANDN(1.)
      CX(J)=CMPLX(A,B)
      CMU(J)=CMU(J)+CX(J)/10.
      DO 4 J1=1,J
      CC(J,J1)=CX(J)*CONJG(CX(J1))/20.+CC(J,J1)
      CC(J1,J)=CONJG(CC(J,J1))
4     CONTINUE
3     CONTINUE
8     CONTINUE
```

(Continued on next page)

```
      WRITE(6,12)
12    FORMAT(1X,'  MEAN  VECTOR  ')
      WRITE(6,5) (CMU(I),I=1,5)
5     FORMAT(1X,10F9.3)
      WRITE(6,6)
6     FORMAT(//,1X,'  MATRIX  XX*  ',/)
      DO 7 I=1,5
      WRITE(6,5) (CC(I,J),J=1,5)
7     CONTINUE
C GENERATE (X-XO)(X-XO)* MATRIX
      DO 13 J=1,5
      DO 14 I=J,5
      CCO(J,I)=CC(J,I)-CMU(J)*CONJG(CMU(I))/2.
      CCO(I,J)=CONJG(CCO(J,I))
14    CONTINUE
13    CONTINUE
      WRITE(6,15)
15    FORMAT(//,1X,'  MATRIX  (X-XO)(X-XO)*  ',/)
      DO 16 I=1,5
      WRITE(6,5) (CCO(I,J),J=1,5)
16    CONTINUE
C COMPUTE DETERMINANTS, RATIO
      CDET=CMPLX(1.,0.)
      CDETO=CDET
      DO 9 I=1,4
      CPIV=CC(I,I)
      CPIVO=CCO(I,I)
      CDET=CDET*CPIV
      CDETO=CDETO*CPIVO
      JMIN=I+1
      DO 10 J=JMIN,5
      DO 11 J1=JMIN,5
      CC(J,J1)=CC(J,J1)-CC(I,J1)*CC(J,I)/CPIV
      CCO(J,J1)=CCO(J,J1)-CCO(I,J1)*CCO(J,I)/CPIVO
11    CONTINUE
10    CONTINUE
9     CONTINUE
      CDET=CDET*CC(5,5)
      CDETO=CDETO*CCO(5,5)
      CRAT=CDET/CDETO
      WRITE(6,17)
17    FORMAT(/,1X,'  DETS  AND  RATIO  ',/)
      WRITE(6,5) CDET,CDETO,CRAT
      STOP
      END
/*
//LKED.SYSLIB DD
//              DD DSN=C591.SUBLIB,DISP=SHR
/*
```

PROGRAM P-6   FULL-SCALE IMPLEMENTATION OF MULTIVARIATE PROCESSING.

```
//D2BIGONE JOB (0591,0000,,,,,,Y,0),MILLER,CLASS=E
/*FETCH
/*NOTIFY
/*NOSETUP
//STEP1 EXEC FORTGCLG
//FORT.SYSIN DD *
      IMPLICIT COMPLEX(C)
      DIMENSION CX(5),CVX(5),CAX(5,5),CBX(5,5),CAXX(5,5),CBXX(5,5)
     1,CAL(5,5),CBL(5,5),CAC(5,5),CBC(5,5),CVXX(5),CVL(5),CVC(5)
      CALL STARTR(65539)
      M=5
      NSAMP=10
      ENSAMP=FLOAT(NSAMP)
      THRX=5.8486
      THRXX=1.9930
      THRL=1.2805
      THRC=1.1271
      WXX=2.*ENSAMP
      WL=5.*ENSAMP
      WC=10.*ENSAMP
      CALL CLRV(CVXX,M)
      CALL CLRV(CVL,M)
      CALL CLRV(CVC,M)
      CALL CLRM(CAXX,M,M)
      CALL CLRM(CBXX,M,M)
      CALL CLRM(CAL,M,M)
      CALL CLRM(CBL,M,M)
      CALL CLRM(CAC,M,M)
      CALL CLRM(CBC,M,M)
      DO 1 I=1,5
      DO 1 J=1,2
      CALL CLRM(CAX,M,M)
      CALL CLRM(CBX,M,M)
      CALL CLRV(CVX,M)
      DO 2 K=1,NSAMP
      DO 2 K1=1,M
      A=RANDN(1.)+.3
      B=RANDN(1.)
      CX(K1)=CMPLX(A,B)
      CVX(K1)=CVX(K1)+CX(K1)/ENSAMP
      DO 2 K2=1,K1
      CAX(K1,K2)=CX(K1)*CONJG(CX(K2))+CAX(K1,K2)
2     CAX(K2,K1)=CONJG(CAX(K1,K2))
      CALL ADDM(CAXX,CAX,M,1.,1.)
      CALL ADDV(CVXX,CVX,M,1.,.5)
      CALL ADDM(CAL,CAX,M,1.,1.)
      CALL ADDV(CVL,CVX,M,1.,.2)
      CALL ADDM(CAC,CAX,M,1.,1.)
      CALL ADDV(CVC,CVX,M,1.,.1)
```

(continued on next page)

```
C
C      TENS DETECT
C
       CALL PRODV(CVX,M,CVX,M,CBX,ENSAMP)
       CALL ADDM(CBX,CAX,M,-1.,1.)
       CALL DETER(CAX,M,CNX)
       CALL DETER(CBX,M,CSX)
       RATX=REAL(CNX/CSX)
       WRITE(6,3) RATX,THRX
3      FORMAT(/,1X,'TENS:RATIO=',F9.4,1X,'THRESHOLD=',F7.4)
C
C      TWENTIES DETECT
C
       IF(J.EQ.1) GOTO 4
       CALL PRODV(CVXX,M,CVXX,M,CBXX,WXX)
       CALL CLRV(CVXX,M)
       CALL ADDM(CBXX,CAXX,M,-1.,1.)
       CALL DETER(CAXX,M,CNXX)
       CALL DETER(CBXX,M,CSXX)
       RATXX=REAL(CNXX/CSXX)
       CALL CLRM(CAXX,M,M)
       CALL CLRM(CBXX,M,M)
       WRITE(6,5) RATXX,THRXX
5      FORMAT(/,1X,'TWENTIES:RATIO=',F9.4,1X,'THRESHOLD=',F7.4)
C
C      FIFTIES DETECT
C
4      IF(I*J.EQ.3.OR.J*I.EQ.10) GOTO 8
       GOTO 1
8      CALL PRODV(CVL,M,CVL,M,CBL,WL)
       CALL CLRV(CVL,M)
       CALL ADDM(CBL,CAL,M,-1.,1.)
       CALL DETER(CAL,M,CNL)
       CALL DETER(CBL,M,CSL)
       RATL=REAL(CNL/CSL)
       CALL CLRM(CAL,M,M)
       CALL CLRM(CBL,M,M)
       WRITE(6,6) RATL,THRL
6      FORMAT(/,1X,'FIFTIES:RATIO=',F9.4,1X,'THRESHOLD=',F7.4)
1      CONTINUE
C
C      HUNDREDS DETECT
C
       CALL PRODV(CVC,M,CVC,M,CBC,WC)
       CALL CLRV(CVC,M)
       CALL ADDM(CBC,CAC,M,-1.,1.)
       CALL DETER(CAC,M,CNC)
       CALL DETER(CBC,M,CSC)
       RATC=REAL(CNC/CSC)
       CALL CLRM(CAC,M,M)
       CALL CLRM(CBC,M,M)
       WRITE(6,7) RATC,THRC
7      FORMAT(/,1X,'HUNDREDS:RATIO=',F9.4,1X,'THRESHOLD=',F7.4)
       STOP
       END
```

(continued on next page)

PROGRAM P-6 (continued)

```
C
C     SUBROUTINES
C
      SUBROUTINE DETER(CMAT,M,CDET)
      IMPLICIT COMPLEX(C)
      DIMENSION CMAT(M,M)
      IMAX=M-1
      CDET=(1.,0.)
      DO 1 I=1,IMAX
      CPIV=CMAT(I,I)
      CDET=CDET*CPIV
      JMIN=I+1
      DO 1 J=JMIN,M
      DO 1 J1=JMIN,M
1     CMAT(J,J1)=CMAT(J,J1)-CMAT(I,J1)*CMAT(J,I)/CPIV
      CDET=CDET*CMAT(M,M)
      RETURN
      END
C
      SUBROUTINE PRODV(CV1,N1,CV2,N2,CMAT,W)
      IMPLICIT COMPLEX(C)
      DIMENSION CV1(N1),CV2(N2),CMAT(N1,N2)
      DO 1 I=1,N1
      DO 1 J=1,N2
1     CMAT(I,J)=W*CV1(I)*CONJG(CV2(J))
      RETURN
      END
C
      SUBROUTINE ADDM(CMAT1,CMAT2,M,W1,W2)
      IMPLICIT COMPLEX(C)
      DIMENSION CMAT1(M,M),CMAT2(M,M)
      DO 1 I=1,M
      DO 1 J=1,M
      CMAT1(I,J)=W1*CMAT1(I,J)+W2*CMAT2(I,J)
1     CONTINUE
      RETURN
      END
C
      SUBROUTINE CLRM(CMAT,M,N)
      IMPLICIT COMPLEX(C)
      DIMENSION CMAT(M,N)
      DO 1 I=1,M
      DO 1 J=1,N
1     CMAT(I,J)=(0.,0.)
      RETURN
      END
C
      SUBROUTINE CLRV(CV,M)
      IMPLICIT COMPLEX(C)
      DIMENSION CV(M)
      DO 1 I=1,M
1     CV(I)=(0.,0.)
      RETURN
      END
```

(continued on next page)

PROGRAM P-6 (continued)

```
C
      SUBROUTINE ADDV(CV1,CV2,M,W1,W2)
      IMPLICIT COMPLEX(C)
      DIMENSION CV1(M),CV2(M)
      DO 1 I=1,M
1     CV1(I)=W1*CV1(I)+W2*CV2(I)
      RETURN
      END
/*
//LKED.SYSLIB DD
//             DD DSN=C591.SUBLIB,DISP=SHR
/*
```

OUTPUT FROM PROGRAM P-6 AS LISTED, SHOWING DETECTIONS (P_{FA} = .01)
(5 sensors, S = .3 or SNR = -13.5 dB)

```
   TENS:RATIO=     3.0335 THRESHOLD= 5.8486
   TENS:RATIO=     1.6548 THRESHOLD= 5.8486
   TWENTIES:RATIO=     1.6419 THRESHOLD= 1.9930
   TENS:RATIO=     3.5713 THRESHOLD= 5.8486
   TENS:RATIO=     3.8445 THRESHOLD= 5.8486
   TWENTIES:RATIO=     1.9891 THRESHOLD= 1.9930
   TENS:RATIO=     3.0560 THRESHOLD= 5.8486
-> FIFTIES:RATIO=     1.5326 THRESHOLD= 1.2805
   TENS:RATIO=     3.0316 THRESHOLD= 5.8486
-> TWENTIES:RATIO=     2.1577 THRESHOLD= 1.9930
   TENS:RATIO=     1.5716 THRESHOLD= 5.8486
   TENS:RATIO=     3.7013 THRESHOLD= 5.8486
   TWENTIES:RATIO=     1.3103 THRESHOLD= 1.9930
   TENS:RATIO=     1.7447 THRESHOLD= 5.8486
   TENS:RATIO=     1.7189 THRESHOLD= 5.8486
   TWENTIES:RATIO=     1.2670 THRESHOLD= 1.9930
   FIFTIES:RATIO=     1.1324 THRESHOLD= 1.2805
-> HUNDREDS:RATIO=     1.2309 THRESHOLD= 1.1271
```

APPENDIX A

DISTRIBUTION OF MULTIVARIATE TEST STATISTIC FOR COMPLEX DATA

(A) Following the approach of Anderson, we note that the sample mean vector and sample covariance are independent. The original data vector samples are

$$\underline{x}_k = \underline{u}_k + j\underline{v}_k, \quad (m \times 1), \tag{A-1}$$

where samples are assumed independent and identically distributed as multivariate normal, denoted

$$\underline{u}_k \sim N(\underline{a}, \Sigma), \ \underline{v}_k \sim N(\underline{b}, \Sigma), \ (\underline{u}_k, \underline{v}_k \text{ independent}). \tag{A-2}$$

The sample covariance $\hat{\Sigma}$ is such that

$$2n\hat{\Sigma} = XX^* - X_0X_0^* \equiv A$$

$$= \sum_{k=1}^{n} \underline{x}_k \underline{x}^*_k - n\underline{\hat{\mu}}\underline{\hat{\mu}}^* , \ \underline{\hat{\mu}} = \frac{1}{n} \sum_{k=1}^{n} \underline{x}_k . \tag{A-3}$$

The data matrix can also be represented in a different coordinate system; let B be an orthogonal, $(n \times n)$ matrix:

$$X = ZB', \quad BB' = I_n; \quad B \equiv (\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_n). \text{ (real)} \tag{A-4}$$

That is, there is another set of vectors such that

$$\underline{z}_k = X\underline{b}_k = \sum_{i=1}^{n} b_{ik}\underline{x}_i$$

and these vectors are also independent. Let the nth column of B have all components equal to $1/\sqrt{n}$. Then

$$\underline{z}_n = \sqrt{n}\,\underline{\hat{\mu}} \qquad \begin{cases} \underline{z}_n \sim N(\sqrt{n}\,\underline{\mu}, \Sigma) \\ \underline{z}_k \sim N(\underline{0}, \Sigma), \ k \neq n \end{cases} \tag{A-5}$$

$$A = ZB'BZ^* - n\,\underline{\hat{\mu}}\,\underline{\hat{\mu}}^* = Z\,Z^* - \underline{z}_n\,\underline{z}_n^*$$

$$= \sum_{k=1}^{n} \underline{z}_k\,\underline{z}_k^* - \underline{z}_n\,\underline{z}_n^* = \sum_{k=1}^{n-1} \underline{z}_k\,\underline{z}_k^*. \qquad \text{(A-6)}$$

Therefore $\underline{\hat{\mu}}$ is independent of A because $\underline{z}_n$ is independent of the other $\{\underline{z}_k\}$.

(B) Using these new vectors, the test statistic is

$$n\underline{\hat{\mu}}^*\,A^{-1}\,\underline{\hat{\mu}} = \underline{z}_n^*\left(\sum_{k=1}^{n-1} \underline{z}_k\,\underline{z}_k^*\right)^{-1}\underline{z}_n$$

$$= \mathrm{tr}\left[\left(\sum_{k=1}^{n-1} \underline{z}_k\,\underline{z}_k^*\right)^{-1}\underline{z}_n\,\underline{z}_n^*\right]. \qquad \text{(A-7)}$$

Now the $\{\underline{z}_k\}$ are transformed by diagonalization:

$$\underline{y}_k \triangleq D\underline{z}_k \quad \text{where } D\Sigma D' = I_m \qquad \text{(A-8)}$$

so that $n\,\underline{\hat{\mu}}^*\,A^{-1}\,\underline{\hat{\mu}} = \mathrm{tr}\left[\left(\sum_{k=1}^{n-1} \underline{y}_k\,\underline{y}_k^*\right)^{-1}\underline{y}_n\,\underline{y}_n^*\right]$ (A-9)

where the $\{\underline{y}_k\}$ have identity covariance matrices.

(C) A third transformation is performed using a unitary matrix (complex orthogonal) Q such that $QQ^* = I$ and resulting in the data vectors

$$\underline{w}_k = Q\underline{y}_k \qquad \text{(A-10)}$$

with the constraint

$$\underline{w}_n = Q\underline{y}_n = (|\underline{y}_n|, 0, 0, \ldots, 0)'. \qquad \text{(A-11)}$$

This makes the matrix $\underline{y}_n\,\underline{y}_n^*$ in (A-9) become a matrix with only one nonzero component, causing the trace to be

$$n\, \underline{\hat{\mu}}^* A^{-1} \underline{\hat{\mu}} = \left[\left(\sum_{k=1}^{n-1} \underline{w}_k\, \underline{w}_k^*\right)^{-1}\right]_{11} \left|\underline{w}_n\right|^2. \tag{A-12}$$

The element b^{11} of the inverse of $B = \sum_{k=1}^{n-1} \underline{w}_k \underline{w}_k^*$

is given by

$$\frac{1}{b^{11}} = b_{11} - \underline{b}^*_{(1)}\, B_{22}^{-1}\, \underline{b}_{(1)} \tag{A-13}$$

where B has been partitioned into

$$B = \begin{bmatrix} b_{11} & \underline{b}^*_{(1)} \\ b_{(1)} & B_{22} \end{bmatrix}. \tag{A-14}$$

Therefore the test statistic is given by

$$n\underline{\hat{\mu}}^* A^{-1} \underline{\mu} = \frac{\left|\underline{w}_n\right|^2}{b_{11} - \underline{b}^*_{(1)} B_{22}^{-1} \underline{b}_{(1)}} \tag{A-15}$$

The numerator can be seen as a noncentral chi-squared variate

$$\left|\underline{w}_n\right|^2 \sim \chi^2(2m;\lambda)$$

with noncentrality parameter

$$= E\left\{\underline{w}_n^*\right\} E\left\{\underline{w}_n\right\} = E\left\{\underline{z}_n^*\right\} D'Q^*\, Q\, D\, E\left\{\underline{z}_n\right\}$$

$$= n\, \underline{\mu}^*\, D'\, D\, \underline{\mu} = n\, \underline{\mu}^*\, \Sigma^{-1}\, \underline{\mu} \tag{A-16}$$

By Theorem 4.3.3 of Anderson the denominator of (A-15) is seen to be (central) chi-squared, independent of the numerator:

$$1/b^{11} \sim \chi^2(2n-2m). \tag{A-17}$$

Thus the ratio is an F-statistic, denoted

$$n\underline{\hat{\mu}}^* A^{-1} \underline{\hat{\mu}} \sim \frac{\chi^2(2m;\lambda)}{\chi^2(2n-2m)} = \frac{m}{n-m} F_{2m,2(n-m)}(\lambda). \tag{A-18}$$

Appendix B

Moments of the Random Signal Test Statistic

(A completely analogous derivation for real data is given by Anderson [2].)

For n samples from an m-dimensional distribution of zero-mean, complex, jointly Gaussian variables with covariance matrix Σ the sample covariance

$$A = 2n\,\hat{\Sigma} \triangleq \sum_{k=1}^{n} \underline{x}_k \underline{x}_k^* \tag{B-1}$$

has the Wishart distribution [2, 3]

$$p(A;n) = K_m(\Sigma,n)\ |A|^{n-m}\ \mathrm{etr}\left\{-\frac{1}{2}\,\Sigma^{-1}A\right\},$$

where

$$K_m(\Sigma,n)^{-1} \equiv 2^{mn}\ \pi^{m(m-1)/2}\ \Gamma(n)\Gamma(n-1)\ldots(n-m+1)|\Sigma|^{n}. \tag{B-2}$$

The moments therefore of

$$z = |A_0|/|A|\ ,\quad A_0 \equiv \mathrm{diag}\left(a_{11},\ a_{22},\ldots a_{mm}\right) \tag{B-3}$$

are given by

$$\begin{aligned} E\left\{z^h\right\} &= \int dA|A_0|^h\ |A|^{-h}\ p(A;n) \\ &= \int dA\ K_m(\Sigma,n)|A_0|^h\ |A|^{n-m-h}\ \ \mathrm{etr}\left\{-\frac{1}{2}\,\Sigma^{-1}A\right\} \\ &= \frac{K_m(\Sigma,\ n)}{K_m(\Sigma,\ n-h)}\int dA\ |A_0|^h\ p(A;n-h). \end{aligned} \tag{B-4}$$

Integrating first over the off-diagonal elements yields

$$E\left\{z^h\right\} = \frac{K_m(\Sigma,\ n)}{K_m(\Sigma,\ n-h)}\int dA_0\ |A_0|^h\ p_0(A_0;n-h), \tag{B-5}$$

where $p_0(\ )$ is the joint pdf of the diagonal elements of A. For Σ non-diagonal this density is extremely complex in expression [9, 10]. However, for

$$\Sigma = \Sigma_0 = \mathrm{diag}\left(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{mm}\right), \tag{B-6}$$

these elements are independently distributed as

$$p_1\left(a_{ii};\ n-h\right) = K_1\left(\sigma_{ii},\ n-h\right)\ a_{ii}^{\ n-1-h}\ \exp\left\{-a_{ii}/2\sigma_{ii}\right\}, \tag{B-7}$$

and the integration is simple, producing

$$E\left\{z^h\right\} = \frac{K_m\left(\Sigma_0, n\right)}{K_m\left(\Sigma_0, n-h\right)} \prod_{k=1}^{m} \int_0^\infty da_{ii}\ K_i\left(\sigma_{ii}, n-h\right) a_{ii}^{\ n-1} \exp\left\{-a_{ii}/2\sigma_{ii}\right\}$$

$$= \frac{K_m\left(\Sigma_0, n\right)}{K_m\left(\Sigma_0, n-h\right)} \prod_{i=1}^{m} \frac{K_1\left(\sigma_{ii}, n-h\right)}{K_1\left(\sigma_{ii},\ n\right)}$$

$$= \frac{\Gamma(n-h)\Gamma(n-h-1)\ldots\Gamma(n-h-m+1)}{\Gamma(n)\Gamma(n-1)\ldots\Gamma(n-m+1)} \left[\frac{\Gamma(n)}{\Gamma(n-h)}\right]^m . \tag{B-8}$$

This expression may be further developed as

$$E\left\{z^h\right\} = \prod_{i=2}^{m} \frac{\Gamma(n-h-i+1)\Gamma(n)}{\Gamma(n-i+1)\Gamma(n-h)}$$

$$= \prod_{i=2}^{m} \frac{B(n-h-i+1,\ i-1)}{B(n-i+1,\ i-1)}, \tag{B-9}$$

where the beta function is

$$B(a,\ b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{B-10}$$

Appendix C

Effect of Correlation and Phase on Multivariate Detection Parameter

From (4-10), the multivariate detection parameter (generalized SNR) is given by

$$\lambda_M = n \sum_{i=1}^{m} \sum_{v=1}^{m} \sigma^{ir} S_i S_r \cos(\theta_i - \theta_r), \tag{C-1}$$

where $\left\{\sigma^{ir}\right\} = \Sigma^{-1}$ is the inverse of the intersensor noise covariance matrix and the $\left\{S_i, \theta_i\right\}$ are the amplitudes and phases of the signals arriving at the sensors (or magnitude and phase of the DFT bin output).

If the phases are independent and uniformly distributed on some interval $(-\Delta, \Delta)$, then

$$E\left\{\cos(\theta_i - \theta_r)\right\} = \frac{\sin^2\Delta}{\Delta^2}. \tag{C-2}$$

Thus we have

$$E_\theta\left\{\lambda_M\right\} = n\sum_i S_i^2\sigma^{ii} + n\,\frac{\sin\Delta^2}{\Delta^2} \sum_{i\neq r} S_i S_r\, \sigma^{ir}, \tag{C-3}$$

and the second term vanishes for $\Delta = k\pi$. A random sensor placement therefore, such as shown in Figure C-1, could cause the second term to drop out regardless of the covariance matrix structure.

If the covariance matrix Σ can be represented by the case in which the intersensor correlations are all equal, then

$$\Sigma = \left\{\rho\sigma_i\sigma_r\right\} = D_\sigma R\, D_\sigma, \tag{C-4}$$

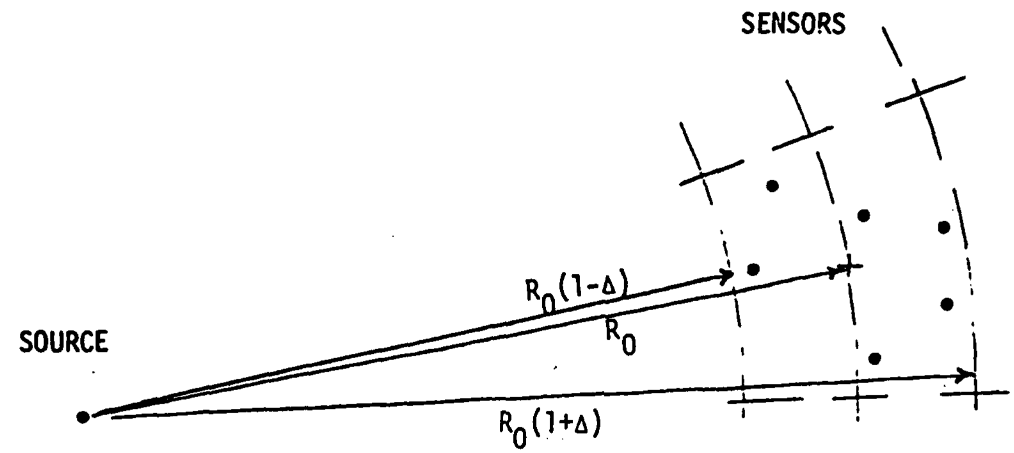


FIGURE C-1 RANDOM SENSOR PLACEMENT

using

$$D_\sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m) \tag{C-5}$$

and

$$R = \left\{r_{ir}\right\}, \begin{cases} r_{ii} = 1, & i = r \\ r_{ir} = \rho, & i \neq r \end{cases} . \tag{C-6}$$

For this special case

$$\Sigma^{-1} = D_\sigma^{-1} R^{-1} D_\sigma^{-1} \tag{C-7}$$

and

$$\sigma^{ir} = \begin{cases} \beta\sigma_i^{-2}, & i = r \\ \alpha/\sigma_i\sigma_r, & i \neq r \end{cases} \tag{C-8}$$

where

$$\beta = \frac{1+(m-2)\rho}{(1-\rho)[1+(m-1)\rho]} \tag{C-9a}$$

and

$$\alpha = \frac{-\rho}{(1-\rho)[1+(m-1)\rho]} . \tag{C-9b}$$

In this instance the average detection parameter has the form

$$E_\sigma\left\{\lambda_M;\rho\right\} = n\beta \sum_i S_i^2/\sigma_i^2 + n\alpha\left(\frac{\sin\Delta}{\Delta}\right)^2\left(\sum_i S_i/\sigma_i\right)^2$$

$$= 2nm\ \mathrm{SNR}_{avg}$$

$$\times\left[\beta + (m-1)\alpha\left(\frac{\sin\Delta}{\Delta}\right)^2\right]. \tag{C-10}$$

Appendix D

Distribution of Sample Correlation Coefficient (Squared)

The elements of the sample covariance matrix $2n\hat{\Sigma} = A$ for m=2 have the Wishart distribution

$$p_1(A) = K(\Sigma,n)\ |A|^{n-2}\ \mathrm{etr}\left\{-\frac{1}{2}\Sigma^{-1}A\right\} \tag{D-1}$$

where

$$K^{-1}(\Sigma,n) = 2^{2n}\ \pi|\Sigma|^{n}\Gamma(n)\Gamma(n-1). \tag{D-2}$$

Explicitly (D-1) is written

$$p_1(A) = \frac{\left(a_{11}a_{22}-|a_{12}|^2\right)^{n-2}\exp\left\{-\frac{1}{2}\ \frac{1}{1-\rho^2}\left[\frac{a_{11}}{\sigma_1^2} - 2\rho\frac{a_{12R}}{\sigma_1\sigma_2} + \frac{a_{22}}{\sigma_2^2}\right]\right\}}{4^n\pi\left[\sigma_1^2\sigma_2^2(1-\rho^2)\right]^n\ \Gamma(n)\Gamma(n-1)}. \tag{D-3}$$

Let a transformation of variables be defined by

$$z = |a_{12}|^2/a_{11}a_{22}$$

$$\left.\begin{aligned} u\cos\theta &= a_{12R} \\ u\sin\theta &= a_{12I} \\ v &= a_{22} \end{aligned}\qquad |J| = \frac{u^3}{vz^2}\right\} \tag{D-4}$$

The joint pdf of the new variables is

$$p_2(z,u,\theta,v) = K\,\frac{u}{v}^{2(n-2)+3}\frac{(1-z)^{n-2}}{z^n}\exp\left\{-\frac{\beta u^2}{vz} + \frac{\rho u\cos\theta}{\sigma_1\sigma_2(1-\rho^2)} - \alpha v\right\}. \tag{D-5}$$

In the next steps the unwanted variables are integrated out:

$$\int_0^{2\pi} p_2(z,u,\theta,v)d\theta = 2\pi K\,\frac{u}{v}^{2(n-2)+3}\ \frac{(1-z)^{n-2}}{z^n}$$

$$\cdot\ I_0\left[\frac{\rho u}{\sigma_1\sigma_2(1-\rho^2)}\right]e^{-\frac{\beta u^2}{vz}-\alpha v} = p_3(z,u,v) \tag{D-6}$$

$$\int_0^\infty du\ p_3(z,u,v) = \frac{\pi K}{v}\frac{(1-z)^{n-2}}{z^n} e^{-\alpha v}\int_0^\infty dx\ x^{n-1}\ I_0\left[\gamma\sqrt{x}\right] e^{-\beta x/vz}$$

$$= \Gamma(n)\pi K \frac{(1-z)^{n-2}}{\beta^n} v^{n-1}\ e^{-\alpha v}\ {}_1F_1\left(n;1;\ \frac{\gamma^2 vz}{4\beta}\right) = p_4(z,v) \tag{D-7}$$

$$\int_0^\infty dv\ p_4(z,v) = \frac{\pi K(1-z)^{n-2}}{\beta^n}\ \frac{\Gamma(n)\Gamma(n)}{\alpha^n}\ {}_2F_1\left(n,n;1;\ \frac{\gamma^2 z}{4\beta\alpha}\right)$$

$$= \frac{\Gamma(n)}{\Gamma(n-1)}\ (1-\rho^2)^n\ (1-z)^{n-2}\ {}_2F_1(n,n;1;\rho^2 z) = p_5(z) \tag{D-8}$$

The complementary cumulative distribution function for z is computed as follows:

$$\Pr\left\{z \geq \eta\right\} = \int_\eta^1 dz\ p(z)$$

$$= \frac{(1-\rho^2)^n}{(n-2)!}(n-1)!\int_\eta^1 dz\ (1-z)^{n-2}\ {}_2F_1(n,n;1;\rho^2 z)$$

$$= \frac{(1-\rho^2)^n}{(n-2)!}(n-1)!\ \sum_{k=0}^\infty \frac{\rho^{2k}}{k!k!}\ (n)_k(n)_k \int_\eta^1 dz\ z^k(1-z)^{n-2}$$

$$= \frac{(1-\rho^2)^n}{(n-2)!}(n-1)!\sum_{k=0}^\infty \frac{\rho^{2k}}{k!k!}\ (n)_k(n)_k\ B(k+1,\ n-1) I_{1-\eta}(n-1\ ,\ k+1) \tag{D-9}$$

or

$$Q_z(\eta) = (1-\rho^2)^n \sum_{k=0}^\infty \frac{\rho^{2k}}{k!}\ (n)_k\ I_{1-\eta}(n-1,\ k+1) \tag{D-10}$$

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Glossary of Notation

$\underline{a}$	real part of mean vector
A	sample covariance matrix ($2n\hat{\Sigma}$)
$\underline{b}$	imaginary part of mean vector
$B_r(x)$	Bernoulli polynomial
$B(a,b)$	beta function
det(B)	determinant of the matrix B
DFT	discrete Fourier transform
$E\{x\}$	mathematical expectation of quantity x
etr(B)	exp{tr(B)}
F	matrix of DFT components
F_{ν_1,ν_2}	F-statistic with degrees of freedom ν_1 and ν_2
${}_1F_1(a;b;x)$	*confluent hypergeometric function*
${}_2F_1(a,b;c;x)$	Gaussian hypergeometric function
H_0, H_1	hypotheses
I	identity matrix
$I_x(a,b)$	Pearson's incomplete beta function
j	$\sqrt{-1}$
$\Lambda(X)$	likelihood ratio value
λ	noncentrality parameter, detection parameter
m	number of sensors
MDS	minimum detectable signal
ML	maximum likelihood
MPG	multivariate processing gain
$\underline{\mu}$	mean vector
n	number of samples
pdf	probability density function
P_D	probability of detection

PFA	probability of false alarm
Pr{A}	probability of event A
Q (z)	probability integral
ROC	receiver operating characteristics
ρ	correlation coefficient
Σ	covariance matrix
SNR	signal-to-noise ratio
tr (B)	trace of matrix B
$\underline{u}$, U	real part of data vector, matrix
$\underline{v}$, V	imaginary part of data vector, matrix
$\underline{x}$, X	data vector, matrix
z, z_θ	statistic, test value
$\hat{}$	estimated value
$\underline{b}$	column vector
$\underline{b}'$, B'	transpose of vector, matrix
$\underline{b}^*$, B*	conjugate transpose of vector, matrix
$\|B\|$	determinant of matrix B
$\|\underline{b}\|$	magnitude of vector