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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) UNCLASSIFIED

LEVEL II

| (19) REPORT DOCUMENTATION PAGE | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|
| 1. REPORT NUMBER (18) AFOSR-TR-80-1164 — 2. GOVT ACCESSION NO. AD-A092 191 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) (6) ASYMPTOTIC DISTRIBUTIONS OF FUNCTIONS OF THE EIGENVALUES OF THE REAL AND COMPLEX NONCENTRAL WISHART MATRICES. | 5. TYPE OF REPORT & PERIOD COVERED (9) Interim rept. |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) (10) C. Fang and P. R. Krishnaiah | 8. CONTRACT OR GRANT NUMBER(s) (15) F49620-79-C-0161 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Pittsburgh Department of Mathematics and Statistics Pittsburgh, PA 15260 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F (16) 2304 (17) A5 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office ofScientific Research /NM Bolling AFB, Washington, DC 20332 | 12. REPORT DATE (11) July 1980 (12) 40 |
| | 13. NUMBER OF PAGES 39 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) (14) TR-80-11 | 15. SECURITY CLASS. (of this report) UNCLASSIFIED |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

DTIC ELECTE NOV 28 1980 S D E

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymptotic Distributions, Noncentral Wishart Matrix, Functions of Eigenvalues Comples Matrices, Mixtures of Populations, Quadratic Forms, Perturbation Theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the authors derived asymptotic expressions for the joint desities of various functions of the noncentral real and comples Wishart matrices. Applications of the above results are also discussed in the area of reduction of dimensionality and testing for the structure of interaction in two-way classification with one observation per cell.

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AFOSR-TR- 80-1164

# ASYMPTOTIC DISTRIBUTIONS OF FUNCTIONS OF THE EIGENVALUES OF THE REAL AND COMPLEX NONCENTRAL WISHART MATRICES*

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July 1980

Technical Report No. 80-11

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# 1. Introduction

The distributions of functions of the eigenvalues of the real and complex Wishart matrices are very useful in studying the structures of the covariance matrices of the real and complex multivariate normal distributions respectively and other problems. Krishnaiah and Lee (1977) derived the joint asymptotic distributions of the linear functions as well as the ratios of the linear functions of the roots of the central Wishart matrix when the population covariance matrix has simple roots. Fujikoshi (1978) derived an asymptotic expression for the distribution of a function of the roots of the central Wishart matrix when the roots have multiplicity whereas Krishnaiah and Lee (1979) obtained corresponding expressions for the joint density of the functions of the roots. In this paper, we obtain asymptotic expressions for the joint densities of various functions of the noncentral real and complex Wishart matrices. These expressions are in terms of multivariate normal density and multivariate Hermite polynomials. Percentage points of some test statistics are computed by using the above asymptotic expressions and these percentage points are compared with the results obtained by simulation. Applications of the above results are also discussed in problems of studying the structure of interactions, mixtures of multivariate normal populations, and reduction of dimensionality. The results obtained on the joint distribution of the functions of the eigenvalues of the real Wishart matrix are generalized to the case of multivariate quadratic forms. Finally, the joint asymptotic distribution of the functions of the roots of the complex Wishart matrix is derived.

## 2. Perturbation Technique

Let $\ell_1 \geq \ldots \geq \ell_p$ be the eigenvalues of the symmetric matrix $T: p\times p$, and $\lambda_1 \geq \ldots \geq \lambda_p$ are the eigenvalues of the symmetric matrix $V: p\times p$, where

$$T(\varepsilon) = V + \varepsilon V^{(1)} + \varepsilon^2 V^{(2)} + \ldots \qquad (2.1)$$

Then, there exist orthogonal matrices $\Gamma$ and $G$ such that $T = GLG'$ and $V = \Gamma\Lambda\Gamma'$, where $L = \text{diag}(\ell_1,\ldots,\ell_p)$, $\Lambda = \text{diag}(\lambda_1,\ldots,\lambda_p)$. The columns of $\Gamma$ and $G$ consist of the eigenvectors of $V$ and $T$ respectively.

Lawley (1956), Mallows (1961), Izenman (1972) and Fujikoshi (1978) have approximated the eigenvalues and eigenvectors of $T$ in various papers. The authors have either assumed that $\lambda_i$'s do not have multiplicity or the approximations were established by tacitly assuming that the eigenvalues and eigenvectors admit series expansions in the infinitesimal parameter $\varepsilon$ as follows:

$$\begin{aligned} \ell_j &= \lambda_j + \varepsilon\lambda_j^{(1)} + \varepsilon^2\lambda_j^{(2)} + \ldots \\ \underset{\sim}{G}_j &= \Gamma_j + \varepsilon\underset{\sim}{\Gamma}_j^{(1)} + \varepsilon^2\underset{\sim}{\Gamma}_j^{(2)} + \ldots \end{aligned} \qquad (2.2)$$

and no attempts were made to prove that the series converge. A more insight treatment to settle this question of convergence is found in Kato (1976).

Now $T(\varepsilon)$ and $V$ are linear transformations which operate on the p-dimensional complex vector field $C^p$, $\varepsilon$ is also complex, $\lambda_1 \geq \ldots \geq \lambda_p$ are the eigenvalues of $V: p\times p$ such that

$$\lambda_{q_1+\ldots+q_{\alpha-1}+1} = \ldots = \lambda_{q_1+\ldots+q_\alpha} = \theta_\alpha \tag{2.3}$$

for $\alpha = 1,2,\ldots,r$ , $q_1+\ldots+q_r = p$ , $q_o = 0$ and let $J_\alpha$ $(\alpha=1,\ldots,r)$ denote the set of integers $q_1+\ldots+q_{\alpha-1}+1, \ \ldots, \ q_1+\ldots+q_\alpha$. We need the following lemma in the sequel.

<u>Lemma 2.1</u>. For Hermitian matrices $T(\varepsilon)$ and $V$ as defined before,

$$T(\varepsilon) = V + \varepsilon V^{(1)} + \varepsilon^2 V^{(2)} + \ldots$$

and $V$ is diagonalized as $V = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_p)$. Then the mean eigenvalue of $T(\varepsilon)$ corresponding to $\theta_\alpha$ which is the eigenvalue of $V$ with multiplicity $q_\alpha$ , is

$$\bar{\ell}_\alpha \ell(\varepsilon) = \theta_\alpha + \varepsilon\, \bar{\ell}_\alpha^{(1)} + \varepsilon^2\, \bar{\ell}_\alpha^{(2)} + \ldots \tag{2.4}$$

where

$$\bar{\ell}_\alpha^{(1)} = \frac{1}{q_\alpha} \operatorname{tr} V_{\alpha\alpha}^{(1)}$$

$$\bar{\ell}_\alpha^{(2)} = \frac{1}{q_\alpha} \operatorname{tr}[V_{\alpha\alpha}^{(2)} + \sum_{\beta\neq\alpha} (\theta_\alpha - \theta_\beta)^{-1} V_{\alpha\beta}^{(1)} V_{\beta\alpha}^{(1)}] \tag{2.5}$$

$$\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$$

with

$$V^{(i)} = \begin{pmatrix} V_{11}^{(i)} & V_{12}^{(i)} & \cdots & V_{1r}^{(i)} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ V_{r1}^{(i)} & V_{r2}^{(i)} & \cdots & V_{rr}^{(i)} \end{pmatrix}$$

and $V_{\alpha\beta}^{(i)}$ is of order $q_\alpha \times q_\beta$.

When $q_1 = \ldots q_r = 1$, the above lemma was proved in Lawley (1956). When $q_\alpha \geq 1$ $(\alpha = 1,\ldots,r)$, the lemma was given implicitly in Kato (1976). For $q_\alpha = 1$ the normalized eigenvector of $T(\varepsilon)$ corresponding to $\theta_\alpha$ is $G_\alpha(\varepsilon) = (G_{1\alpha}(\varepsilon),\ldots,G_{p\alpha}(\varepsilon))'$, with

$$G_{i\alpha}(\varepsilon) = \frac{a_{i\alpha}(\varepsilon)}{(\theta_\alpha - \lambda_i)} + \sum_{j \neq \alpha} \frac{a_{ij}(\varepsilon) a_{j\alpha}(\varepsilon)}{(\theta_\alpha - \lambda_i)(\theta_\alpha - \lambda_j)} - \frac{a_{i\alpha}(\varepsilon) a_{\alpha\alpha}(\varepsilon)}{(\theta_\alpha - \lambda_i)^2} + \ldots \quad i \neq \alpha$$

$$G_{\alpha\alpha}(\varepsilon) \; 1 - \frac{1}{2} \sum_{j \neq \alpha} \frac{a_{im}(\varepsilon) \; a_{mi}(\varepsilon)}{(\theta_\alpha - \lambda_j)(\theta_\alpha - \lambda_j)} + \ldots \tag{2.6}$$

where

$$A(\varepsilon) = T(\varepsilon) - V = (a_{ij}(\varepsilon))$$

The series (2.4), (2.6) are convergent for

$$|\varepsilon| < \left(\frac{2c_1}{d} + c_2\right)^{-1} \tag{2.7}$$

where $c_1, c_2 \geq 0$ such that $||V^{(j)}|| \leq c_1 c_2{}^{j-1}$ for $j = 1,2,\ldots$, and $d = \min(|\theta_\alpha - \theta_{\alpha-1}|, |\theta_\alpha - \theta_{\alpha+1}|)$.

## 3. Asymptotic Joint Distributions of Functions of the Roots of Noncentral Wishart Matrix

Let the columns of X: p×n be distributed as multivariate normal with covariance matrix $\Sigma = (\sigma_{ij})$ and means given by $E(X) = M = (\underset{\sim}{\mu}_1, \cdots, \underset{\sim}{\mu}_n)$, where $\underset{\sim}{\mu}_j' = (\mu_{j1}, \cdots, \mu_{jp})$. Then, $S = XX' = (S_{ij})$ is distributed as the central or noncentral Wishart matrix $W_p(n, \Sigma, M)$ with n degrees of freedom according as $M = 0$ or $M \neq 0$ where $M = \sum_{j=1}^{n} \underset{\sim}{\mu}_j \underset{\sim}{\mu}_j' = n(\nu_{ij})$. Now, let $\ell_1 \geq \cdots \geq \ell_p$ denote the eigenvalues of S/n whereas $\lambda_1 \geq \cdots \geq \lambda_p$ denote the eigenvalues of $E(S/n) = \Sigma + M/n = \Lambda$. Without loss of generality, we assume that $\Lambda = \text{diag.}(\lambda_1, \ldots, \lambda_p)$. Also, let

$$\lambda_{q_1+\cdots+q_{\alpha-1}+1} = \cdots = \lambda_{q_1+\cdots+q_\alpha} = \theta_\alpha \tag{3.1}$$

for $\alpha = 1, 2, \cdots, r$, $q_1 + \cdots + q_r = p$, and $q_0 = 0$.

In this section, we derive the joint asymptotic distribution of $L_1, \cdots, L_k$ where $L_j = \sqrt{n}\,\{T_j(\ell_1, \cdots, \ell_p) - T_j(\lambda_1, \ldots, \lambda_p)\}$ and $T_j(\ell_1, \ldots, \ell_p)$ satisfy the following assumptions:

(i) $T_j(\ell_1, \cdots, \ell_p)$ is analytic about $\lambda_1, \cdots, \lambda_p$

(ii) $$\left.\frac{\partial T_i(\ell_1, \cdots, \ell_p)}{\partial \ell_{j_1}}\right|_{\underset{\sim}{\ell} = \underset{\sim}{\lambda}} = c_{ij_1} = a_{i\alpha}$$

$$\left.\frac{\partial^2 T_i(\ell_1, \cdots, \ell_p)}{\partial \ell_{j_2}\, \partial \ell_{j_1}}\right|_{\underset{\sim}{\ell} = \underset{\sim}{\lambda}} = c_{ij_1j_2} = a_{i\alpha\beta}$$

$$\left.\frac{\partial^3 T_i(\ell_1,\cdots,\ell_p)}{\partial\ell_{j_3}\;\partial\ell_{j_2}\;\partial\ell_{j_1}}\right|_{\underset{\sim}{\ell}=\underset{\sim}{\lambda}} = c_{ij_1j_2j_3} = a_{i\alpha\beta\gamma} \tag{3.2}$$

for $j_1 \varepsilon J_\alpha$, $j_2 \varepsilon J_\beta$, $j_3 \varepsilon J_\gamma$, $\underset{\sim}{\lambda}' = (\lambda_1,\cdots,\lambda_p)$, $\underset{\sim}{\ell}' = (\ell_1,\cdots,\ell_p)$ and $J_\alpha$ denotes the set of integers $q_1 +\cdots+q_{\alpha-1} + 1,\cdots,q_1+\cdots+ q_\alpha$ for $\alpha = 1,2,\cdots,r$.

Expanding $T_i(\ell_1,\cdots,\ell_p)$ as the Taylor series, we obtain

$$T_i(\ell_1,\cdots,\ell_p) = T_i(\lambda_1,\cdots,\lambda_p) + \sum_{\alpha=1}^{r} a_{i\alpha} \sum_{j_1\epsilon J_\alpha} (\ell_{j_1} - \theta_\alpha)$$

$$+ \frac{1}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{i\alpha\beta} \sum_{j_1\varepsilon J_\beta} \sum_{j_2\varepsilon J_\beta} (\ell_{j_1}-\theta_\alpha)(\ell_{j_2}-\theta_\beta)$$

$$+ \frac{1}{6} \sum_{\alpha\beta\gamma}\sum\sum a_{i\alpha\beta\gamma} \sum_{j_1\varepsilon J_\alpha} \sum_{j_2\varepsilon J_\beta} \sum_{j_3\varepsilon J_\gamma} (\ell_{j_1}-\theta_\alpha)(\ell_{j_2}-\theta_\beta)(\ell_{j_3}-\theta_\gamma)$$

$$+ \text{terms of higher degree.} \tag{3.3}$$

Now, let

$$S/n = \Lambda + \frac{1}{\sqrt{n}} V \tag{3.4}$$

where

$$V = \sqrt{n}\,(\frac{S}{n} - \Lambda) = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1r} \\ V_{21} & V_{22} & \cdots & V_{2r} \\ \vdots & \vdots & & \vdots \\ V_{r1} & V_{r2} & \cdots & V_{rr} \end{pmatrix}.$$

So by applying Lemma 2.1 on (3.4), we obtain

$$L_i = \sum_{\alpha=1}^{r} a_{i\alpha} \operatorname{tr} Z_\alpha^{(1)} + \frac{1}{\sqrt{n}} \{ \sum_{\alpha=1}^{r} a_{i\alpha} \operatorname{tr} Z_\alpha^{(2)} + \frac{1}{2} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} a_{i\alpha\beta}$$

$$(\operatorname{tr} Z_\alpha^{(1)})(\operatorname{tr} Z_\beta^{(1)})\} + O(n^{-1}) \qquad (3.5)$$

where $Z_\alpha^{(1)} = V_{\alpha\alpha}$, $Z_\alpha^{(2)} = \sum_{\beta\neq\alpha} \theta_{\alpha\beta}^{-1} V_{\alpha\beta} V_{\beta\alpha}$ and $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$.

Also,

$$\operatorname{tr} Z_\alpha^{(1)} = \sqrt{n} \sum_{j_1 \varepsilon J_\alpha} \left( \frac{S_{j_1 j_1}}{n} - \lambda_{j_1} \right),$$

$$\operatorname{tr} Z_\alpha^{(2)} = \frac{1}{n} \sum_{\beta\neq\alpha} \theta_{\alpha\beta}^{-1} \sum_{j_1 \varepsilon J_\alpha} \sum_{j_2 \varepsilon J_\beta} S_{j_1 j_2}^2 .$$

After some algebraic manipulations, we obtain the following expression for the joint characteristic function of $L_1, \cdots, L_k$:

$$\Psi(t_1, \cdots, t_k) = E\{\exp(i \sum_{j=1}^{k} t_j L_j)\}$$

$$= E[\exp(i \sum_{j=1}^{k} \sum_{\alpha=1}^{r} t_j a_{j\alpha} \operatorname{tr} Z_\alpha^{(1)}) \qquad (3.6)$$

$$\times\{1 + \frac{1}{\sqrt{n}} (i \sum_{j=1}^{k} \sum_{\alpha=1}^{r} t_j a_{j\alpha} \operatorname{tr} Z_\alpha^{(2)} + \frac{i}{2} \sum_{j=1}^{k} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} t_j a_{j\alpha\beta}$$

$$\operatorname{tr} Z_\alpha^{(1)} \operatorname{tr} Z_\beta^{(1)})$$

$$+ O(n^{-1})\}] = E_1(\underset{\sim}{t}) + E_2(\underset{\sim}{t}) + E_3(\underset{\sim}{t}) + O(n^{-1})$$

where $\underset{\sim}{t}' = (t_1, \cdots, t_k)$. In Eq. (3.6),

$$E_1(\underset{\sim}{t}) = E[\exp(i \sum_{i=1}^{k} \sum_{\alpha=1}^{r} t_i a_{i\alpha} \operatorname{tr} Z_\alpha^{(1)})]$$

$$= \operatorname{etr}(-i\sqrt{n}\,B)\,|\,I-2i\,B\,\Sigma/\sqrt{n}\,|^{-n/2}$$

$$\times \exp\{i\,\operatorname{tr}[M(I-2i\,B\,\Sigma/\sqrt{n})^{-1}B/\sqrt{n}\,)\} \qquad (3.7)$$

where $B = \sum_{i=1}^{k} t_i \operatorname{diag}(c_{i1},\cdots,c_{ip})$. Also,

$$E_2(\underset{\sim}{t}) = E[\frac{i}{\sqrt{n}} \sum_{i=1}^{k} \sum_{\alpha=1}^{r} t_i a_{i\alpha} \operatorname{tr} Z_\alpha^{(2)} \times \exp\{i \sum_{i_1=1}^{k} \sum_{\alpha_1=1}^{r} t_{i_1} a_{i_1\alpha_1} \times \operatorname{tr} Z_{\alpha_1}^{(1)}\}]$$

$$= E_1(\underset{\sim}{t}) \frac{i}{\sqrt{n}} \sum_{i=1}^{k} \sum_{\alpha=1}^{r} \sum_{\beta\neq\alpha} \sum_{j_1\varepsilon J_\alpha} \sum_{j_2\varepsilon J_\beta} t_i a_{i\alpha} \theta_{\alpha\beta}^{-1}$$

$$\times \frac{1}{n} \{ \sum_{j=1}^{n} (\sigma^*_{j_1j_1}\sigma^*_{j_2j_2} + \sigma^{*2}_{j_1j_2} + \sigma^*_{j_1j_1}\xi^2_{jj_2} + 2\sigma^*_{j_1j_2}\xi_{jj_1}\xi_{jj_2}$$

$$+ \sigma^*_{j_2j_2}\xi^2_{jj_1})$$

$$+ [\sum_{j=1}^{n} (\sigma^*_{j_1j_2} + \xi_{jj_1}\xi_{jj_2})]^2\} \qquad (3.8)$$

$$E_3(\underset{\sim}{t}) = E[\frac{i}{2\sqrt{n}} \sum_{i=1}^{k} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} t_i a_{i\alpha\beta} (\operatorname{tr} Z_\alpha^{(1)})(\operatorname{tr} Z_\beta^{(1)})$$

$$\times \exp\{i \sum_{i_1=1}^{k} \sum_{\alpha_1=1}^{r} t_{i_1} a_{i_1\alpha_1} \operatorname{tr} Z_{\alpha_1}^{(1)}\}]$$

$$= E_1(\underset{\sim}{t}) \frac{i}{2\sqrt{n}} \sum_{i=1}^{k} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{j_1\varepsilon J_\alpha} \sum_{j_2\varepsilon J_\beta} t_i a_{i\alpha\beta}$$

$$\times \{\frac{1}{n} \sum_{j=1}^{n} (2\sigma^{*2}_{j_1j_2} + 4\sigma^*_{j_1j_2}\xi_{jj_1}\xi_{jj_2})$$

$$+ \frac{1}{n} \sum_{j=1}^{n} \sum_{m=1}^{n} (\sigma^*_{j_1j_1} + \xi^*_{jj_1})(\sigma^*_{j_2j_2} + \xi^2_{mj_2}) - \lambda_{j_1} \sum_{j=1}^{n} (\sigma^*_{j_2j_2} + \xi^2_{jj_2})$$

$$- \lambda_{j_2} \sum_{j=1}^{n} (\sigma^*_{j_1j_1} + \xi^2_{jj_1}) + n\, \lambda_{j_1} \lambda_{j_2} \} \qquad (3.9)$$

where

$$\Sigma^* = \Sigma\left(I - \frac{2iB_1\Sigma}{\sqrt{n}}\right)^{-1} = (\sigma^*_{ij})$$

$$\underset{\sim}{\xi}_j = \left(I - \frac{2i\Sigma B_1}{\sqrt{n}}\right)^{-1} \underset{\sim}{\mu}_j = (\xi_{j1}, \cdots, \xi_{jp})' \qquad (3.10)$$

Using the expansion that

$$|I-A|^{-\beta} = \exp \beta\left(\sum_{j=1}^{\infty} \frac{\mathrm{tr}\, A^j}{j}\right)$$

in (3.7) and

$$(I-A)^{-1} = \sum_{j=0}^{\infty} A^j$$

in (3.10), we obtain

$$\sigma^*_{j_1j_2} = \sigma_{j_1j_2} + \frac{2i}{\sqrt{n}} \sum_{i=1}^{k} \sum_{j=1}^{p} t_i c_{ij}\, \sigma_{j_1j} \sigma_{j_2j} + O(n^{-1})$$

$$\xi_{jj_1} = \mu_{jj_1} + \frac{2i}{\sqrt{n}} \sum_{i=1}^{k} \sum_{m=1}^{p} t_i c_{im}\, \sigma_{j_1m}\, \mu_{jm} + O(n^{-1})$$

Eq. (3.7), (3.8), (3.9) lead to

$$\psi(\underset{\sim}{t}) = \exp(-\tfrac{1}{2}\, \underset{\sim}{t}'Q\underset{\sim}{t})$$

$$\times \{1 + \frac{1}{\sqrt{n}} \sum_{i=1}^{k} i\, t_i(d_1+d_2) + \frac{1}{\sqrt{n}} \sum_{i_1i_2i_3}^{k} (i^3 t_{i_1} t_{i_2} t_{i_3})(g_1+g_2+g_3)\}$$

$$+ O(n^{-1}) \qquad (3.11)$$

where $Q = (Q_{i_1 i_2})$, $Q_{i_1 i_2} = 2\text{tr}\, R^{(i_1)} R^{(i_2)} + 4\text{tr}\, R^{(i_1)} \psi^{(i_2)}$,

and Q is assumed to be nonsingular. Also,

$$d_1 = \sum_{\alpha=1}^{r} \sum_{\beta\neq\alpha}^{r} \sum_{j_1 \varepsilon J_\alpha} \sum_{j_2 \varepsilon J_\beta} a_{i\alpha}\, \theta_{\alpha\beta}^{-1} (\sigma_{j_1 j_1}\sigma_{j_2 j_2} + \sigma_{j_1 j_2}^2 + 2\sigma_{j_1 j_2}\nu_{j_1 j_2} + \sigma_{j_1 j_1}\nu_{j_2 j_2} + \sigma_{j_2 j_2}\nu_{j_1 j_1})$$

$$d_2 = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{j_1 \varepsilon J_\alpha} \sum_{j_2 \varepsilon J_\beta} a_{i\alpha\beta} (\sigma_{j_1 j_2}^2 + 2\sigma_{j_1 j_2}\nu_{j_1 j_2})$$

$$g_1 = \frac{4}{3}\, \text{tr}\, R^{(i_1)} R^{(i_2)} R^{(i_3)} + 4\, \text{tr}\, R^{(i_1)} R^{(i_2)} \psi^{(i_3)}$$

$$g_2 = 4 \sum_{\alpha=1}^{r} \sum_{\beta\neq\alpha} \sum_{j_1 \varepsilon J_\alpha} \sum_{j_2 \varepsilon J_\beta} a_{i_1\alpha}\, \theta_{\alpha\beta}^{-1} (\Xi_{j_1 j_2}^{(i_2)} + T_{j_1 j_2}^{(i_2)} + T_{j_2 j_1}^{(i_2)}) \times (\Xi_{j_1 j_2}^{(i_3)} + T_{j_1 j_2}^{(i_3)} + T_{j_2 j_1}^{(i_3)})$$

$$g_3 = 2 \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{j_1 \varepsilon J_\alpha} \sum_{j_2 \varepsilon J_\beta} a_{i_1\alpha\beta} (\Xi_{j_1 j_1}^{(i_2)} + 2T_{j_1 j_1}^{(i_2)}) (\Xi_{j_2 j_2}^{(i_3)} + 2T_{j_2 j_2}^{(i_3)})$$

(3.12)

We define here $M/n - (\nu_{ij})$

$$C^{(i)} = \text{diag}(c_{i1}, \cdots, c_{ip})$$

$$R^{(i)} = C^{(i)}\Sigma,\ \psi^{(i)} = C^{(i)}\frac{M}{n},\ \Xi^{(i)} = \Sigma C^{(i)}\Sigma,\ T^{(i)} = \frac{M}{n}C^{(i)}\Sigma$$

(3.13)

where $A_{ij}$ denotes the (i,j)th element of matrix $A = (A_{ij})$.

Now inverting (3.11) we obtain the following expansion for density of $\underset{\sim}{L} = (L_1, \cdots, L_k)$

$$f(L_1, \cdots, L_k) = N(\underset{\sim}{L}, Q) \times$$

$$[1 + \frac{1}{\sqrt{n}} \sum_{i=1}^{k} H_i(\underset{\sim}{L})(d_1 + d_2)$$

$$+ \frac{1}{\sqrt{n}} \sum_{i_1, i_2, i_3}^{k} H_{i_1 i_2 i_3}(\underset{\sim}{L})(g_1 + g_2 + g_3)] + O(n^{-1}) \quad (3.14)$$

where $\quad N(\underset{\sim}{L}, Q) = \frac{1}{(2\pi)^{k/2} |Q|^{1/2}} \exp(-\frac{1}{2} \underset{\sim}{L}' Q^{-1} \underset{\sim}{L})$

$$H_{j_1, \cdots, j_s}(\underset{\sim}{L}) N(\underset{\sim}{L}, Q) = (-1)^s \frac{\partial^s}{\partial L_{j_1}, \cdots, \partial L_{j_s}} N(\underset{\sim}{L}, Q) \quad (3.15)$$

Now, let $T_i(\ell_1, \cdots, \ell_p) = \ell_i$. Then $L_i = \sqrt{n}\,(\ell_i - \lambda_i)$. Using Eq. (3.14), we obtain the following expression for the joint density of the roots $\ell_1, \cdots, \ell_p$ when $\Sigma = \sigma^2 I$:

$$f(L_1, \cdots, L_p) = N(\underset{\sim}{L}, Q)\{1 + \frac{1}{\sqrt{n}} \sum_{i=1}^{p} H_i(\underset{\sim}{L}) \sum_{j \neq i} \theta_{ij}^{-1}(\sigma^2 \lambda_i + \sigma^2 \lambda_j - \sigma^4)$$

$$+ \frac{4\sigma^4}{\sqrt{n}} \sum_{i=1}^{p} H_{iii}(\underset{\sim}{L})(\lambda_i - \frac{2}{3}\sigma^2)\} + O(n^{-1})$$

$$(3.16)$$

where $Q = \mathrm{diag}(Q_1, \cdots, Q_p)$ and $Q_i = 2\sigma^2(2\lambda_i - \sigma^2)$. When $\Sigma = \sigma^2 I$ and $\lambda_1 > \ldots > \lambda_t = \lambda_{t+1} = \ldots \lambda_p = \sigma^2$, Carter and Srivastava (1979) obtained an alternative expression for the joint density of $\ell_1, \ldots, \ell_p$ by using a different method.

The general k-dimensional Hermite polynomial of order $s \geq 0$ is denoted by $H_{i_1,i_2,\ldots,i_s}(\underset{\sim}{x};\Delta)$ where $\underset{\sim}{x} = (x_1,\ldots,x_k)'$ is the polynomial variate and $\Delta = (\delta_{ij})$ is a positive-definite k×k matrix, $0 \leq i_j \leq k$ for $j=1,\ldots,s$ and

$$H_{i_1,i_2,\ldots,i_s}(\underset{\sim}{x},\Delta)\ N(\underset{\sim}{x},Q) = (-1)^s \frac{\partial^s}{\partial x_{i_1}\cdots\partial x_{i_s}} N(\underset{\sim}{x},Q) \tag{3.17}$$

where

$$N(\underset{\sim}{x},Q) = \frac{1}{(2\pi)^{k/2}|Q|^{\frac{1}{2}}} \exp(-\tfrac{1}{2}\,\underset{\sim}{x}'Q^{-1}\,\underset{\sim}{x}) \tag{3.18}$$

and

$$\Delta = Q^{-1}$$

For dimension k=1, $Q=\tau^2$ is a scalar and $\Delta=\delta=\frac{1}{\tau^2}$

$$\begin{aligned} H_1(x,\delta) &= x\delta \\ H_{111}(x,\Delta) &= x^3\delta^3 - 3\,x\,\delta^2 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \int_{-\infty}^{\tau a} H_1(x,\Delta)\ N(x,Q)dx &= -\frac{1}{\tau}\ \frac{1}{\sqrt{2\pi}}\ \exp\left(\frac{-a^2}{2}\right) \\ \int_{-\infty}^{\tau a} H_{111}(x,\Delta)\ N(x,Q)dx &= \frac{1}{\tau^3}\,(1-a^2)\,\frac{1}{\sqrt{2\pi}}\ \exp\left(\frac{-a^2}{2}\right) \end{aligned} \tag{3.20}$$

For the dimension k=2, let $Q = \begin{pmatrix} \tau_1^2 & \rho\tau_1\tau_2 \\ \rho\tau_1\tau_2 & \tau_2^2 \end{pmatrix}$. Then

$$\Delta = Q^{-1} = \frac{1}{\tau_1^2\tau_2^2-(\rho\tau_1\tau_2)^2} \begin{pmatrix} \tau_2^2 & -\rho\tau_1\tau_2 \\ -\rho\tau_1\tau_2 & \tau_1^2 \end{pmatrix} = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix} \tag{3.21}$$

Also, we have

$$H_1(\underset{\sim}{x},\Delta)=x_1\,\delta_{11} + x_2\delta_{12}$$

$$H_{111}(\underset{\sim}{x},\Delta) = x_1^3\,\delta_{11}^3 + 3\,x_1^2 x_2\delta_{11}^2\delta_{12} + 3\,x_1\,x_{12}^2\delta_{11}\delta_{12}^2$$

$$+ x_2^3\,\delta_{12}^3 - 3\,x_1\,\delta_{11}^2 - 3\,x_2\,\delta_{11}\,\delta_{12} \qquad (3.22)$$

$$H_{112}(\underset{\sim}{x},\Delta) = x_1^3\,\delta_{11}^2\,\delta_{12} + x_1^2\,x_2(2\delta_{11}\delta_{12}^2+\delta_{11}^2\delta_{22})$$

$$+ x_1x_2^2(\delta_{12}^3 + 2\delta_{11}\delta_{12}\delta_{22}) + x_2^3\delta_{12}^2\,\delta_{22}$$

$$- 3\,x_1\,\delta_{11}\,\delta_{12} - x_2(2\delta_{12}^2+\delta_{11}\,\delta_{22}).$$

Similar equations for $H_2(\underset{\sim}{x},\Delta)$ , $H_{222}(\underset{\sim}{x},\Delta)$ and $H_{122}(\underset{\sim}{x},\Delta)$ are obtained by interchanging subscript 1, 2 in $H_1(\underset{\sim}{x},\Delta)$, $H_{111}(\underset{\sim}{x},\Delta)$ and $H_{112}(\underset{\sim}{x},\Delta)$ respectively.

Now

$$\int_{-\infty}^{\tau_2 b}\int_{-\infty}^{\tau_1 a} x_1^r\,x_2^s\;N(\underset{\sim}{x},Q)dx_1dx_2 = \int_{-\infty}^{b}\int_{-\infty}^{a}\tau_1^r\,x_1^r\,\tau_2^s\,x_2^s\,N(\underset{\sim}{x},R)dx_1dx_2 \qquad (3.23)$$

by changing of variables and $R = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ is the correlation matrix.

Define

$$\mu_{r,s} = \int_{-\infty}^{b}\int_{-\infty}^{a} x_1^r\,x_2^s\,N(\underset{\sim}{x},R)dx_1dx_2 \qquad (3.24)$$

Integrating by parts, we have

$$\mu_{1,0} = -\,(\phi(a)\Phi(B) + \rho\phi(b)\Phi(A)) \qquad (3.25)$$

$$\mu_{1,1} = -\rho b\phi(b)\Phi(A) - \rho a\phi(a)\Phi(B) + (1-\rho^2)\,N(a,b;R)$$

$$+ \rho \mu_{0,0}$$

$$\mu_{2,0} = \mu_{0,0} - a\,\phi(a)\,\Phi(B) - \rho^2 b\,\phi(b)\,\Phi(A)$$
$$+ \rho(1-\rho^2)\,N(a,b;R)$$

In general, the recursive relation is

$$\mu_{r,s+1} - b\,\mu_{r,s} = (1-\rho^2)s\,\mu_{r,s-1} + \mu_{r+1,s}\,\rho$$
$$- (1-\rho^2)\,b(s-1)\mu_{r,s-2} - b\rho\mu_{r+1,s} \qquad (3.26)$$

$$\mu_{r+1,s} - a\,\mu_{r,s} = (1-\rho^2)r\mu_{r-1,s} + \mu_{r,s+1}\,\rho - (1-\rho^2)a\,(r-1)$$
$$\times\,\mu_{r-2,s} - a\,\rho\,\mu_{r-1,s+1}$$

and

$$\mu_{3,0} = -(a^2+2)\,\phi(a)\,\Phi(B) + \rho(\rho^2-\rho^2b^2-3)\,\phi(b)\,\Phi(A)$$
$$+ (a+\rho b)\,\rho(1-\rho^2)\,N(a,b;R) \qquad (3.27)$$

$$\mu_{2,1} = -\rho(a^2+2)\,\phi(a)\,\Phi(B) - (1+\rho^2+\rho^2 b^2)\,\phi(b)\,\Phi(A)$$
$$+ (1-\rho^2)(a+\rho b)\,N(a,b;R)$$

where

$$A = (1-\rho^2)^{-\frac{1}{2}}(a-\rho b), \quad B = (1-\rho^2)^{-\frac{1}{2}}(b-\rho a)$$

$$\phi(a) = \frac{1}{\sqrt{2\pi}} \exp \frac{-a^2}{2} \qquad (3.28)$$

$$\Phi(A) = \int_{-\infty}^{A} \phi(t)dt$$

$\mu_{o,1}$, $\mu_{0,2}$, $\mu_{0,3}$ and $\mu_{1,2}$ are obtained by interchanging a with b and A with B in $\mu_{1,0}$, $\mu_{2,0}$, $\mu_{3,0}$ and $\mu_{2,1}$ respectively.

## 4. Applications in Investigation of the Structures of Interactions

In this section, we discuss some applications of the results of Section 3 in studying the power functions of various tests for the hypothesis of no interaction in two-way classification model with one observation per cell.

Consider the model

$$x_{ij} = \mu + \alpha_i + \beta_j + \eta_{ij} + \varepsilon_{ij} \qquad (4.1)$$

for $i = 1,\ldots,u$, $j = 1,2,\ldots,s$. Here $x_{ij}$ denotes the observation in i-th row and j-th column, $\mu$ is the general mean, $\alpha_i$ denotes the effect due to i-th row, $\beta_j$ denotes the effect due to j-th column and $\eta_{ij}$ denotes the interaction of i-th row and j-th column. Also, we assume that $\varepsilon_{ij}$'s are distributed independently and normally with mean 0 and variance $\sigma^2$. Now, let $d_{ij} = x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..}$ ,where $s\bar{x}_{i.} = \sum_{j=1}^{s} x_{ij}$, $u\bar{x}_{.j} = \sum_{i=1}^{u} x_{ij}$ and $us\bar{x}_{..} = \sum_{i=1}^{u}\sum_{j=1}^{s} x_{ij}$, Also, let $D = (d_{ij})$, $X = (x_{ij})$, $W = C_u' X C_s C_s' X' C_u$ where $C_u$ is chosen such that $C_u'C_u = I_{u-1}$ and $C_uC_u' = I_u - \frac{1}{u} J_u$ where $J_u$ is the $u\times u$ matrix with all its elements equal to unity. The non-zero eigenvalues of $DD'$ are the same (e.g., see Johnson and Graybill (1972)) as the nonzero eigenvalues of W. Also, the columns of $C_u'X$ are distributed independently as multivariate normal with mean vectors given by $E(C_u'X) = C_u'M_o$ and a common covariance matrix $\sigma^2 I_{u-1}$ where $M_o = (m_{ij})$ and $m_{ij} = \mu + \alpha_i + \beta_j + \eta_{ij}$. In addition,

$$E(W/(s-1)) = \sigma^2 I_{u-1} + \{C_u' M_o C_s C_s' M_o C_u/(s-1)\}$$
$$= \Sigma_0 \tag{4.2}$$

and $C_u' M_o C_s C_s' M_o C_u = C_u' \eta\eta' C_u = \Omega$ where $\eta = (\eta_{ij})$. So, W is distributed as the noncentral Wishart matrix with (s-1) degrees of freedom and noncentrality matrix $\Omega$. When $\eta=0$, W is distributed as the central Wishart matrix with (s-1) degrees of freedom.

Let $\ell_1 \geq \cdots \geq \ell_{u-1}$ be the eigenvalues of $W/(s-1)$ and let $\lambda_1 \geq \cdots \geq \lambda_{u-1}$ be the nonzero roots of $\Sigma_0$. Then, the problem of testing the hypothesis H: $\Omega = 0$ is equivalent to testing the hypothesis that the eigenvalues of $\Sigma_0$ are equal. Suppose $\eta = \lambda \, \underset{\sim}{\alpha}\underset{\sim}{\beta}'$ where $\underset{\sim}{\alpha}' = (\alpha_1, \ldots, \alpha_u)$ and $\underset{\sim}{\beta}' = (\beta_1, \ldots, \beta_s)$. Then $\Omega = \lambda^2 C_u' \underset{\sim}{\alpha}\underset{\sim}{\beta}' \underset{\sim}{\beta}\underset{\sim}{\alpha}' C_u$, and the nonzero root of $\Omega$ is $\lambda \underset{\sim}{\beta}' \underset{\sim}{\beta} \, \underset{\sim}{\alpha}' \underset{\sim}{\alpha}$. Next, we will assume that the rank of $\eta$ is c. Then, using the well-known singular value decomposition of the matrix, we can write $\eta$ as

$$\eta = \lambda_1 \underset{\sim}{w}_1 \underset{\sim}{v}_1' + \ldots + \lambda_c \underset{\sim}{w}_c \underset{\sim}{v}_c'. \tag{4.3}$$

The nonzero eigenvalues of $\eta\eta'$ are $\lambda_1^2, \ldots, \lambda_c^2$ and the associated eigenvectors are $\underset{\sim}{w}_1, \ldots, \underset{\sim}{w}_c$. The eigenvectors of $\eta'\eta$ corresponding to the eigenvalues $\lambda_1^2, \ldots, \lambda_c^2$ are $\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_c$. The nonzero eigenvalues of $\Omega$ are $\lambda_1^2, \ldots, \lambda_c^2$.

The problem of testing the hypothesis of no interaction in two-way classification with one observation per cell was studied by several authors (e.g., see Tukey (1949) and Williams (1951)). The statistic proposed by Tukey is given by $(\hat{\underset{\sim}{\alpha}}'\hat{\underset{\sim}{\eta}}\hat{\underset{\sim}{\beta}})^2/(\hat{\underset{\sim}{\alpha}}'\hat{\underset{\sim}{\alpha}})(\hat{\underset{\sim}{\beta}}'\hat{\underset{\sim}{\beta}})$ where $\hat{\underset{\sim}{\alpha}}' = (\hat{\alpha}_1,\ldots,\hat{\alpha}_u)$, $\hat{\underset{\sim}{\beta}}' = (\hat{\beta}_1,\ldots,\hat{\beta}_s)$, $\hat{\eta} = (\hat{\eta}_{ij})$, $\hat{\alpha}_i = \bar{x}_{i.} - \bar{x}_{..}$, $\hat{\beta}_j = \bar{x}_{.j} - \bar{x}_{..}$ and $\hat{\eta}_{ij} = x_{ij} - \bar{x}_{i.} - \bar{x}_{.j} + \bar{x}_{..}$. Gollob (1968) and Mandel (1969) considered the problem of testing the hypotheses $\lambda_j = 0$ individually under the model (4 .3) by using the statistics $F_j = \ell_j/(\ell_1+\ldots+\ell_{u-1})$. Gollob treated $\ell_1,\ldots,\ell_{u-1}$ as independent chi-square variables to get an approximation to the distribution of $F_j$. But $\ell_i$'s are neither independent nor distributed as chi-square variables. Corsten and Van Eijnsbergen (1972) showed that the likelihood ratio statistic for testing the hypothesis $\lambda_1=\ldots=\lambda_c = 0$ under the model (4 .3) is $(\ell_1+\ldots+\ell_c)/(\ell_{c+1}+\ldots+\ell_{u-1})$. When c=1, this statistic was derived independently by Johnson and Graybill (1972). Schuurmann, Krishnaiah and Chattopadhyay (1973) and Krishnaiah and Schuurmann (1974) discussed the problem of testing the hypotheses $\lambda_i = 0$ simultaneously by applying the simultaneous tests of Krishnaiah and Waikar (1971) for the equality of the eigenvalues of the covariance matrix of the multivariate normal population. Ghosh and Sharma (1963) studied the power function of Tukey's test for $\eta_{ij} = 0$ against the alternative that $\eta_{ij} = \lambda\, \alpha_i\, \beta_j$. Yochmowitz and Cornell (1978) derived the likelihood ratio test for $\lambda_1 = \ldots = \lambda_a = 0$ against the alternative that $\lambda_a \neq 0$ and $\lambda_{a+1}=\ldots=\lambda_c = 0$. We now compare the power functions of various procedures for testing the hypothesis of no interaction.

Let $T_1 = \ell_1/\ell_{u-1}$, $T_2 = (\mathrm{tr}\, W/u-1)^{u-1}/|W|$, $T_3 = (u-c-1) \times \ell_1/(\ell_{c+1}+\ldots+\ell_{u-1})$, $T_4 = (u-c-1)(\ell_1+\ldots+\ell_c)/c(\ell_{c+1}+\ldots+\ell_{u-1})$. When $\sigma^2$ is unknown, we can use any of the above statistics for testing the hypothesis of no interaction.

If we use $T_i$, we accept of reject $H_0$ according as

$$T_i \lessgtr c_\alpha \tag{4.4}$$

where

$$P[T_i \le c_\alpha | H_0] = (1-\alpha) \tag{4.5}$$

The test statistics $T_1$ is based upon the statistic considered by Krishnaiah and Waikar (1971) for testing the sphericity, whereas the test statistic $T_2$ is based upon the likelihood ratio test statistic for sphericity. The statistic $T_4$ is the likelihood ratio test statistic (see Corsten and Van Eijnsbergen (1972)) for testin[illegible]e hypothesis of no interaction of multiplicative compone[illegible] model (4.3).

Table 1 gives a comparison of the power functions of various procedures for testing the hypothesis of no interaction when $\sigma^2$ is known. The rows corresponding to S denote the simulated values. The multivariate normal deviates are generated by the IMSL subroutine GGNRM, and 10,000 trials are performed for each case, the 95% confidence limit for each value is then $1.96\{\hat{p}(1-\hat{p})/10{,}000\}^{\frac{1}{2}}$, where $\hat{p}$ is the actural value from the empirical trials.

The rows corresponding to N denote the values corresponding to the first term in the asymptotic expansion. The rows corresponding to $N + O(n^{-\frac{1}{2}})$ give the values corresponding to the first two terms of the expansion.

TABLE 1

Comparison of the Power Functions of the Tests for no Interaction When $\sigma^2$ is Unknown

$p = 3,\ \alpha = 0.05$

| n | $(\lambda_1, \lambda_2, \lambda_3)$ | Type of Approximation | $\ell_1/\ell_p$ | $\frac{(trW/p)^p}{\lvert W \rvert}$ | $(p-1)\ell_1/(\ell_2+\ell_3+\ldots+\ell_p)$ | $(p-2)(\ell_1+\ell_2)/2\ell_3$ |
|---|---|---|---|---|---|---|
| | | N | 0.57 | 0.49 | | 0.59 |
| 10 | (12,6,1) | $N+0(n^{-\frac{1}{2}})$ | 0.83 | 0.78 | | 0.83 |
| | | S | 0.81 | 0.77 | | 0.82 |
| | | N | 0.57 | 0.63 | | 0.44 |
| 10 | (12,3,1) | $N+0(n^{-\frac{1}{2}})$ | 0.85 | 0.90 | | 0.71 |
| | | S | 0.82 | 0.87 | | 0.73 |
| | | N | 0.57 | 0.55 | | 0.72 |
| 10 | (12,10,1) | $N+0(n^{-\frac{1}{2}})$ | 0.88 | 0.83 | | 0.95 |
| 10 | (7,1,1) | N | | | 0.79 | |
| | | $N+0(n^{-\frac{1}{2}})$ | | | 0.95 | |
| | | N | | | | 0.76 |
| 10 | (12,12,1) | $N+0(n^{-\frac{1}{2}})$ | | | | 0.99 |
| | | N | 0.51 | 0.54 | | 0.68 |
| 25 | (4,3.5,1) | $N+0(n^{-\frac{1}{2}})$ | 0.82 | 0.79 | | 0.85 |
| | | S | 0.79 | 0.77 | | 0.84 |
| | | N | | | 0.89 | |
| 25 | (4,1,1) | $N+0(n^{-\frac{1}{2}})$ | | | 0.98 | |
| | | N | | | | 0.74 |
| 25 | (4,4,1) | $N+0(n^{-\frac{1}{2}})$ | | | | 0.90 |
| | | S | | | | 0.89 |

TABLE 1 (continued)

| n | $(\lambda_1, \lambda_2, \lambda_3)$ | Type of Approximation | $\ell_1/\ell_p$ | $\frac{(trW/p)^p}{\|W\|}$ | $(p-1)\ell_1/(\ell_2+\ell_3)$ | $(p-2)(\ell_1+\ell_2)/2\ell_3$ |
|---|---|---|---|---|---|---|
| 50 | (2.5,1.7,1) | N | 0.46 | 0.42 | | 0.45 |
| | | $N+O(n^{-\frac{1}{2}})$ | 0.66 | 0.68 | | 0.62 |
| | | S | 0.68 | 0.68 | | 0.64 |
| 50 | (2.5,1,1) | N | | | 0.83 | |
| | | $N+O(n^{-\frac{1}{2}})$ | | | 0.92 | |
| | | S | | | 0.92 | |
| 50 | (2.5,2.5,1) | N | | | | 0.72 |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | 0.86 |
| 100 | (2,1.5,1) | N | 0.57 | 0.54 | | 0.57 |
| | | $N+O(n^{-\frac{1}{2}})$ | 0.75 | 0.77 | | 0.72 |
| | | S | 0.75 | 0.76 | | 0.73 |
| 100 | (2,1.15,1) | N | 0.57 | 0.65 | | 0.32 |
| | | $N+O(n^{-\frac{1}{2}})$ | 0.82 | 0.86 | | 0.60 |
| | | S | 0.82 | 0.85 | | 0.63 |
| 100 | (2,1,1) | N | | | 0.88 | |
| | | $N+O(n^{-\frac{1}{2}})$ | | | 0.95 | |
| 100 | (2,2,1) | N | | | | 0.82 |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | 0.92 |

TABLE 1 (continued)

$p = 4, \quad \alpha = 0.05$

| $n$ | $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ | Type of Approximation | $\ell_1/\ell_p$ | $\frac{(trW/p)^p}{\lvert W \rvert}$ | $\frac{(p-1)\ell_1}{\ell_2+\ell_3+\ell_4}$ | $\frac{(p-1)\ell_1}{\ell_3+\ell_4}$ | $\frac{(p-2)(\ell_1+\ell_2)}{2(\ell_3+\ell_4)}$ | $\frac{(p-3)(\ell_1+\ell_2+\ell_3)}{3\ell_4}$ |
|---|---|---|---|---|---|---|---|---|
| 100 | (3,2.5,2,1) | N | 0.45 | 0.36 | | | | 0.59 |
| | | $N+O(n^{-\frac{1}{2}})$ | 0.78 | 0.76 | | | | 0.80 |
| | | S | 0.79 | 0.76 | | | | 0.81 |
| 100 | (3,2.5,1,1) | N | | | | 0.80 | 0.90 | |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | 1.00 | 1.00 | |
| 100 | (3,2.5,2.5,1) | N | | | | | | 0.70 |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | | | 0.88 |
| 100 | (3,1,1,1) | N | | | 0.95 | | | |
| | | $N+O(n^{-\frac{1}{2}})$ | | | 1.00 | | | |
| 100 | (3,3,2,1) | N | | | | | | 0.70 |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | | | 0.88 |
| 100 | (3,3,3,1) | N | | | | | | 0.83 |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | | | 0.98 |
| | | S | | | | | | 0.96 |
| 100 | (3,3,1,1) | N | | | | | 0.96 | |
| | | $N+O(n^{-\frac{1}{2}})$ | | | | | 1.00 | |

TABLE 1 (continued)

$p = 4, \quad \alpha = 0.05$

| $n$ | $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ | Type of Approximation | $\ell_1/\ell_p$ | $\frac{(trW/p)^p}{\lvert W\rvert}$ | $\frac{(p-1)\ell_1}{\ell_2+\ell_3+\ell_4}$ | $\frac{(p-1)\ell_1}{\ell_3+\ell_4}$ | $\frac{(p-2)(\ell_1+\ell_2)}{2(\ell_3+\ell_4)}$ | $\frac{(p-3)(\ell_1+\ell_2+\ell_3)}{3\ell_4}$ |
|---|---|---|---|---|---|---|---|---|
| 100 | (3,2.5,2,1) | N | 0.45 | 0.36 | | | | 0.59 |
| | | $N+0(n^{-\frac{1}{2}})$ | 0.78 | 0.76 | | | | 0.80 |
| | | S | 0.79 | 0.76 | | | | 0.81 |
| 100 | (3,2.5,1,1) | N | | | | 0.80 | 0.90 | |
| | | $N+0(n^{-\frac{1}{2}})$ | | | | 1.00 | 1.00 | |
| 100 | (3,2.5,2.5,1) | N | | | | | | 0.70 |
| | | $N+0(n^{-\frac{1}{2}})$ | | | | | | 0.88 |
| 100 | (3,1,1,1) | N | | | 0.95 | | | |
| | | $N+0(n^{-\frac{1}{2}})$ | | | 1.00 | | | |
| 100 | (3,3,2,1) | N | | | | | | 0.70 |
| | | $N+0(n^{-\frac{1}{2}})$ | | | | | | 0.88 |
| 100 | (3,3,3,1) | N | | | | | | 0.83 |
| | | $N+0(n^{-\frac{1}{2}})$ | | | | | | 0.98 |
| | | S | | | | | | 0.96 |
| 100 | (3,3,1,1) | N | | | | | 0.96 | |
| | | $N+0(n^{-\frac{1}{2}})$ | | | | | 1.00 | |

The radius of convergence is

$$\gamma_0 = \frac{d}{2||V||} \qquad (4.6)$$

where

$$V = \sqrt{n}\ (S/n-M)$$

We will choose n such that $1/\sqrt{n} < \gamma_0$ Now $||V|| = CL(\sqrt{n}(S/n-M))$, where $CL(\cdot)$ denotes the largest root of $\sqrt{n}((S/n)-M)$, and is approximately distributed with mean 0 and variance $2\sigma^2(\lambda_1-\sigma^2)$.

In Table 1, consider the entry when p=3, $\lambda_1=12$, $\lambda_2 = 6$ and $\lambda_3 = 1$. In this case, $\sigma^2 = 1, d = \lambda_2 - \lambda_3 = 5$, with one standard deviation,

$$\frac{d}{2\sqrt{2\sigma^2(2\lambda_1-\sigma^2)}} \sim \frac{1}{\sqrt{7.5}}$$

where n=10 is chosen.

When the entry is p = 4, $\lambda_1 = 2.5$, $\lambda_2 = 1.7$ and $\lambda_3 = 1$, then $\sigma^2 = 1$, $\lambda_1 = 2.5$, $d = \lambda_1 - \lambda_2 = 0.8$ and n = 50.

The table reveals that results based on normal approximations are not sufficiently accurate for n as large as 100, while the asymptotic expression taking the term of order $n^{-\frac{1}{2}}$ achieves numerical accuracy for moderate sample sizes. This suggests that care should be given for the statistical inferences which are based on the normal approximations.

Next, consider the model (4.1) when $\sigma^2$ is known. In this case, we accept or reject the hypothesis $\lambda_1=\dots=\lambda_c = 0$ according as

$$\frac{\ell_1}{\sigma^2} \lessgtr d_{2\alpha} \qquad (4.7)$$

where

$$P\left[\frac{\ell_1}{\sigma^2} \le d_{2\alpha} \mid H\right] = (1-\alpha) \tag{4.8}$$

When H is rejected, the hypothesis $\theta_i = 0$ is accepted or rejected according as $(\ell_i/\sigma^2) \lessgtr d_{2\alpha}$. When H is true, $\ell_1$ is the largest eigenvalue of the central Wishart matrix. Exact distribution of this statistic is given in Krishnaiah and Chang (1971) and exact percentage points are given in Krishnaiah (1980). When H is not true, an asymptotic expression for the distribution of $\ell_1$ can be obtained as a special case of (3.14) if $\lambda_1$ is different from $\theta_2,\ldots,\theta_r$.

When $\lambda_1>\ldots>\lambda_2>0$, Srivastava and Carter (1980) obtained asymptotic expression of $\log(\ell_1/\ell_1+\ldots+\ell_{u-1})$ and $T_2^{1/(u-1)}$ by a different method. For a review of the literature on tests for no interaction in two way classification model with one observation per cell, the reader is referred to Krishnaiah and Yochmowitz (1980).

## 5. Applications in Cluster Analysis and Reduction of Dimensionality

Let $X_1, \ldots, X_N$ be independent p-dimensional random variables. We consider using the sample covariance matrix

$$S = \sum_{i=1}^{N} (\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})(\underset{\sim}{X}_i - \bar{\underset{\sim}{X}})'$$

where $\bar{\underset{\sim}{X}} = N^{-1}(\underset{\sim}{X}_1 + \ldots + \underset{\sim}{X}_N)$. We wish to test the hypothesis that $\underset{\sim}{X}_i$'s come from a single multivariate normal population with covariance $\Sigma$ against the hypothesis that they come from a mixture of $k \leq p$ such populations, differing in means. We assume $\Sigma = \sigma^2 I$. For k=2 the null hypothesis $H_1$ and the alternative hypothesis $H_2$ are given by

$$H_1: \quad \underset{\sim}{X}_i \sim N(\underset{\sim}{\mu}, \sigma^2 I)$$

$$H_2: \quad \underset{\sim}{X}_i \sim \pi N(\underset{\sim}{\mu}_1, \sigma^2 I) + (1-\pi)\, N(\underset{\sim}{\mu}_2, \sigma^2 I)$$

where $\pi$ is the mixing probability. Under $H_2$ it is known (e.g. see Bryant (1975)) that

$$S \sim \sum_{j=0}^{N} \binom{N}{j} \pi^j (1-\pi)^{N-j} W_p(N-1, \sigma^2 I, M_j)$$

$$M_j = N^{-1} j(N-j)(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)'$$

$M_j$ is of rank 1. Now, let $\Delta^2 = (\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)'(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2)/\sigma^2$ which is proportional to the largest root of $M_j/(N-1)$. When the null hypothesis is true, we know that

$$S \sim W_p(N-1, \sigma^2 I, 0).$$

Let the test statistics $T_1$ and $T_2$ be given as below:

$$T_1 = \ell_1/\sigma^2 \tag{5.1}$$

$$T_2 = \frac{(p-1)\ell_1}{\ell_2+\ldots+\ell_p} \tag{5.2}$$

where $\ell_1 \geq \ldots \geq \ell_p$ are eigenvalues of $S/(N-1)$.

If we use the statistics $T_i$ to test $H_1$, then we accept or reject $H_1$ according as

$$T_i \lessgtr c_{\alpha i} \tag{5.3}$$

where

$$P\{ T_i \leq c_{\alpha i} \,|H_1\} = 1-\alpha \tag{5.4}$$

Let $f_j(\cdot)$ be the asymptotic density of a function of the eigenvalues of $S/(N-1)$ when j of the samples come from population 1. Under this condition $S \sim W_p(N-1,\sigma^2 I,M_j)$. So the unconditional asymptotic density function of the function of eigenvalue of S is

$$\sum_{j=0}^{N} \binom{N}{j} \pi^j (1-\pi)^{N-j} f_j(\cdot)$$

The following table gives a comparison of the asymptotic power value with the simulated value of tests of $H_1$ against $H_2$ for $p=4$, $\alpha=0.05$, $\pi$ = mixing probability, $\Delta = ||\underset{\sim}{\mu}_1-\underset{\sim}{\mu}_2||/\sigma$ , N = sample size.

| | Test | | $\pi = .25$ | | | $\pi = .50$ | |
|---|---|---|---|---|---|---|---|
| | | | $\Delta$=1 | 2 | 3 | $\Delta$=1 | 2 | 3 |
| | $T_1$ | | .02 | .42 | .95 | .02 | .63 | 1.00 |
| N= 51 | | Simu. | .07 | .45 | .94 | .07 | .62 | 1.00 |

| | Test | | $\pi = .25$ | | | $\pi = .50$ | | |
|---|---|---|---|---|---|---|---|---|
| | | | $\Delta=1$ | 2 | 3 | $\Delta=1$ | 2 | 3 |
| N=51 | $T_2$ | | .07 | .68 | .99 | .11 | .88 | 1.00 |
| | | Simu. | .11 | .70 | .98 | .14 | .86 | 1.00 |
| | $T_1$ | | .05 | .76 | 1.00 | .08 | .96 | 1.00 |
| | | Simu. | .13 | .78 | 1.00 | .21 | .96 | 1.00 |
| N=101 | $T_2$ | | .13 | .93 | 1.00 | .22 | 1.00 | 1.00 |
| | | Simu. | .13 | .93 | 1.00 | .27 | 1.00 | 1.00 |

When $\Lambda = 1$, the largest roots of $M_j/(N-1)$ are close to zero, the radius of convergence for perturbation approximation of eigenvalues is small, and the asypmtotic expression does not give good approximation. Note, that if the means under $H_2$ are separated by more than two or three standard derivations, that is, for $\Delta = 2,3$, one may reasonably expect to detect the presence of two components, while if they are separated by less than two standard derivations the detection generally will not be good.

Consider k p-variate normal populations with unknown mean vectors $\underset{\sim}{\mu}_1, \ldots, \underset{\sim}{\mu}_k$ and a common known covariance matrix $\Sigma$. We assume that $n_i$ observations are available from i-th population and the sample mean vector and corrected sum of squares and cross-products (SP) matrix based on these observations are respectively given by $\underset{\sim}{\bar{X}}_{i.}$ and $S_j$. The SP matrix which explain the variation between groups is given by

$$B = \sum_{i=1}^{k} n_i(\underset{\sim}{\bar{X}}_{i.} - \underset{\sim}{\bar{X}}_{..})(\underset{\sim}{\bar{X}}_{i.} - \underset{\sim}{\bar{X}}_{..})'$$

$$n_i \bar{\underset{\sim}{X}}_{i.} = \sum_{j=1}^{n_i} \underset{\sim}{X}_{ij}, \quad n\bar{\underset{\sim}{X}}_{..} = \sum_{i=1}^{k} \sum_{j=1}^{n_i} \underset{\sim}{X}_{ij} \tag{5.5}$$

and $n = n_1+\ldots+n_k$. Now, let $\ell_1 \geq \ldots \geq \ell_p$ denote the eigenvalues of $B\Sigma^{-1}$. Fisher (1939) studied the problem of whether there are any structural relations between the p components of mean vectors. This is equivalent to testing the hypothesis $H_0$ where

$$H_0 \; : \; C\underset{\sim}{\mu}_i = \underset{\sim}{\xi} \tag{5.6}$$

for $i = 1,2,\ldots,k$ where C: s×p and $\underset{\sim}{\xi}$ are unknown and the rank of C is s. We assume that $t<k-1$ where $t = p-s$ The likelihood ratio statistic for testing $H_0$ is given by $U_1 = (\ell_{t+1}+\ldots+\ell_p)$. A detailed discussion of the above procedure was given in Rao (1965). The statistic $B\Sigma^{-1}$ is distributed as the noncentral Wishart matrix with $\gamma$ degrees of freedom and

$$E(B\Sigma^{-1}) = (k-1)\; I + \Omega\Sigma^{-1} = \Sigma_* \tag{5.7}$$

where $\gamma = k-1$

$$\Omega = \sum_{i=1}^{k} n_i(\underset{\sim}{\mu}_i-\bar{\underset{\sim}{\mu}})(\underset{\sim}{\mu}_i-\bar{\underset{\sim}{\mu}}.)'. \tag{5.8}$$

and $n\bar{\underset{\sim}{\mu}} = n_1\underset{\sim}{\mu}_1+\ldots+n_k\underset{\sim}{\mu}_k$. Also, let $\lambda_1 \geq \cdots \geq \lambda_p$ denote the roots of $\Sigma_*$. The distribution of $U_1$ can be obtained as a special case of (3.14) and so the power of the likelihood ratio test for $H_0$ can be studied using the results in this paper.

## 6. Asymptotic Joint Distribution of Functions of the Eignevalues of Multivariate Quadratic Form

In this section we shall derive the joint asymptotic distributions of functions of eigenvalues of the multivariate quadratic form $S = XGX'$, where we assume that $G^s = O(1)$, for whatever power s raised on G. G is a symmetric matrix and the columns of $X$: $p\times n$ are distributed independently as multivariate normal with covariance matrix $\Sigma = (\sigma_{ij})$ and means $E(X) = U = (\underset{\sim}{\mu}_1,\ldots,\underset{\sim}{\mu}_n)$. Then

$$E(S/n) = \frac{\mathrm{tr}G}{n}\Sigma + \frac{UGU'}{n} = \Lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_p), \tag{6.1}$$

Also assume that

$$\lambda_{q_1+\ldots+q_{\alpha-1}+1} = \ldots = \lambda_{q_1+\ldots+q_\alpha} = \theta_\alpha \tag{6.2}$$

for $\alpha=1,2,\ldots,r$, $q_1+\ldots+q_r = p$, and $q_0=0$ and let $\ell_1 \geq \ldots \geq \ell_p$ be the eigenvalues of $S/n$.

We consider the joint asymptotic distribution of $L_1,\ldots,L_k$ where $L_j = \sqrt{n}\,\{T_j(\ell_1,\ldots,\ell_p) - T_j(\lambda_1,\ldots,\lambda_p)\}$, which satisfy assumptions of (3.2). Let

$$S/n = \Lambda + V/\sqrt{n} \tag{6.3}$$

Using the Taylor expansion of $T_i(\ell_1,\ldots,\ell_p)$ for $i=1,2,\ldots,k$, as in (3.3) and the application of Lemma 2.1 we obtain the same equations as (3.5), (3.6) and the characteristic function

$$\Psi(t_i,\ldots,t_k) = E_1(\underset{\sim}{t}) + E_2(\underset{\sim}{t}) + E_3(\underset{\sim}{t}) + O(n^{-1})$$

$$E_1(\underset{\sim}{t}) = \exp(-i\ \sqrt{n}\ \mathrm{tr}\ B_1\Lambda)\ \left|I - 2i(G \otimes B_1\Sigma/\sqrt{n})\right|^{-\frac{1}{2}}$$

$$\exp[i\ \mathrm{tr}\ \underset{\sim}{\xi}\underset{\sim}{\xi}'(I - 2i\ G\otimes B_1\Sigma/\sqrt{n})^{-1}\ G\otimes B_1/\sqrt{n}]$$

$$E_2(\underset{\sim}{t}) = E_1(\underset{\sim}{t})\ \frac{i}{\sqrt{n}} \sum_{i=1}^{k} \sum_{\alpha=1}^{r} \sum_{\beta\neq\alpha} \sum_{j_1\in J_\alpha} \sum_{j_2\in J_\beta} t_i\ a_{i\alpha}\ \theta^{-1}_{\alpha\beta}$$

$$\times \frac{1}{n} \sum_{k_1,k_2,k_3,k_4}^{n} g_{k_1k_2}\ g_{k_3k_4}\ (\sigma^*_{i_1i_2}\sigma^*_{i_3i_4}$$

$$+ \sigma^*_{i_1i_3}\sigma^*_{i_2i_4} + \sigma^*_{i_1i_4}\ \sigma^*_{i_2i_3} \tag{6.4}$$

$$+ \sigma^*_{i_1i_2}\ \pi_{i_3}\pi_{i_4} + \sigma^*_{i_1i_3}\ \pi_{i_2}\pi_{i_4} + \sigma^*_{i_1i_4}\ \pi_{i_2}\pi_{i_3} + \sigma^*_{i_2i_3}\ \pi_{i_1}\pi_{i_4}$$

$$+ \sigma^*_{i_2i_4}\ \pi_{i_1}\pi_{i_3} + \sigma^*_{i_3i_4}\ \pi_{i_1}\pi_{i_2} + \pi_{i_1}\ \pi_{i_2}\ \pi_{i_3}\ \pi_{i_4})$$

$$E_3(t) = E_1(t)\ \frac{i}{2\sqrt{n}} \sum_{i=1}^{k} \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{j_1\in J_\alpha} \sum_{j_2\in J_\beta} t_i\ a_{i\alpha\beta}$$

$$\times [\frac{1}{n} \sum_{k_1,k_2,k_3,k_4}^{n} g_{k_1k_2}\ g_{k_3k_4}\ (\sigma^*_{m_1m_2}\sigma^*_{m_3m_4} + \sigma^*_{m_1m_3}\sigma^*_{m_2m_4}$$

$$+ \sigma^*_{m_1m_4}\ \sigma^*_{m_2m_3}$$

$$+ \sigma^*_{m_1m_2}\ \pi_{m_3}\pi_{m_4} + \sigma^*_{m_1m_3}\ \pi_{m_2}\pi_{m_4} + \sigma^*_{m_1m_4}\ \pi_{m_2}\pi_{m_3}$$

$$+ \sigma^*_{m_2m_3}\ \pi_{m_1}\pi_{m_4}$$

$$+ \sigma^*_{m_2m_4}\ \pi_{m_1}\pi_{m_3} + \sigma^*_{m_3m_4}\ \pi_{m_1}\pi_{m_2} + \pi_{m_1}\pi_{m_2}\pi_{m_3}\pi_{m_4})$$

$$- \lambda_{j_2} \sum_{k_1,k_2} g_{k_1k_2}\ (\sigma^*_{m_1m_2} + \pi_{m_1}\pi_{m_2}) - \lambda_{j_1} \sum_{k_3,k_4}$$

$$g_{k_3k_4}\ (\sigma^*_{m_3m_4} + \pi_{m_3}\pi_{m_4}) + n\ \lambda_{j_1}\lambda_{j_2}]$$

where $\otimes$ is the Kronecker product, and as in (3.7)

$$B_1 = \sum_{i=1}^{k} t_i \ \mathrm{diag}(c_{i1}, \ldots, c_{ip})$$

$$G = (g_{ij})$$

$$\underset{\sim}{\xi} = (\underset{\sim}{\mu}_1', \underset{\sim}{\mu}_2', \ldots, \underset{\sim}{\mu}_n')' \tag{6.5}$$

$$\underset{\sim}{\pi} = [I - \frac{2i}{\sqrt{n}} (I \otimes \Sigma)(G \otimes B_1)]^{-1} \underset{\sim}{\xi} = (\pi_1, \pi_2, \ldots, \pi_{np})'$$

$$\Sigma^* = (I \otimes \Sigma)[I - \frac{2i}{\sqrt{n}} (G \otimes B_1)(I \otimes \Sigma)]^{-1} = (\sigma_{ij}^*)$$

and

$$i_1 = p \times (k_1 - 1) + j_1 \qquad m_1 = p \times (k_1 - 1) + j_1$$

$$i_2 = p \times (k_3 - 1) + j_1 \qquad m_2 = p \times (k_2 - 1) + j_1$$

$$i_3 = p \times (k_2 - 1) + j_2 \qquad m_3 = p \times (k_3 - 1) + j_2$$

$$i_4 = p \times (k_4 - 1) + j_2 \qquad m_4 = p \times (k_4 - 1) + j_2$$

Use the expansion that

$$[I - \frac{2i}{\sqrt{n}} (I \otimes \Sigma)(G \otimes B_1)]^{-1} = \sum_{s=0}^{\infty} [\frac{2i}{\sqrt{n}} (I \otimes \Sigma)(G \otimes B_1)]^s$$

$$|I - \frac{2i}{\sqrt{n}} (G \otimes B_1 \Sigma)|^{-\frac{1}{2}} = \exp \tfrac{1}{2} \left( \sum_{s=1}^{\infty} \frac{(2i)^s \mathrm{tr}\, G^s \cdot \mathrm{tr}(B_1 \Sigma)^s}{s \sqrt{n}^{\,s}} \right) \tag{6.6}$$

and $\mathrm{tr}\ \underset{\sim}{\xi}\underset{\sim}{\xi}' (G \otimes B_1) = \mathrm{tr}\ B_1 U G U'$.

We obtain the characteristic function as

$$\psi(t_1, \ldots, t_k) = \exp(-\frac{1}{2} \underset{\sim}{t}' Q \underset{\sim}{t})$$

$$\{1 + \frac{1}{\sqrt{n}} \sum_{i=1}^{k} i\ t_i (h_1 + h_2) + \frac{1}{\sqrt{n}} \sum_{i_1, i_2, i_3}^{k} (i^3 t_{i_1} t_{i_2} t_{i_3})$$

$$(h_3+h_4+h_5)] + O(n^{-1}) \tag{6.7}$$

where

$$Q = (Q_{i_1 i_2}), \quad Q_{i_1 i_2} = 2\,\frac{trG^2}{n}\, tr\, R^{(i_1)} R^{(i_2)} + 4\, tr\, R^{(i_1)} \psi^{(2,i_2)}$$

$$h_1 = \sum_{\alpha=1}^{r} \sum_{\beta\neq\alpha} \sum_{\substack{j_1\in J_\alpha \\ j_2\in J_\beta}} a_{i\alpha}\, \theta^{-1}_{\alpha\beta} \Big(\frac{trG^2}{n}\, (\sigma_{j_1j_1}\sigma_{j_2j_2} + \sigma^2_{j_1j_2})$$

$$+ 2\,\sigma_{j_1j_2} T^{(2)}_{j_1j_2} + \sigma_{j_1j_1} T^{(2)}_{j_2j_2} + \sigma_{j_2j_2} T^{(2)}_{j_1j_1}\Big)$$

$$h_2 = \sum_{\alpha,\beta=1}^{\gamma} \sum_{\substack{j_1\in J_\alpha \\ j_2\in J_\beta}} a_{i\alpha\beta} \Big(\frac{trG^2}{n}\, \sigma^2_{j_1j_2} + 2\, T^{(2)}_{j_1j_2}\, \sigma_{j_1j_2}\Big)$$

$$h_3 = \frac{4}{3}\,\frac{trG^3}{n}\, tr\, R^{(i_1)} R^{(i_2)} R^{(i_3)} + 4\, tr\, R^{(i_1)} R^{(i_2)} \psi^{(3,i_3)} \tag{6.8}$$

$$h_4 = 4 \sum_{\alpha=1} \sum_{\beta\neq\alpha} \sum_{\substack{j_1\in J_\alpha \\ j_2\in J_\beta}} a_{i_1\alpha}\, \theta^{-1}_{\alpha\beta} \Big(\frac{trG^2}{n}\, \Xi^{(i_2)}_{j_1j_2} + \Omega^{(i_2)}_{j_1j_2} + \Omega^{(i_2)}_{j_2j_1}\Big)$$

$$\Big(\frac{trG^2}{n}\, \Xi^{(i_3)}_{j_1j_2} + \Omega^{(i_3)}_{j_1j_2} + \Omega^{(i_3)}_{j_2j_1}\Big)$$

$$h_5 = 2 \sum_{\alpha,\beta} \sum_{\substack{j_1\in J_\alpha \\ j_2\in J_\beta}} a_{i\alpha\beta} \Big(\frac{trG^2}{n}\, \Xi^{(i_2)}_{j_1j_1} + 2\, \Omega^{(i_2)}_{j_1j_1}\Big)$$

$$\Big(\frac{trG^2}{n}\, \Xi^{(i_3)}_{j_2j_2} + 2\, \Omega^{(i_3)}_{j_2j_2}\Big)$$

$$T^{(2)} = \frac{UG^2U'}{n}, \quad R^{(i)} = C^{(i)}\Sigma, \quad C^{(i)} = diag(c_{i1},\dots,c_{ip})$$

$$\psi(j,i) = C^{(i)}\,\frac{UG^jU'}{n}, \quad \Xi^{(i)} = \Sigma R^{(i)}, \quad \Omega^{(i)} = T^{(2)} R^{(i)} \tag{6.9}$$

where $A_{ij}$ denotes the (i,j)th elements of matrix $A = (A_{ij})$.

Inverting the characteristic function, we obtain the following expression for the asymptotic joint distribution of $\underset{\sim}{L} = (L_1, \ldots, L_k)'$.

$$f(L_1, \ldots, L_k) = N(\underset{\sim}{L}, Q)$$

$$\times \Big[1 + \frac{1}{\sqrt{n}} \sum_{i=1}^{k} H_i(\underset{\sim}{L})(h_1 + h_2) \tag{6.10}$$

$$+ \frac{1}{\sqrt{n}} \sum_{i_1, i_2, i_3}^{k} H_{i_1 i_2 i_3}(\underset{\sim}{L})(h_3 + h_4 + h_5)\Big] + O(n^{-1})$$

where $N(\underset{\sim}{L}, Q)$ and $H_i(\underset{\sim}{L})$, $H_{i_1 i_2 i_3}(\underset{\sim}{L})$ are as defined in Eq.(3.15)

## 7. Asymptotic Distributions of Functions of the Roots of the Complex Wishart Matrix

Let $Z = Z_1 + i\, Z_2$ be a $p\times n$ matrix and let the rows of $(Z_1' : Z_2')$ be distributed independently as multivariate normal with covariance matrix

$$\begin{pmatrix} \Sigma_1 & \Sigma_2 \\ -\Sigma_2 & \Sigma_1 \end{pmatrix}.$$

and let the mean vector of j-th row of $(Z_1' : Z_2')$ be $\underset{\sim}{\mu}_j' = (\underset{\sim}{\mu}_j^{(1)'}, \underset{\sim}{\mu}_j^{(2)'})$. Also let $\tilde{S} = Z\bar{Z}'$ where $\bar{Z}$ denotes the complex conjugate of Z. Then, the distribution of $\tilde{S}$ is known to be central or noncentral complex Wishart matrix with $n$ degrees of freedom according as $\tilde{M} = 0$ or $\tilde{M} = \underset{\sim}{0}$. where $\tilde{M} = \tilde{U}\bar{\tilde{U}}'$, $E(Z) = \tilde{U}$. The expected value of S is given by

$$E(S) = 2n(\Sigma_1 - i\Sigma_2) + \tilde{M}$$

The matrix S is Hermitian and the eigenvalues of S/n are denoted by

$$\ell_1 \geq \cdots \geq \ell_p.$$

In the sequel, we assume that $E(\tilde{S}) = n \text{ diag. } (\lambda_1, \ldots, \lambda_p)$ and $\lambda_1 \geq \cdots \geq \lambda_p$. In addition, we assume that $\lambda_i$'s have multiplicity as in (3.1). Now, let

$$L_j = \sqrt{n}\, \{T_j(\ell_1, \ldots, \ell_p) - T_j(\lambda_1, \ldots, \lambda_p)\} \tag{7.1}$$

for $j = 1,2,\ldots,k$ and the function $T_j(\ell_1, \ldots, \ell_p)$ satisfy the assumptions (3.2) and (3.3). Then, following the same lines as in Section 3 for the real case, we obtain the following asymptotic expression for the joint density of $L_1, L_2, \ldots, L_k$:

$$f_1(L_1,\ldots,L_k) = N(\underset{\sim}{L},\tilde{Q})\,[1+\frac{1}{\sqrt{n}}\sum_{i=1}^{k} H_i(L)(\tilde{d}_1+\tilde{d}_2)$$

$$+\frac{1}{\sqrt{n}}\sum_{i_1,i_2,i_3=1}^{k} H_{i_1i_2i_3}(\underset{\sim}{L})(\tilde{g}_1+\tilde{g}_2+\tilde{g}_3)]$$

$$+\,O(n^{-1}) \qquad (7.2)$$

where

$$\tilde{Q} = (\tilde{Q}_{i_1i_2}),\tilde{Q}_{i_1i_2} = 4\ \mathrm{tr}\ \tilde{R}_1^{(i_1)}\tilde{R}_1^{(i_2)} + 4\ \mathrm{tr}\ \tilde{R}_2^{(i_1)}\tilde{R}_2^{(i_2)}$$

$$+\ 8\ \mathrm{tr}\ \tilde{R}_1^{(i_1)}\tilde{\psi}^{(i_2)}$$

and $\tilde{Q}$ is assumed to be nonsingular. Also,

$$\tilde{d}_1 = 4\sum_{\alpha=1}^{r}\sum_{\beta\neq\alpha}^{r}\sum_{j_1\in J_\alpha}\sum_{j_2\in J_\beta} a_{i\alpha}\theta_{\alpha\beta}^{-1}(\tilde{\sigma}^{(1)}_{j_1j_1}\tilde{\sigma}^{(1)}_{j_2j_2}+\tilde{\sigma}^{(1)}_{j_1j_1}\tilde{\nu}_{j_2j_2}+\tilde{\sigma}^{(1)}_{j_2j_2}\tilde{\nu}_{j_1j_1}) \qquad (7.3)$$

$$\tilde{d}_2 = 2\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}\sum_{j_1\in J_\alpha}\sum_{j_2\in J_\beta} a_{i\alpha\beta}\,(\tilde{\sigma}^{(1)^2}_{j_1j_2} + 2\tilde{\sigma}^{(1)}_{j_1j_2}\,\tilde{\nu}_{j_1j_2} - \tilde{\sigma}^{(2)^2}_{j_1j_2}) \qquad (7.4)$$

$$\tilde{g}_1 = \frac{8}{3}\ \mathrm{tr}\ \tilde{R}_1^{(i_1)}\tilde{R}_1^{(i_2)}\tilde{R}_1^{(i_3)} + 8\ \mathrm{tr}\ \tilde{R}_1^{(i_1)}R_1^{(i_2)}\tilde{\psi}^{(i_3)}$$

$$+\ 8\ \mathrm{tr}\ \tilde{R}_2^{(i_1)}\tilde{R}_2^{(i_2)}\tilde{R}_1^{(i_3)} - 8\ \mathrm{tr}\ \tilde{R}_2^{(i_1)}\tilde{R}_2^{(i_2)}\tilde{\psi}^{(i_3)} \qquad (7.5)$$

$$\tilde{g}_2 = 16\sum_{\alpha=1}^{r}\sum_{\beta\neq\alpha}^{r}\sum_{j_1\in J_\alpha}\sum_{j_2\in J_\beta} a_{i_1\alpha}\,\theta^{-1}_{\alpha\beta}[(\tilde{\Xi}^{(i_2)}_{j_1j_2} + \tilde{T}^{(i_2)}_{j_1j_2} + \tilde{T}^{(i_2)}_{j_2j_1} + \tilde{G}^{(i_2)}_{j_1j_2})$$

$$\times (\tilde{\Xi}^{(i_3)}_{j_1j_2} + \tilde{T}^{(i_3)}_{j_1j_2} + \tilde{T}^{(i_3)}_{j_2j_1} + \tilde{G}^{(i_3)}_{j_1j_2}$$

$$+ (\tilde{U}^{(i_2)}_{j_1j_2} - \tilde{U}^{(i_2)}_{j_2j_1})(\tilde{U}^{(i_3)}_{j_1j_2} - \tilde{U}^{(i_3)}_{j_2j_1})]$$

$$\tag{7.6}$$

$$\tilde{g}_3 = 8 \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} \sum_{j_1\in J_\alpha} \sum_{j_2\in J_\beta} a_{i_1\alpha\beta} (\tilde{\Xi}^{(i_2)} + 2\tilde{T}^{(i_2)}_{j_1j_1} + \tilde{G}^{(i_2)}_{j_1j_1})$$

$$\times (\tilde{\Xi}^{(i_3)}_{j_2j_2} + 2\tilde{T}^{(i_3)}_{j_2j_2} + \tilde{G}^{(i_3)}_{j_2j_2}) \tag{7.7}$$

where $C^{(i)} = \text{diag}\ (c_{i1},\ldots,c_{ip})$, $\Sigma_1 = (\tilde{\sigma}^{(1)}_{i_1i_2})$, $\Sigma_2 = (\tilde{\sigma}^{(2)}_{i_1i_2})$

$$\tilde{M} = \sum_{j=1}^{n} (\underset{\sim}{\mu}_j^{(1)}\underset{\sim}{\mu}_j^{(1)'} + \underset{\sim}{\mu}_j^{(2)}\underset{\sim}{\mu}_j^{(2)'})/\ 2n = (\tilde{\nu}_{j_1j_2})$$

$$\tilde{R}_1^{(i)} = C^{(i)}\Sigma_1,\ \tilde{R}_2^{(i)} = C^{(i)}\Sigma_2,\ \tilde{\psi}^{(i)} = C^{(i)}\tilde{M}$$

$$\tilde{\Xi}^{(i)} = \Sigma_1 C^{(i)}\Sigma_1,\ \tilde{G}^{(i)} = \Sigma_2\ C^{(i)}\ \Sigma_2$$

$$\tilde{T}^{(i)} = \tilde{M}\ C^{(i)}\Sigma_1,\ \tilde{U}^{(i)} = \tilde{M}\ C^{(i)}\ \Sigma_2$$

Krishnaiah and Lee (1977) derived the asymptotic joint distributions of the linear combinations and ratios of the linear combinations of the eigenvalues of the central complex Wishart matrix when the roots are simple. These results are special cases of the results in this section.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>AFOSR-TR- 80-1164 | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>Asymptotic Distributions of Functions of the Eigenvalues of the Real and Complex Noncentral Wishart Matrices | | 5. TYPE OF REPORT & PERIOD COVERED<br>Interim Report |
| | | 6. PERFORMING ORG. REPORT NUMBER<br>U.P. Stat. Rept. #80-11 |
| 7. AUTHOR(s)<br>C. Fang and P. R. Krishnaiah | | 8. CONTRACT OR GRANT NUMBER(s)<br>F49620-79-C-0161 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>University of Pittsburgh<br>Department of Mathematics & Statistics<br>Pittsburgh, PA. 15260 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>61102F 2304/A5 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>AFOSR<br>BOLLING AFB WASHINGTON DC 20332 | | 12. REPORT DATE<br>July 1980 |
| | | 13. NUMBER OF PAGES<br>38 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>Unclassified |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Asymptotic Distributions, Noncentral Wishart Matrix, Functions of Eigenvalues, Complex Matrices, Mixtures of Populations, Quadratic Forms, Perturbation Theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper, the authors derived asymptotic expressions for the joint densities of various functions of the noncentral real and complex Wishart matrices. Applications of the above results are also discussed in the area of reduction of dimensionality and testing for the structure of interaction in two way classification with one observation per cell.

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