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SURFACES IN THREE-DIMENSIONAL
DIGITAL IMAGES

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ABSTRACT

This is one of a series of reports on the digital geometry
of three-dimensional images, such as those produced by computed
tomography. In this report we define simple surface points and
simple closed surfaces, and show that any connected collection
of simple surface points form a simple closed surface, thus
proving a three-dimensional analog of the two-dimensional
Jordan curve theorem. We also show that the converse is not
a theorem (in contrast to the two-dimensional case) and dis-
cuss more complex surface types.

1. Introduction

This report is one of a series [1-3] on the digital geometry of three-dimensional images. Three-dimensional images are routinely produced in computed tomography (CT) where values (CT numbers) are assigned to volume elements (voxels), which are rectangular parallelepipeds filling a portion of three-dimensional space. In this report we consider binary-valued images, as might be obtained by applying a threshold to an image produced by CT. This series of reports provides a theoretical basis for the three-dimensional analogs of various processing algorithms, such as segmentation, thinning, connected component labelling and counting.

In this report we define simple surface points and simple closed surfaces, and show that any connected collection of simple surface points forms a simple closed surface, thus proving a three-dimensional analog of the two-dimensional Jordan curve theorem. We also show that the converse is not a theorem (in contrast to the two-dimensional case), and discuss more complex surface types. The concepts introduced conform as closely as possible to the corresponding concepts used in the topology and geometry of continuous three-dimensional space.

The approach here is fundamentally different from that of Artzy, Frieder, and Herman [4] and Herman and Webster [5] in that we construe surfaces to be sets of voxels, rather than

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This is one of a series of reports on the digital geometry of three-dimensional images, such as those produced by computed tomography. In this report we define simple surface points and simple closed surfaces, and show that any connected collection of simple surface points form a simple closed surface, thus proving a three-dimensional analog of the two-dimensional Jordan curve theorem. We also show that the converse is not a theorem (in contrast to the two-dimensional case) and discuss more complex surfaces types.

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of faces of voxels. The approach of representing the boundary between an object and its surrounding by a set of faces separating pairs of voxels may be used to describe the surface of any object which is "connected" in some appropriate sense, but has the disadvantage of not providing a natural framework for processes such as thinning. Our approach, which treats surfaces as "thin" objects, is complementary to theirs. Algorithms such as thinning are simplified (a paper on the theory of 3-D thinning is in preparation) but it is not true that such a surface can be used to describe the border of any object. We will indicate later how surfaces of faces may be encoded by surfaces of voxels.

There is a well developed theory of geometry and topology for subsets of two-dimensional arrays [6]. Some early work on 3-D digital geometry was done by Gray [7] and Park [8]. A more complete set of references is given in [1].

We begin with a short discussion of connectivity and components in 3D; a more detailed discussion of these topics, as well as distance, curves, surroundedness, borders, and genus, are given in [1].

2. Connectivity and components

A 3D digital image Σ is a three-dimensional lattice of elements called voxels defined by triples of Cartesian coordinates (x,y,z) which we may take to be integer valued. We will consider two types of neighbors of a point $p=(x,y,z)$:

   (a) the neighbors (u,v,w) such that $|x-u|+|y-v|+|z-w|=1$

   (b) the neighbors (u,v,w) such that $\max[|x-u|,|y-v|,|z-w|]=1$

We refer to the neighbors of type (a) as 6-neighbors of p (the face neighbors) and to the neighbors of type (b) as the 26-neighbors of p (the face, edge, and corner neighbors). The 6-neighbors are said to be 6-adjacent to p, and the 26-neighbors 26-adjacent to p.

By a path π of length $n>0$ from p to q in Σ, we mean a sequence of points $p=p_0,\ldots,p_n=q$ of Σ such that p_i is adjacent to p_{i-1}, $1 \le i \le n$. Any point alone is a path of length 0. We thus speak of 6-paths and 26-paths depending on the type of adjacency used.

Let S be a non-empty subset of Σ. To avoid special cases we assume that S does not meet the border of Σ. We say p and q are connected in S if there exists a path from p to q consisting entirely of points of S. Connectivity is an equivalence relation, since a path of length 0 is a path, the reversal of a path is a path, and the concatenation of two paths is a path. The equivalence classes under this relation are called components of S. Again, we have 6-connectivity, 26-connectivity, 6-components, and 26-components.

Similarly we can consider the components of the complement $\bar{S}$ of S. Evidently, exactly one of these contains the border of Σ; we call this component the background of S. All other components of $\bar{S}$, if any, are called cavities in S. If S has no cavities, it is called simply connected. To avoid ambiguous situations we shall assume that opposite types of connectivity are used for S and for $\bar{S}$.

Finally, we shall use a special type of path called a run along a principal half-line. In the 6-connected case a north half-line emanating from a point $p=(x,y,z)$ is the set of points $h_p = \{(u,v,w) | u=x, w=z, v \geq y\}$, and similarly for east, west, south, up, and down half-lines. In the 26-connected case the principal half-lines include those along the various diagonals (such as $\{(u,v,w) | u=x+i, v=y+i, w=z+i, i>0\}$). Thus, for 6-connectedness there are six principal half-lines, and for 26-connectedness there are 26. A run π along a principal half-line is the path formed by points along the half-line emanating from p such that no point occurs twice on the path. To simplify the discussion below we will assume both in the 26-case and the 6-case that we are talking about the north half-line emanating from a point.

Let p be a point of S. We let $N_{27}(p)$ denote the 26 points in the 3x3x3 neighborhood of p excluding p (these are the 26-neighbors of p), and we let $N_{125}(p)$ denote the 124 points in the 5x5x5 neighborhood centered at p excluding p.

3. Surfaces

A point $p \in S$ is a __simple surface point__ if the following conditions are all satisfied:

(i) $S \cap N_{27}(p)$ has exactly one component adjacent to p (in the S sense); call this component together with p A_p.

(ii) $\overline{S} \cap N_{27}(p)$ has exactly two components adjacent to p (in the $\overline{S}$ sense); call these components B_p and C_p.

(iii)   For every $q \in S$ adjacent to p (in the S sense), q is adjacent (in the $\overline{S}$ sense) to some point in B_p and to some point in C_p.

When confusion will not arise, we will call a simple surface point a surface point.

__Proposition 1.__   There are at most two components of $\overline{S} \cap N_{125}(p)$ adjacent (in the $\overline{S}$ sense) to a surface point p.

__Proof:__   There are exactly two components in $\overline{S} \cap N_{27}(p)$ adjacent to p, and no points in $N_{125}(p) - N_{27}(p)$ are adjacent to p. $\square$

Thus there are either one or two components in $\overline{S} \cap N_{125}(p)$ adjacent to a surface point p. Now suppose that all $q \in A_p$ are also surface points (so that p is not near an "edge"). When there are two components of $\overline{S} \cap N_{125}(p)$ adjacent to p we say that (the surface at) p is __orientable__ and call A_p a __disk__. When there is only one component in $\overline{S} \cap N_{125}(p)$ adjacent to p we say (the surface at) p is __non-orientable__ and call A_p a __cross-cap__. When A_p is a disk we call the two components of $\overline{S} \cap N_{125}(p)$ adjacent to p B'_p and C'_p, where $B_p \subseteq B'_p$ and $C_p \subseteq C'_p$. Clearly every $q \in A_p$ is adjacent to some point in B'_p and to some point in C'_p.

Let p and q be adjacent surface points. Then we call the component of $\overline{S} \cap N_{27}(q)$ which contains the point of B_p adjacent to q B_{qp}, and the component which contains the point of C_p adjacent to q C_{qp}, although B_{qp} and C_{qp} are not necessarily distinct components. However, whenever A_p is a disk it is easily seen that $B_q = B_{qp} \subseteq B'_p$ and $C_q = C_{qp} \subseteq C'_p$, where B_q and C_q (in some order) are the two components of $\overline{S} \cap N_{27}(q)$ adjacent to q. [Proof: Since B'_p and C'_p are distinct, so are B_{qp} and C_{qp} for any q.]

<u>Proposition 2</u>. Let $\pi = p_1, \ldots, p_n$ be any path of (not necessarily orientable) surface points. There exist connected subsets (in the $\overline{S}$ sense) B_π and C_π of $\overline{S} \cap [\underset{p \in \pi}{\cup} N_{27}(p)]$ such that every point of π is adjacent to some point of B_π and to some point of C_π (in the $\overline{S}$ sense).

<u>Proof</u>: Let $B_1 = B_{p_1}$ and $C_1 = C_{p_1}$; clearly B_1 and C_1 are each connected subsets of $\overline{S} \cap N_{27}(p_1)$, and p_1 is adjacent to some point in B_1 and to some point in C_1. Now for each i>1 let $B_i = B_{i-1} \cup B_{p_i p_{i-1}}$ and $C_i = C_{i-1} \cup C_{p_i p_{i-1}}$. If B_{i-1} is a connected subset of $\overline{S} \cap [\underset{1 \leq j \leq i-1}{\cup} N_{27}(p_j)]$, then B_i is a connected subset of $\overline{S} \cap [\underset{1 \leq j \leq i}{\cup} N_{27}(p_j)]$ by definition of $B_{p_i p_{i-1}}$, and similarly for C_i. Also, if every p_j, $1 \leq j \leq i-1$ is adjacent to B_{i-1} and to C_{i-1}, then every p_j, $1 \leq j \leq i$ is adjacent to B_i and C_i, since $B_{p_i p_{i-1}}$ and $C_{p_i p_{i-1}}$ are each adjacent to p_i. Then $B_\pi \equiv B_n$ and $C_\pi \equiv C_n$ are the desired subsets.□

Proposition 3. Let $\pi = p_1, \ldots, p_n$ be any path of orientable surface points. There exist connected subsets B_π' and C_π' of $\overline{S} \cap [\bigcup_{p \in \pi} N_{125}(p)]$ such that every $q \in [\bigcup_{p \in \pi} A_p]$ is adjacent to some point in B_π' and to some point in C_π' (in the $\overline{S}$ sense).

Proof: The proof parallels that of the previous proposition. Let $A_1 = A_{p_1}$, $B_1' = B_{p_1}'$ and $C_1' = C_{p_1}'$; clearly B_1' and C_1' are each connected subsets of $\overline{S} \cap N_{125}(p_1)$, and every $q \in A_1$ is adjacent to B_1' and to C_1'. For every $i > 1$ let $A_i = A_{i-1} \cup A_{p_i}$, $B_i' = B_{i-1}' \cup B_{p_i}'$, and $C_i' = C_{i-1}' \cup C_{p_i}'$. Since $B_{p_i p_{i-1}} \subseteq B_{p_{i-1}}' \subseteq B_{i-1}'$ and $B_{p_i p_{i-1}} \subseteq B_{p_i}'$, B_i' is a connected subset of $\overline{S} \cap [\bigcup_{1 \leq j \leq i} N_{125}(p_j)]$ if B_{i-1}' is a connected subset of $\overline{S} \cap [\bigcup_{1 \leq j \leq i-1} N_{125}(p_j)]$, and similarly for C_i'. Also, if every $q \in A_{i-1}$ is adjacent to B_{i-1}' and C_{i-1}', then every $q \in A_i$ is adjacent to B_i' and C_i' since $A_i - A_{i-1} \subseteq A_{p_i}$. Thus $B_\pi' \equiv B_n'$ and $C_\pi' \equiv C_n'$ are the desired subsets. $\square$

Proposition 4. Let $\pi = p_1, \ldots, p_n$ be a run of orientable surface points along a (say) north half-line. Then B_π' and C_π' are distinct components in $\overline{S} \cap [\bigcup_{p \in \pi} N_{125}(p)]$.

Proof: We follow the construction of B_π' and C_π' in the above proof. Clearly B_1 and C_1 are distinct components in $\overline{S} \cap N_{125}(p_1)$. Now we note that $B_i' = B_{i-1}' \cup [B_{p_i}' - B_{i-1}']$ and $C_i' = [C_{i-1}' - C_{p_i}'] \cup C_{p_i}'$. Since $[B_{p_i}' - B_{i-1}'] \subseteq B_{p_i}'$ the points in $[B_i' - B_{i-1}'] = [B_{p_i}' - B_{i-1}']$ are nowhere adjacent to points of C_{p_i}'. Because the p_i are along a half-line the points in $[B_{p_i}' - B_i']$ are nowhere adjacent to the points in $[C_{i-1}' - C_{p_i}']$. Hence, points in $[B_i' - B_{i-1}']$ are nowhere adjacent to points in C_i'.

In the same fashion, starting with $B_i' = [B_{i-1}' - B_{p_i}'] \cup B_{p_i}'$

and $C_i' = C_{i-1}' \cup [C_{p_i}' - C_{i-1}']$, we can show that points in $[C_i' - C_{i-1}']$

are nowhere adjacent to points in B_i'. Thus, if B_{i-1}' and C_{i-1}'

are nowhere adjacent to each other (induction hypothesis) then

neither are B_i' and C_i'. By Proposition 3 B_π' and C_π' are each

connected subsets of $\overline{S} \cap [\underset{p \in \pi}{\cup} N_{125}(p)]$, so they form distinct

components. $\square$

Proposition 5. Let $\pi = p_1, \ldots, p_n$ be a run of orientable surface

points along a (say) north half-line. Then B_π and C_π are

distinct components in $\overline{S} \cap [\underset{p \in \pi}{\cup} N_{27}(p)]$.

Proof: Note that by the construction of B_π, C_π, B_π', and C_π',

we have $B_\pi \subseteq B_\pi'$ and $C_\pi \subseteq C_\pi'$, so that the connected subsets B_π and

C_π are nowhere adjacent to each other. $\square$

Remark. In Propositions 4 and 5 we could let π be any path

such that the points being added at the ith step in the con-

structions are nowhere adjacent to those already considered

except inside $N(p_i)$. In particular, when 6-connectedness is

used for S we can use any of the six principal half-lines, and

when 26-connectedness is used for S we can use any of the 26

principal half-lines. Also, paths that turn are not strictly

disallowed in the 26-connected case.

Let $\pi = p_1, \ldots, p_n$ be a run of (not necessarily orientable)

surface points along a principal (say north) half-line h_p

emanating from $p \in \overline{S}$ such that p_0 and p_{n+1} (the points preceding

and following π along the half-line) are both in $\overline{S}$. Clearly

p_0 and p_{n+1} are in $B_\pi \cup C_\pi$. If p_0 is connected to p_{n+1} in $B_\pi \cup C_\pi$

then we say that h_p _touches_ S in π. If p_0 is not connected to p_{n+1} in $B_\pi \cup C_\pi$ we say that h_p _crosses_ S in π. Clearly, if π consists solely of orientable surface points and h_p touches S in π, then p_0 and p_{n+1} are either both in B_π or both in C_π. We call p_0 the _head_ and p_{n+1} the _tail_ of π. If h_p crosses S an odd number of times in runs $\pi_1, \ldots, \pi_m$ we say that p is _inside_ S. When h_p crosses S an even number of times in runs $\pi_1, \ldots, \pi_m$ we say that p is _outside_ S.

Let p and q be adjacent points not in S, and let $A^*_{p,q} (=A^*_{q,p})$ be a component (in the S sense) of $S \cap [h_p \cup h_q]$, where h_p and h_q are north half-lines emanating from p and q, and no other points on h_p are connected to $A^*_{p,q}$ in $S \cap [h_p \cup h_q]$. Clearly $A^*_{p,q}$ is a union of runs π_i along h_p and ρ_j along h_q.

Proposition 6. If $A^*_{p,q}$ consists solely of orientable surface points, then $\bar{B} = [\underset{i}{\cup} B_{\pi_i}] \cup [\underset{j}{\cup} B_{\rho_j}]$ is a connected set, and $\bar{C} = [\underset{i}{\cup} C_{\pi_i}] \cup [\underset{j}{\cup} C_{\rho_j}]$ is a connected set.

Proof: Where π_i meets ρ_j, say at $s \in \pi_i, t \in \rho_j$, s adjacent to t, we have distinct components B_δ and C_δ along the run $\delta = s,t$ by Proposition 5 and the ensuing   remark, such that $B_{s,t} \subseteq B_\delta$, $B_{t,s} \subseteq B_\delta$, and $C_{s,t} \subseteq C_\delta$, $C_{t,s} \subseteq C_\delta$, so that $B_{\pi_i} \cup B_{\rho_j}$ and $C_{\pi_i} \cup C_{\rho_j}$ are each connected sets. The proposition follows from induction on the number of places where a π_i meets a ρ_j. $\square$

Proposition 7. If h_p crosses S an even number of times in $A^*_{p,q}$, where $A^*_{p,q}$ consists solely of orientable surface points, then the head of the first π_i is connected (in the $\bar{S}$ sense) to the tail of the last π_i in $[\underset{i}{\cup} [\underset{p \in \pi_i}{\cup} N_{27}(p)]] \cup [\underset{j}{\cup} [\underset{q \in \rho_j}{\cup} N_{27}(q)]]$.

<u>Proof</u>: Let π_i and π_j be runs along h_p that are consecutive crossings, and (w.ℓ.g.) let the head of π_i be in B_{π_i}. Since h_p crosses S in π_i, B_{π_i} and C_{π_i} are distinct components in $[\underset{p\in\pi_i}{\cup} N_{27}(p)]$ and the tail of π_i is in C_{π_i}. By Proposition 6 we know that C_{π_i} and C_{π_j} are connected and B_{π_i} and B_{π_j} are connected. If any runs π_k occur between π_i and π_j, h_p must touch S in π_k since π_i and π_j are consecutive crossings, so that the head and tail of π_k are connected in $[B_{\pi_k}\cup C_{\pi_k}]$. Thus it is clear that the tail of π_i is connected to the head of π_j (between runs tails are connected to heads in $\overline{S}\cap h_p$). Then the head of π_j is in C_{π_j}, and so its tail is in B_{π_j}.

To finish the proof requires an induction on the number of pairs of consecutive crossings. If h_p never crosses S in $A^*_{p,q}$, then the head of the first π_i is connected to the tail of the last π_i. After every two consecutive crossings we see that if the head of the first crossing π_1 is in B_{π_1}, then the tail of the last crossing π_n is in B_{π_n}. By Proposition 6 these two points are connected in $[\cup[\underset{i\ p\in\pi_i}{\cup} N_{27}(p)]]\cup[\cup[\underset{j\ q\in\rho_j}{\cup} N_{27}(q)]]$. $\square$

To establish that the head of the first π_i is not connected to the tail of the last one in $[\cup[\underset{i\ p\in\pi_i}{\cup} N_{27}(p)]]\cup[\cup[\underset{j\ q\in\rho_j}{\cup} N_{27}(q)]]$ when h_p crosses S an odd number of times in $A^*_{p,q}$, we will need orientability to show that $[\underset{i}{\cup}B_{\pi_i}]\cup[\underset{j}{\cup}B_{\rho_j}]$ and $[\underset{i}{\cup}C_{\pi_i}]\cup[\underset{j}{\cup}C_{\rho_j}]$ are distinct components, since at a cross-cap these would become connected.

<u>Proposition 8.</u> If every point of $A^*_{p,q}$ is orientable, and if h_p crosses S an odd number of times in $A^*_{p,q}$, then the head of the first π_i is not connected to the tail of the last one in

$$[\cup[\ \underset{p\in\pi_i}{\cup}\ N_{27}(p)]]\cup[\cup[\ \underset{q\in\rho_j}{\cup}\ N_{27}(q)]].$$

<u>Proof:</u> Clearly the head(call it x) of the first π_i and the tail of the last one (call it y) occur in $[\underset{i}{\cup}(B_{\pi_i}\cup C_{\pi_i})]\cup[\underset{j}{\cup}(B_{\rho_j}\cup C_{\rho_j})]$.

Since $[\underset{i}{\cup}B_{\pi_i}]\cup[\underset{j}{\cup}B_{\rho_j}]$ and $[\underset{i}{\cup}C_{\pi_i}]\cup[\underset{j}{\cup}C_{\rho_j}]$ are each connected sets, they must be distinct components in $[\cup[\ \underset{p\in\pi_i}{\cup}\ N_{27}(p))]]\cup[\cup[\ \underset{q\in\rho_j}{\cup}\ N_{27}(q)]]$ if x and y are not connected. Let us suppose then that these are not distinct components. By Proposition 5 these two sets are not connected in the neighborhood of any single run π_i or ρ_j. Thus, they must be connected where some π_i meets some ρ_j. That is, at some $p_r\in\pi_i$ we have B_{π_i} connected to B_{ρ_j} and C_{π_i} connected to C_{ρ_j}, but at some other $p_s\in\pi_i$ we have B_{π_i} connected to C_{ρ_j} and C_{π_i} connected to B_{ρ_j}. But this violates Proposition 4 along the run from p_r to p_s in h_p, a contradiction.

Having established that $[\underset{i}{\cup}B_{\pi_i}]\cup[\underset{j}{\cup}B_{\rho_j}]$ and $[\underset{i}{\cup}C_{\pi_i}]\cup[\underset{j}{\cup}C_{\rho_j}]$ are distinct components, the proposition follows from an induction on the number of crossings. If x is in B_{π_i}, where π_i is the first run in $A^*_{p,q}\cap h_p$, and if π_k is the first crossing, then the tail of π_k is in C_{π_k} by the argument used in the proof of Proposition 7. In the portion of $A^*_{p,q}$ beyond π_k there remain an even number of crossings, so by Proposition 7, y is in C_{π_ℓ}, where ℓ is the last run in $A^*_{p,q}\cap h_p$. Thus, $x\in[\underset{i}{\cup}B_{\pi_i}]\cup[\underset{j}{\cup}B_{\rho_j}]$ is not connected to $y\in[\underset{i}{\cup}C_{\pi_i}]\cup[\underset{j}{\cup}C_{\rho_j}]$.□

We now define a (simple) closed surface as a connected set S consisting entirely of orientable surface points. Let S be a simple closed surface.

Proposition 9. Any two adjacent $p,q \in \overline{S}$ are either both inside or both outside S.

Proof: Notice that for any component of $S \cap [h_p \cup h_q]$ we can write $A^*_{p,q} = A^*_{q,p}$. For each such component the heads of the first π_i and ρ_j are connected in $[\cup [\underset{i}{\cup} [\underset{p \in \pi_i}{\cup} N_{27}(p)]]] \cup [\cup [\underset{j}{\cup} [\underset{q \in \rho_j}{\cup} N_{27}(q)]]]$, as are the tails of the last π_i and ρ_j. Now suppose h_p crosses S an even number of times in $A^*_{p,q}$, so that the head of the first π_i is connected to the tail of the last π_i. Then by Proposition 8 it cannot be that h_p crosses S an odd number of times in $A^*_{q,p}$. Suppose next that h_p crosses S an odd number of times in $A^*_{p,q}$, so that the head of the first π_i is not connected to the tail of the last π_i. Then by Proposition 7 it cannot be the case that h_q crosses S an even number of times in $A^*_{q,p}$. Thus h_p and h_q both cross S in $A^*_{p,q}$ either an odd number of times or an even number of times. Since this is true for every $A^*_{p,q} \subseteq S \cap [h_p \cup h_q]$ it follows that p and q are either both inside or both outside.□

Proposition 10. Points connected in $\overline{S}$ are either both inside or both outside S.

Proof: Suppose there is a path $p_1, \ldots, p_n$ from p to q in $\overline{S}$ where p is inside and q is outside. Then there exist two consecutive points p_i, p_{i+1} on the path such that p_i is inside and p_{i+1} is outside, a contradiction to Proposition 9. □

<u>Proposition 11</u>. The inside and outside of S are both non-empty.

<u>Proof</u>: The border of Σ consists of outside points. Let P be the northmost plane that meets S, and P_n and P_s the planes immediately to the north and south of P, and let $p \in P \cap S$. Since $N_{27}(p) \cap P_n$ is all in $\bar{S}$, it must be that (say) B_p lies entirely in P_s, while C_p contains $P_n \cap N_{27}(p)$. Let $q \in B_p$; it must have a point $t \in S$ as its north neighbor (it could be that t=p), since otherwise q would be connected to $P_n \cap N_{27}(p) \subseteq C_p$. Then h_q crosses S in $\pi = t$, so that q is inside.$\square$

<u>Proposition 12</u>. S-{p} has no cavities, where p is any point of the closed surface S.

<u>Proof</u>: Let q and r be in distinct components of $\bar{S} \cup \{p\}$, so that every path from q to r contains at least one point in S-{p}. Let $\delta = t_1, \ldots, t_k$ be such a path, and let t_i, t_j be the first and last points of δ in S-{p}. Notice that since there is exactly one component adjacent to p in $S \cap N_{27}(p)$, deleting p cannot leave S-{p} disconnected. Thus there are paths $\pi = p_1, \ldots, p_n$ from t_1 to p and $\rho = q_1, \ldots, q_m$ from t_j to p lying entirely in S, where p_n and q_m are each the first occurrence of p on π and ρ. Along the composite path in S $(\pi\rho) = p_1, \ldots, (p_n = q_m), \ldots, q_1$ (i.e., with ρ reversed) there exist connected subsets $B_{(\pi\rho)}$ and $C_{(\pi\rho)}$ of $\bar{S}$. Further, p is adjacent to each of these, so that $B_{(\pi\rho)} \cup C_{(\pi\rho)} \cup \{p\}$ is a connected subset of $\bar{S} \cup \{p\}$. Clearly $t_{i-1} \in B_{(\pi\rho)} \cup C_{(\pi\rho)} \cup \{p\}$ and $t_{j+1} \in B_{(\pi\rho)} \cup C_{(\pi\rho)} \cup \{p\}$, so that there exists a path from q to r in $\bar{S} \cup \{p\}$, a contradiction. $\square$

Proposition 13. A simple closed surface has at most one cavity.

Proof: Deleting p from a closed surface S leaves S-{p} with no cavities. Since every point in S has exactly two components adjacent to it in $\overline{S} \cap N_{27}(p)$, deleting p merges at most two components. If at most two were merged, and only one remains, there were at most two to start with. $\square$

Proposition 14. A simple closed surface has exactly one cavity.

Proof: By Propositions 10 and 11 it has at least one, and by Proposition 13 it has at most one. $\square$

Proposition 14 is the 3D analog of the Jordan curve theorem for connected sets of simple surface points. The definition given for a simple surface point is modeled after the standard definition in continuous space, namely that a surface point is one whose neighbhorhood is homeomorphic with the inside of a circle on the plane. Thus, every point in a small enough neighborhood of a point must be adjacent to either side of the surface.

Similarly, the concepts of orientability and cross-caps are modeled after the corresponding concepts used in the topology of continuous space. A cross-cap is homeomorphic with a Mobius strip, and may be visualized by deforming the edge of the strip to a circle in a plane. Thus, while each point on the face of the strip appears as a surface point, there is only one side (face) in the collection of points. We use the requirement on the 125-neighbhorhood of a surface point to guarantee that such phenomena do not occur (at least locally).

This raises the question of the realizability of cross-caps in the 3D lattice. That is, are the definitions of connectedness, together with the definition of simple surface point, strong enough to imply that cross-caps do not exist? From a theoretical standpoint an affirmative answer to this question would simplify the definition of simple closed surface, and from a practical viewpoint it would lessen the computational cost of detecting simple closed surfaces. While various properties such as symmetries may be used to reduce the effort needed to answer this question, the answer ultimately rests on a case analysis of the 2^{124} different configurations in the 125-neighborhood of a point p∈S.

The above definition of simple closed surface may be termed a local one, in that except for the connectivity requirement the conditions on the points are local. In two dimensions the converse of the Jordan curve theorem shows that curves are actually characterized by the global specification of the theorem; namely, if S is connected, $\overline{S}$ has exactly two components, and every point of S is adjacent to both these components, then S is a curve. The following proposition shows that in 3D no such characterization of simple closed surfaces is possible.

Proposition 15. Let S be connected, $\overline{S}$ have exactly two components, and every point of S be adjacent to each component of $\overline{S}$; then S is not necessarily a simple closed surface.

Proof: Consider the following sets S:

26-connectivity

1st plane	2nd plane	3rd plane	4th plane	5th plane
0 0 0 0 0	0 1 1 1 0	0 1 0 1 0	0 1 1 1 0	0 0 0 0 0
0 1 1 1 0	1 0 0 0 1	1 0 $\underline{1}$ 0 1	1 0 0 0 1	0 1 1 1 0
0 0 0 0 0	0 1 1 1 0	0 1 0 1 0	0 1 1 1 0	0 0 0 0 0

The central point in the third plane (underlined) is not a
simple surface point, since it is adjacent to three components
in its 27-neighborhood.

6-connectivity

1st plane	2nd plane	3rd plane	4th plane	5th plane
1 1 1	1 1 1	1 1 1	1 1 1	1 1 1
1 1 1 1	1 0 1 1	1 0 1 1	1 0 1	1 0 1 1
1 1 1 1 1	1 1 0 1 1	1 1 $\underline{1}$ 1 1	1 1 ▢ 1 1	1 1 $\underline{1}$ 1 1
1 1 1 1	1 1 0 1	1 1 0 1	1 0 1	1 1 0 1
1 1 1	1 1 1	1 1 1	1 1 1	1 1 1

6th plane	7th plane
1 1 1	1 1 1
1 0 1 1	1 1 1 1
1 1 0 1 1	1 1 1 1 1
1 1 0 1	1 1 1 1
1 1 1	1 1 1

The central point in the fourth plane is locally adjacent to
four components in $\overline{S}$. (The central points in the third and
fifth planes are adjacent to outside points in the fourth plane.)□

We see then that the converse of Proposition 14 fails because surfaces (not simple closed surfaces) may touch themselves without globally affecting connectivity. Thus, in addition to simple surface points for which $\overline{S} \cap N_{27}(p)$ has two components adjacent to p, we see that there are non-simple surface points for which $\overline{S} \cap N_{27}(p)$ has three or more components adjacent to p. One might wonder then if an analog of Proposition 14 might be given for connected sets of simple and non-simple surface points. The following example shows that this is not possible.

26-connectivity

```
0 0 0 0 0 0 0    0 1 1 1 1 1 0    0 1 0 0 0 1 0    0 1 0 1 0 1 0
0 1 1 1 1 1 0    1 0 0 0 0 0 1    1 0 1 1 1 0 1    1 0 1 0 1 0 1
0 0 0 0 0 0 0    0 1 1 1 1 1 0    0 1 0 0 0 1 0    0 1 0 1 0 1 0


          0 1 0 0 0 1 0    0 1 1 1 1 1 0    0 0 0 0 0 0 0
          1 0 1 1 1 0 1    1 0 0 0 0 0 1    0 1 1 1 1 1 0
          0 1 0 0 0 1 0    0 1 1 1 1 1 0    0 0 0 0 0 0 0
```

6-connectivity

```
1 1 1                1 1 1                  1 1 1
1 1 1 1              1 0 1 1                1 0 1 1
1 1 1 1 1            1 1 0 1 1              1 1 1 1 1
  1 1 1 1 1            1 1 0 1 1              1 1 0 1 1
    1 1 1 1 1            1 1 0 1 1              1 1 1 1 1
      1 1 1 1            1 1 0 1                1 1 0 1
        1 1 1              1 1 1                  1 1 1
```

```
1 1 1           1 1 1           1 1 1
1 0 1           1 0 1           1 0 1
1 1 1 1 1       1 1 1 1 1       1 1 1 1 1
    1 1 1           1 0 1           1 1 1
      1 1 1 1 1       1 1 1 1 1       1 1 1 1 1
          1 0 1           1 0 1           1 0 1
            1 1 1           1 1 1           1 1 1


1 1 1           1 1 1           1 1 1
1 0 1 1         1 0 1 1         1 1 1 1
1 1 1 1 1       1 1 0 1 1       1 1 1 1 1
  1 1 0 1 1       1 1 0 1 1       1 1 1 1 1
    1 1 1 1 1       1 1 0 1 1       1 1 1 1 1
      1 1 0 1         1 1 0 1         1 1 1 1
        1 1 1           1 1 1           1 1 1
```

In each case $\overline{S}$ has three components: the outside and a ring
of 0's surrounding a central component of 0's (underlined).

Proposition 16. No closed surface is both a 6-surface and
a 26-surface.

Proof: Let S be a closed 6-surface. Clearly there exist points
p,q$\in$S with p in the northmost plane P of S and q a 6-neighbor of
p in the plane P_s just south of P (since all points of S are
connected by 6-paths). We denote the two components in
$\overline{S} \cap N_{27}(p)$ by B_p^6 and C_p^6 when 6-adjacency is used for S (so that
B_p^6 and C_p^6 are 26-components). Let C_p^6 be the component containing
$P_n \cap N_{27}(p)$, where P_n is the plane just north of P. Then B_p^6
lies entirely in $P_s \cap N_{27}(p)$. If p is also a simple 26-surface

point (so that B_p^{26} and C_p^{26} exist), B_p^{26} must be 6-adjacent to p;

but B_p^{26} too must lie entirely in P_s, and thus cannot be 6-

adjacent to p, since q is p's south neighbor. $\square$
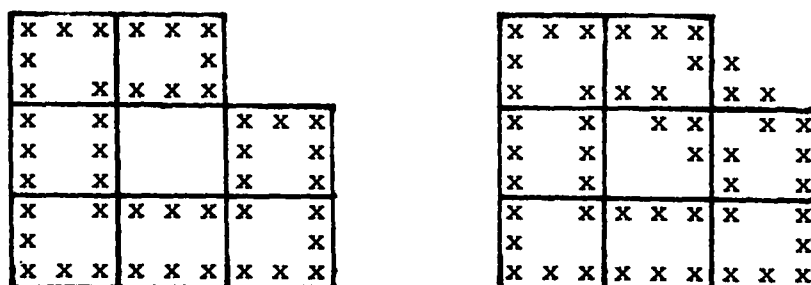
4. Concluding remarks

We have proposed definitions for simple surface points
and simple closed surfaces in discrete three-dimensional space,
and have shown that any connected collection of simple surface
points forms a simple closed surface. We can now make several
immediate generalizations of these ideas.

The definition of simple surface point refers explicitly
to the types of connectivity and adjacency of S and its comple-
ment $\bar{S}$. While we have assumed that 6- and 26-adjacency are
used, and opposite types for S and $\bar{S}$, this is not strictly
necessary. All of the results of Section 3 rest solely on adja-
cencies which are guaranteed to exist by hypothesis, e.g., p
is a surface point. Thus, we are free to use any kind of adja-
cencies for S and $\bar{S}$ (including the choice of using one type of
adjacency for both) although it may no longer be the case that
surface points exist.

By the remarks of the previous paragraph, then, we are free
also to define adjacency between points which are not even "near"
each other. Such alternate adjacencies may be useful in, for
example, noisy images, where noisy data creates gaps between
otherwise "connected" objects.

Secondly, we are free to define adjacency on data of any
dimensionality. We may thus speak of a simple n-dimensional
closed hyper-surface as a connected set of simple n-dimensional
hypersurface points each of which is orientable in n dimensions.

We noted earlier that simple closed surfaces as presented here cannot be used to describe the borders of arbitrary objects, as can be done with the approach based on faces of voxels. However, by effectively tripling the resolution of the image we can encode the voxel pairs which constitute faces as single points in the high resolution image, so that the simple closed surfaces defined here are equivalent to those defined in terms of faces. For example, below we show (by x's) the voxels (pixels in this example) of the 6-surfaces (4-curves in this example) of the high resolution images when 6- and 26-connectivity are used for the low resolution object whose faces (edges in this example) are shown by lines.

```
x x x x x x          x x x x x x
x         x          x         x x
x       x x x x      x       x x x   x x
x       x     x x x  x       x x     x x
x       x     x   x  x         x x   x   x
x       x     x   x  x       x       x   x
x       x x x x x x  x       x x x x x   x
x             x      x                   x
x x x x x x x x x x  x x x x x x x x x x x
```

Similarly each object may be encoded as a 26-surface.

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