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A POPULATION OF LINEAR, SECOND ORDER, ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS ON RECTANGULAR DOMAINS - PART I

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ABSTRACT

We present a population of 56 linear, two-dimensional elliptic partial differential equations (PDEs) suitable for evaluating numerical methods and software. Forty two of the PDEs are parameterized which allows much larger populations to be made; 189 specific cases are presented here along with solutions (some are only approximate). Many of the PDEs are artifically created so as to exhibit various mathematical behaviors of interest; the others are taken from "real world" problems in various ways. The population has been structured by introducing measures of complexity of the operator, boundary conditions, solution and problem. The PDEs are first presented in mathematical terms along with contour plots of the 189 specific solutions. Machine readable descriptions are given in Part 2, MRC Technical Summary Report #2079; many of the PDEs involve lengthy expressions and about a dozen involve extensive tabulations of approximate solutions.

AMS (MOS) Subject Classification: 65N99

Key Words: elliptic partial differential equations, numerical methods, software evaluation, population of problems, linear, second order

Work Unit Number 7 (Numerical Analysis)

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SIGNIFICANCE AND EXPLANATION

A population of 56 linear, two-dimensional elliptic partial differential equations is given. Forty two of them are parameterized and 189 specific cases are presented in mathematical terms, with contour plots and in machine readable form. Some of the equations are very complicated and over 8800 lines are needed for the complete, machine readable definitions of the problems. The objective is to provide a population for the scientific evaluation of the effectiveness of numerical methods for solving such equations. The population has been structured by introducing complexity measures of various problem features. It is anticipated that the structuring along with the expandability (due to the parameterizations) will allow this population to be and as the basis for statistical or systematic evaluations of numerical methods and/or software over a wide range of situations.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

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LINEAR, SECOND ORDER, ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS
ON RECTANGULAR DOMAINS
PART I

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1. INTRODUCTION

The motivation for creating this PDE population is for use in the evaluation of numerical methods and PDE software. The need and rationale for a systematic approach to such evaluations is given in [Rice, 1979], [Houstis and Rice, 1980], [Crowder, Dembo and Mulvey, 1979]; it suffices here to say that a properly chosen problem population is an essential ingredient for a sound evaluation of numerical methods and software.

A useful population of PDEs is inevitably very lengthy and this one is no exception as one sees from the last two appendices. Thus in the body of this paper we discuss the sources of the PDEs, how they are described in the appendices and how a structure has been created in the population through the use of quantitative (but subjective) measures of features.

It is important that one be able to create relevant subpopulations as one inevitably wants to evaluate methods for particular subclasses of PDEs (e.g., separable, with singularities or with mixed boundary conditions). Experience shows that no one universal method is best for all PDEs (even in this rather restricted context) and one of the important tasks of research is to create and/or identify methods that are especially efficient for particular classes of PDEs. Once one embarks on such a task one sees that this population, which originally might seem large and bulky, is actually rather small for the uses to be made of it. It is only the fact that it can be substantially expanded in various directions through the parameterization that gives one hope that it is adequate for a wide variety of evaluations.

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2. CHARACTERISTICS OF THE PROBLEMS.

A source parameter is assigned to each PDE which ranges from 0 (artificial problem) to 100 (actual real world problem). This feature, as the others introduced later, is subjective in nature and the values given must be taken as approximate indications of our intuitive feelings. The PDE $u_{xx}+u_{yy} = 1$ might be completely artificial for one person and be the actual applications PDE for another. We have at least tried to be consistant in these values.

2.1. Sources

Many problems have been normalized so the maximum value of the solution is 1.0 and all have this value between .1 and 100. Many of the domains have been standardized to the unit square, $0 \leq x,y \leq 1$. The sources of the PDEs are:

A. Problems used in previous studies. Nine problems are included which were used by [Eisenstat and Schultz, 1973] or [Houstis et al, 1975 and 1978]. Subsets of this population have been used by [Houstis and Papatheodorou, 1977 and 1979] and [Lynch and Rice, 1978]. Some of these PDEs have had parameters added and all have been normalized so the maximum value of the solution is about 1.0.

B. Artifical Problems. Many problems have been created just to exhibit various mathematical behaviors of interest (e.g. singularities, oscillations or wave fronts). Such behaviors are important for theory or application (or both) and one needs to have them present in the population in an easily identifiable manner.

C. Problems adapted from the "real world". A persistent difficulty is the desire to have PDEs which represent the "real world" and the necessity to know their true solutions. Among the strategies to adapt real world problems we have used:

(i) choosing explicit functions which model the physical solutions and then determining appropriate boundary conditions and/or right side to make this the true solution.

(ii) using truncated series expansions (of high accuracy) with appropriate small modifications in the boundary conditions or right side.

(iii) solving nonlinear problems approximately, then substituting the tabulated numerical solution into the operator (using quadratic interpolation from a 10 by 10 grid) to obtain a linear problem which is, in turn, solved approximately. In these cases the true solution is not known, but the machine readable population contains tabulated values of a hopefully accurate numerical solution.

2.2 Problem Features and Complexity Classifications. We identify as problem features the smoothness and local variation of operator, the boundary conditions and the solution. These features are quantified on a one-dimensional scale of 0 to 100 even though there are rather independent properties that can be called smoothness or local variation. These features are measured subjectively from the following descriptions of the scale.

Smoothness. This refers to the mathematical properties of the functions or operators involved. Key points on the scale are:

00 = entire functions or constants

10 = analytic; very well behaved

30 = very smooth, some higher derivative (5 or so) discontinuity possible

50 = still smooth, third derivative discontinuity possible

70 = not rough to the eye, but possibly only 1 continuous derivative

80 = continuous, functions might be theoretically smooth but rough on a gross scale

90 = possibly discontinuous, nearly singular functions or operators

100 = strong singularities like $1/x$ or $1/x^2$.

Local variation. This refers to how much a function changes (relative to its size) in a small part of its domain. These variations might be oscillations, wave fronts, peaks or boundary layers. Key points on the scale are:

00 = very smooth, uniform

10 = mild variation, probably convex, some non-uniformity, e.g. $\sin(2x)$, e^{3x} on $[0,1]$

25 = modest variation of oscillation; mild wave front or peak, e.g. $\sin(6x)$, $1/(1+100x^4)$ on $[0,1]$

40 = considerable peak or oscillation; change of magnitude occurs within 10-15% of domain

60 = sharp peaks, wave fronts, boundary layers or oscillations; 100% change in magnitude occurs within 5% of domain

75 = practically a discontinuity in magnitude; continuity observable only with a fine scale examination

90 = actual discontinuity in magnitide; extreme oscillation, step functions, e.g. $SIN(300x)$ on $[0,1]$

The overall problem complexity is represented by the average of the above six feature measures. The problems in this population do not have complexities exceeding 58 (only one exceeds 50), a level which might be interpreted as "rather messy with one or two substantial complications". The problem feature measures are included in the descriptions along with the source parameter.

Appendix 1 presents some summary information about the population. Tables are given which

A. group the PDEs according to types of the operator and boundary conditions (e.g. Helmholtz and Dirichlet or constant coefficients and mixed)

B. list the 56 PDEs with abbreviated feature descriptions

C. group the PDEs according to the smoothness of the operator and right side

D. group the PDEs according to the smoothness of the solution.

A figure is also given which displays the overall problem complexity for the 189 specific PDEs.

3. FORMAT OF PROBLEM DESCRIPTIONS

Appendix 2 contains a mathematical description of each PDE along with contour plots for each specific instance included in the set of 189 PDEs. An example is shown in Figure 1. The description begins with a problem number and source followed by a mathematical description of the PDE. Then brief comments are given for the operator, right side, boundary conditions, solution and parameters (if any). Sometimes functions appearing in the mathematical description are defined in these comments.

Four generic functions are used:

$f(x,y)$ = right side of PDE determined so that the given true solution is correct.

$f(x),g(y)$ = right sides of boundary conditions determined so that the given true solution is correct.

$T(x,y)$ = the true solution, used in the coefficients of some PDEs derived from nonlinear problems.

$r(x,y)$ = an approximate solution used in some PDEs whose true solution is unknown.

Contour plots are given for one or more particular PDEs for each problem. The border of the plots contains the following information:

(i) values of the parameters

(ii) maximum and minimum values of the solution; the contours are equispaced between these values.

(iii) the classification parameters in the form

S.P 01.02 B1.B2 S1.S2

where

S = source parameter P = problem complexity

$\alpha 1$ = smoothness feature $\alpha 2$ = local variation feature

and $\alpha = 0$ for the operator, B for the boundary contitions,

S for the solution.

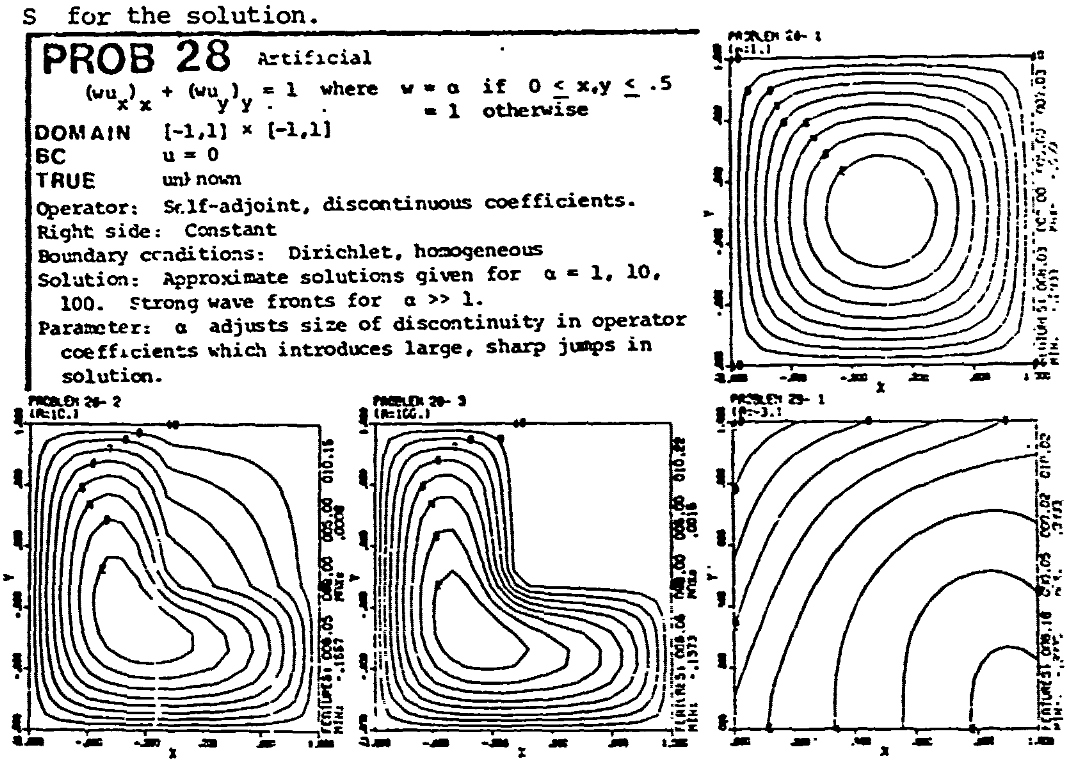

Figure 1. An example of the mathematical description of a PDE along with some contours.

The machine readable description of the PDE population consists of two files: EQNFIL and MACFIL. EQNFIL has 189 entries which are either complete statements of the PDE in the ELLPACK language (see [Boisvert, Houstis and Rice, 1979]) or a reference to an entry in MACFIL with values given for parameters. See Figure 2 for a short example. The information given starts with the problem number, feature parameter values and a code for various attributes of the PDE which are used within the ELLPACK system. Then ELLPACK language code is given for the operator and boundary conditions; this code should be self explanatory once one sees that UXX$ represents u_{xx}, etc. Finally, there is a Fortran code for any functions that appear in the operator, right side or boundary conditions. This latter code averages about 20 lines and can be as much as 150 lines (excluding tables that are part of some problems). These descriptions are given in Part 2 of this report, MRC Technical Summary Report No. 2079

MACFIL entries are just like EQNFIL descriptions of a PDE except that the places where parameter values are to be substituted are indicated by &A, &B, etc. A refers to the first parameter, B the second and so on. There are somewhat more than 8500 lines in these two files.

```
*EOR
*EOF
***************
* PROBLEM   2 *
***************
*EOR
*
*         000.04     000.00     004.05     010.02
*         20002000000020
1         TWO DIMENSIONS
1         UXX$ + (1.+Y*Y)UYY$ - UX$ - (1.+Y*Y)UY$ = F(X,Y)
2         MIXED
2         X=0. , MIXED = (1.)U + ( 1.)UX  =  0.27*EXP(Y)
2         X=1. , MIXED = (1.)U + (-1.)UX  =  0.
2         Y=0. , MIXED = (1.)U + ( 1.)UY  =  0.27*EXP(X)
2         Y=1. , MIXED = (1.)U + (-1.)UY  =  0.135*(ALOG(2.)-1.)*(X*X-X)**2
3     FUNCTION TRUE(X,Y)
3     TRUE = 0.135*(EXP(X+Y)+(X*X-X)**2*ALOG(1.+Y*Y))
3     RETURN
3     END
3     FUNCTION F(X,Y)
3     F = 0.135*( (-4.*X*X*X+18.*X*X-14.*X+2.)*ALOG(1.+Y*Y)
3    $    - 2.*((X*X-X)**2)*(Y*Y+Y**3+Y-1.)/(1.+Y*Y) )
3     RETURN
3     END
```

```
*EOR
*EOF
***************
* PROBLEM   3 *
***************
*EOR
*PARAMETER SET   1(A=1.5)
*         000.43     030.60     000.00     070.40
EXPAND 3/1.5/
*EOR
*PARAMETER SET   2(A=2.5)
*         000.35     030.50     000.00     060.20
EXPAND 3/2.5/
*EOR
*PARAMETER SET   3(A=3.5)
*         000.28     070.30     000.00     050.15
EXPAND 3/3.5/
*EOR
*PARAMETER SET   4(A=4.5)
*         000.23     055.20     000.00     040.20
EXPAND 3/4.5/
```

Figure 2. A sample from EQNFIL showing a short PDE description in machine readable form and a reference to a similar description in MACFIL

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JRR:ENH:WPD/db

Table 1

Classifications of Problems According to Operator and Boundary Conditions

Operator	Constant Coefficients			Non-Constant Coefficients		
	Dirichlet	Neumann	Mixed	Dirichlet	Neumann	Mixed
Laplace	3, 4, 7, 8, 10, 11, 17, 33, 34, 35, 47, 50		4, 31, 35, 38, 55			
Helmholtz Type	9, 41, 53			6, 20, 39, 44, 45, 48, 49		
Self-Adjoint	5			1, 13, 22, 25, 28, 54		1, 19, 23, 52
General	14, 46	42	43	12, 15, 16, 18, 21, 26, 27, 29, 30, 32, 36, 37, 56		2, 23, 24, 40, 51

Note that problems 1, 4, and 35 appear in two places in the table since they have boundary conditions of the form

$$u + \alpha u_N = g$$

and hence have Dirichlet boundary conditions for $\alpha=0$.

Table 2

Problem Characteristics

The principal characteristics are tabulated below using the following encodings:

A Analytic
BL Boundary Layer
C Constant (coefficients)
CC Computationally Complex
D Dirichlet Boundary Condition
E Entire
H Homogeneous
J Jump Discontinuity
M Mixed Boundary Condition
N Neumann Boundary Condition
NS Nearly Singular
O Oscillatory
P Parameterized or Peaked
S Singular (infinite)
SD Singular Derivative
U Unknown
VS Variable Smoothness
WF Wave Front

Problem Number	Operator	Right Side	Solution	Boundary Conditions	Domain
1P	A	E	E	M	Unit Square
2	E	A	A	M,H	Unit Square
3P	C	S,SD	S,SD	D,H	Unit Square
4P	C	E	E	M	Unit Square
5P	C	E	E	D,H	Unit Square
6	E,NS	A	A,O	D,H	Unit Square
7	C	C	SD	D,H	Unit Square
8P	C	SD	SD,WF	D	Unit Square
9P	C,NS	E,NS	E,BL	D	Unit Square
10P	C	E,P	E,P	D,H	Unit Square
11P	C	A,O	A,O	D	Unit Square
12P	E,O	E,O	E,O	D	Unit Square
13	J	S	SD	D	Unit Square
14P	C	S	S	D	Unit Square
15P	A,NS	S	SD	D	Unit Square
16P	A,NS	C	U,BL	D,H	Variable Square
17P	C	A,NS	A,NS,WF	D	Unit Square
18P	E	A,NS	A,NS,WF	D	Unit Square
19P	S	S	E	M,H	Square
20P	NS,P,CC	P	E,P	D	Rectangle
21	E	E	E	D	Unit Square
22	SD	S	E	D	Unit Square
23P	SD	SD	SD,WF	M,H	Unit Square
24P	S,NS	S,NS	U,P	M,H	Square
25P	SD	S	E	D,H	Unit Square

Table 2

Problem Characteristics

Problem Number	Operator	Right Side	Solution	Boundary Conditions	Domain
26P	A	A	U, SD	D, H	Variable Square
27	A, NS	C	U, BL	D, H	Square
28P	J	C	U, WF	D, H	Square
29P	S	H	U, VS, BL	D	Unit Square
30P	A, CC	A, CC	A, NS	D	Unit Square
31	C	C	E, (SD)	M	Square
32	A	A	E	D, H	Rectangle
33	C	E	E, O	D	Rectangle
34	C	C	E, (SD)	D	Square
35P	C	H	E, O, BL	M	Square
36P	S	S	A, BL	D	Unit Square
37	E	E	E	D	Unit Square
38P	C	H	E, O, VS	D	Rectangle
39P	CC, S	CC, S	U, BL	D, C	Unit Square
40P	E	A	A	M	Unit Square
41P	C, NS	SD, NS	SD	D, H	Square
42P	C	H	A, O	N	Variable Rectangle
43	C	H	E	M	Square
44P	CC	CC	U, BL	D, H	Unit Square
45P	C, NS	H	U, BL	D	Unit Square
46P	C, NS	H	U, BL	D	Variable Rectangle
47P	C	S	SD, VS	D	Unit Square
48P	CC	CC	U	U	Unit Square
49P	CC	CC	U, SD, BL	D, C	Unit Square
50	C	H	E, O	D	Rectangle
51P	S	C	U, SD, WF	M, H	Unit Square
52P	CC	H	U, O	M, C	Unit Square
53P	C, NS	E, O	E, O	D	Unit Square
54P	E, CC	S, CC	SD, VS	D	Unit Square
55P	C	H	S, VS, BL	M	Rectangle
56P	S	CC	U, O, (SD)	M	Rectangle

Table 3

Classifications of Problems
According to Smoothness of the Operator and Right–Side

(A=Analytic; C=Constants; CC=Computationally Complicated; DD=Discontinuous Derivatives; E=Entire; O=Oscillatory; P=Peak; S=Singular)

Smoothness Operator	Smoothness Right-Side	Problem Numbers
C	C	7, 31, 34, 35, 38, 42, 43, 45, 46, 50, 55
C	E	4, 5, 9, 10, 33, 53
C	A	11, 17
C	DD	3, 8, 41
C	S	3, 14, 47
C	O	6, 11, 53
C	P	10
E	E	12, 21, 37
E	A	2, 6, 18, 40
E	S	54
A	C	16, 27
A	E	1
A	A	26, 30, 32
A	S	15
DD	C	28
DD	DD	13, 23, 25
DD	S	22, 25
S	C	29, 51, 56
S	S	19, 24, 36
O	A	6
O	O	12
C	CC	17, 18
S	CC	56
CC	C	52
CC	P	20
CC	CC	30, 39, 44, 48, 49, 54

Table 4

Classifications of Problems According to Smoothness of the Solution

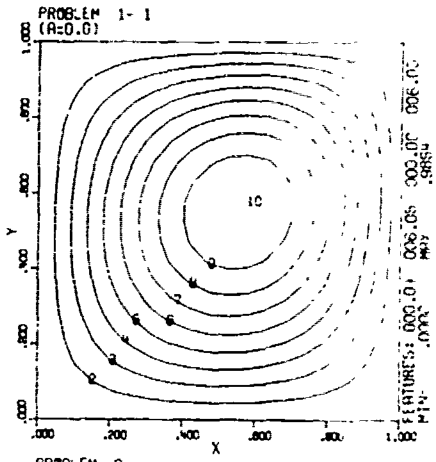
Solution Smoothness	Problem Numbers
Entire	1, 4, 5, 9, 10, 12, 19, 20, 21, 22, 25, 31, 32, 33, 34, 35, 37, 38, 43, 50, 53
Analytic	2, 6, 11, 17, 18, 30, 36, 40, 42
Singular Derivatives	3, 7, 13, 14, 15, 41, 47, 51, 54, 56
Oscillatory	6, 11, 12, 33, 35, 38, 42, 50, 53, 56
Wave Front	8, 17, 18, 23, 28, 51
Discontinuous Derivatives	8, 23
Singular	54, 55
Boundary Layer	7, 9, 15, 16, 27, 29, 44, 45, 46, 49
Peak	10, 20, 24
Tabled Solution	16, 24, 26, 27, 28, 29, 39, 44, 45, 46, 48, 49, 51, 52

PROB 1 Artificial [7,12,13]

$$(e^{xy}u_x)_x + (e^{-xy}u_y)_y - u/(1+x+y) = f$$

DOMAIN unit square
BC $u + \alpha u_N = g$
TRUE $.75e^{xy}\sin(\pi x)\sin(\pi y)$

Operator: Self-adjoint, analytic
Right side: Entire
Boundary conditions: Mixed except for $\alpha = 0$.
Solution: Entire, independent of α.
Parameter: α introduces normal derivative into boundary conditions.

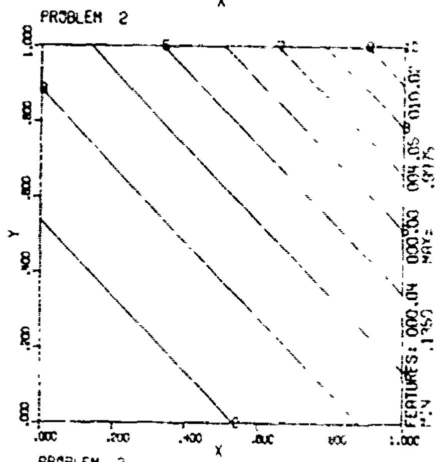


PROB 2 Artificial [12,13]

$$u_{xx} + (1+y^2)u_{yy} - u_x - (1+y^2)u_y = f$$

DOMAIN unit square
BC $u - u_N = g$
TRUE $.135(e^{x+y} + (x^2 - x)^2\log(1+y^2))$

Operator: Entire
Right side: Analytic
Boundary conditions: Mixed
Solution: Analytic
Parameter: None

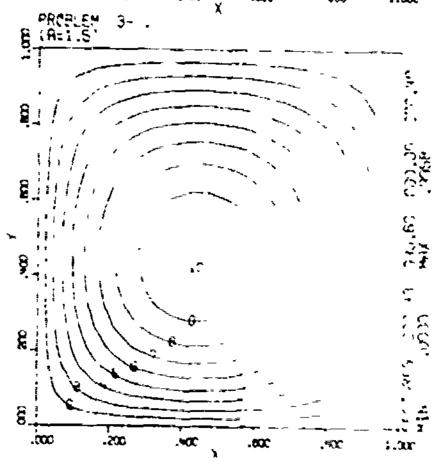


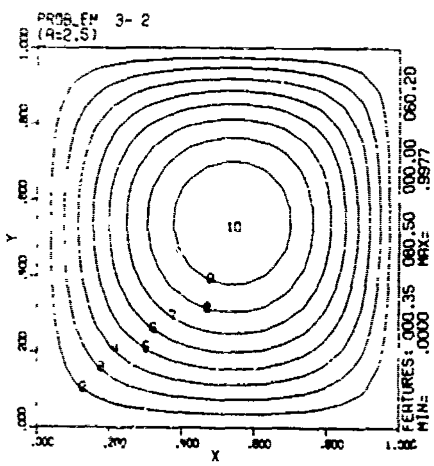
PROB 3 Artificial [13]

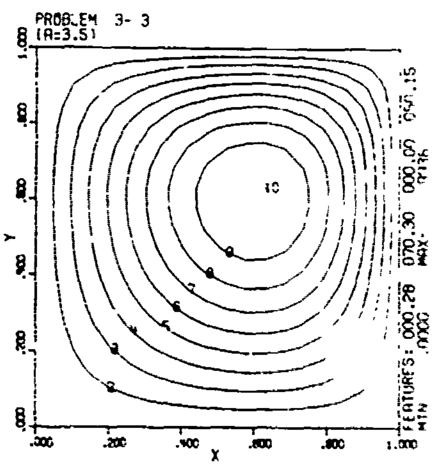
$$u_{xx} + u_{yy} = f$$

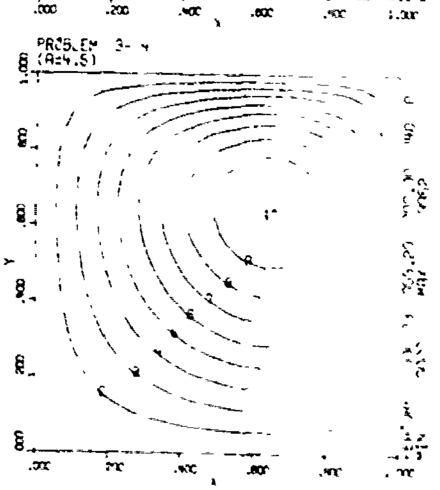
DOMAIN unit square
BC $u = 0$
TRUE $c(x^{\alpha/2} - x)(y^{\alpha/2} - y)$, $c = 1/(\alpha^{\alpha/(1-\alpha)} - \alpha^{1/(1-\alpha)})^2$

Operator: Laplace
Right side: singular for $\alpha \le 3$
Boundary condition: Dirichlet, homogeneous
Parameter: $1 < \alpha \le 5$ adjusts singularity strength







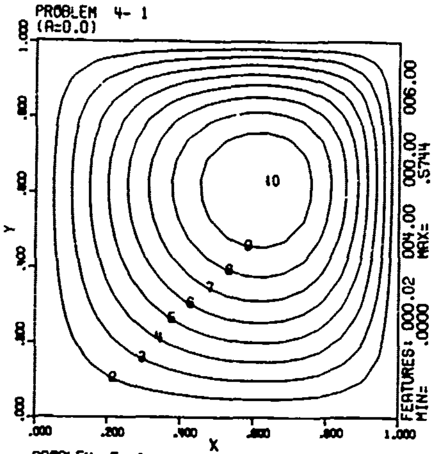


PROB 4 Artificial [7,12,13]

$$u_{xx} + u_{yy} = 6xy\, e^{x+y}(xy + x + y - 3)$$

DOMAIN unit square
BC $u = 0$ for $x \neq 0$; $u - \alpha(y - y^2)u_x = g$ for $x = 0$
TRUE $3e^{x+y}(x - x^2)(y - y^2)$

Operator: Laplace
Right side: Entire
Boundary conditions: Mixed except for $\alpha = 0$
Solution: Entire, independent of α
Parameter: α introduces normal derivative into boundary conditions

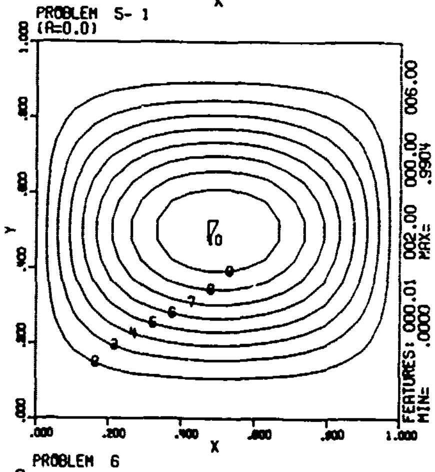


PROB 5 Artificial [13,14]

$$4u_{xx} + u_{yy} - \alpha u = f$$

DOMAIN unit square
BC $u = 0$
TRUE $2(x^2 - x)(\cos(2\pi y) - 1)$

Operator: Constant coefficient, separable
Right side: Entire
Boundary conditions: Dirichlet, homogeneous
Parameter: α makes operator more singular without affecting solution



PROB 6 Stratospheric physics [13,14,16]

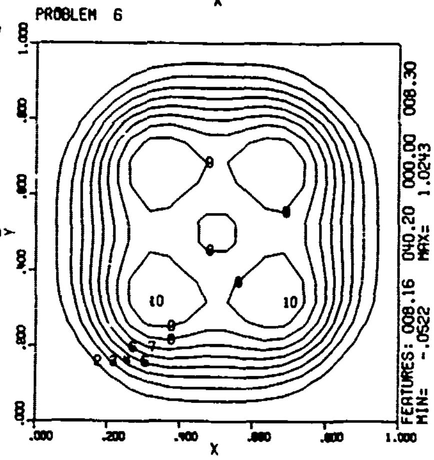
$$u_{xx} + u_{yy} - (100 + \cos(2\pi x) + \sin(3\pi y))u = f$$

DOMAIN unit square
BC $u = 0$
TRUE

$$-0.31(5.4-\cos(4\pi x))\sin(\pi x)(y^2-y)(5.4-\cos(4\pi y))(1/(1+\varphi^4)-.5)$$

$$\varphi = 4(x-.5)^2 + 4(y-.5)^2$$

Operator: Entire, oscillatory, somewhat singular
Right side: Analytic
Boundary conditions: Dirichlet, homogeneous
Parameter: None

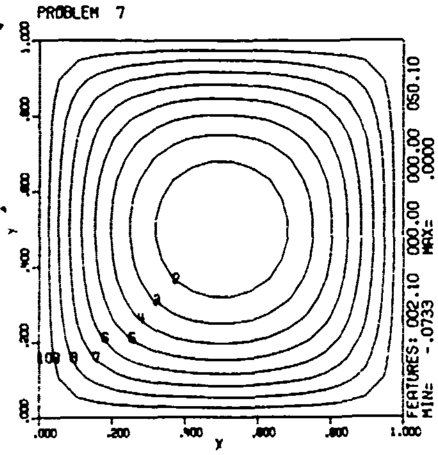


PROB 7 Artificial [illegible]

$$u_{xx} + u_{yy} = 1$$

DOMAIN unit square
BC $u = 0$
TRUE Approximate series solution gives 10^{-9} accuracy.

Operator: Laplace
Right side: Constant
Boundary condition: Dirichlet, homogeneous
Solution: Has logarithmic singularities at corners in second derivatives; approximate solution is a polynomial.
Parameter: None



PROB 8 Artificial [13]

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square

BC $u = g$

TRUE $\phi(x)\phi(y)$ where $\phi(x) = 1$ ... for $x > .5 + a$ and $\phi(x)$ is a ... polynomial for $.5 - a \le x \le .5 + a$ so ϕ has two continuous derivatives.

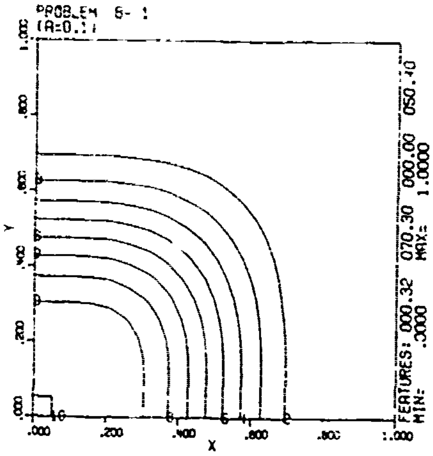
Operator: Laplace

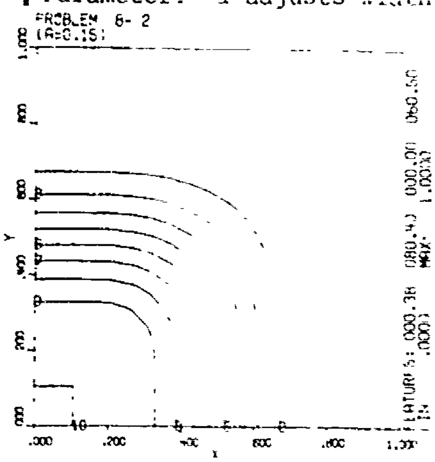
Right side: Just continuous with a right angle ridge.

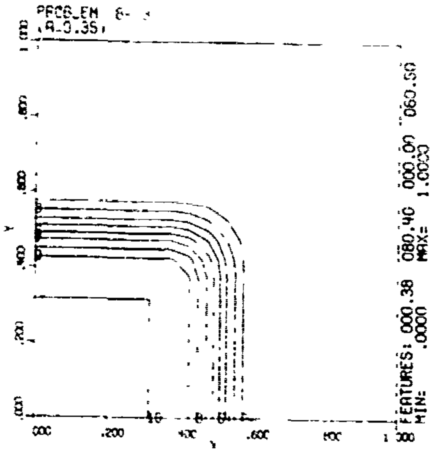
Boundary conditions: Dirichlet

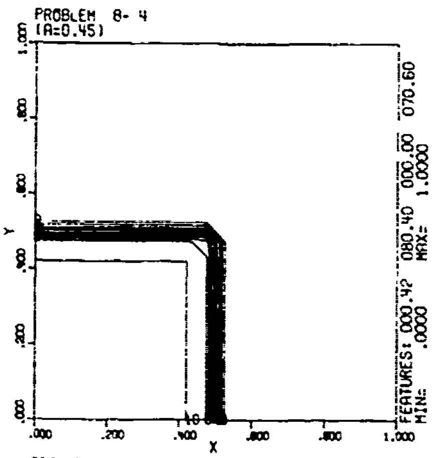
Solution: Wave front along a right angle joining two regions where it is constant.

Parameter: a adjusts width and sharpness of wave front.









PROB 9 Artificial [13]

$$u_{xx} + u_{yy} - 100u = .5(\ ^2 - 100)\cosh(\ \ )\ldots$$

DOMAIN unit square

BC $u = g$

TRUE .5(cosh 10x/cosh 10 + cosh ... y/cosh ...)

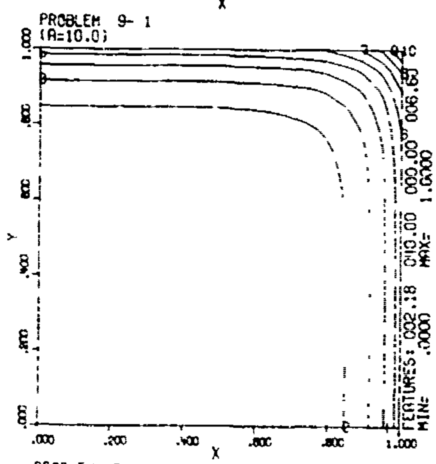
Operator: Helmhotz, constant coefficients, somewhat singular.

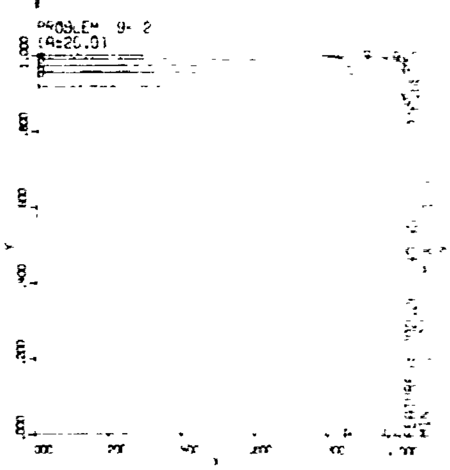
Right side: Entire but nearly singular for ... = 10.

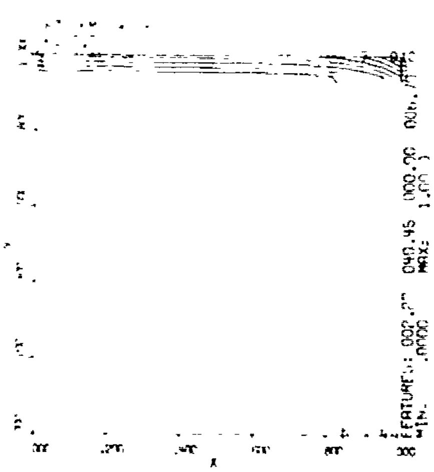
Boundary conditions: ...

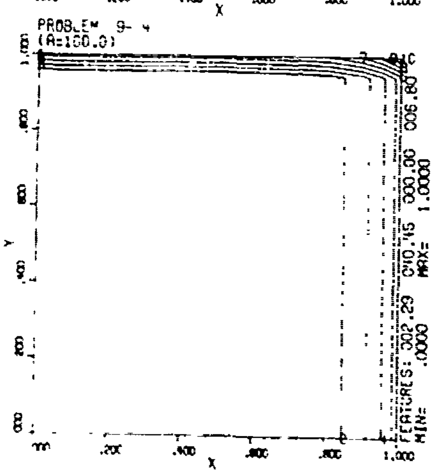
Solution: ... ular.

Parameter: ... de boundary layer.









PROB 10 Artificial [13]

$$u_{xx} + u_{yy} = f$$

DOMAIN unit square

BC $u = 0$

TRUE $e^{-\alpha[(x-.5)^2+(y-\beta)^2]}(x^2 - x)(y^2 - y)$

Operator: Laplace

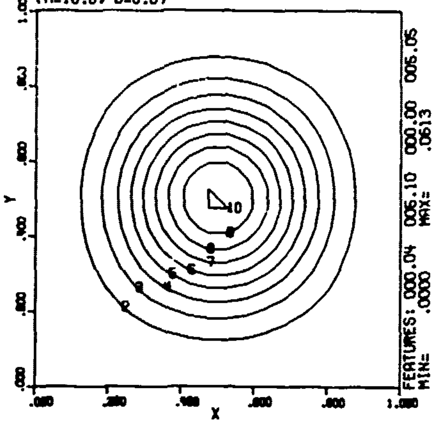
Right side: Strongly peaked if α large, but entire.

Boundary condition: Dirichlet, homogeneous

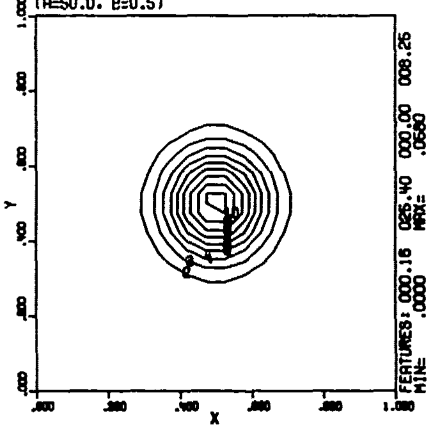
Solution: Strongly peaked for large α.

Parameters: α adjusts strength of the peak, β moves it in the y-direction.

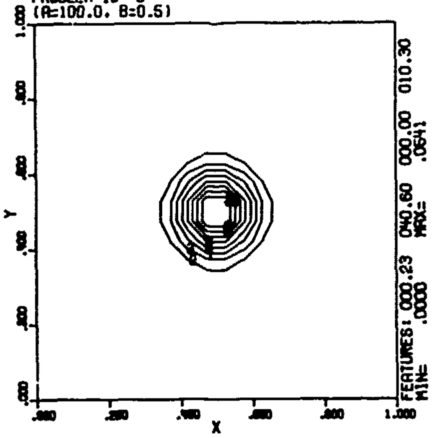
PROBLEM 10- 1
(A=10.0, B=0.5)



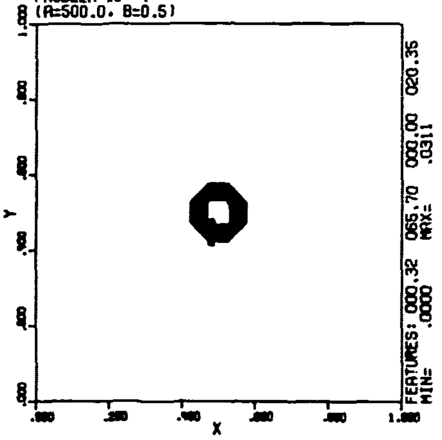
PROBLEM 10- 2
(A=50.0, B=0.5)



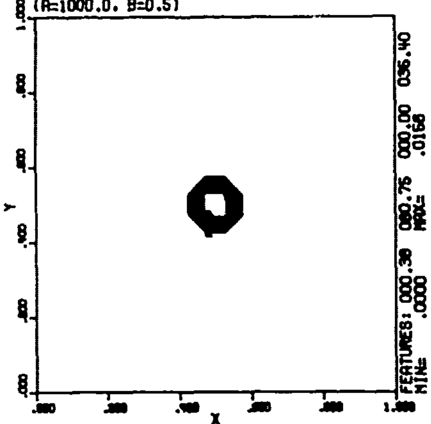
PROBLEM 10- 3
(A=100.0, B=0.5)



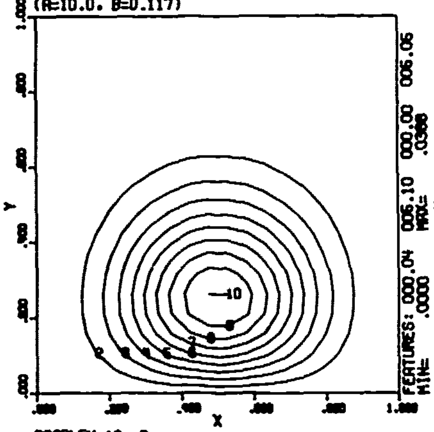
PROBLEM 10- 4
(A=500.0, B=0.5)



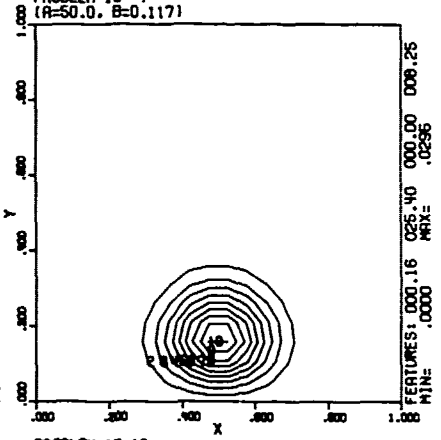
PROBLEM 10- 5
(A=1000.0, B=0.5)



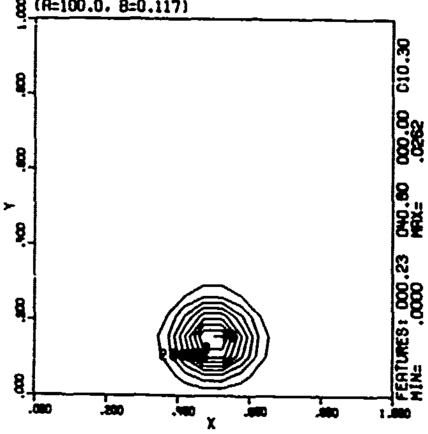
PROBLEM 10- 6
(A=10.0, B=0.117)



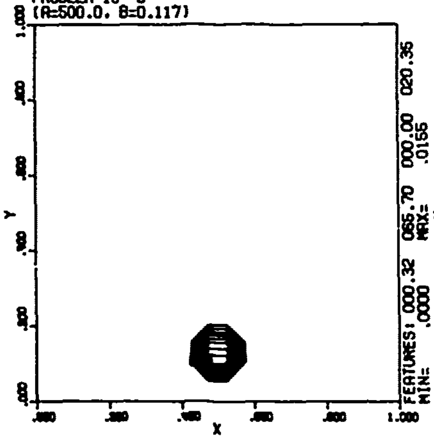
PROBLEM 10- 7
(A=50.0, B=0.117)



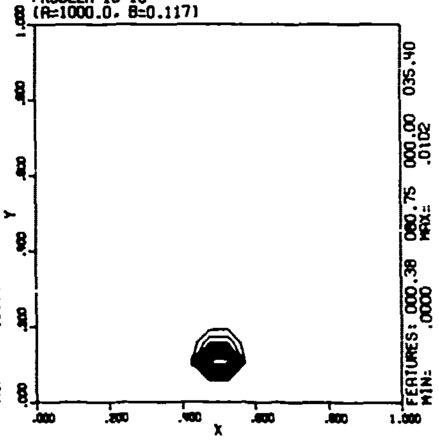
PROBLEM 10- 8
(A=100.0, B=0.117)



PROBLEM 10- 9
(A=500.0, B=0.117)



PROBLEM 10-10
(A=1000.0, B=0.117)



PROB 11 Artificial

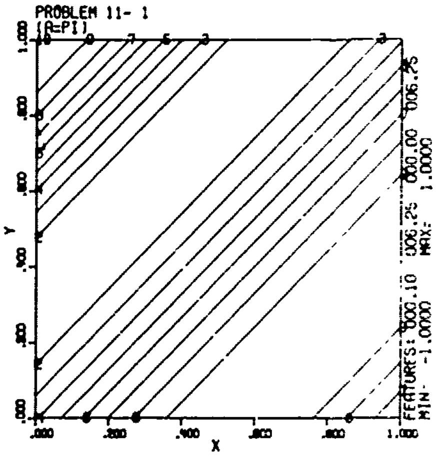
$$u_{xx} + u_{yy} = f$$

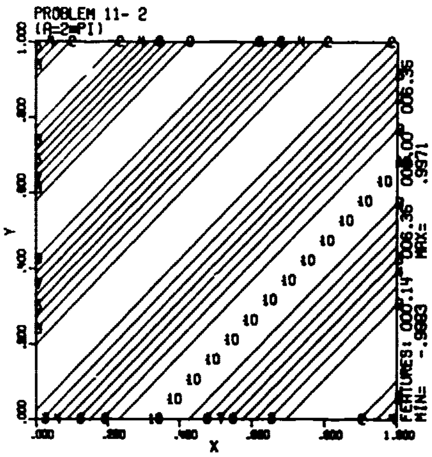
DOMAIN unit square

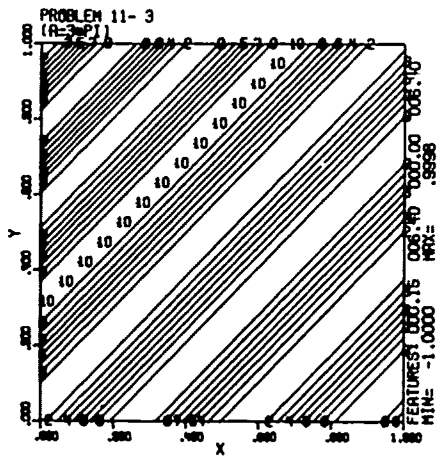
BC $u = g$

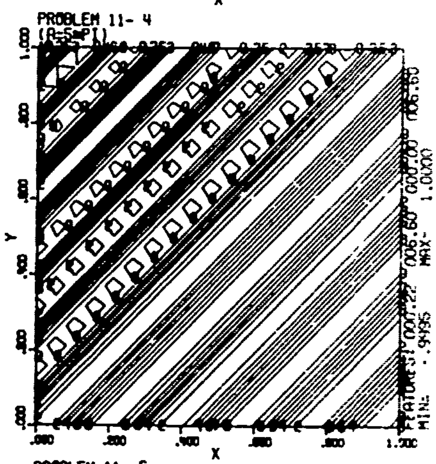
TRUE $\sin[\alpha(x - y + 2)^5/(1 + (x - y + 2)^4)]$

Operator: Laplace
Right side: Oscillatory, analytic
Boundary conditions: Dirichlet
Solution: Oscillatory
Parameter: α adjusts frequency of oscillations









PROB 12 Artificial

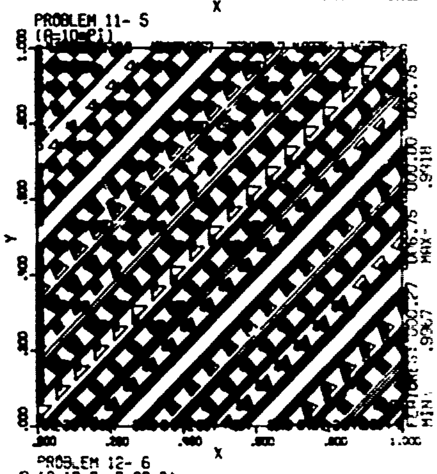
$$u_{xx} + u_{yy} + (1 + \sin(\alpha x))u_x - \cos(\alpha y)u = f$$

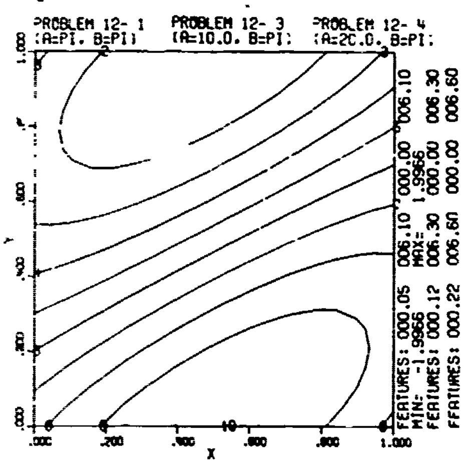
DOMAIN unit square

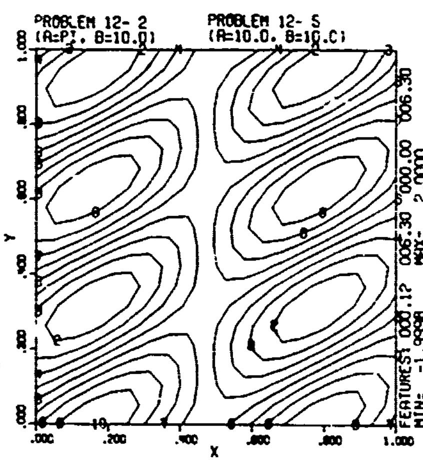
BC $u = g$

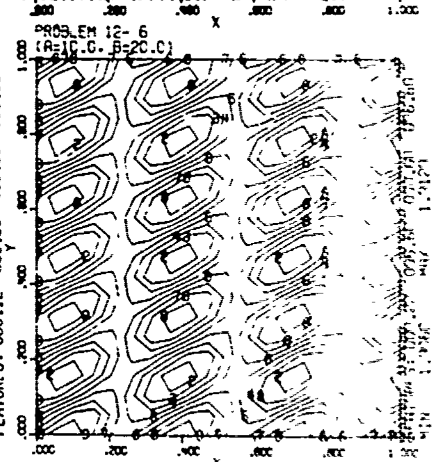
TRUE $\cos(\beta y) + \sin\beta(x - y)$

Operator: Oscillatory, Laplacian plus lower terms
Right side: Oscillatory, analytic
Boundary conditions: Dirichlet
Solution: Oscillatory, entire
Parameter: α adjusts oscillation in PDE coefficients.
β adjust oscillation in the solution.









PROB 13 Artificial

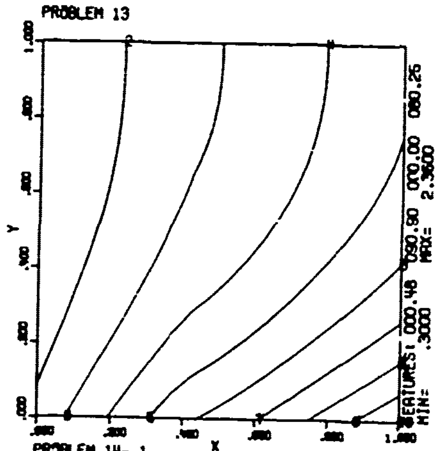
$$((1 + (x - .4)^0_+ u_x)_x + u_{yy} = f$$

DOMAIN unit square

BC $u = g$

TRUE $\min[x+.3, .7+.5(x-.4)+(x-.4)^2/(1+x^2)](1+(y-1)^2 e^{-y})$

Operator: Self-adjoint, discontinuous coefficients.
Right side: Line of singularities along $x = 0.4$
Boundary conditions: Dirichlet
Solution: Derivative in x is singular.
Parameter: None



PROB 14 Artificial

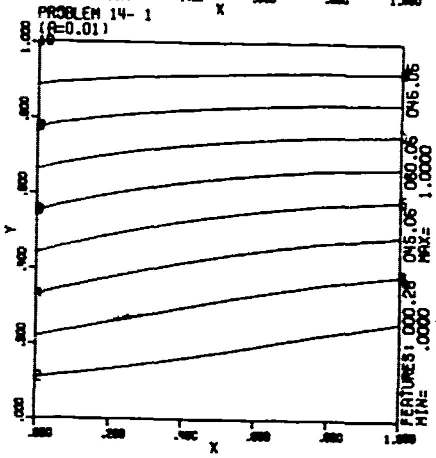
$$u_{xx} + 2u_{yy} + 3u_x - 4u_y - u = f$$

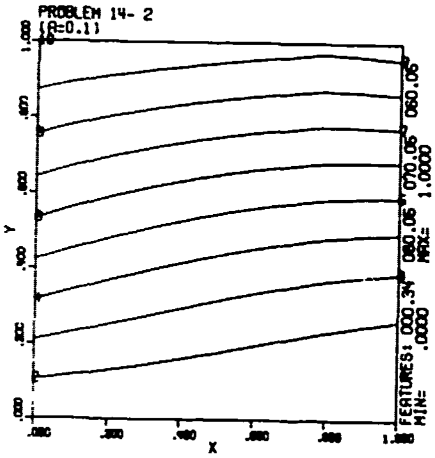
DOMAIN unit square

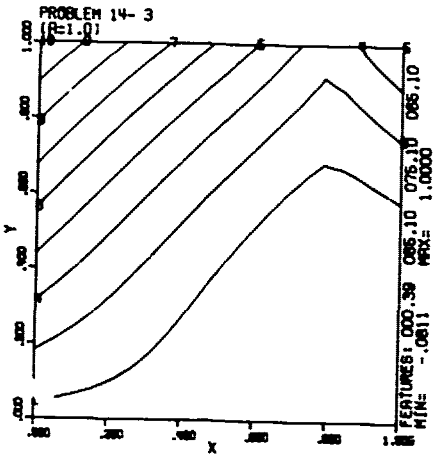
BC $u=0$, $y=0$; $u=y$, $x=0$; $u=g$, $x=1$; $u=1-.8\alpha+\alpha|x-.8|$ for $y=1$.

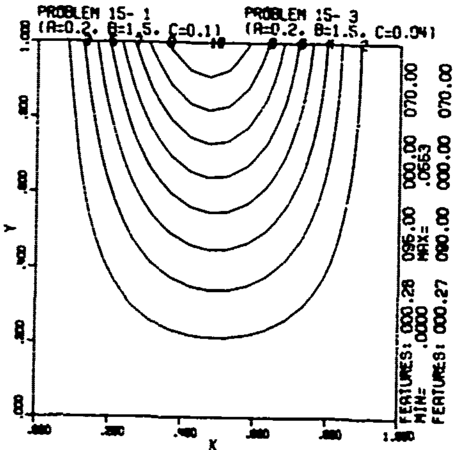
TRUE $y(1 - .8\alpha^{2-y} + \alpha|x - .8|^{2-y}) + xye^{-xy}(y - 1)$

Operator: Constant coefficients
Right side: Line of singularities at $x = .8$
Boundary conditions: Dirichlet, discontinuous derivative
Solution: Line of singularities of variable strength along $x = .8$.
Parameter: α adjusts strength of the singularity









PROB 15 Artificial

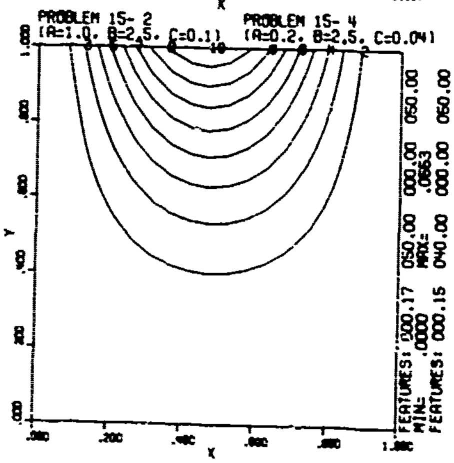
$$u_{xx} + u_{yy} + \alpha/(y + \gamma)u_y = f$$

DOMAIN unit square

BC $u = g$

TRUE $[y^\beta + \cos(xy^2) - 1]x^2(x - 1)^2$

Operator: Laplace plus nearly singular derivative term
Right side: Singularity in $\beta - 1$ y-derivative, nearly singular for small ..
Boundary conditions: Dirichlet
Solution: B =0, derivative singular.
Parameters: or singularity, β adjusts
unrelated ity in solution.

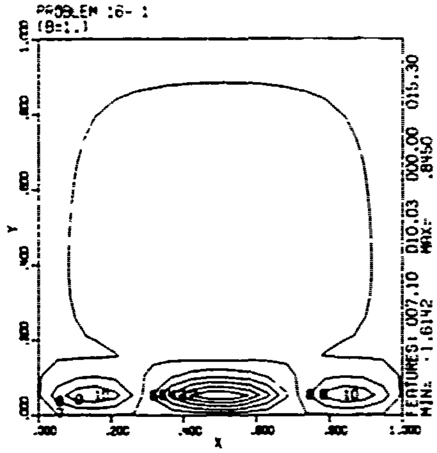


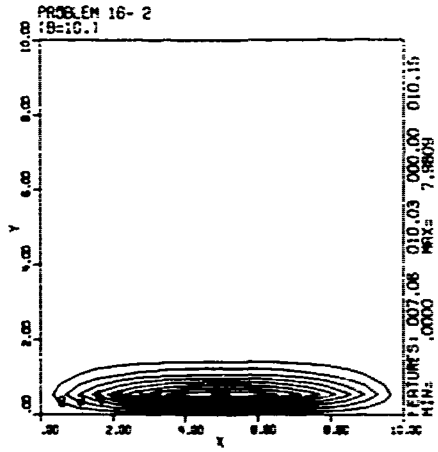
PROB 16 Tension in a spring [3]

$$u_{xx} + u_{yy} - \frac{500}{1 - 250y} u_y = -1/\beta^2$$

DOMAIN $[0,\beta] \times [0,\beta]$
BC $u = 0$
TRUE unknown

Operator: Laplace plus nearly singular u_y term.
Right side: Constant, domain dependent.
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solutions given for $\beta = 1, 10$.
Parameter: β adjusts the size of the domain and right side.



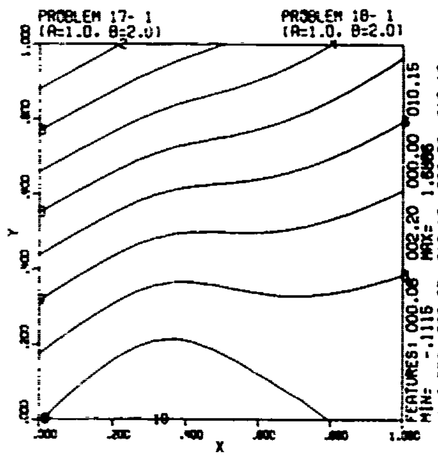


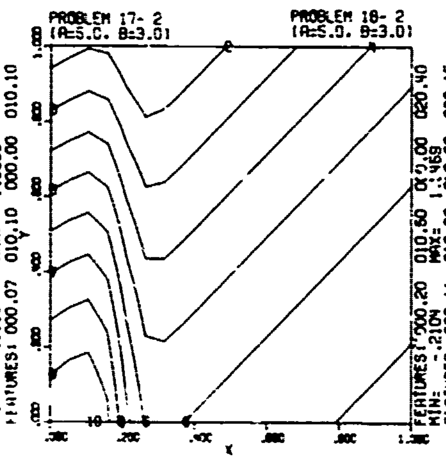
PROB 17 Artificial

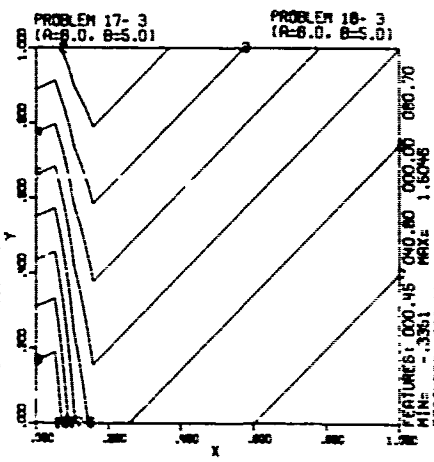
$$u_{xx} + u_{yy} = f$$

DOMAIN unit square
BC $u = g$
TRUE $e^{[y^2+(\alpha(\beta x)^3/(1+(\beta x)^3))^2]} + \sin(x - y + .5)$

Operator: Laplace
Right side: Large values for x near .15
Boundary conditions: Dirichlet
Solution: Sharp wave front near $x = .15$, entire.
Parameters: α, β adjust the strength and shape of the wave front.







PROB 18 Artificial

$$u_{xx} + (1 + xy)u_{yy} + \cos(x)u_x - e^{-x}u_y + 3u = f$$

DOMAIN unit square
BC $u = g$
TRUE $e^{-[y^2+(\alpha(\beta x)^3/(1+(\beta x)^3))^2]} + \sin(x - y + .5)$

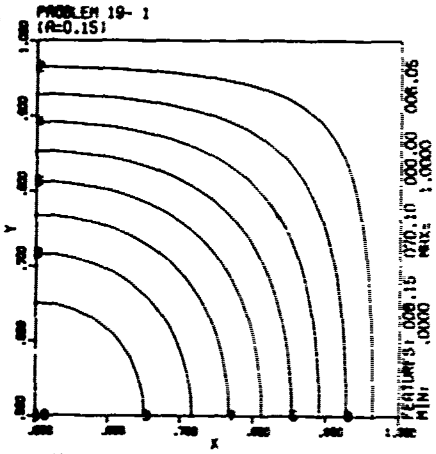
Operator: Entire
Right side: Large values for x near .15
Boundary conditions: Dirichlet
Solution: Sharp wave front near $x = .15$, entire. Solution is the same as in the preceding problem.
Parameters: α, β adjust the strength and shape of the wave front.

PROB 19 Nonlinear laminar, non-Newtonian flow [1]

$$(wu_x)_x + (wu_y)_y = f, \quad w = [T_x^2 + T_y^2]^\alpha$$

DOMAIN $[.5,1] \times [.5,1]$
BC $u = 0$ for $x,y = 1$; $u_x = 0$ for $x = .5$, $u_y = 0$ for $y = .5$
TRUE $\sin(\pi x)\sin(\pi y)$

Operator: Self-adjoint, possibly singular.
Right side: Possibly singular.
Boundary conditions: Mixed, homogeneous.
Solution: Entire, T similar to that of non-linear problem.
Parameter: $-.5 \le \alpha \le 1$ is a physical parameter.

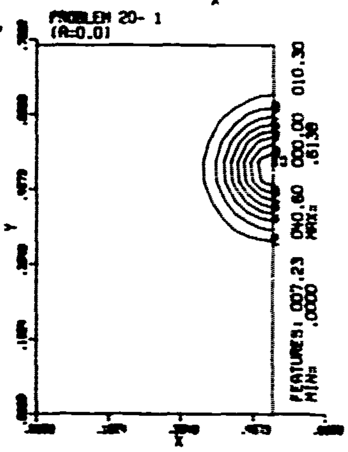


PROB 20 From $u_{xx} + u_{yy} = e$ [1]

$$u_{xx} + u_{yy} - wu = f, \quad w = e^T$$

DOMAIN $[0,.5] \times [0,.75]$
BC $u = g$
TRUE $10\varphi(x)\varphi(y) + \alpha$ where $\varphi(x) = e^{-100(x-.5)^2}(x^2 - x)$

Operator: Helmholtz type, approximates nonlinear operator.
Right side: Sharp, large values near $x = y = .5$.
Boundary conditions: Dirichlet, homogeneous.
Solution: T has a peak at $x = y = .5$.
Parameter: α adjusts singularity of operator.

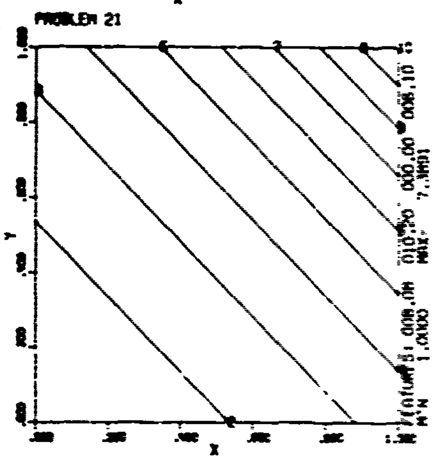


PROB 21 Artificial

$$Au_{xx} + Bu_{xy} + Cu_{yy} = f, \quad A = C = 1 + T^2, \quad B = -2T^2$$

DOMAIN unit square
BC $u = g$
TRUE e^{x+y}

Operator: Entire, has mixed derivative term.
Right side: Entire
Boundary conditions: Dirichlet
Solution: T is entire
Parameter: None

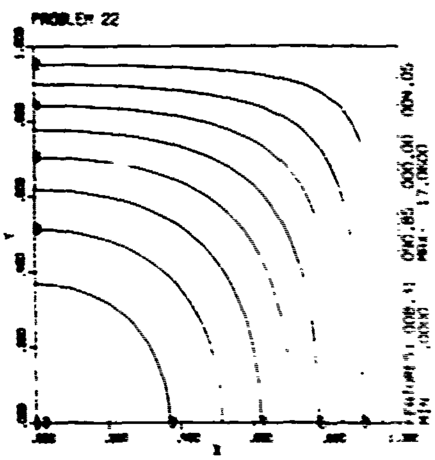


PROB 22 Elastic-plastic torsion [15]

$$w(u_{xx} + u_{yy}) + w_x u_x + w_y u_y = f, \quad w \text{ defined below}$$

DOMAIN unit square
BC $u = g$
TRUE $[17.06 + 3.62(x^2 + y^2)](x^2 - 1)(y^2 - 1)$

Operator: Expanded form of self-adjust problem. discontinuous coefficients. $w = 1/7996$ if $A \le .0025$
$w = 1/(236 + 19.4/A)$ if $A > .0025$ where $A = \sqrt{T_x^2 + T_y^2}$
Right side: Singular
Boundary conditions: Dirichlet
Solution: T is a quartic polynomial
Parameter: None



PROB 23 Nonlinear laminar, non-Newtonian flow [1]

$$w(u_{xx} + u_{yy}) + w_x u_y + w_y u_y = f, \quad \text{see below for } w$$

DOMAIN unit square

BC $u_x=0, x=0,1$; $u=2\cos(\pi x)$ for $y=0$; $u=\cos(\pi x)$ for $y=1$.

TRUE $(\varphi(y) + 1)\cos(\pi x)$ where $\varphi(y) = 1$ for $y \le .5 - \delta$, $= 0$ for $y \ge .5 + \delta$ and $\varphi(y)$ is a quintic polynomial for $.5 - \delta \le y \le .5 + \delta$ so φ has two continuous derivatives.

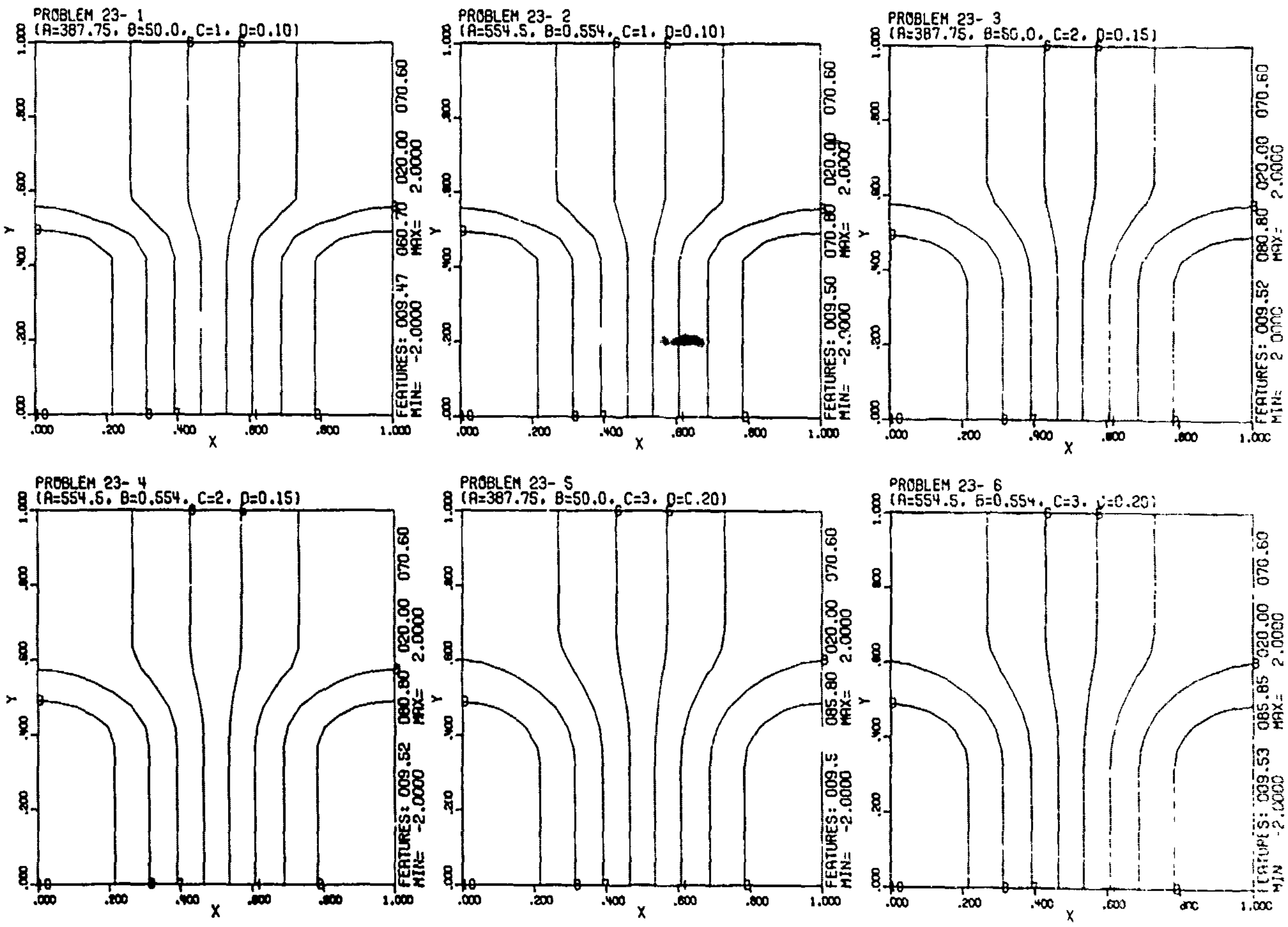
Operator: Expanded from self-adjoint problem, analytic.
Right side: Analytic
Boundary conditions: Mixed
Solution: Has jumps in third y-derivatives.
Parameters: Three cases for w given in terms of

$\sqrt{A = T_x^2 + T_y^2}$, $c = 1.$ $w = 1/(\alpha + \beta A)$

$c = 2.$ $w = e^{[A/(\alpha+\beta A)]}/A$

$c = 3.$ $w = \alpha\tanh(\beta A)/A$

Physical parameters α, β of $(387.75, 50)$ and $(554.5, .544)$ have been used in practice.



PROB 24 Friction in a brake shoe [3]

$$u_{xx} + u_{yy} + \frac{3h_x}{h} u_x = \gamma h_x/h^3, \quad h = \sin(\alpha\pi xy)$$

DOMAIN $[.1,1] \times [.1,1]$

BC $\beta u + u_x + u_y = 0$

TRUE Unknown

Operator: Laplacian plus u_x term which is possibly singular.

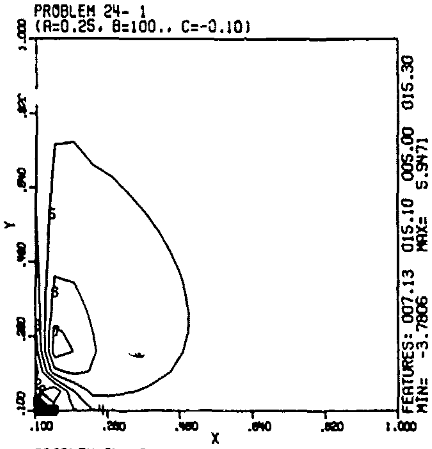
Right side: Analytic, possibly nearly singular.

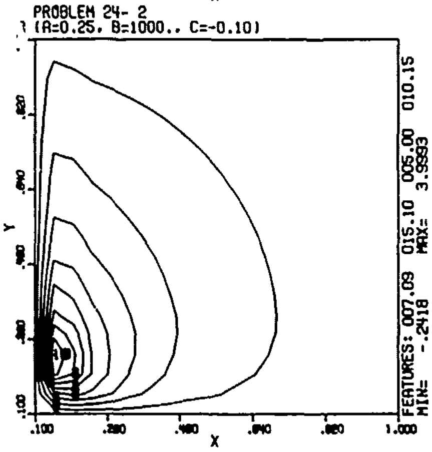
Boundary conditions: Mixed, homogeneous.

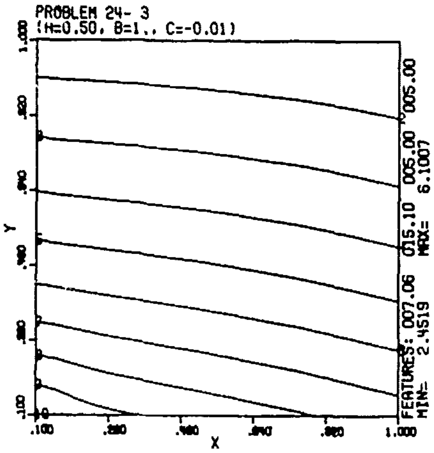
Solution: Approximate solutions given for 8 cases.

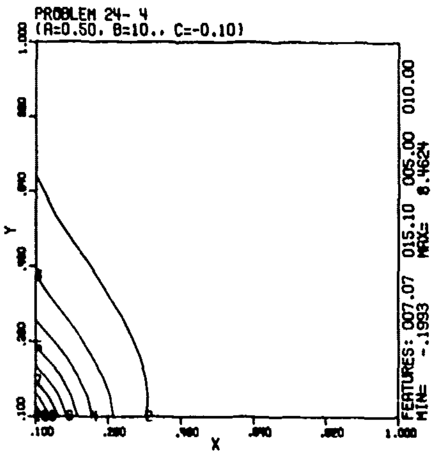
Parameters: α, β and γ are physical parameters,

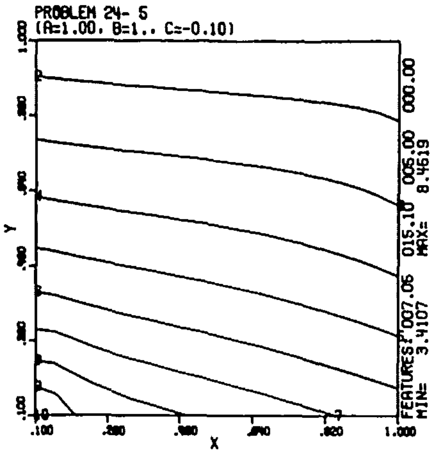
1. $\alpha = .25$ $\beta = 100$ $\gamma = -.1$
2. $\alpha = .25$ $\beta = 1000$ $\gamma = .1$
3. $\alpha = .5$ $\beta = 1$ $\gamma = -.01$
4. $\alpha = .5$ $\beta = 10$ $\gamma = -.1$
5. $\alpha = 1$ $\beta = 1$ $\gamma = -.1$
6. $\alpha = 1$ $\beta = 10$ $\gamma = -.1$
7. $\alpha = 1$ $\beta = 100$ $\gamma = -1.$
8. $\alpha = 1$ $\beta = 1000$ $\gamma = -1.$

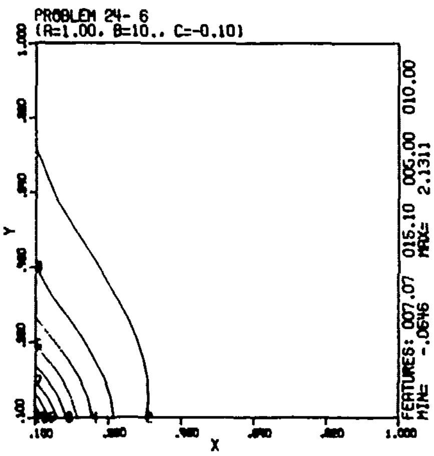


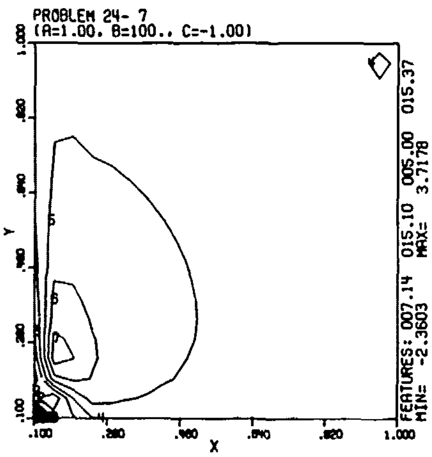


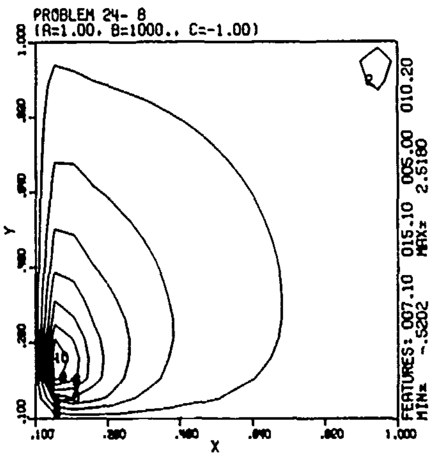










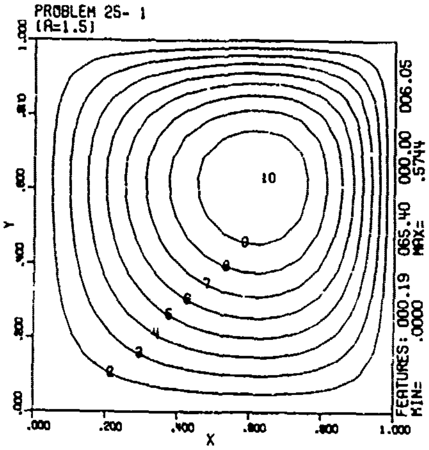


PROB 25 Artificial

$$-x^{\alpha}u_{xx} - y^{\alpha}u_{yy} - \alpha x^{\alpha-1}u_x - \alpha y^{\alpha-1}u_y + (xy)^{\alpha}u = f$$

DOMAIN unit square
BC $u = 0$
TRUE $3e^{x+y}(x - x^2)(y - y^2)$

Operator: Variable smoothness, expanded self-adjoint
Right side: Variable smoothness
Boundary conditions: Dirichlet, homogeneous
Solution: Entire, does not depend on parameter α
Parameter: α affects smoothness of operator and right side without affecting solution.

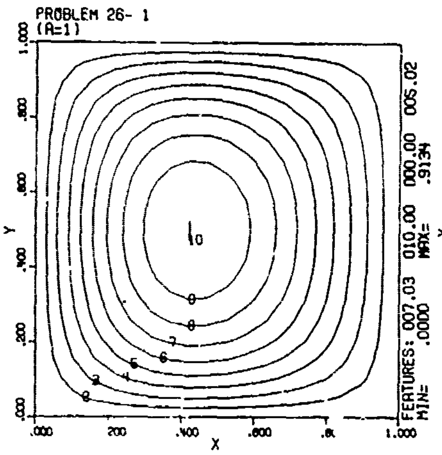


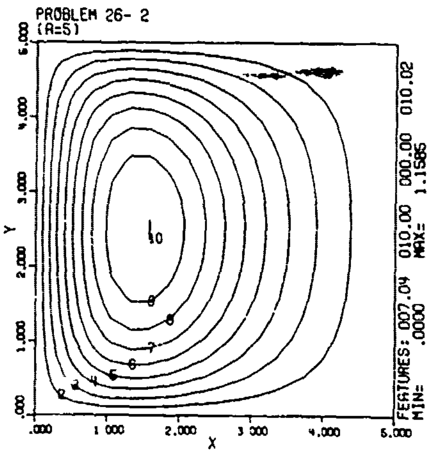
PROB 26 Viscous flow [3]

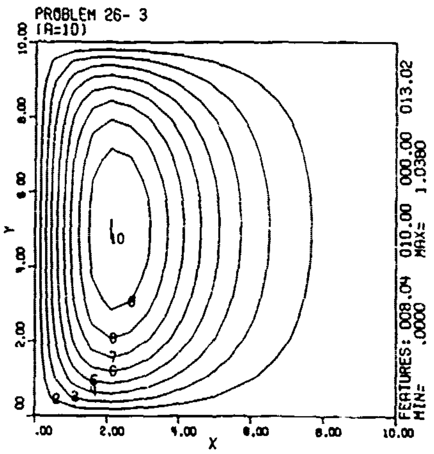
$$u_{xx}+u_{yy}+Au_x=-60\alpha x/B \text{ where } B=(\alpha+x^2)^3,\ A=6x(1+x^2)^2/B$$

DOMAIN $[0,\alpha] \times [0,\alpha]$
BC $u = 0$
TRUE unknown

Operator: Laplacian plus u_x term. For $\alpha = 1$ it is expansion of a self-adjoint operator.
Right side: Analytic
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solutions found for $\alpha = 1,5$ and 10.
Parameter: α is a physical parameter adjusting the domain and entering the coefficients.





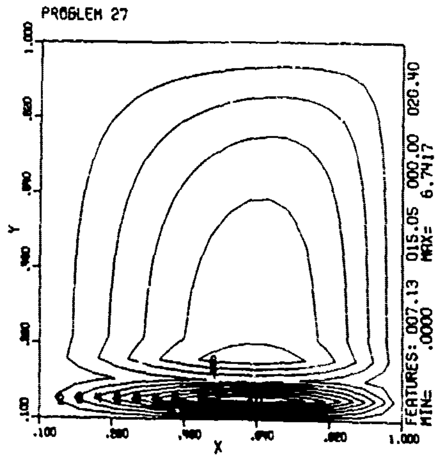


PROB 27 Distribution of diffused particles [3]

$$u_{xx} + \frac{2}{x}u_x + \frac{1}{x^2}u_{yy} + \frac{1}{x^2}(\cot y)^3 u_y = -100$$

DOMAIN $[.1,1] \times [.1,1]$
BC $u = 0$
TRUE unknown

Operator: Nearly singular, analytic
Right side: Constant
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solution given.
Parameter: None



PROB 28 Artificial

$$(wu_x)_x + (wu_y)_y = 1 \quad \text{where} \quad w = \alpha \quad \text{if} \quad 0 \leq x,y \leq .5$$
$$= 1 \quad \text{otherwise}$$

DOMAIN $[-1,1] \times [-1,1]$

BC $u = 0$

TRUE unknown

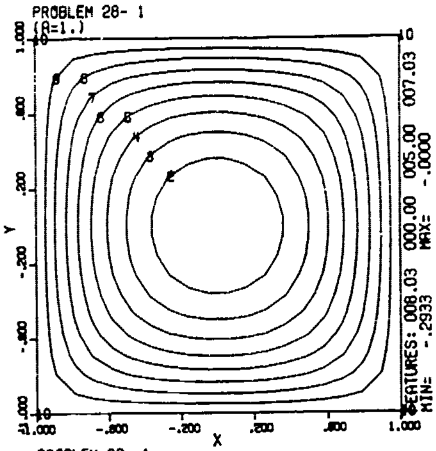
Operator: Self-adjoint, discontinuous coefficients.

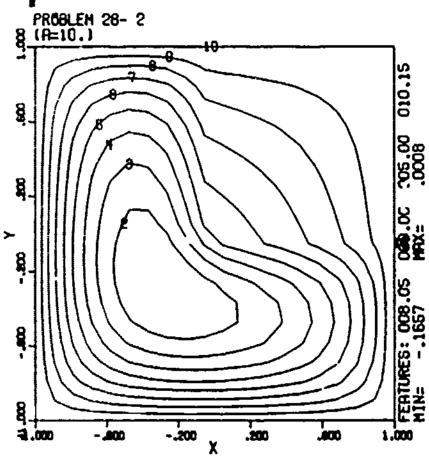
Right side: Constant

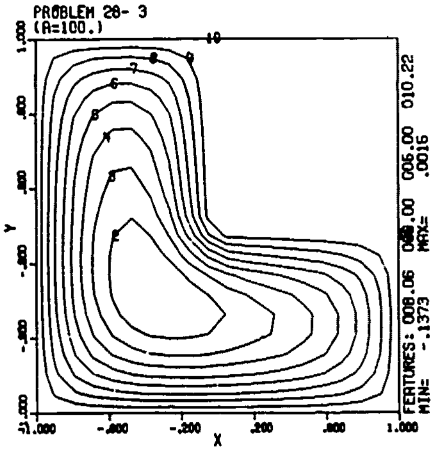
Boundary conditions: Dirichlet, homogeneous

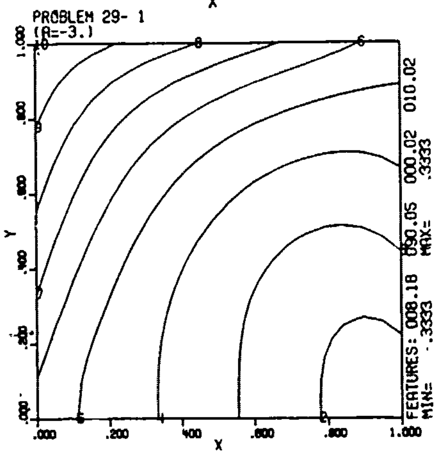
Solution: Approximate solutions given for $\alpha = 1, 10, 100$. Strong wave fronts for $\alpha >> 1$.

Parameter: α adjusts size of discontinuity in operator coefficients which introduces large, sharp jumps in solution.









PROB 29 Many physical interpretations [10]

$$u_{xx} + u_{yy} + \frac{\alpha}{y} u_y = 0$$

DOMAIN unit square

BC $u = (x - y)/\alpha$

TRUE unknown

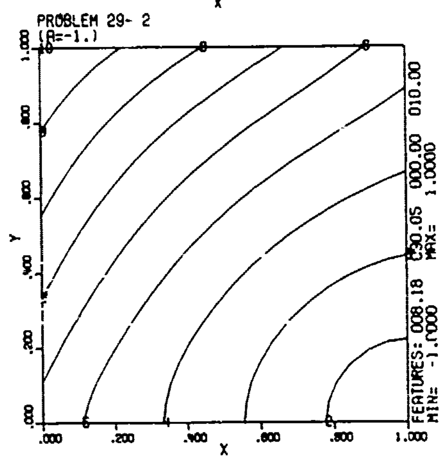
Operator: Laplace plus singular u_y term.

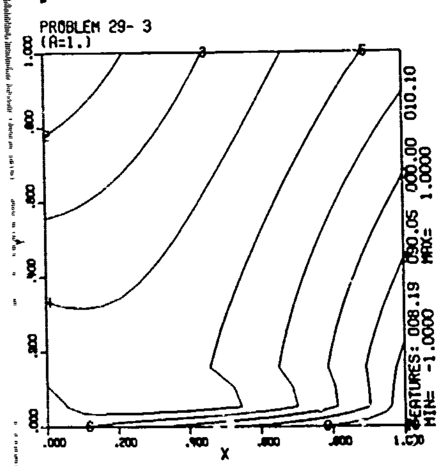
Right side: Homogeneous

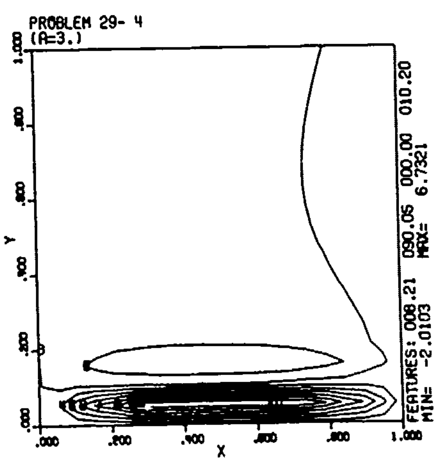
Boundary conditions: Dirichlet

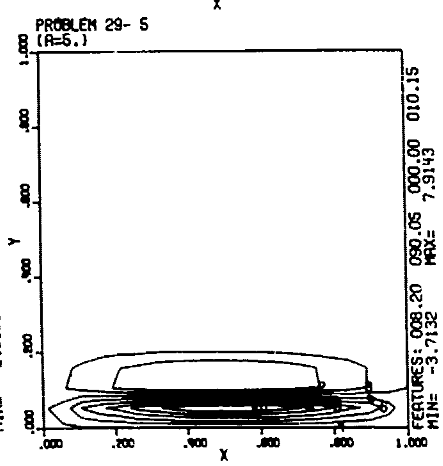
Solution: Five approximate solutions given, some are difficult.

Parameter: α changes physical application: $\alpha = 1$, potentials; $\alpha = -1$ streamlines; $\alpha = 3$, torsion and $\alpha = -3$ or 5, stresses.









PROB 30 Artificial

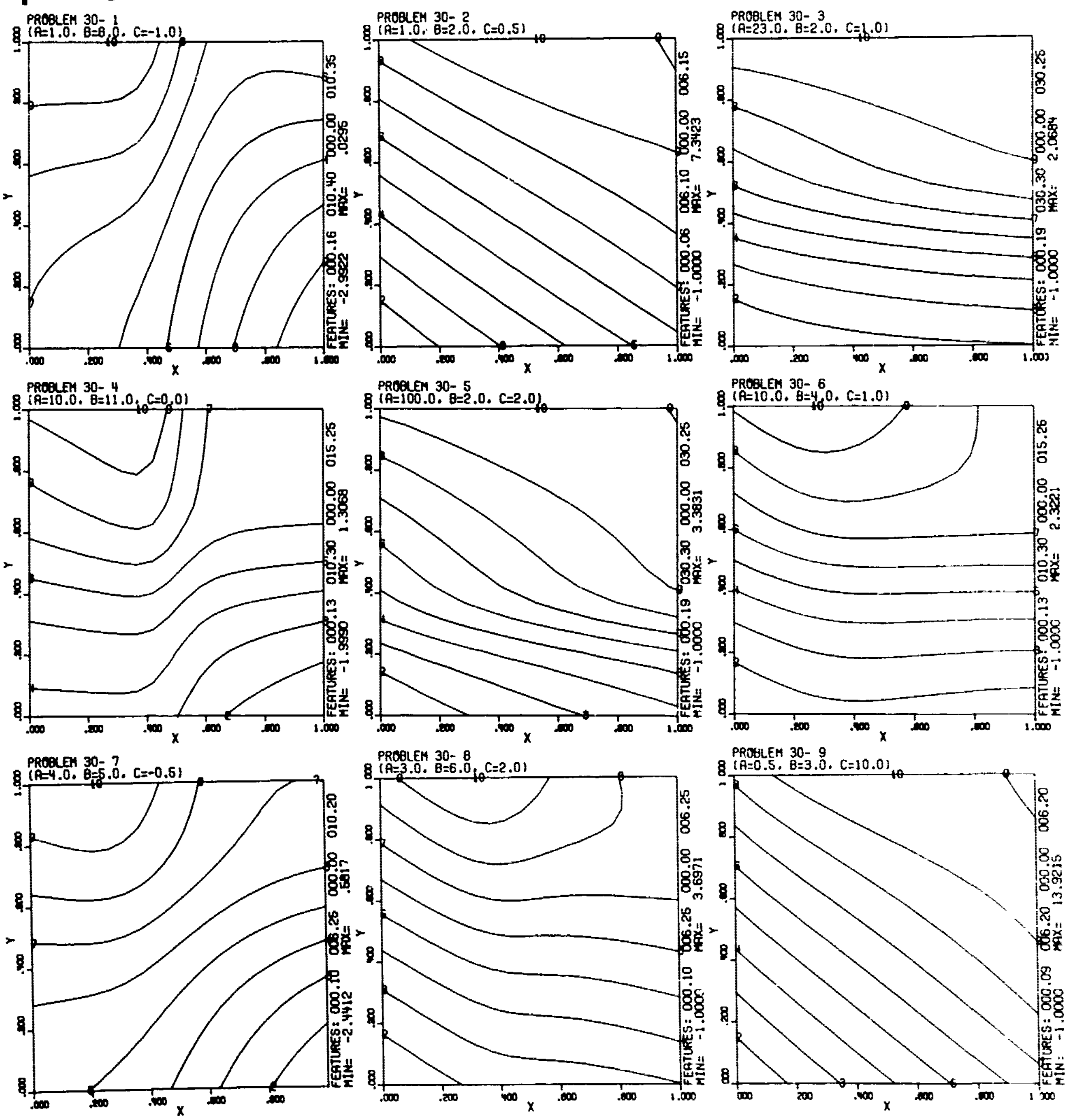
$$[2+(y-1)e^{-\alpha y^4}]u_{xx}+[1+\frac{1}{1+(2x)^{\beta}}]u_{yy}+\gamma[x(x-1)+(y-.3)(6-.7)]u=f$$

DOMAIN unit square

BC $u = g$

TRUE $$\frac{x+y^2}{1+(2x)^{\beta-1}} + (y-1)(1+x)e^{-\alpha y^4} + \gamma(x+y)\cos(xy)$$

Operator: Coefficients may be widely varying, singular.
Right side: Complicated behavior
Boundary conditions: Dirichlet
Solution: Complicated behavior, with wave fronts. etc.
Parameters: α, β, γ adjust the contribution of 3 independent complexities of the problem.



PROB 31 Temperature distribution [5]

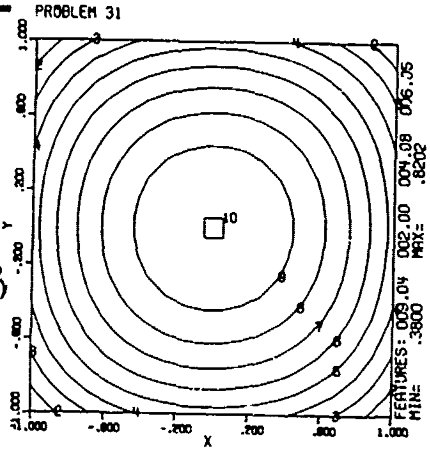
$$u_{xx} + u_{yy} = -1$$

DOMAIN $[-1,1] \times [-1,1]$

BC $u + u_N = g$

TRUE $-(x^2+y^2)/4 + .821564 + .01440(x^4-6x^2y^2+y^4)$
$+ .0000493(x^8-28x^6y^2+70x^4y^4-28x^2y^6+y^8)$
$-.00000064(x^{12}-66x^{10}y^2+495x^8y^4-924x^6y^6+495x^4y^8-66x^2y^{10}+y^{12})$

Operator: Laplace
Right side: Constant
Boundary conditions: Mixed
Solution: Harmonic poly. expansion for homo. BC.
Parameter: None



PROB 32 Stress in helical spring [5]

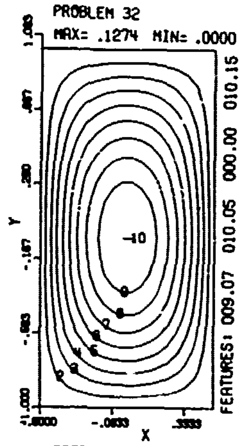
$$u_{xx} + u_{yy} + \frac{3}{5-y} u_y = f$$

DOMAIN $[-.5,.5] \times [-1,1]$

BC $u = 0$

TRUE $(1-y^2)(1-4x^2)(5-y^3)(.0004838y + .0010185)$

Operator: Analytic
Right side: Analytic
Boundary conditions: Dirichlet, homogeneous
Solution: Polynomial obtained by Ritz method for a physical problem.
Parameter: The 5 in the operator is a value of a physical parameter.



PROB 33 Torsion on a shaft [5]

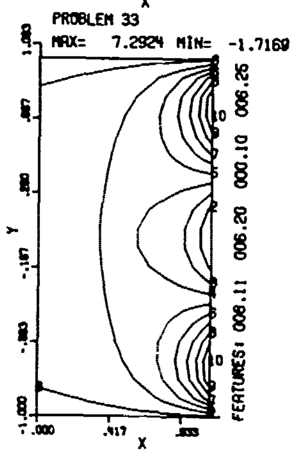
$$u_{xx} + u_{yy} = f$$

DOMAIN $[0,1] \times [-1,1]$

BC $u = g$

TRUE $p = 14 + \sqrt{133}$, $q = 14 - \sqrt{133}$, $r = (7-q)/(r\sqrt{133})$,
$t(y) = 1-y^2$, $C(x) = e^{\sqrt{px}} - e^{\sqrt{qx}}$, $B(x) = (7-p)r/16C(x)$,
$A(x) = rC(x) + e^{\sqrt{qx}}$, TRUE = $t(y)[A(x) + t(y)B(x)]$

Operator: Laplace
Right side: Entire
Boundary conditions: Dirichlet
Solution: Entire



PROB 34 From infinite region problem [5]

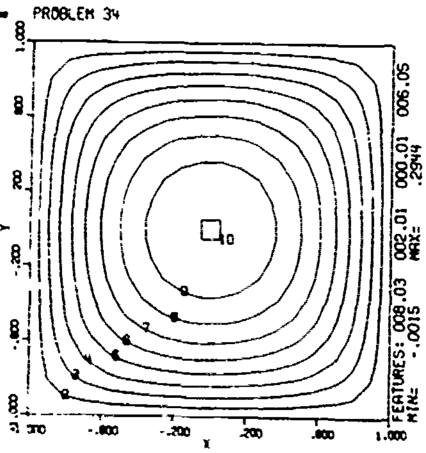
$$u_{xx} + u_{yy} = -1$$

DOMAIN $[-1,1] \times [-1,1]$

BC $u = g$

TRUE $.295776 - (x^2+y^2)/4 - 14476(x^4-6x^2y^2+y^4)/319424$
$+ 429(x^8-28x^6y^2+70x^4y^4-28x^2y^6+y^8)/319424$

Operators: Laplace
Right side: Constant
Boundary conditions: Dirichlet
Solution: Harmonic polynomial expansion for homogeneous boundary conditions.
Parameter: None



PROB 35 Torsion for a beam [5]

$$u_{xx} + u_{yy} = 0$$

DOMAIN $[-1,1] \times [-1,1]$

BC $u = g$ for $y = \pm 1$, $(1+\alpha)u + \alpha u_N = g$ for $x = \pm 1$

TRUE $1.1786 - .1801p + (.006)q$

$p(x,y) = x^4 - 6x^2y^2 + y^4$, $q(x,y) = x^8 - 28x^6y^2 + 70x^4y^4 - 28x^2y^6 + y^8$

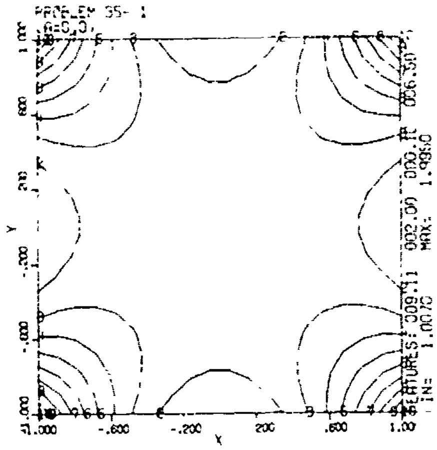
Operator: Laplace, homogeneous

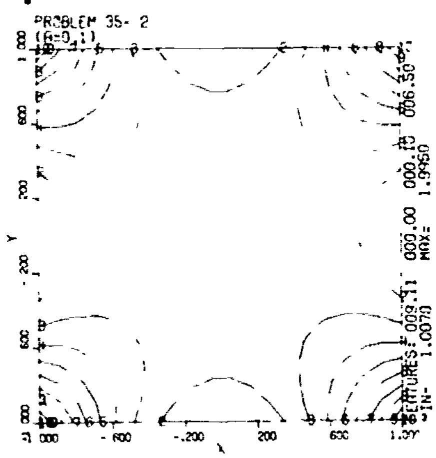
Right side: Zero

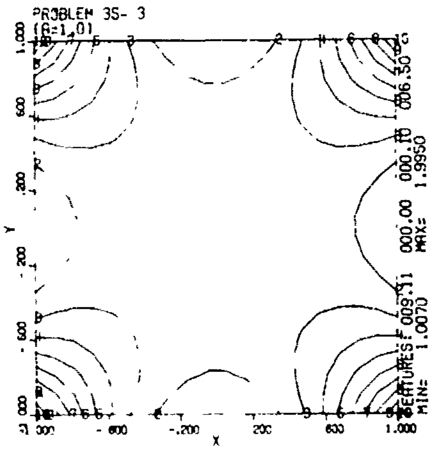
Boundary conditions: Mixed, Dirichlet for $\alpha = 0$.

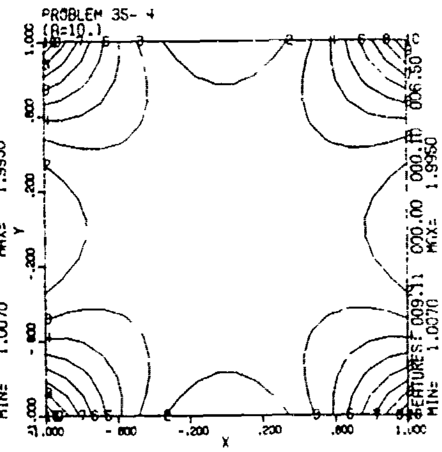
Solution: Harmonic polynomial combination.

Parameter: α adjusts contribution of mixed boundary condition; $\alpha = 0$ is the physical problem.









PROB 36 Adapted from Problem 27

$$(1+\beta)u_{xx} + \frac{2}{x+\alpha}u_x + \frac{1}{(x+\alpha)^2}u_{yy} + \frac{\cot y}{x+\alpha}u_y = f$$

DOMAIN unit square

BC $u = g$

TRUE $(1-\beta)e^{x+y} + \beta \log_e(x+\alpha)$

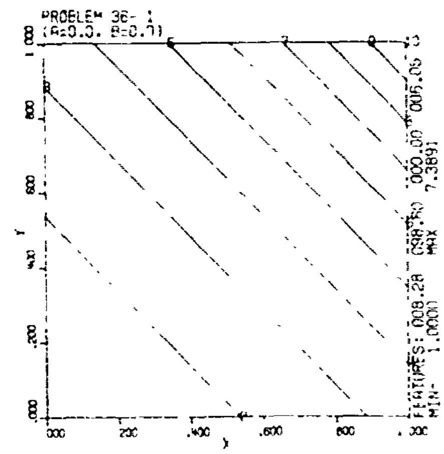
Operator: Possibly singular coefficients for $\alpha = 0$.

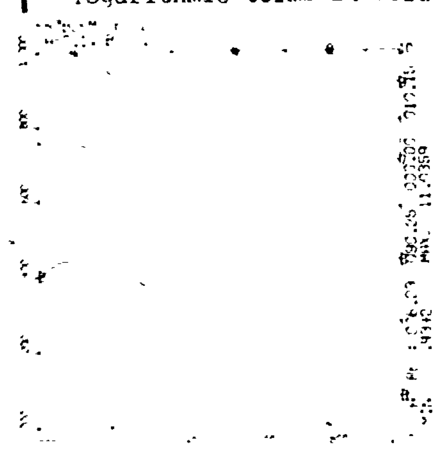
Right side: Analytic except for $\alpha = 0$; then singular.

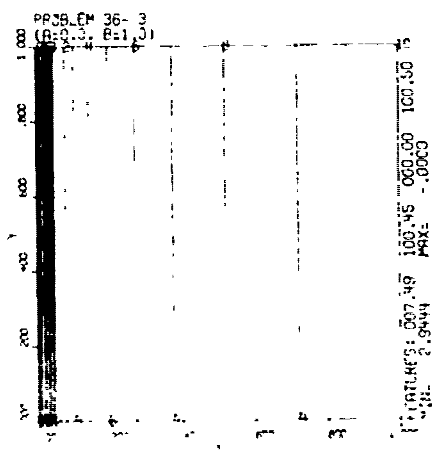
Boundary conditions: Dirichlet

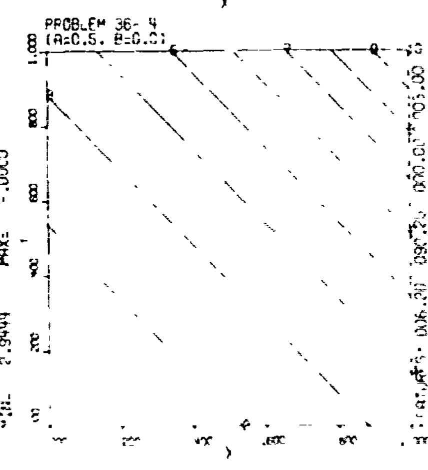
Solution: Logarithmic singularity for $\alpha = 0$.

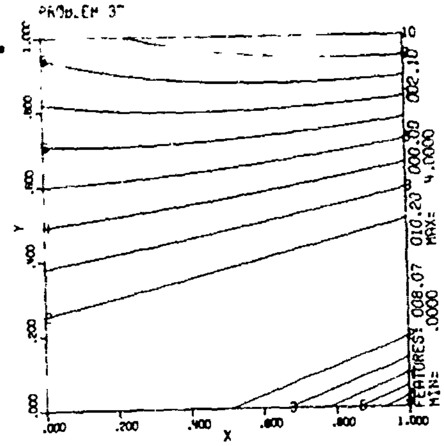
Parameters: α adjusts distance of singularity from domain, β adjusts relative size of exponential and logarithmic terms in solution.











PROB 37 From nonlinear minimal surface [1]

$$Au_{xx} + Bu_{xy} + Cu_{yy} = f,\ A = (1+T_y)^2,\ B = -2T_xT_y,\ C = (1+T_x)^2$$

DOMAIN unit square

BC $u = g$

TRUE $T(x,y) = (x - 3y)^2 e^{x-y}$

Operator: Includes cross derivative.
Right side: Entire
Boundary conditions: Dirichlet
Solution: Entire
Parameter: None

PROB 38 Electrostatics [11]

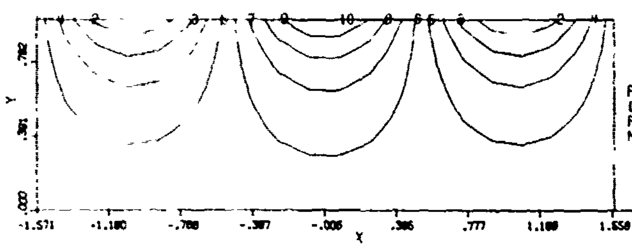
$$u_{xx} + u_{yy} = 0$$

DOMAIN $[-\pi/2, \pi/2] \times [0,1]$

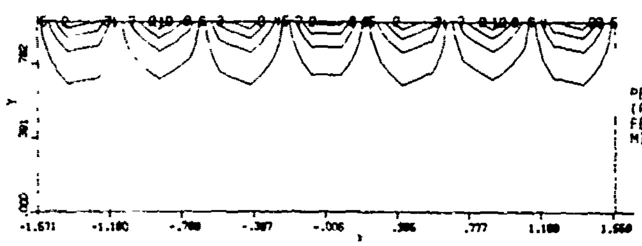
BC $u = g$ for $x = \pm\pi/2$, $y = 1$; $u_y = g$ for $y = 0$

TRUE $e^{-\sqrt{2\alpha+1}} \cos[(2\alpha+1)x]\sinh[(2\alpha+1)y]/(2\alpha+1)$

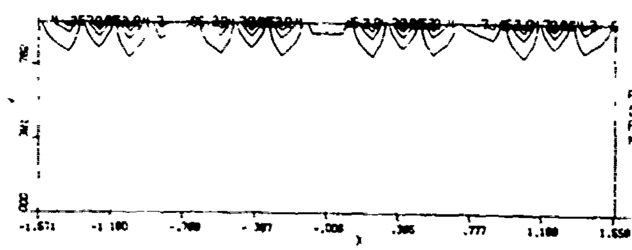
Operator: Laplace, homogeneous
Right side: Zero
Boundary conditions: Mixed
Solution: Entire, may be oscillatory.
Parameter: α adjusts the oscillations.



PROBLEM 38- 1
(A=1.0)
FEATURES: 000.09 000.00 000.20 008.25
MIN= -.5888 MAX= .5727



PROBLEM 38- 2
(A=3.0)
FEATURES: 000.13 000.00 000.30 010.40
MIN= -5.3877 MAX= 5.5388



PROBLEM 38- 3
(A=7.0)
FEATURES: 000.26 000.00 000.60 015.80
MIN= -2075.2416 MAX= 2258.3685

PROB 39 From nonlinear problem [4]

$$u_{xx} + u_{yy} + [1-h(x)^2 w(x,y)^2]/\beta u = 0$$

DOMAIN unit square

BC $u = 1$

TRUE unknown

Operator: Helmholtz type, homogeneous

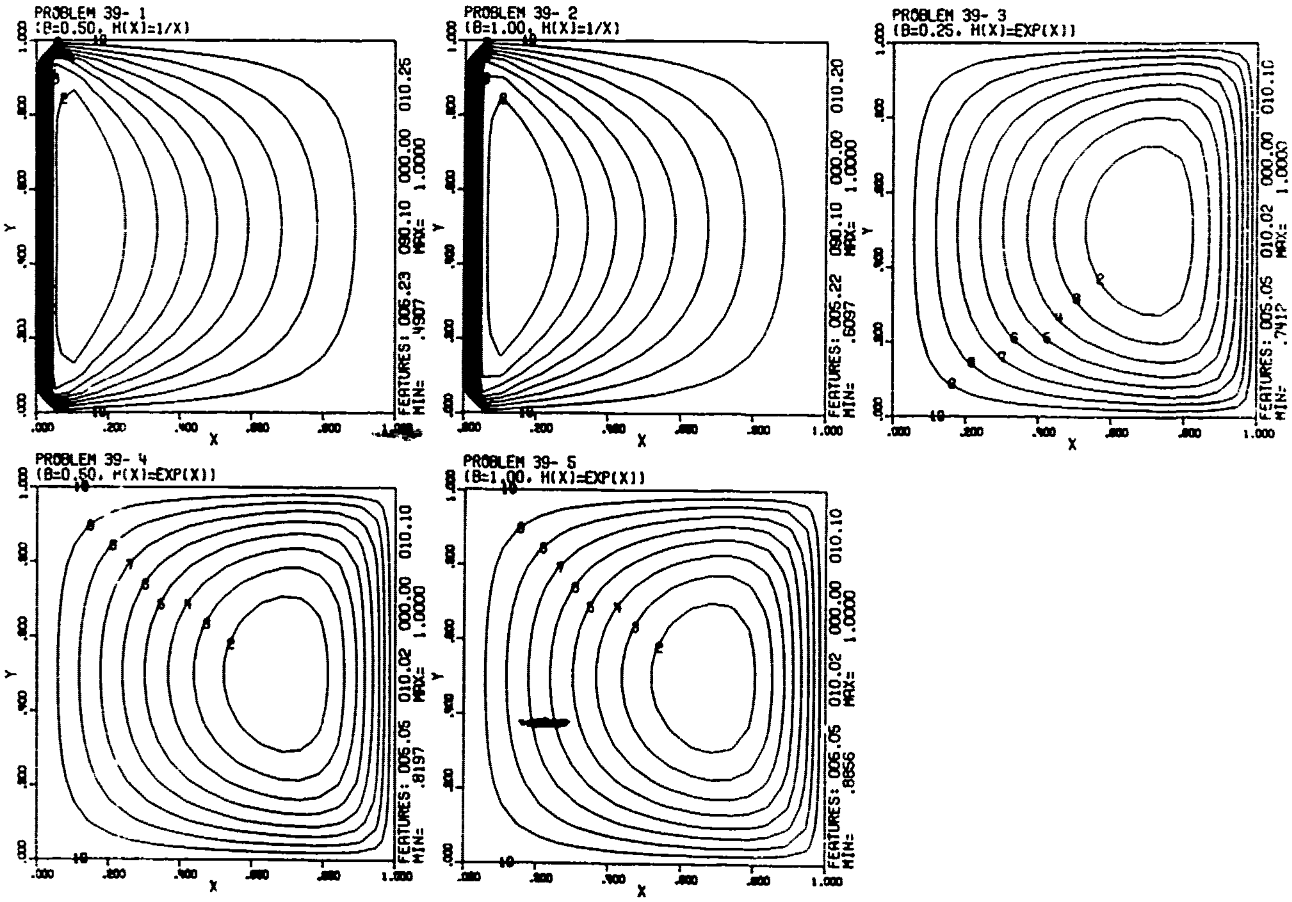
Right side: Zero

Boundary conditions: Dirichlet, constant

Solution: Approximate solution $w(x,y)$ calculated and tabulated for 5 cases.

Parameters: $h(x) = 1/x$ for $\beta = .5,1$ (Cases 1 and 2)

$h(x) = e^x$ for $\beta = .25,.5,1$ (Cases 3, 4 and 5)



PROB 40 Hadamard's example [17]

$$u_{xx} + (1+x^2)u_{yy} - yu_x = f$$

DOMAIN unit square

BC $u=g$ for $y=0$ or 1, $\alpha u + \beta u_x = g$ for $x=0$ or 1.

TRUE $\log_{10}[(x+1)/(y+1)] + e^{2(x+y)}/(2+x-y)-2$

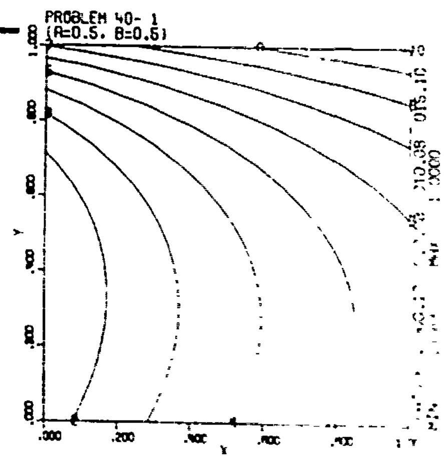
Operator: Entire

Right side: Analytic

Boundary conditions: Mixed

Solution: Analytic

Parameters: α and β adjust the contributions to the mixed boundary condition on two sides.



PROB 41 Artificial [20]

$$u_{xx} + u_{yy} + \alpha u = f$$

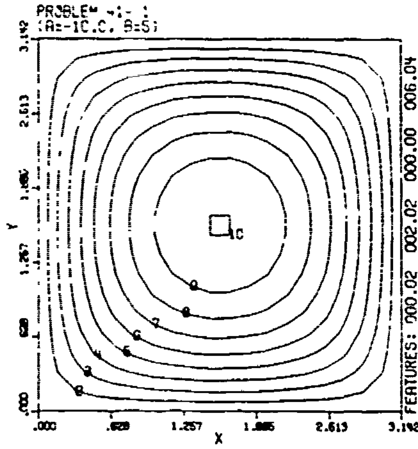
DOMAIN $[0,\pi] \times [0,\pi]$

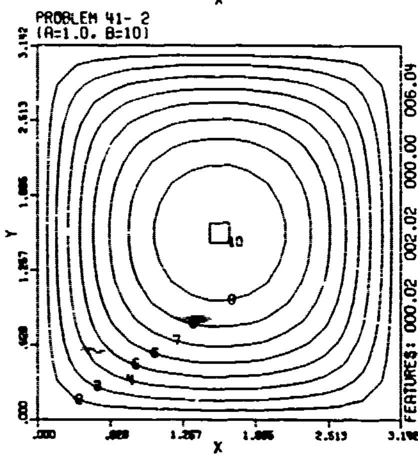
BC $u = 0$

TRUE approximate solution accuracy depends on β

$$\frac{x(\pi-x)}{2} - \frac{4}{\pi} \sum_{k=1}^{\beta} \frac{\sin[(2k-1)x]\cosh[(2k-1)(y-\pi/2)]}{(2k-1)^3\cosh[(2k-1)\pi/2]}$$

Operator: Helmholtz
Right side: Series for function with singularities.
Boundary conditions: Dirichlet, homogeneous.
Solution: Infinite series converging like $1/k^3$. The solution has derivative singularities.
Parameters: α adjust u term, possibly makes operator nearly singular. β is number of terms in series.





PROB 42 Artificial [20]

$$u_{xx} + u_{yy} + u_y - u = 0$$

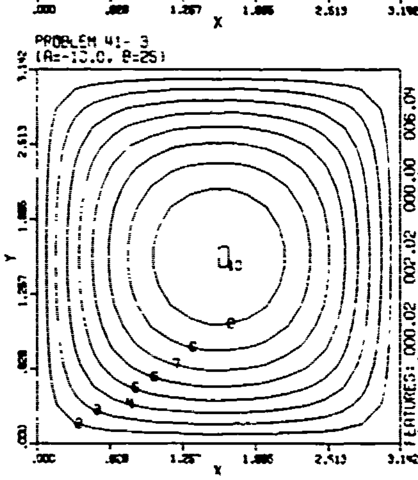
DOMAIN $[\alpha,\beta] \times [0,1]$

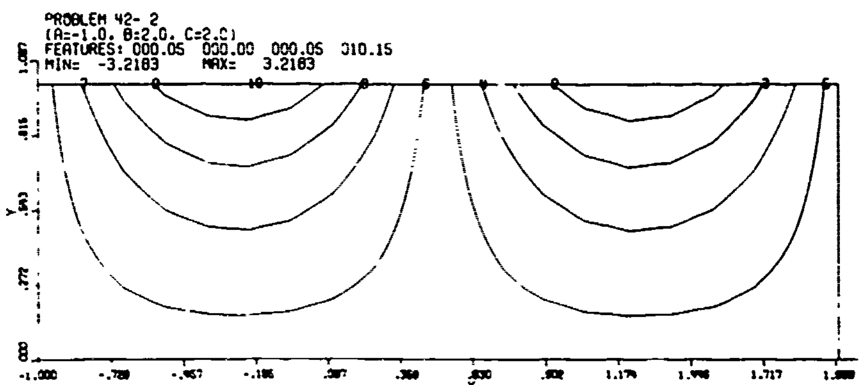
BC $u_N = g$

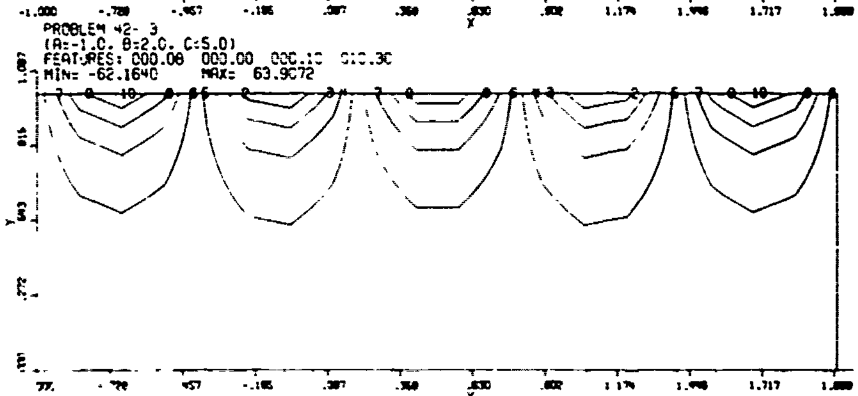
TRUE

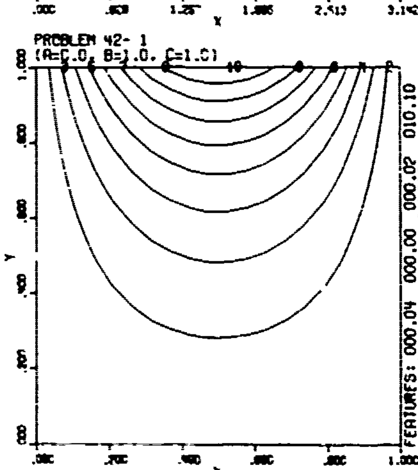
$$e^{-y/2} \sinh\sqrt{\frac{5}{4} + \frac{\gamma^2\pi^2}{(\beta-\alpha)^2}}\, y \sin\left[\frac{\gamma\pi(x-\alpha)}{(\beta-\alpha)}\right]$$

Operator: Constant coefficients, homogeneous.
Right side: Zero
Boundary conditions: Neumann (but PDE solution unique)
Solution: Analytic, can oscillate as γ increases.
Parameters: (α,β) for domain, γ adjusts oscillations.









PROB 43 Artificial [20]

$$u_{xx} + u_{yy} + u_x = 0$$

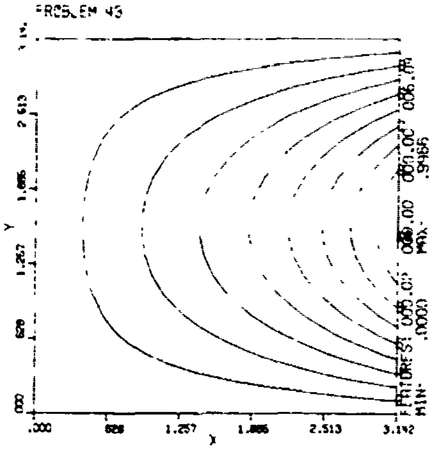
DOMAIN $[0,\pi] \times [0,\pi]$

BC $u=0$ for $x=0$, $y=0,\pi$; $u-u_y = \sqrt{2}\sin(y+\pi/4)$ for $x=\pi$

TRUE

$$\frac{e^{(\pi-x)/2}\sinh(\sqrt{5}x/2)\sin y}{\sinh(\sqrt{5}\pi/2)}$$

Operator: Constant coefficient, homogeneous.
Right side: Zero
Boundary conditions: Mixed
Solution: Entire
Parameters: None



PROB 44 From nonlinear problem [20]

$$u_{xx} + u_{yy} + wu = w$$

DOMAIN unit square

BC $u = 0$

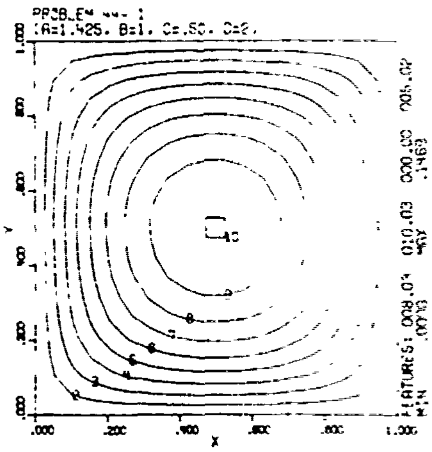
TRUE unknown

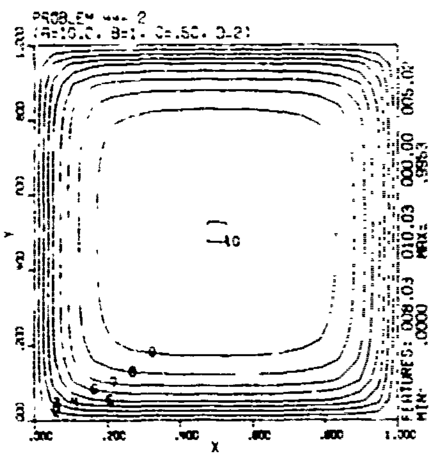
Operator: Helmholtz type
Right side: Complicated
Boundary conditions: Dirichlet, homogeneous
Solution: Approximate solution given for $r = r(x,y)$ tabulated from a solution to the nonlinear problem; r should be u,

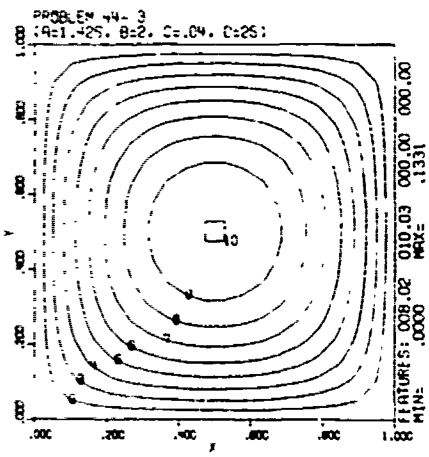
$$w(x,y) = -\alpha^2(1-r)^{\beta-1}e^{[\gamma\delta r/(1+\gamma r)]}.$$

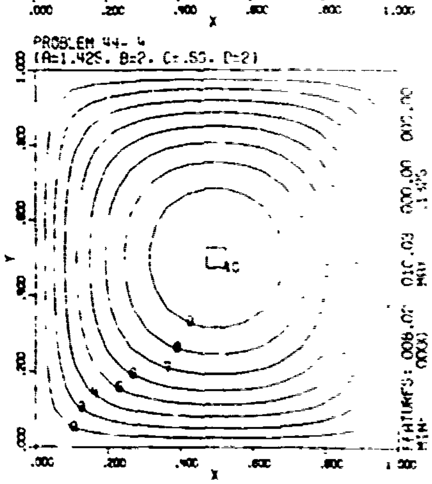
Parameters: α, β, γ and δ are physical parameters. Four cases are given:

1. $\alpha = 1.425$ $\beta = 1$ $\gamma = .5$ $\delta = 2$
2. $\alpha = 10$ $\beta = 1$ $\gamma = .5$ $\delta = 2$
3. $\alpha = 1.425$ $\beta = 2$ $\gamma = .04$ $\delta = 25$
4. $\alpha = 1.425$ $\beta = 2$ $\gamma = .5$ $\delta = 2$







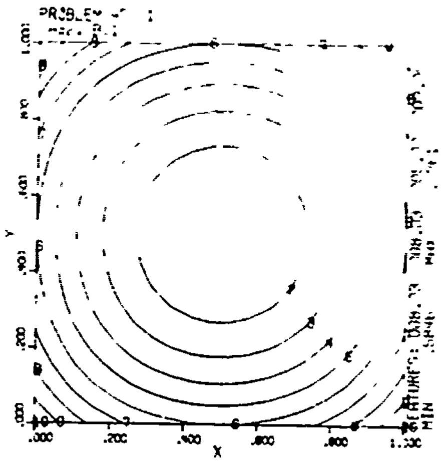


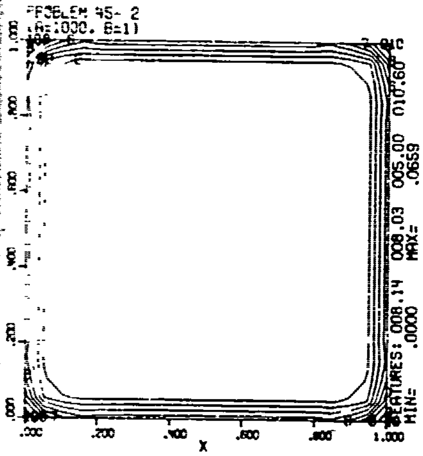
PROB 45 Nonlinear pth order reaction [20]

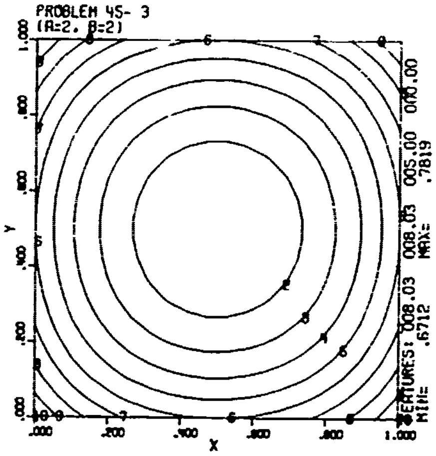
$$u_{xx} + u_{yy} - \alpha r^{\beta-1} u = 0$$

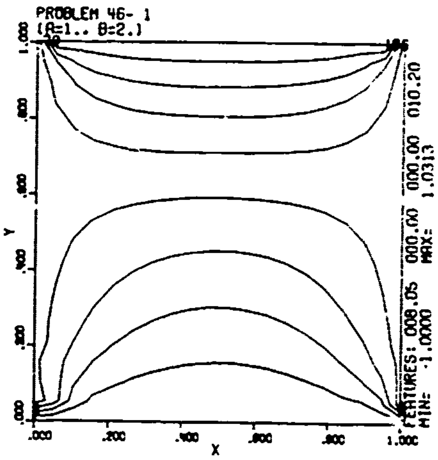
DOMAIN unit square
BC $u = g$
TRUE Unknown

Operator: Helmholtz type, homogeneous.
Right side: Zero
Boundary conditions: Dirichlet
Solution: Approximate solution given for $r = r(x,y)$ tabulated from a solution to the nonlinear problem; r should be u.
Parameters: α and β are physical parameters. Three cases are given: (1,2), (1,1000) and (2,2).







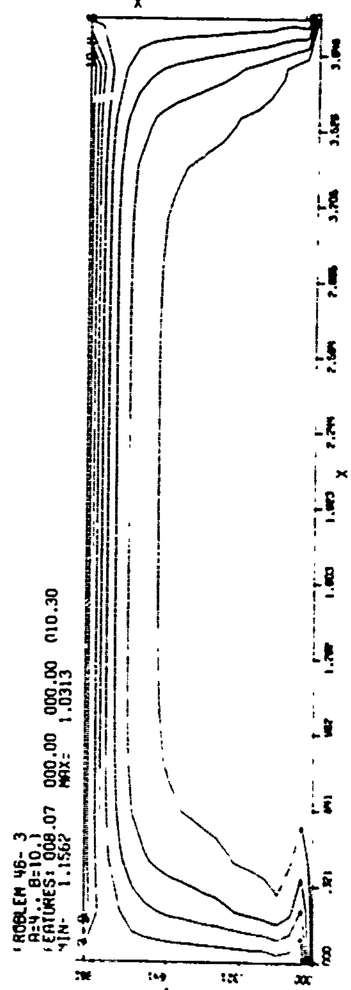


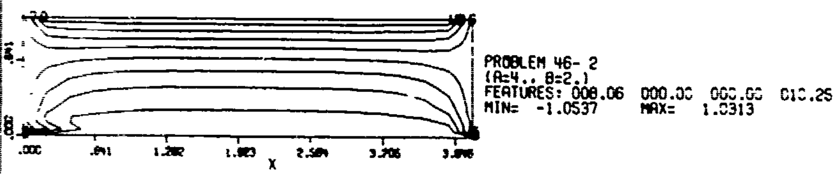
PROB 46 Magnetohydrodynamics [19]

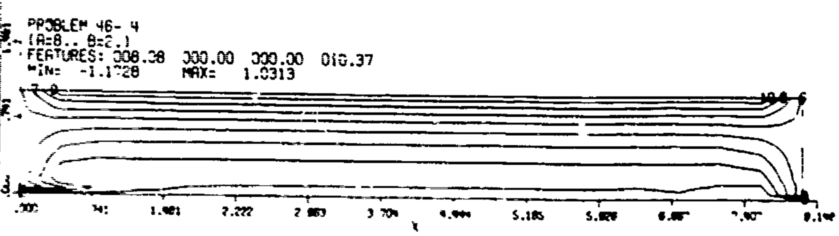
$$u_{xx} + u_{yy} - \beta u_y = 0$$

DOMAIN $[0,\alpha] \times [0,1]$
BC $u=0$ for $x=0,\alpha$; $u=1$ for $y=1$; $u=-1$ for $y=0$
TRUE Unknown

Operator: Constant coefficients, homogeneous.
Right side: Zero
Boundary conditions: Dirichlet
Solution: Approximate solution given for 4 cases: $\alpha = 1$ and $\beta = 2$, $\alpha = 4$ and $\beta = 2$, $\alpha = 4$ and $\beta = 10$, $\alpha = 8$ and $\beta = 2$.





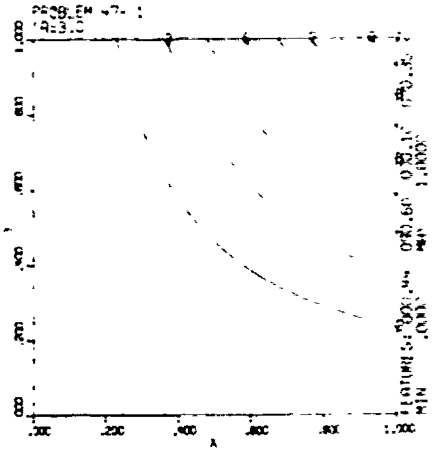


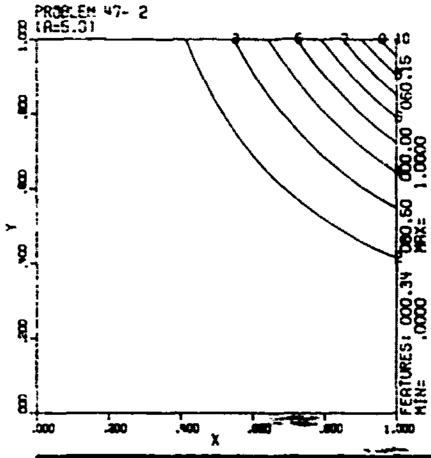
PROB 47 Artificial

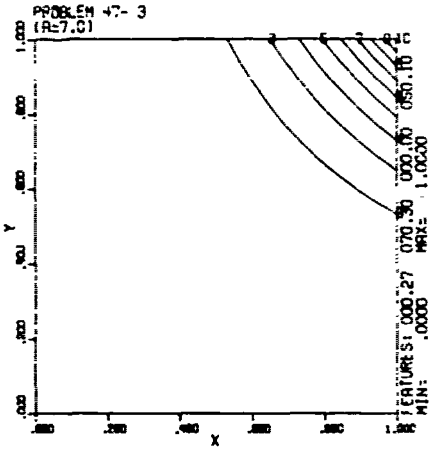
$$u_{xx} + u_{yy} = f$$

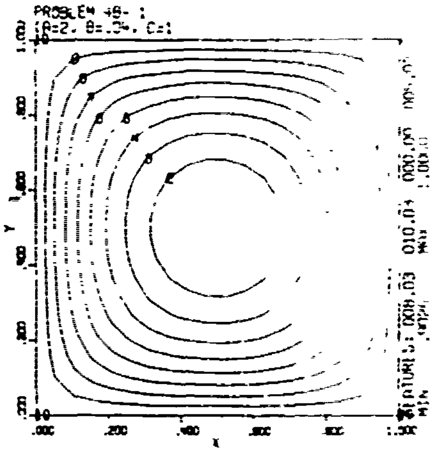
DOMAIN unit square
BC $u = g$
TRUE $(xy)^{\alpha/2}$

Operator: Laplace
Right side: Variable singularities
Boundary conditions: Dirichlet
Solution: Singularity of variable strength.
Parameter: α adjusts singularity strength.







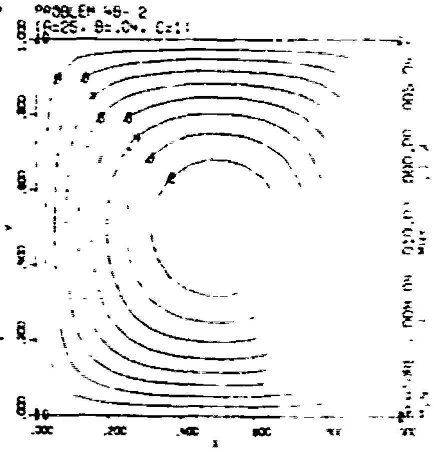


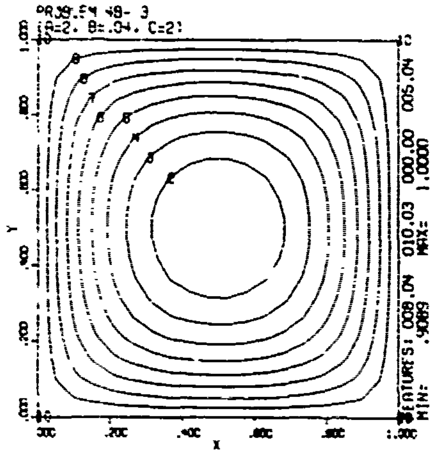
PROB 48 Nonlinear diffusion in catalysts [2]

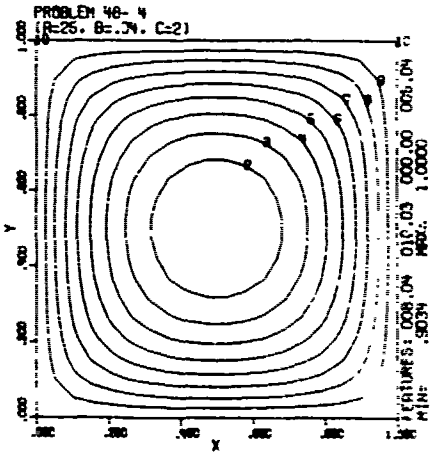
$$u_{xx} + u_{yy} - 1.425r^{(\gamma-1)}e^{(\alpha\beta(1-r))/(1+\beta(1-r))}u = 0$$

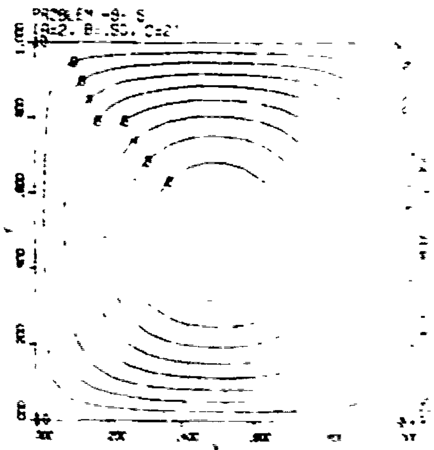
DOMAIN unit square
BC $u = 1$
TRUE Unknown

Operator: Helmholtz type, homogeneous
Right side: Zero
Boundary conditions: Dirichlet
Solution: Approximate solution given for $r = r(x,y)$ tabulated from a nonlinear PDE solver; r should be u.
Parameters: (α,β,γ) are physical parameters. 5 cases given: (1,.04,2), (1,.04,25), (2,.04,2), (2,.04,25) and (2,.5,2).









PROB 49 Nonlinear diffusion in catalysts [2]

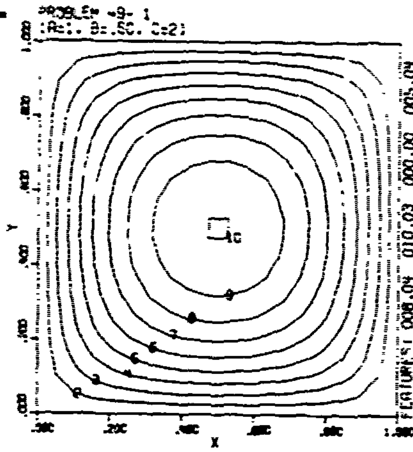
$$u_{xx} + u_{yy} \cdot u = f$$

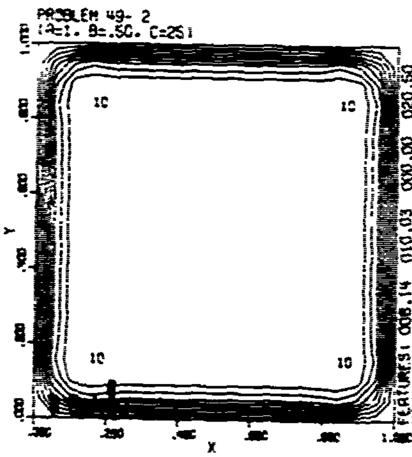
DOMAIN unit square
BC $u = 1$
TRUE Unknown

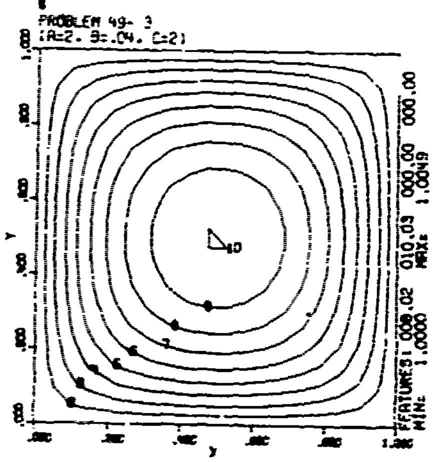
Operator: Helmholtz type
Right side: Complicated
Boundary conditions: Dirichlet
Solution: Approximate solution given for $r = r(x,y)$ tabulated from a solution to the nonlinear problem; r should be u,

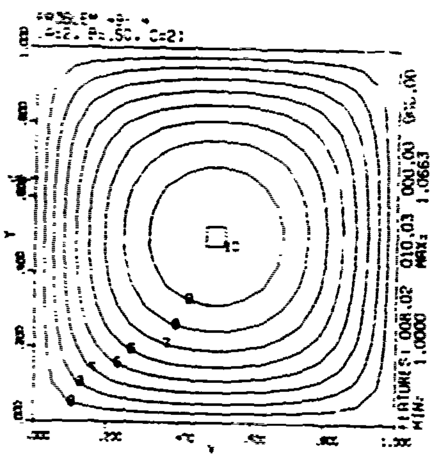
$$w(x,y) = -(1.425)^2 [(1+\beta-r)/\beta]^{\alpha-1} e^{\gamma(r-1)/r}$$

Parameters: (α,β,γ) are physical parameters. Four cases are given: (1,.5,2), (1,.5,25), (2,.04,2) and (2,.5,2).









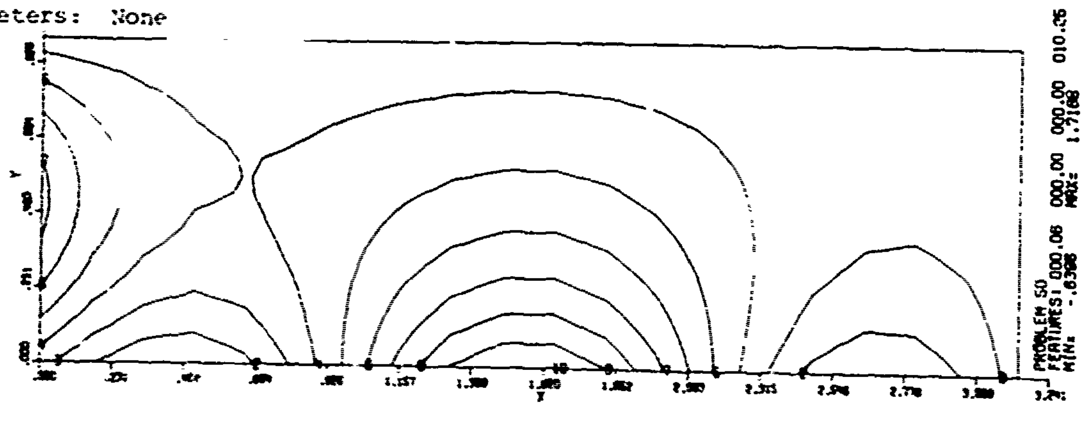
PROB 50 Artificial [20]

$$u_{xx} + u_{yy} = 0$$

DOMAIN $[0,\pi] \times [0,1]$
BC $u=3\sin(x)/4-\sin(3x)$, $y=0$; $u=0$, $x=\pi$, $y=1$; $u=\sin\pi y$, $x=0$
TRUE

$$\frac{3\sinh(1-y)\sin x}{4\sinh 1} - \frac{\sinh 3(1-y)\sin 3x}{\sinh 3} + \frac{\sinh\pi(1-x)\sin\pi y}{\sinh\pi^2}$$

Operator: Laplace, homogeneous
Right side: Zero
Boundary conditions: Dirichlet
Solution: Entire
Parameters: None



PROB 51

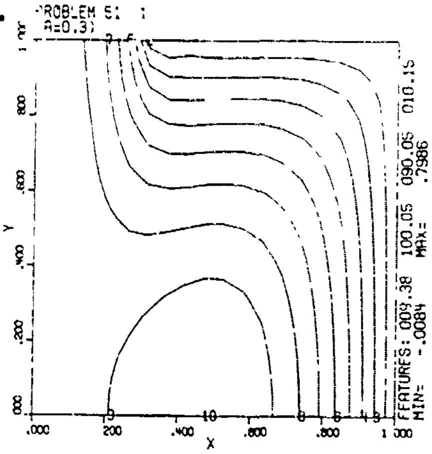
$$u_{xx} + \frac{1}{x} u_x + \frac{1}{x^2} u_{yy} = -10$$

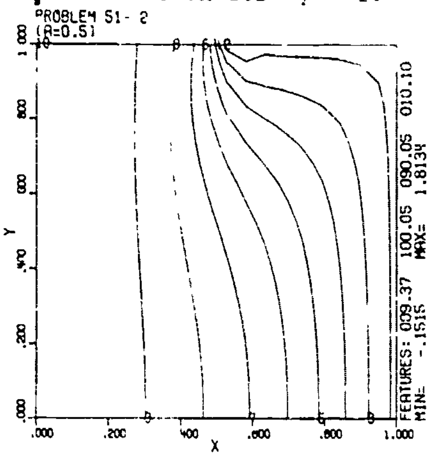
DOMAIN unit square
BC $u = 0$ for $x = 1$; $u_N = 0$ for $x, y = 0$; $Au_y + Bu = 0$ for $y = 1$
TRUE Unknown

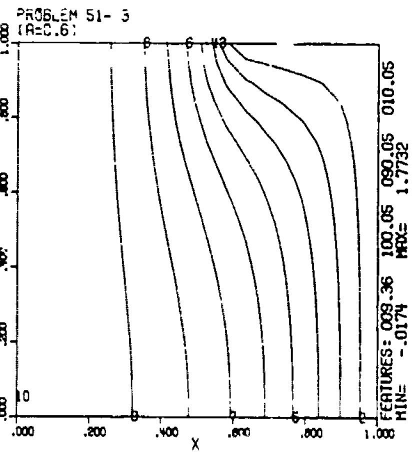
Operator: Singular coefficients
Right side: Constant
Boundary conditions: Mixed

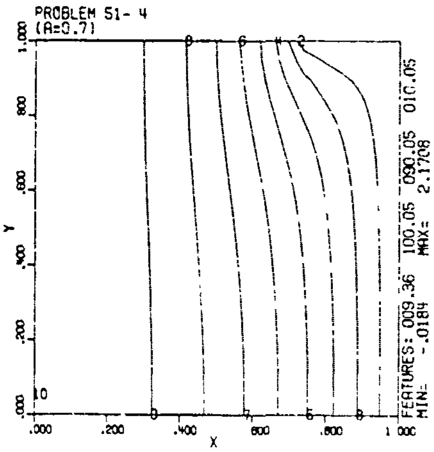
$$A(x) = \begin{cases} 0 & x > \alpha \\ 1 & x \le \alpha \end{cases} \qquad B(x) = \begin{cases} 1 & x > \alpha \\ 0 & x \le \alpha \end{cases}$$

Solution: Has singularity, unusual behavior.
Parameters: α adjusts position of change in boundary condition for $y = 1$.







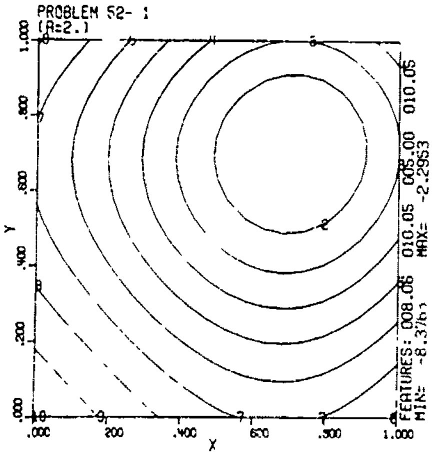


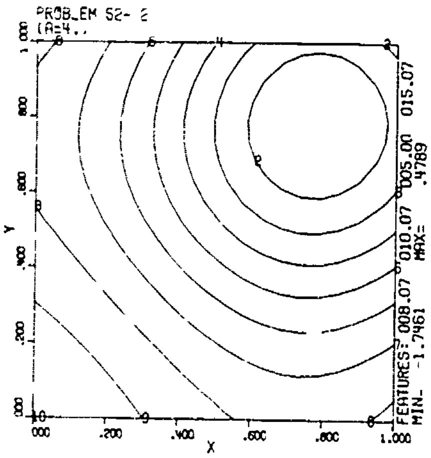
PROB 52 Nonlinear reaction [2]

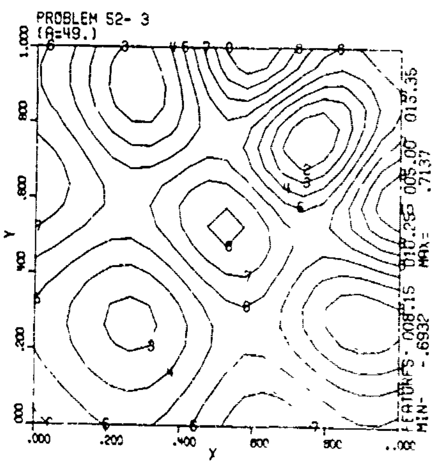
$$r(u_{xx} + u_{yy}) + r_1 u_x + r_2 u_y - \alpha u = 0$$

DOMAIN unit square
BC $u + u_N = 1$
TRUE Unknown

Operator: Expanded from self-adjoint PDE, homogeneous.
Right side: Zero
Boundary conditions: Mixed
Solution: Approximate solution for $r(x,y)$ tabulated from nonlinear PDE solver; r should be $11/(1 + 10u)$; r_1, r_2 are finite differences for r_x, r_y.
Parameter: α = physical parameter.





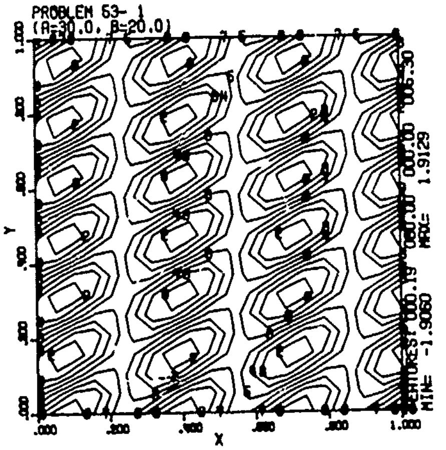


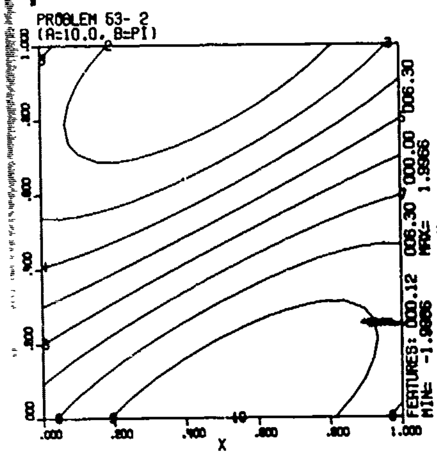
PROB 53 Artificial

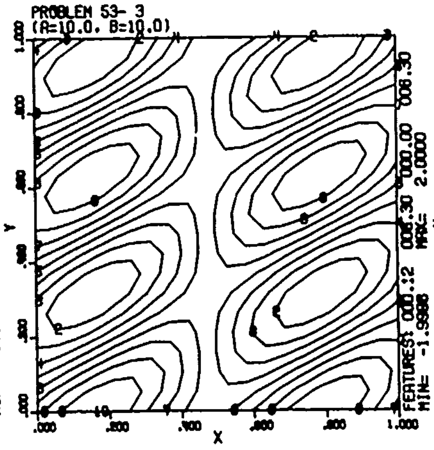
$$u_{xx} + u_{yy} - \alpha u = f$$

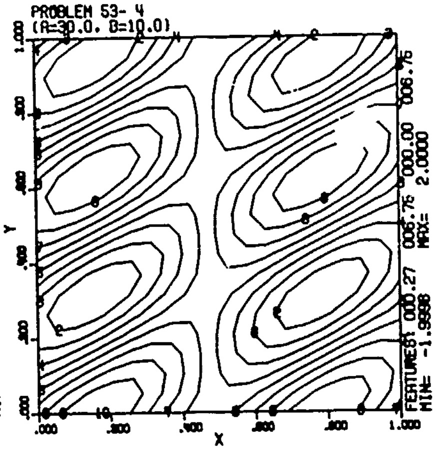
DOMAIN unit square
BC $u = g$
TRUE $\cos(\beta y)\sin(\beta(x-y))$

Operator: Helmholtz
Right side: Entire
Boundary conditions: Dirichlet
Solution: Entire, oscillatory
Parameters: α can make operator nearly singular. β adjusts the oscillations of the solution.









PROB 54 Artificial

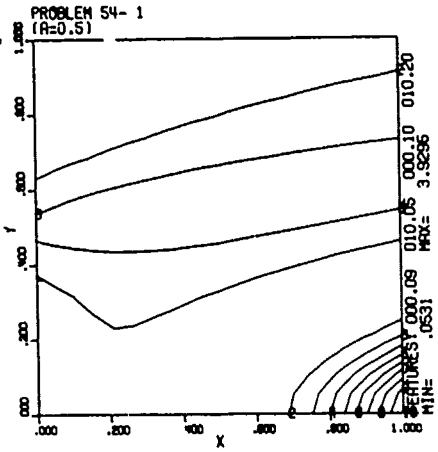
$$(1+x^2)u_{xx}+(1+A^2)u_{yy}+2xu_x+16yAu_y-(1+(8y-x-4)^2)u = f$$
$$A(y) = 4y^2 + \alpha$$

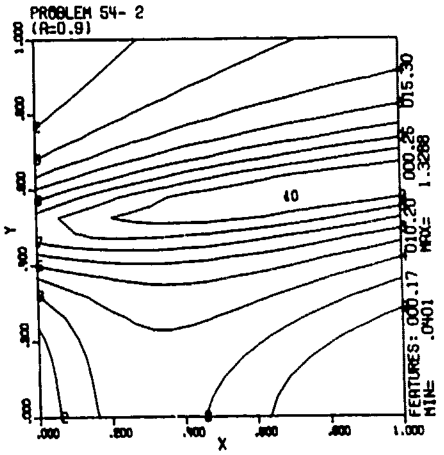
DOMAIN unit square
BC $u = g$
TRUE $B = \max[0,(3-x/A(y))^3]$, $C = \max[0,x-A(y)]$

$D = 0$ if $C < .02$, $D = e^{-B/C}$ if $C \geq .02$

$$u(x,y) = 2.25x(x-A(y))^2(1-D)/(4A(y)^3)+1/(1+(8y-x-4)^2)$$

Operator: Expanded form of self-adjoint operator. Analytic.
Right side: Complicated with possible wild behavior.
Boundary conditions: Dirichlet
Solution: Wildly behaving for α possible, has singularities for $x - 4y^2 = \alpha$ or $4y^2 = -\alpha$.





PROB 55 Conducting fluid in magnetic field [?]

$$u_{xx} + u_{yy} = 0$$

DOMAIN $[0,6] \times [0,1]$

BC $u_x=0$ for $x=0$; $u=f$ for $x=6$; $Au+Bu_y=g$ for $y=0$,
$Cu+Du_y=h$ for $y=1$

TRUE Unknown

Operator: Laplace, homogeneous

Right side: Zero

Boundary conditions: Mixed and complicated:

$$f(y) = e^{\pi(\alpha-y)/2}\sin(\pi y/2)/\alpha^3$$

$$A(x) = \begin{cases} 1 & x < \alpha \\ x & x \geq \alpha \end{cases} \qquad B(x) = \begin{cases} 0 & x < \alpha \\ 1 & x \geq \alpha \end{cases}$$

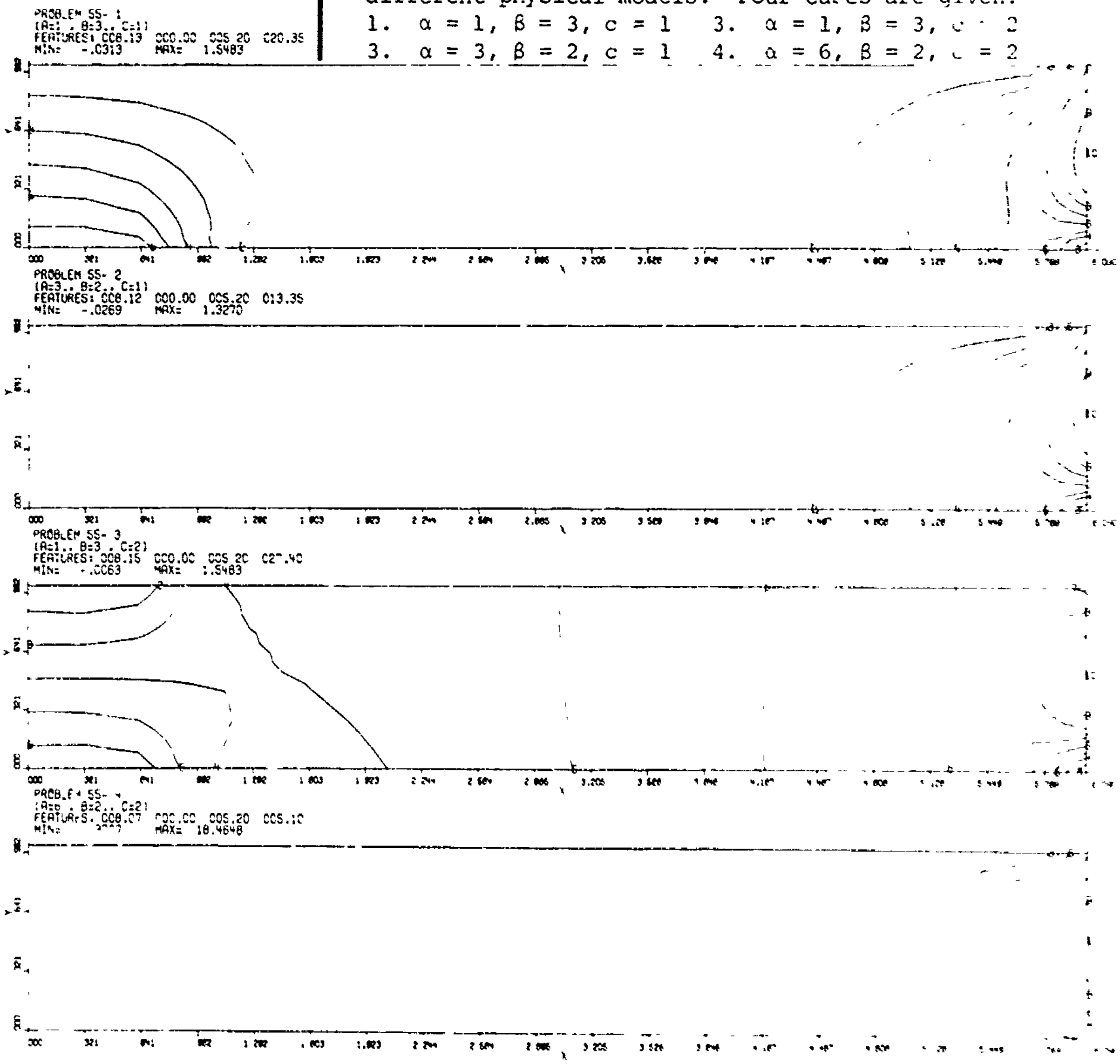
$$C(x) = \begin{cases} 1 & x < \alpha \text{ or } k=1 \\ 0 & x \geq \alpha \text{ and } k=2 \end{cases} \qquad D(x) = \begin{cases} 0 & x < \alpha \text{ or } k=1 \\ 1 & x \geq \alpha \text{ and } k=2 \end{cases}$$

$$G(x) = \begin{cases} 1/\alpha^3 & x<\alpha \\ e^{2(\alpha-x)}/\alpha^3 & x>\alpha \end{cases} \qquad H(x) = \begin{cases} 0 & x<\alpha \text{ or } k=1 \\ e^{3(\alpha-x)}/\alpha^3 & x\geq\alpha \text{ and } k=2 \end{cases}$$

Solution: Has singularities at boundaries, widely varying behavior.

Parameters: α, β are physical parameters, c selects different physical models. Four cases are given.

1. $\alpha = 1$, $\beta = 3$, $c = 1$ 3. $\alpha = 1$, $\beta = 3$, $c = 2$
3. $\alpha = 3$, $\beta = 2$, $c = 1$ 4. $\alpha = 6$, $\beta = 2$, $c = 2$



PROB 56 Artificial

$$u_{xx} + \frac{1}{x} u_x + \frac{1}{x^2} u_{yy} = 0$$

DOMAIN $[0,1] \times [0,2\pi]$

BC $u = g$

TRUE $\sum_{k=0}^{\alpha} w_k [e^{-z_k x\sin y} \cos(z_k x\cos y) + e^{-z_k x\cos y} \cos(z_k x\sin y)] + \sum_{n=1}^{\beta} a_n x^{2n} \cos(2ny)$

Gauss weights w_k and points z_k depend on α, $a_n = (1,.5,$ [illegible] $1/6,-1/10,-1/15,1/30,-1/50)$

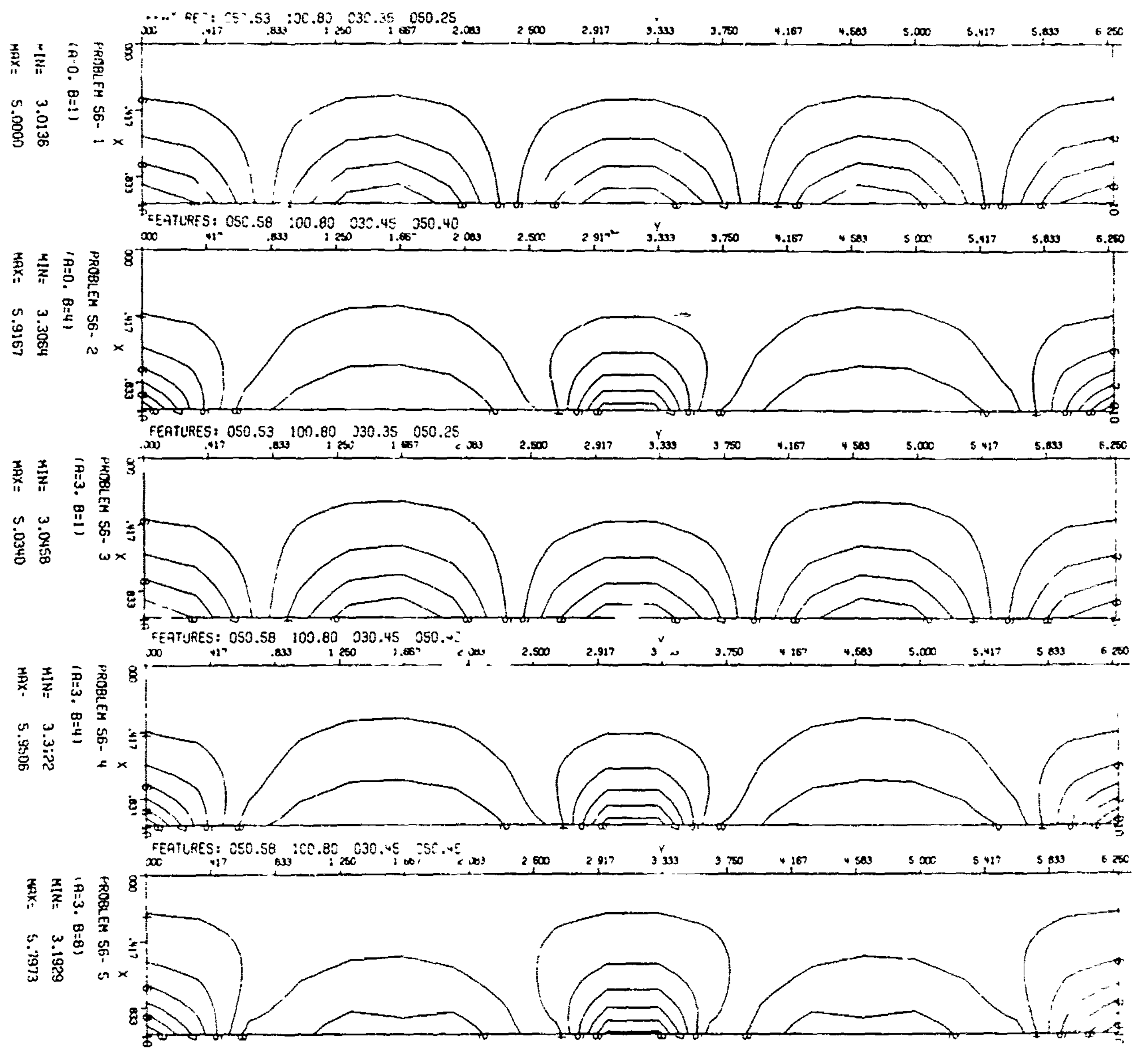
Operator: Singular coefficients, homogeneous

Right side: Constant

Boundary conditions: Mixed

Solution: Series expansion approximates electrostatics solution.

Parameters: α = order of Gauss quadrature for integral, β = no. terms in expansion.



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9) Technical summary rept.

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Elliptic partial differential equations, numerical methods, software evaluation, population of problems, linear, second order.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We present a population of 56 linear, two-dimensional elliptic partial differential equations (PDEs) suitable for evaluating numerical methods and software. Forty two of the PDEs are parameterized which allows much larger populations to be made; 189 specific cases are presented here along with solutions (some are only approximate). Many of the PDEs are artifically created so as to exhibit various mathematical behaviors of interest; the others are taken from "real world" problems in various ways. The population has been structured by introducing measures of complexity of the operator, boundary conditions, solution and

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20. Abstract (continued)

problem. The PDEs are first presented in mathematical terms along with contour plots of the 189 specific solutions. Machine readable descriptions are given in Part 2, MRC Technical Summary Report # 2079; many of the PDEs involve lengthy expressions and about a dozen involve extensive tabulations of approximate solutions.