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STATE SPACE MODELS FOR GAUSSIAN STOCHASTIC PROCESSES.

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ABSTRACT: A comprehensive theory of stochastic realization for
multivariate stationary Gaussian processes is presented. It is
coordinate-free in nature, starting out with an abstract state
space theory in Hilbert space, based on the concept of splitting
subspace. These results are then carried over to the spectral do-
main and described in terms of Hardy functions. Each state space
is uniquely characterized by its structural function, an inner
function which contains all the systems theoretical characteristics
of the corresponding realizations. Finally coordinates are intro-
duced and concrete differential-equation-type representations are
obtained. This paper is an abridged version of a forthcoming
paper, which in turn summarizes and considerably extends results
which have previously been presented in a series of preliminary
conference papers.

1. INTRODUCTION

In recent years there has been a considerable interest in var-
ious versions of the so-called *stochastic realization problem*
[1-51], which, loosely speaking, can be described as the problem
of finding (a suitable class of) stochastic dynamical systems,
called *realizations*, all having a given random process $\{y(t); t \in T\}$
as its output. (Here T is the index set, which usually is the real
line R or the set Z of integers.) In the past it has often been

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assumed that y is a stationary (or stationary increment) process with rational spectral density, thus insuring the existence of finite dimensional realizations.

The early papers on the subject [1-3] consider a deterministic version of the problem, the objective being to realize (in the deterministic sense [32,33]) the spectral factors of the given process which is defined up to second-order properties only. The probabilistic aspects of the stochastic realization problem were subsequently clarified in [8-10]. In all these papers the states of the realizations are represented in a fixed coordinate system to avoid trivial questions of uniqueness.

However, the most natural approach to the stochastic realization problem is coordinate free: Begin by constructing families $\{\chi_t; t \in T\}$ of *state spaces* which evolve in time in a Markovian manner. These state spaces should be as small as possible, but large enough to contain the essential information for determining the temporal evolution of the given process. Then, for each such family, concrete realizations can be obtained by introducing suitable bases in the state spaces. This line of study was initiated in [4-6], where a restricted version of the problem of this paper was studied, considering only state spaces contained in the closed span of the past (or, symmetrically, the future) of the given process. With such a state space approach we need not restrict the analysis to processes with rational spectral density, since the framework will also accommodate infinite dimensional state spaces.

During the last couple of years we have been developing a state space theory of stochastic realization which is now in a reasonably complete form. Part of our work has been reported in a series of preliminary conference papers [11-15]; a more complete account will appear in a forthcoming paper [16], which is now under preparation. Some results in the first phase of this work were obtained in cooperation with Ruckebusch [17], who parallelly developed his own geometric state space theory [20,21]. The present paper is an attempt to summarize the results presented in [16]. Due to page limitations, not all topics of [16] will be discussed. Also we have left out the proofs of the theorems, instead providing the reader with references for the proofs. For simplicity, only realizations of continuous-time stationary processes will be discussed, but it should be understood that our basic geometric theory holds also for stationary increment processes and discrete-time processes, and that the subsequent spectral theory can be appropriately modified to take care of these cases also.

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2. PROBLEM FORMULATION

Let $\{y(t); t \in R\}$ be a real stationary m -dimensional Gaussian process which is purely nondeterministic, mean-square continuous and centered, and let H be the Gaussian space [34] generated by y , i.e. the linear span of the stochastic variables $\{y_k(t); t \in R, k = 1, 2, \dots, m\}$ closed in L_2 norm. The space H is a Hilbert space when endowed with the inner product $(\xi, \eta) = E\{\xi\eta\}$, where $E\{\cdot\}$ stands for mathematical expectation. For any two subspaces A and B of H (which are always taken to be closed), $A \vee B$ denotes the closed linear hull of A and B , E^A denotes the orthogonal projection on A , and $\bar{E}^A B$ signifies the closure of $E^A B$. Moreover, let $A^\perp$ denote the orthogonal complement of A in H , and let $A \ominus B$ be the orthogonal complement of B in A , implicitly implying that A is a subspace of B . Since y is a stationary process, there is strongly continuous group $\{U_t; t \in R\}$ of unitary operators $H \rightarrow H$ such that $y_k(t) = U_t y_k(0)$ for all t and $k = 1, 2, \dots, m$ [35]. In the sequel we shall also consider the two semigroups $\{U_t; t \geq 0\}$ and $\{\bar{U}_t; t \geq 0\}$ obtained by setting $U_t := U_t$ and $\bar{U}_t := U_{-t}$ for $t \geq 0$.

The first problem at hand is to determine families $\{X_t; t \in R\}$ of subspaces of H , such that

$$y_k(t) \in X_t \quad ; \quad k = 1, 2, \dots, m \quad (2.1)$$

for all t , which are *Markovian* in the sense that

$$E^{X_t} \bar{X}_\lambda = E^{X_t} X_\lambda \quad \text{for all } \lambda \in X_t^+ \quad (2.2)$$

where $X_t^- := \bigvee_{T \leq t} X_T$ and $X_t^+ := \bigvee_{T \geq t} X_T$, and which are *stationary*, i.e. satisfy the condition

$$X_t = U_t X_0 \quad \text{for all } t \in R. \quad (2.3)$$

The subspaces $\{X_t\}$ will be called *state spaces*. This is a generalization of the following more concrete problem: Find a vector-valued stationary Markov process $\{x(t); t \in R\}$ and a matrix C so that $y(t) = Cx(t)$ for all $t \in R$. However, as we shall see in Section 9, (in a strict sense) the latter problem generally makes sense only if the spectral density of y is rational, in which case there are finite dimensional Markov processes x . To circumvent this difficulty we would have to consider *weak* Hilbert space-valued Markov processes. Problem formulation (2.1)-(2.3), on the other hand, is *coordinate free* and makes sense without further restrictions or modifications.

In view of condition (2.3), it is enough to require that (2.1) and (2.2) hold for one t , say $t = 0$; then they will automatically hold for all $t \in R$. For simplicity we shall drop the index

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and write X instead of X_0 . Now define the *past space* H^- and the *future space* H^+ as the closed linear hulls in H of the stochastic variables $\{y_k(t); t \leq 0, k = 1, 2, \dots, m\}$ and $\{y_k(t); t \geq 0, k = 1, 2, \dots, m\}$ respectively. Then, it follows from (2.1) that $H^- \vee X \subset X^-$ and that $H^+ \subset X^+$. Consequently applying the projection $E^{H^- \vee X}$ to (2.2) with $t = 0$ we obtain

$$E^{H^- \vee X} \lambda = E^X \lambda \quad \text{for all } \lambda \in H^+. \quad (2.4a)$$

It is easy to show [15] that the symmetric condition

$$E^{H^+ \vee X} \lambda = E^X \lambda \quad \text{for all } \lambda \in H^- \quad (2.4b)$$

is equivalent to (2.4a). A subspace satisfying one of conditions (2.4) is called a *splitting subspace*. Loosely speaking, such a subspace contains all information about the past needed in predicting the future, or, equivalently, all the information about the future required to estimate the past.

PROPOSITION 2.1. [14]. *Let X be a subspace of H and let $S := H^- \vee X$ and $\tilde{S} := H^+ \vee X$. Then the family $\{U_t X; t \in \mathbb{R}\}$ satisfies conditions (2.1) and (2.2) if and only if X is a splitting subspace such that*

$$\bar{U}_t S \subset S \quad \text{for all } t \geq 0 \quad (2.5a)$$

and

$$U_t \tilde{S} \subset \tilde{S} \quad \text{for all } t \geq 0. \quad (2.5b)$$

In view of this proposition we shall say that a splitting subspace is *Markovian* if it satisfies conditions (2.5). Hence we can instead consider the problem of finding Markovian splitting subspaces. This problem formulation has the advantage of also covering situations which are not discussed in this paper, e.g. realization of processes with stationary increments and discrete-time processes, for which problem formulation (2.1)-(2.3) is too restrictive (since it does not allow for observation noise, for example). Hence the basic geometric theory developed below has a wider applicability, as we shall demonstrate in some subsequent papers.

Obviously the whole space H is a Markovian splitting subspace, and so are H^- and H^+ , but they are too large for our purposes. Indeed, the whole idea is to obtain "data reduction." Therefore we shall be particularly interested in (Markovian) splitting subspaces X which are *minimal* in the sense that there is no proper subspace of X which is also a (Markovian) splitting subspace. The following proposition implies that a minimal Markovian splitting subspace is the same thing as a Markovian minimal splitting subspace.

PROPOSITION 2.2. [16]. *Any minimal Markovian splitting subspace is a minimal splitting subspace.*

When the Markovian splitting subspaces have been adequately characterized, there remains the problem of obtaining *concrete* dynamical representations of the given process y based on these state spaces, if possible of differential equation type. Note that we are only considering representations for which the state spaces are contained in the Hilbert space H generated by the given process (or its increments), i.e. *internal* realizations. Our theory could be modified to accommodate realizations containing exogeneous random elements (*external* realizations), but this is outside the scope of this paper, being a bit unnatural in the present setting.

3. THE GEOMETRY OF SPLITTING SUBSPACES

Our first problem will be to determine the set of all splitting subspaces, and, in particular, those which are minimal.

The *predictor space*

$$X_- := \bar{E}^{H^-} H^+ \quad (3.1)$$

is a splitting subspace. To see this, note that, for all $\lambda \in H^+$, $\bar{E}^{H^-} \lambda \in X_-$; hence $\bar{E}^{H^-} \lambda = E^{X_-} \bar{E}^{H^-} \lambda = E^{X_-} \lambda$. Moreover, all splitting subspaces $X \subset H^-$ contain X_- . In fact, by (2.4a), $\bar{E}^{H^-} H^+ = E^{X_-} H^+$ which is contained in X . Hence X_- is a minimal splitting subspace. By symmetry, we see that the *backward prediction space*

$$X_+ := \bar{E}^{H^+} H^- \quad (3.2)$$

is also a minimal splitting subspace. (The reader is urged to carefully distinguish between X_- and X_+ and X^- and X^+ defined in Section 2. The reason for the former notation will be clear from what follows.)

PROPOSITION 3.1. [14]. *The spaces H , H^- and H^+ have the orthogonal decompositions*

$$H = N^- \oplus H^\square \oplus N^+, \quad (3.3a)$$

$$H^- = N^- \oplus X_- \quad \text{and} \quad H^+ = N^+ \oplus X_+, \quad (3.3b)$$

where $H^\square := X_- \vee X_+$, $N^- := H^- \cap (H^+)^\perp$ and $N^+ := H^+ \cap (H^-)^\perp$.

The space $H^\square$ is called the *frame space* and N^- and N^+ are called the (past respectively the future) *junk spaces*. These notations are suggested by the following result.

PROPOSITION 3.2. [14]. *Let X be a minimal splitting subspace. Then*

$$H^- \cap H^+ \subset X \subset H^0. \quad (3.4)$$

Hence the frame space H^0 is the closed linear hull of all minimal splitting subspaces, and consequently it contains all "information" needed for state space construction. On the other hand, the junk spaces contain no useful information and could be discarded. It is not hard to see that the frame space is itself a (generally nonminimal) splitting subspace. The following proposition illustrates the importance of these concepts in filtering theory. It should be compared with the corresponding result in [20], which is weaker.

PROPOSITION 3.3. [16]. *Let X be a splitting subspace. Then*

$$E^{H^-} X = X_- \quad (3.5a)$$

if and only if $X \perp N^-$, and symmetrically

$$E^{H^+} X = X_+ \quad (3.5b)$$

if and only if $X \perp N^+$.

We shall say that the given process y is *noncyclic* if it has nontrivial junk spaces, i.e. $N^- \neq 0$ and $N^+ \neq 0$, and *strictly noncyclic* if N^- and N^+ are both full range. (A subspace $A \subset H$ is *full range* if the closed linear hull of $\{U_t A; t \in \mathbb{R}\}$ in H is all of H .) In the scalar case ($m = 1$) strict noncyclicity is the same as noncyclicity. Clearly the problem of state space construction is not very interesting unless we have noncyclicity, since otherwise there will be no "data reduction," H^- and H^+ being minimal splitting subspaces.

In order to describe the set of splitting subspaces, we need to introduce the concept of *perpendicular intersection*. Two subspaces of H , A and B , are said to *intersect perpendicularly* if

$$\bar{E}^A B = A \cap B \quad (3.6a)$$

or equivalently [15]

$$\bar{E}^B A = A \cap B. \quad (3.6b)$$

If A and B together span all of H , we have the following characterization of perpendicular intersection.

PROPOSITION 3.4. [15]. Let A and B be subspaces of H such that $A \vee B = H$. Then A and B intersect perpendicularly if and only if $B^\perp \subset A$ or, equivalently, $A^\perp \subset B$.

Now, if H^- and H^+ intersect perpendicularly,[†] $H^0 = H^- \cap H^+$, and consequently there is a unique minimal splitting subspace, namely $H^- \cap H^+$ (Proposition 3.2). Hence the problem of finding the minimal splitting subspaces is trivial. In general, however, H^- and H^+ do not intersect perpendicularly, but, by appropriately extending H^- and H^+ so that the extended spaces intersect perpendicularly, we can still describe each splitting subspace as the intersection between two subspaces.

THEOREM 3.1. [15]. The subspace $X \subset H$ is a splitting subspace if and only if

$$X = S \cap \tilde{S} \quad (3.7)$$

for some perpendicularly intersecting subspaces S and $\tilde{S}$ such that $S \supset H^-$ and $\tilde{S} \supset H^+$. The correspondence $X \leftrightarrow (S, \tilde{S})$ is one-one, the pair $(S, \tilde{S})$ being uniquely determined by relations

$$S = H^- \vee X \quad (3.8a)$$

and

$$\tilde{S} = H^+ \vee X. \quad (3.8b)$$

COROLLARY 3.1. A subspace $X \subset H$ is a splitting subspace if and only if there are subspaces $S \supset H^-$ and $\tilde{S} \supset H^+$ such that one of the following four equivalent conditions hold

$$X = \tilde{E}^{\tilde{S}} S \quad (3.9a)$$

$$X = \tilde{E}^S \tilde{S} \quad (3.9b)$$

$$X = S \ominus \tilde{S}^\perp \quad (3.9c)$$

$$X = \tilde{S} \ominus S^\perp. \quad (3.9d)$$

A subspace S such that $S \supset H^-$ ($S \supset H^+$) will be called an *augmented past (future) space*. Hence, each (minimal or nonminimal) splitting subspace is uniquely characterized by two perpendicularly intersecting subspaces S and $\tilde{S}$, one being an augmented past space

[†]For a process y with rational spectral density ϕ , H^- and H^+ intersect perpendicularly if and only if ϕ has no zeros, i.e. y is a "purely autoregressive" process.

and one an augmented future space. We shall write $X \sim (S, \bar{S})$ to recall this correspondence. For example, $X_- \sim (S_-, \bar{S}_-)$ where $S_- = H^- \vee X_- = H^-$ and $\bar{S}_- = H^+ \vee X_- = (N^-)^\perp$ (Proposition 3.1). Likewise, $X_+ \sim (S_+, \bar{S}_+)$ where $S_+ = (N^+)^\perp$ and $\bar{S}_+ = H^+$.

To have X minimal we clearly need to make S and $\bar{S}$ as small as possible. Given an S , the smallest $\bar{S}$ which both contains H^+ and intersects S perpendicularly is

$$\bar{S} = H^+ \vee S^\perp \quad (3.10a)$$

(Proposition 3.4). Likewise, given $\bar{S}$,

$$S = H^- \vee \bar{S}^\perp \quad (3.10b)$$

is the smallest subspace containing H^- which intersects $\bar{S}$ perpendicularly. It is not hard to see that the two conditions not only characterize minimality but also the splitting property.

THEOREM 3.2. [15]. *Let $S \supset H^-$ and $\bar{S} \supset H^+$ be two subspaces, and set $X = S \cap \bar{S}$. Then X is a minimal splitting subspace if and only if both conditions (3.10) hold.*

In view of Proposition 3.2, (3.3a) and (3.8), it is clearly necessary that

$$S \subset (N^+)^\perp \quad (3.11a)$$

and that

$$\bar{S} \subset (N^-)^\perp \quad (3.11b)$$

in order that $X \sim (S, \bar{S})$ be minimal, but not sufficient; in fact, any subspace of H^0 satisfies (3.11). However, in Theorem 3.2., conditions (3.10) can be replaced by (3.10a) + (3.11a) or by (3.10b) + (3.11b), as seen from the following pair of propositions.

PROPOSITION 3.5. [14]. *Let $S \supset H^-$ and $\bar{S} \supset H^+$ be two subspaces satisfying (3.10a). Then (3.10b) holds if and only if (3.11a) holds.*

PROPOSITION 3.6. [14]. *Let $S \supset H^-$ and $\bar{S} \supset H^+$ be two subspaces satisfying (3.10b). Then (3.10a) holds if and only if (3.11b) holds.*

It follows from Theorem 3.2 and Proposition 3.5 that X is a minimal splitting subspace if and only if $\bar{S} := H^+ \vee X$ is given by (3.10a) and $S := H^- \vee X$ satisfies

$$H^- \subset S \subset (N^+)^\perp, \quad (3.12)$$

in which case $X = \bar{S}^{\perp} H^+$, as can be seen from (3.9a) and (3.10a).

Consequently the minimal splitting subspaces are in one-one correspondence with subspaces S satisfying (3.12). For this reason, we shall call a subspace satisfying (3.12) a *minimal augmented past space*. The set of such subspaces form a complete lattice, where the partial ordering is induced by the $\subset$ operation. Consequently the set of minimal splitting subspaces also form a complete lattice, in which X_- is the minimum element and X_+ is the maximum. In fact, as we have seen above, $S_- = H^-$ and $S_+ = (N^+)^{\perp}$.

Symmetrically, there is a one-one correspondence between the minimal splitting subspaces and subspaces $\bar{S}$ such that

$$H^+ \subset \bar{S} \subset (N^-)^{\perp}. \quad (3.13)$$

We shall call such a subspace a *minimal augmented future space*. In terms of it, the minimal splitting subspace has the representation $X = \bar{E} \bar{S} H^-$.

4. OBSERVABILITY, CONSTRUCTIBILITY AND MINIMALITY

Relation (2.4a), defining a splitting subspace $X \sim (S, \bar{S})$, can be written

$$E^S|_{H^+} = E^S|_X \circ E^X|_{H^+}, \quad (4.1a)$$

where $|_A$ denotes restriction to the domain A . (Here the first operator on the right-hand side is merely an insertion map, insuring that the range spaces match.) Likewise, the alternative definition (2.4b) can be written

$$E^{\bar{S}}|_{H^-} = E^{\bar{S}}|_X \circ E^X|_{H^-}. \quad (4.1b)$$

Define $G^+ := E^S|_{H^+}$ and $G^- := E^{\bar{S}}|_{H^-}$. Then the splitting property (2.4) is equivalent to either of the two *Hankel operators* G^+ and G^- having a factorization through X described by the commutative diagrams

$$\begin{array}{ccc} H^+ & \xrightarrow{G^+} & S \\ & \searrow O & \nearrow R \\ & X & \end{array} \quad \begin{array}{ccc} H^- & \xrightarrow{G^-} & \bar{S} \\ & \searrow C & \nearrow \bar{R} \\ & X & \end{array} \quad (4.2)$$

respectively, where $O := E^X|_{H^+}$, $C := E^X|_{H^-}$, $R = E^S|_X$ and $\bar{R} := E^{\bar{S}}|_X$. Such a factorization is said to be *canonical* [32] if the first factor (here R or $\bar{R}$) is *one-one* and the second factor (here O or C)

maps onto a dense subset of X ; if the second factor maps onto X we say that the factorization is *exactly canonical*.

Since the insertion map R is trivially one-one, the first of diagrams (4.2) is canonical if and only if

$$\bar{E}^X H^+ = X . \quad (4.3a)$$

A splitting subspace with this property is said to be *observable*; the mapping O is called the *observability operator*. Likewise, the second factorization (4.2) is canonical if and only if

$$\bar{E}^X H^- = X . \quad (4.3b)$$

If this condition holds, we say that X is *constructible*; we call C the *constructibility operator* [20]. If one of the factorizations is exactly canonical, the closure bar over the E in the corresponding relation (4.3) can be removed; then we say that X is *exactly observable* ($E^X H^+ = X$) or *exactly constructible* ($E^X H^- = X$) respectively.

It follows from (3.9a) and the splitting property (2.4a) that a splitting subspace X is observable if and only if

$$\bar{E}^S \bar{S} = \bar{E}^S H^+ . \quad (4.4)$$

But this condition is equivalent to

$$\bar{S} = H^+ \vee S^\perp . \quad (4.5)$$

In fact, since $E^S \lambda = 0$ for $\lambda \in S^\perp$, it is easy to see that (4.5) implies (4.4). To see that (4.5) is a consequence of (4.4), first note that, since S and $\bar{S}$ intersect perpendicularly (Theorem 3.1), $\bar{S} \supset H^+ \vee S^\perp$ (Proposition 3.4). But $Z := \bar{S} \ominus (H^+ \vee S^\perp) \subset S \cap (H^+)^{\perp}$, and therefore (4.4) cannot hold unless $Z = 0$. In the same way we see that a splitting subspace X is constructible if and only if

$$S = H^- \vee \bar{S}^\perp , \quad (4.6)$$

and hence we have proven the following theorem.

THEOREM 4.1. *Let $X \sim (S, \bar{S})$ be a splitting subspace. Then X is observable if and only if (4.5) holds and constructible if and only if (4.6) holds.*

We can now tie together the concepts of observability and constructibility with that of minimality, discussed in Section 3. It follows from Theorems 3.2 and 4.1 that a splitting subspace is

minimal if and only if it is both observable and constructible. This point can also be illustrated in the following way: Apply the projector E^{H^-} to (2.4a) to obtain $E^{H^-}\lambda = E^{H^-}E^X\lambda$ for all $\lambda \in H^+$, i.e. the diagram

$$\begin{array}{ccc} H^+ & \xrightarrow{E^{H^-}|_{H^+}} & H^- \\ & \searrow 0 \quad \nearrow C^* & \\ & X & \end{array} \quad (4.7)$$

commutes if X is a splitting subspace, where $C^* = E^{H^-}|_X$ is the adjoint of the constructibility operator $C = E^X|_{H^-}$. It is not hard to see that C^* is one-one if and only if C maps onto a dense subset of X . (Cf. [36; p.89].) Consequently, in view of the equivalence between minimality and observability plus constructibility proven above, the factorization (4.7) is canonical if and only if X is minimal. Of course, in the same way, if X is a splitting subspace, the dual diagram

$$\begin{array}{ccc} H^- & \xrightarrow{E^{H^+}|_{H^-}} & H^+ \\ & \searrow C \quad \nearrow O^* & \\ & X & \end{array} \quad (4.8)$$

commutes, and it is canonical if and only if X is minimal.

Finally we summarize the connections between observability, constructibility and minimality provided by Theorems 3.2 and 4.1 and Propositions 3.5 and 3.6.

THEOREM 4.2. *Let $X \sim (S, \bar{S})$ be a splitting subspace. Then the following conditions are equivalent:*

- (i) X is minimal
- (ii) X is observable and constructible
- (iii) X is observable and S is minimal
- (iv) X is constructible and $\bar{S}$ is minimal.

5. MARKOVIAN SPLITTING SUBSPACES

Let $X \sim (S, \bar{S})$ be a splitting subspace. Then, by Proposition 2.1, X is Markovian if and only if the two conditions

$$\bar{U}_t S \subset S \quad (\text{left invariance}) \quad (5.1a)$$

and

$$U_t \bar{S} \subset \bar{S} \quad (\text{right invariance}) \quad (5.1b)$$

both hold. It is immediately seen that $X_- \sim (H^-, (N^-)^\perp)$ and $X_+ \sim ((N^+)^\perp, H^+)$ satisfy these conditions, and consequently X_- and X_+ are Markovian splitting subspaces. Hence all minimal Markovian splitting subspaces form a complete sublattice of the lattice defined in Section 3, with X_- being the minimum and X_+ the maximum element.

If (5.1a) holds, $\{\bar{U}_t|_S; t \geq 0\}$ is a strongly continuous semigroup on S , and the same holds true for the adjoints

$$U_t(S) := E^S U_t|_S \quad ; \quad t \geq 0 \quad (5.2a)$$

Similarly, if (5.1b) holds, $\{U_t|\bar{S}; t \geq 0\}$ and the adjoints

$$\bar{U}_t(\bar{S}) := E^{\bar{S}} \bar{U}_t|_{\bar{S}} \quad ; \quad t \geq 0 \quad (5.2b)$$

both form strongly continuous semigroups on $\bar{S}$. Operators of type (5.2) are called *compressions* of the shifts U_t and $\bar{U}_t$ respectively. Compressions with respect to subspaces other than S and $\bar{S}$ will be denoted analogously. It can be shown that a Markovian splitting subspace $X \sim (S, \bar{S})$ is invariant for $U_t(S)$ and $\bar{U}_t(\bar{S})$. More precisely we have:

PROPOSITION 5.1. [16]. Let $X \sim (S, \bar{S})$ be a splitting subspace. Then the conditions (5.1a) and

$$\bar{U}_t(\bar{S})X \subset X \quad (5.3a)$$

are equivalent. Similarly (5.1b) is equivalent to

$$U_t(S)X \subset X. \quad (5.3b)$$

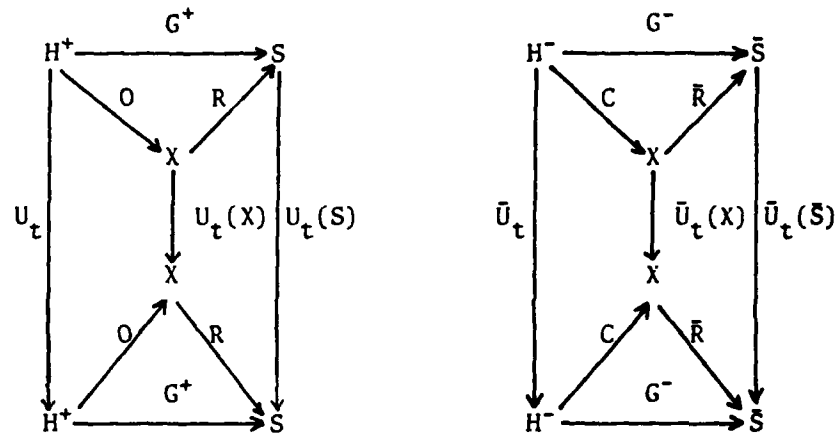
Conditions (5.3) imply that for each $\xi \in X$, $U_t(S)\xi = U_t(X)\xi$ and $\bar{U}_t(\bar{S})\xi = \bar{U}_t(X)\xi$, where $U_t(X)$ and $\bar{U}_t(X)$ are defined as in (5.2).

The operators $U_t(X) : X \rightarrow X$ and $\bar{U}_t(X) : X \rightarrow X$ will play a very important role in what follows.

THEOREM 5.1. [16]. *The splitting subspace X is Markovian if and only if $\{U_t(X); t \geq 0\}$ [$\{\bar{U}_t(X); t \geq 0\}$] is a strongly continuous semigroup.*

Hence, we shall call $\{U_t(X); t \geq 0\}$ the *forward Markov semigroup* and $\{\bar{U}_t(X); t \geq 0\}$ the *backward Markov semigroup* of the Markovian splitting subspace X . The following theorem describes how these shift operators intertwine the Hankel, observability and constructibility operators introduced in Section 4.

THEOREM 5.2. [16]. *Let $X \sim (S, \bar{S})$ be a Markovian splitting subspace. Then the following diagrams commute.*



(For simplicity we write U_t and $\bar{U}_t$ in place of $U_t|_{H^+}$ and $\bar{U}_t|_{H^-}$ respectively.)

As a corollary of this theorem, we obtain the factorization

$$G^+ U_t = R U_t(X) O \quad (5.4a)$$

and its backward counterpart

$$G^- \bar{U}_t = \bar{R} \bar{U}_t(X) C, \quad (5.4b)$$

which should be compared with the corresponding factorizations in deterministic realization theory [32]. Relations (5.4) will be used in Section 9.

A Markovian splitting subspace $X \sim (S, \bar{S})$ is said to be *proper* if both S and $\bar{S}$ are purely nondeterministic. [A subspace Z is *purely nondeterministic* if $\bigcap_{t \in \mathbb{R}} U_t Z = 0$.] If X is proper, neither S

nor $\tilde{S}$ has a doubly invariant subspace, i.e. a subspace which satisfies both conditions (5.1), and, moreover, X is a proper subspace of both S and $\tilde{S}$. In fact, if $X = S$ (say), (3.7) implies that $S \subset \tilde{S}$, and hence we must have $\tilde{S} = H$, which contradicts the purely nondeterministic assumption. Consequently properness of X insures effective data reduction.

PROPOSITION 5.2. [14]. *Let y be strictly noncyclic. Then all splitting subspaces $X \subset H^{\square}$ (i.e., in particular, the minimal ones) are proper.*

Any proper splitting subspace is also purely nondeterministic [but the opposite is not true; $H^{\square} \sim (H^{\square}, H)$ could serve as a counterexample], and therefore the following result can be applied.

PROPOSITION 5.3. [16]. *Let X be a purely nondeterministic splitting subspace. Then $U_t(X)$ and $\tilde{U}_t(X)$ tend strongly to zero as $t \rightarrow \infty$.*

6. SPECTRAL REPRESENTATION OF PROPER MARKOVIAN SPLITTING SUBSPACES

Since the given process y is stationary, mean-square continuous and purely nondeterministic, it has a spectral representation

$$y(t) = \int e^{st} d\hat{y}(s), \quad (6.1)$$

where integration is over the imaginary axis I and $d\hat{y}$ is an orthogonal stochastic vector measure such that

$$E\{d\hat{y}(i\omega)d\hat{y}(i\omega)^*\} = \frac{1}{2\pi} \Phi(i\omega)d\omega, \quad (6.2)$$

Φ being the $m \times m$ matrix-valued spectral density of y [35]. (Asterisk (*) here denotes conjugation plus transpose.) Moreover, y being purely nondeterministic implies that Φ has a constant rank $p \leq m$ and that it admits a factorization [35; p.114].

A full-rank spectral factor is any $m \times p$ -matrix solution of

$$W(s)W(-s) = \Phi(s) \quad (6.3)$$

such that $\text{rank } W = p$. To any such spectral factor we may associate a p -dimensional Wiener process on R

$$u(t) = \int \frac{e^{st} - 1}{s} d\hat{u}(s) \quad ; \quad d\hat{u} = W^{-L} d\hat{y}, \quad (6.4)$$

where $E\{d\hat{u}(i\omega)d\hat{u}(i\omega)^*\} = \frac{1}{2\pi} I d\omega$, and W^{-L} is a left inverse of W .

Despite the fact that, in general, W has more than one left inverse, it can be shown [16] that $\hat{u}$, and hence u , is uniquely defined. Let U denote the class of all such Wiener processes, and let $H(du)$, $H^-(du)$ and $H^+(du)$ be defined as the closed linear hulls in H of $\{u_k(t); t \in T, k = 1, 2, \dots, p\}$, where T is R , $\{t \leq 0\}$ and $\{t \geq 0\}$ respectively. As these notations suggest, we are merely interested in the increments of the processes $u \in U$, the assumption $u(0) = 0$, contained in (6.4), being for convenience only.

It can be shown [14] that each $u \in U$ spans all of H , i.e. $H(du) = H$. Consequently our basic Hilbert space H consists precisely of the random variables

$$\eta = \int_{-\infty}^{\infty} f(-t) du(t), \quad (6.5a)$$

where f varies over the space $L_2^p(R)$ of p -dimensional row-vector functions square-integrable on the real line (with respect to the Lebesgue measure). This representation can be transformed to the spectral domain to read

$$\eta = \int \hat{f}(i\omega) d\hat{u}(i\omega), \quad (6.5b)$$

where $\omega \mapsto \hat{f}(i\omega)$ is the Fourier-transform of f , defined in the L_2 sense. To conform with formulations prevalent in the systems sciences, formally we use the double-sided Laplace-transform, $\hat{f}$ belonging to the space $L_2^p(I, \frac{1}{2\pi} d\omega)$ of p -dimensional row-vector functions which are square-integrable on the imaginary axis I ; we shall write $L_2^p(I)$ for short. Also, we shall adopt the convention of writing Ff to denote $\hat{f}$. Then F is a unitary operator from $L_2^p(R)$ to $L_2^p(I)$.

For each $u \in U$, (6.5b) defines an isomorphism between H and $L_2^p(I)$. Let $Q_u : H \rightarrow L_2^p(I)$ be the mapping

$$\eta \xrightarrow{Q_u} \hat{f}. \quad (6.6)$$

Then it follows from the definition of stochastic integral that Q_u is a unitary operator. As is evident from (6.5a), $H^-(du)$ consists precisely of those η for which $f \in L_2^p(0, \infty)$, i.e. for which f vanishes on the negative real line. Likewise, $H^+(du)$ consists of those η for which $f \in L_2^p(-\infty, 0)$. Then, defining the Hardy spaces $H_2^+ := FL_2^p(0, \infty)$ and $H_2^- := FL_2^p(-\infty, 0)$, we clearly have $Q_u H^-(du) = H_2^+$ and $Q_u H^+(du) = H_2^-$.

The functions in H_2^+ can be extended to the right complex half-plane and can be seen to be analytic there [37-40]. Likewise, the

H_2^- -functions, properly extended, are analytic in the open left half-plane. Therefore we shall say that a full-rank spectral factor is *stable* if all its rows belong to H_2^+ and *strictly unstable* if its rows belong to H_2^- . Let U^+ and U^- be the subclasses of U corresponding to stable and strictly unstable spectral factors respectively.

LEMMA 6.1. [14]. *There is a one-one correspondence between stable full-rank spectral factors W (determined modulo multiplication with a constant unitary matrix) and left invariant [i.e., satisfying (5.1a)] and purely nondeterministic subspaces $S \supset H^-$. The subspace S is related to W by*

$$S = H^-(du) \quad (6.7)$$

where $u \in U^+$ is the Wiener process corresponding to W .

LEMMA 6.2. [14]. *There is a one-one correspondence between strictly unstable full-rank spectral factors $\bar{W}$ and right invariant [i.e., satisfying (5.1b)] and purely nondeterministic subspaces $S \supset H^+$. The subspace S is related to $\bar{W}$ by*

$$S = H^+(d\bar{u}) \quad (6.8)$$

where $\bar{u} \in U^-$ is the Wiener process corresponding to $\bar{W}$.

Since y is a purely nondeterministic process, H^- and H^+ are purely nondeterministic subspaces, and therefore the lemmas above apply. Hence there is $u_- \in U^+$ such that

$$H^-(du_-) = H^- \quad (6.9)$$

This is the *innovation process* of y . Let W_- be the corresponding spectral factor. In view of (6.1) and the fact that $d\hat{y} = W_- d\bar{u}_-$, $Q_{u_-} a y(t) = e^{i\omega t} a W_-$ for all $t \in \mathbb{R}$. But $Q_{u_-} H^- = H_2^+$, and therefore $\overline{\text{sp}}\{e^{i\omega t} a W_- \mid t \leq 0, a \in \mathbb{R}^m\} = H_2^+$, where $\overline{\text{sp}}\{\cdot\}$ denotes closed span in H . A function W with this property is called *outer* [37-40]; W_- is the unique outer spectral factor. Likewise, there is a $\bar{u}_+ \in U^-$ such that

$$H^+(d\bar{u}_+) = H^+ \quad (6.10)$$

called the *backward innovation process* of y . The corresponding spectral factor $\bar{W}_+$ has the outer property with $t \leq 0$ and H_2^+ exchanged for $t \geq 0$ and H_2^- . Such a function is called *conjugate outer*; $\bar{W}_+$ is the only spectral factor with this property.

By Lemmas 6.1 and 6.2 there is a one-one correspondence between proper Markovian splitting subspaces $X \sim (S, \bar{S})$ and pairs $(W, \bar{W})$ of full-rank spectral factors with W stable and $\bar{W}$ strictly unstable or, equivalently, pairs $(u, \bar{u})$ of Wiener processes with $u \in U^+$ and

$\bar{u} \in U^-$. We shall call $W(\bar{W})$ the *forward (backward) spectral factor* of X and $u(\bar{u})$ the *forward (backward) generating process* of X . For each such pair $(W, \bar{W})$, we define a $p \times p$ matrix function

$$K = \bar{W}^{-L} W, \quad (6.11)$$

which we call the *structural function* of X . Although the left inverse $\bar{W}^{-L}$ is nonunique, it can be shown [16] that K is uniquely defined. In fact,

$$d\hat{u} = K d\bar{u}. \quad (6.12)$$

The structural function K will play a very important part in what follows. Due to certain similarities with the Lax-Phillips scattering operator [41], we shall alternatively call it the *scattering function*. It is not hard to see that $K(i\omega)$ is a unitary matrix for each $\omega \in \mathbb{R}$. Next, we shall show that, in addition, K is bounded and analytic in the open right half-plane. A function with all these properties is called *inner* [37-40].

Now, in view of Theorem 3.1 and Lemmas 6.1 and 6.2, X is a proper Markovian splitting subspace if and only if

$$X = H^-(du) \cap H^+(d\bar{u}) \quad (6.13)$$

for some $u \in U^+$ and $\bar{u} \in U^-$ such that $H^-(du)$ and $H^+(d\bar{u})$ intersect perpendicularly. As pointed out above, the pair $(u, \bar{u})$ is unique, being the pair of generating processes of X .

LEMMA 6.3. [14]. Let $u \in U^+$ and $\bar{u} \in U^-$, and let W and $\bar{W}$ be the corresponding spectral factors. Then $S := H^-(du)$ and $\bar{S} := H^+(d\bar{u})$ intersect perpendicularly if and only if K , defined by (6.11), is inner.

The proof of this lemma, which can be found in [14], is based on the vector version of Beurling's Theorem [37-40].

By Corollary 3.1, (6.13) can be written $X = H^-(du) \ominus H^-(d\bar{u})$, the isomorphic image of which (under Q_u) is $Q_u X = H_2^+ \ominus (H_2^+ K)$. Consequently

$$X = \int (H_2^+ K)^\perp d\hat{u}, \quad (6.14)$$

where $u \in U^+$ is the Wiener process corresponding to W , and the superscript $\perp$ denotes orthogonal complement in H_2^+ . We collect these observations in the following theorem. Representation (6.14) should be compared with the deterministic solutions of [48,49].

THEOREM 6.1. [15,16]. *The subspace X is a proper Markovian splitting subspace if and only if (6.14) holds for some pair $(W, \bar{W})$ of full rank spectral factors such that W is stable, $\bar{W}$ is strictly unstable, and $K := \bar{W}^{-1}W$ is inner.*

In particular, if y is strictly noncyclic, all minimal Markovian splitting subspaces are given by Theorem 6.1 (Proposition 5.2). The minimum and maximum lattice elements X_- and X_+ correspond to the pairs $(W_-, \bar{W}_-)$ and $(W_+, \bar{W}_+)$, where W_- and $\bar{W}_+$ are the outer and conjugate outer spectral factors defined above. The corresponding pairs of generating processes are $(u_-, \bar{u}_-)$ and $(u_+, \bar{u}_+)$. The processes u_- and $\bar{u}_+$ are the forward and backward innovation processes respectively, and $\bar{u}_-$ and u_+ can be defined in terms of the junk spaces through the relations $H^-(d\bar{u}_-) = N^-$ and $H^+(du_+) = N^+$. The following result, which is a generalization to the vector case of a result found in [38], provides a test for noncyclicity in terms of the outer and conjugate outer spectral factor.

PROPOSITION 6.1. [14]. *The process y is strictly noncyclic if and only if there are inner functions J_1, J_2, J_3 and J_4 such that*

$$\bar{W}_+^{-1}W_- = J_1 J_2^{-1} = J_3^{-1} J_4. \quad (6.15)$$

In the scalar case ($m=1$), the structural function K is invariant over the set of (proper) minimal Markovian splitting subspaces, but this is not so in the vector case. This point can be illustrated by a finite dimensional example: The Kronecker structure of a concrete differential-equation representation (of the type derived in Section 9) is uniquely determined by K , but this structure varies with different minimal X [16].

PROPOSITION 6.2. [14]. *Let X be a proper Markovian splitting subspace. Then X is finite dimensional if and only if its structural function K is rational.*

This proposition has two interesting corollaries.

COROLLARY 6.1. [15]. *Suppose that the spectral density ϕ is rational. Then all splitting subspaces X contained in the frame space H^2 are finite dimensional, and all minimal splitting subspaces have the same dimension.*

COROLLARY 6.2. [15]. *Suppose that ϕ is rational. Then y is strictly noncyclic.*

For further discussion of the rational case we refer the reader to Section 7 in [15], where differential-equations representations are derived by factorization of the structural function K , using the ideas of [42].

7. SPECTRAL DOMAIN CRITERIA FOR OBSERVABILITY CONSTRUCTIBILITY AND MINIMALITY

Theorem 6.1 provides us with a procedure to find all proper Markovian splitting subspaces: All possible pairs $(W, \bar{W})$ of full-rank spectral factors with W stable, $\bar{W}$ strictly unstable, and $K = \bar{W}^{-L}W$ inner, inserted into (6.14), generate the whole family of such splitting subspaces. But how can we decide whether such a pair will provide an observable, or a constructible or a minimal splitting subspace? We need to translate the geometric criteria of Section 4 into spectral domain language.

To this end, first note that W is a stable full-rank spectral factor if and only if it can be written

$$W = W_- Q \quad (7.1)$$

where Q is an inner function and W_- is the unique outer spectral factor. Similarly, $\bar{W}$ is a strictly unstable full-rank spectral factor if and only if it has the representation

$$\bar{W} = \bar{W}_+ \bar{Q}, \quad (7.2)$$

where $\bar{Q}$ is conjugate inner (i.e. $\bar{Q}^*$ is inner) and $\bar{W}_+$ is the unique conjugate outer spectral factor [37-40]. The observability condition (4.5) and the constructibility condition (4.6) can now be expressed in terms of the inner functions K , Q and $\bar{Q}^*$.

THEOREM 7.1. [15]. *Let X be a proper Markovian splitting subspace, let (7.1) and (7.2) be the corresponding spectral factors, and let K be the structural function (6.11). Then X is observable if and only if K and $\bar{Q}^*$ are left coprime and constructible if and only if K and Q are right coprime.*

Two inner functions are *left (right) coprime* if they have no common left (right) inner factor, except possibly for a constant unitary matrix. Hence, by Theorem 7.1, X is observable if and only if there is no nontrivial cancellation in the factorization $T = \bar{Q}K$. But according to [43,44] (also see [39]), this is the case if and only if

$$\text{cl}(\text{Im} H_T) = (H_2^+ K)^\perp, \quad (7.3)$$

where $H_T : H_2^- \rightarrow H_2^+$ is the Hankel operator $H_T f = P^{H_2^+} f T$ and $\text{Im} H_T$ denotes the range of H_T , cl the closure, and $P^{H_2^+}$ the orthogonal projection on H_2^+ . In the same way, X is constructible if and only if there are no nontrivial cancellations in $\bar{T} = QK^*$, which statement is equivalent to

$$\text{cl}(\text{Im}H_T) = (H_2^- K^*)^\perp, \quad (7.4)$$

where $H_T : H_2^+ \rightarrow H_2^-$ is the Hankel operator $H_T f = P H_2^- f \bar{T}$ and $\perp$ now denotes orthogonal complement in H_2^- .

To clarify the nature of conditions (7.3) and (7.4), we shall take a closer look at the Hankel operators H_T and $H_{\bar{T}}$, along the lines of [14]. To this end, first note that H_T and $H_{\bar{T}}$ are related to the Hankel operators G^+ and G^- through the commutative diagrams

$$\begin{array}{ccc} H^+ & \xrightarrow{G^+} & S \\ Q_{U+} \downarrow & & \downarrow Q_U \\ H_2^- & \xrightarrow{H_T} & H_2^+ \end{array} \quad \begin{array}{ccc} H^- & \xrightarrow{G^-} & S \\ Q_{U-} \downarrow & & \downarrow Q_U \\ H_2^+ & \xrightarrow{H_{\bar{T}}} & H_2^- \end{array} \quad (7.5)$$

Then it follows from (4.2) that H_T and $H_{\bar{T}}$ factor according to the commutative diagrams

$$\begin{array}{ccc} H_2^- & \xrightarrow{H_T} & H_2^+ \\ \tilde{O} \searrow & & \nearrow \tilde{R} \\ & X & \end{array} \quad \begin{array}{ccc} H_2^+ & \xrightarrow{H_{\bar{T}}} & H_2^- \\ \tilde{C} \searrow & & \nearrow \tilde{R} \\ & \bar{X} & \end{array} \quad (7.6)$$

where $X := Q_{U+} \tilde{O}$, $\bar{X} := Q_{U-} \tilde{C}$, $\tilde{O} f = P^X f \bar{T}$, $\tilde{C} f = P^{\bar{X}} f \bar{T}$, and $\tilde{R}$ and $\bar{R}$ are the insertion maps $\tilde{R} f = f$ and $\bar{R} f = f$.

Since $\tilde{R}$ is merely an insertion, $\text{Im}H_T = \text{Im}\tilde{O}$, and therefore (7.3) is precisely the observability condition (4.3a), transformed as in (7.5), for, by Theorem 6.1, $X = (H_2^+ K)^\perp$. Likewise it is seen that (7.4) is the same as the constructibility condition (4.3b).

However, [43,44] also contain the stronger result that (7.3) holds with the closure operation removed if and only if K and $\bar{Q}^*$ are strongly left coprime, i.e. $\inf_{\text{Re}(s) > 0} \{|aK(s)| + |a\bar{Q}^*(s)|\} > 0$ for every $a \in \mathbb{R}^p$. The analogous statement holds for (7.4) and K and Q . Hence we have the following strong version of Theorem 7.1.

THEOREM 7.2. *Let X , K , Q and $\bar{Q}$ be as in Theorem 7.1. Then X is exactly observable if and only if K and $\bar{Q}^*$ are strongly left coprime and exactly constructible if and only if K and Q are strongly right coprime.*

In order to apply conditions (iii) and (iv) of Theorem 4.2, we also need to characterize minimality of S and $\bar{S}$ in the spectral domain. We shall say that a stable (strictly unstable) full-rank spectral factor is *minimal* if it corresponds to a minimal augmented past (future) space via the correspondence of Lemma 6.1 (Lemma 6.2). Now, assume that y is strictly noncyclic. Then the spectral factors W_+ and $\bar{W}_-$ introduced in Section 6 are well-defined. Let Q_+ and $\bar{Q}_-$ be the inner factors in (7.1) and (7.2) respectively corresponding to these spectral factors. Moreover, let K_- and K_+ be the structural functions of X_- and X_+ .

THEOREM 7.3. [14]. *Suppose y is strictly noncyclic. Let W be a stable full-rank spectral factor, and let $J := W^{-1}W_+$. Then, J is uniquely defined, and the following conditions are equivalent:*

- (i) W is minimal
- (ii) Q is a left inner divisor of Q_+ , i.e. there is an inner function θ such that $Q\theta = Q_+$
- (iii) J is inner.

If one of these conditions holds, J and K_+ are right coprime.

The fact that (i) and (ii) are equivalent was first proven in [19] for the scalar case. The dual version of Theorem 7.3 goes as follows.

THEOREM 7.4. [14]. *Suppose y is strictly noncyclic. Let $\bar{W}$ be a strictly unstable full-rank spectral factor, and let $\bar{J} := \bar{W}^{-1}\bar{W}_-$. Then, $\bar{J}$ is uniquely defined, and the following conditions are equivalent:*

- (i) $\bar{W}$ is minimal
- (ii) $\bar{Q}^*$ is a right inner divisor of $\bar{Q}_-$
- (iii) $\bar{J}^*$ is inner.

If one of these conditions holds, $\bar{J}^$ and K_- are left coprime.*

Note that the coprimeness conditions of Theorems 7.3 and 7.4 are related to T and $\bar{T}$ as follows: $T = K_+J^*$ and $\bar{T} = K_- \bar{J}^*$. This has some significance for state space isomorphism [16].

8. ABSTRACT REALIZATION THEORY IN HARDY SPACE

We shall now translate the results of Section 5 to the spectral domain, thereby laying the ground work for concrete differential-equation type representations to be introduced in the next section.

In view of (6.1), any $\eta \in H$ can be written

$$\eta = \int f d\hat{y} \quad (8.1)$$

for some $f \in L^m_2(I, \Phi d\omega)$. Define the unitary operator $Q_y : H \rightarrow L^m_2(I, \Phi d\omega)$ by $Q_y \eta = \hat{f}$, and introduce the spaces $Y^+ := Q_y H^+$ and $Y^- := Q_y H^-$, which consist of all linear functionals of the future and past respectively of y . Then $Y^+ = \overline{\text{sp}}\{ae^{i\omega t} \mid a \in \mathbb{R}^m; t \geq 0\}$ and $Y^- = \overline{\text{sp}}\{ae^{i\omega t} \mid a \in \mathbb{R}^m; t \leq 0\}$. Clearly, for $t \geq 0$, $Y^+(Y^-)$ is invariant under multiplication by $e^{i\omega t}(e^{-i\omega t})$. For each $t \geq 0$, let $\Sigma_t : Y^+ \rightarrow Y^+$ and $\bar{\Sigma}_t : Y^- \rightarrow Y^-$ be the mappings $\Sigma_t f = e^{i\omega t} f$ and $\bar{\Sigma}_t f = e^{-i\omega t} f$ respectively. Now, for any proper Markovian splitting subspace $X \sim (S, \bar{S})$, let the operators $\hat{G}^+$ and $\hat{G}^-$ be defined in terms of the Hankel operators G^+ and G^- by the commutative diagrams

$$\begin{array}{ccc} H^+ & \xrightarrow{G^+} & S \\ Q_y \downarrow & & \downarrow Q_u \\ Y^+ & \xrightarrow{\hat{G}^+} & H^+_2 \end{array} \quad \begin{array}{ccc} H^- & \xrightarrow{G^-} & \bar{S} \\ Q_y \downarrow & & \downarrow Q_{\bar{u}} \\ Y^- & \xrightarrow{\hat{G}^-} & H^-_2 \end{array} \quad (8.2)$$

Then $\hat{G}^+ f = P^{H^+_2} f W$ and $\hat{G}^- f = P^{H^-_2} f \bar{W}$, i.e. $\hat{G}^+$ and $\hat{G}^-$ are the Hankel operators corresponding to the input-output maps $f \mapsto fW$ and $f \mapsto f\bar{W}$ respectively.

For any $t \geq 0$ and subspace $Z \subset L^p_2(I)$, define the *compressions* $\Sigma_t(Z) := P^Z e^{i\omega t} \upharpoonright_Z$ and $\bar{\Sigma}_t(Z) := P^Z e^{-i\omega t} \upharpoonright_Z$, where we are slightly misusing notations by letting $e^{i\omega t}$ also denote multiplication by the function $\omega \mapsto e^{i\omega t}$. Then, setting $X := Q_u X$ and $\bar{X} := Q_{\bar{u}} X$, it is immediately clear that $Q_u U_t(X) Q_u^{-1} = \Sigma_t(X)$ and $Q_{\bar{u}} \bar{U}_t(X) Q_{\bar{u}}^{-1} = \bar{\Sigma}_t(\bar{X})$ and that $\Sigma_t(H^+_2)$ and $\bar{\Sigma}_t(H^-_2)$ are analogously related to $U_t(S)$ and $\bar{U}_t(\bar{S})$ respectively. Note that $\{\Sigma_t(X); t \geq 0\}$ and $\{\bar{\Sigma}_t(\bar{X}); t \geq 0\}$ are strongly continuous semigroups (Theorem 5.1) which tend strongly to zero as $t \rightarrow \infty$ (Proposition 5.3). We shall call them the *forward* and the *backward spectral semigroups* of X ; X and $\bar{X}$ will be called the *forward* and *backward spectral images* of X respectively. Then, by isomorphism, the following proposition is a corollary of Theorem 5.2.

PROPOSITION 8.1. *Let X be a proper Markovian splitting subspace with generating processes $(u, \bar{u})$ and spectral images $X := Q_u X$ and*

$\bar{X} := Q_{\bar{U}}X$. Then the diagrams

$$\begin{array}{ccc}
 \begin{array}{c}
 \begin{array}{ccccc}
 & & \hat{G}^+ & & \\
 & \nearrow & & \searrow & \\
 y^+ & & \hat{O} & & H_2^+ \\
 & \searrow & & \nearrow & \\
 & & X & & \\
 & \downarrow & & \downarrow & \\
 & & \Sigma_t(X) & & \\
 & \downarrow & & \downarrow & \\
 & & X & & \\
 & \nearrow & & \searrow & \\
 y^+ & & \hat{O} & & H_2^+ \\
 & \nwarrow & & \swarrow & \\
 & & \hat{G}^+ & &
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{ccccc}
 & & \hat{G}^- & & \\
 & \nearrow & & \searrow & \\
 y^- & & \hat{C} & & H_2^- \\
 & \searrow & & \nearrow & \\
 & & \bar{X} & & \\
 & \downarrow & & \downarrow & \\
 & & \bar{\Sigma}_t(\bar{X}) & & \\
 & \downarrow & & \downarrow & \\
 & & \bar{X} & & \\
 & \nearrow & & \searrow & \\
 y^- & & \hat{C} & & H_2^- \\
 & \nwarrow & & \swarrow & \\
 & & \hat{G}^- & &
 \end{array}
 \end{array}
 \quad
 \begin{array}{c}
 \Sigma_t(H_2^+) \\
 \Sigma_t(H_2^-)
 \end{array}
 \quad
 (8.2)
 \end{array}$$

commute, where $\hat{O}f = p^X f_W$, $\hat{C}f = p^{\bar{X}} f_{\bar{W}}$ and $\hat{R}$ and $\hat{\bar{R}}$ are the insertion maps $\hat{R}f = f$ and $\hat{\bar{R}}f = f$.

By isomorphism, X is observable if and only if $\text{cl}\{\text{Im}\hat{O}\} = X$ and constructible if and only if $\text{cl}\{\text{Im}\hat{C}\} = \bar{X}$. Clearly, since $aW \in X$ for all $a \in R^m$ (this is equivalent to $ay(0) \in X$, as can be seen by inserting $d\hat{y} = Wd\hat{u}$ into (6.1)), $\hat{O}e^{i\omega t}a = \Sigma_t(X)M_W a$ for $t \geq 0$, where $M_W : R^m \rightarrow X$ is the mapping $M_W a = aW$. Then, since $Y^+ = \overline{\text{sp}}\{ae^{i\omega t} \mid a \in R^m; t \geq 0\}$, $\text{cl}\{\text{Im}\hat{O}\} = \text{cl}\{U_{t \geq 0} \text{Im}[\Sigma_t(X)M_W]\}$. Likewise, noting that $a\bar{W} \in \bar{X}$ for all $a \in R^m$, $\text{cl}\{\text{Im}\hat{C}\} = \text{cl}\{U_{t \geq 0} \text{Im}[\bar{\Sigma}_t(\bar{X})M_{\bar{W}}]\}$, where $M_{\bar{W}} : R^m \rightarrow \bar{X}$ is defined as the mapping $M_{\bar{W}} a = a\bar{W}$. Hence we have proven

PROPOSITION 8.2. *Let X be a proper Markovian splitting subspace with spectral images X and $\bar{X}$. Then X is observable if and only if*

$$\text{cl}\{U_{t \geq 0} \text{Im}[\Sigma_t(X)M_W]\} = X \quad (8.3)$$

and constructible if and only if

$$\text{cl}\{U_{t \geq 0} \text{Im}[\bar{\Sigma}_t(\bar{X})M_{\bar{W}}]\} = \bar{X}. \quad (8.4)$$

Condition (8.3) is precisely the observability condition of a (deterministic) dynamical system with semigroup $\{\Sigma_t(X)^*; t \geq 0\}$ and read-out map (observation operator) $M_W^* : X \rightarrow R^m$ [45; p.211], and this is precisely the role these operators will play in the differential-equation type representations of Section 9. Likewise, (8.4) is the observability condition of a system with semigroup $\{\bar{\Sigma}_t(\bar{X}); t \geq 0\}$ and observation operation $M_{\bar{W}}^* : \bar{X} \rightarrow R^m$. Since any linear operator A from one Hilbert space V to another Z satisfies the condition

$$\text{cl}\{\text{Im}A\} \oplus \ker A^* = Z \quad (8.5)$$

we may alternatively write the observability condition (8.3) as

$$\bigcap_{t \geq 0} \ker[M_W^* \Sigma_t(X)^*] = 0 \quad (8.6)$$

and the constructibility condition (8.4)

$$\bigcap_{t \geq 0} \ker[M_W^* \bar{\Sigma}_t(\bar{X})^*] = 0, \quad (8.7)$$

where $\ker B$ is the null space (kernel) of B .

It follows from (6.12) that the relationship between the forward and the backward spectral image of X is given by

$$X = \bar{X}K, \quad (8.8)$$

where K is the structural function of X . Let $M_K : \bar{X} \rightarrow X$ be the mapping $f \mapsto fK$. Then M_K is a unitary operator. The following proposition described the interplay between forward and backward.

PROPOSITION 8.3. [16]. *Let X be a proper Markovian splitting subspace. Then the relations between its forward and backward spectral semigroups and its forward and backward observation operators is given by the commutative diagram*

$$\begin{array}{ccc} & \bar{X} & \xrightarrow{\bar{\Sigma}_t(\bar{X})} \bar{X} \\ R^m \nearrow & \downarrow M_K & \downarrow M_K \\ & X & \xrightarrow{\Sigma_t(X)^*} X \end{array}$$

COROLLARY 8.1. *The constructibility condition (8.4) can be written*

$$\text{cl}\{\bigcup_{t \geq 0} \text{Im}[\Sigma_t(X)^* M_W]\} = X \quad (8.9)$$

or equivalently

$$\bigcap_{t \geq 0} \ker[M_W^* \Sigma_t(X)] = 0. \quad (8.10)$$

In the sequel we shall express all relations only in its forward form, referring the reader to Proposition 8.3 for a recipe to obtain the backward counterpart.

This theory also provides us with a natural factorization of the autocorrelation function of y .

PROPOSITION 8.4. [16]. *The $m \times m$ matrix function $\Lambda(t) := E\{y(t)y(0)'\}$ has the factorizations*

$$\Lambda(t) = \begin{cases} M_W^* \Sigma_t(X) * M_W & \text{for } t \geq 0 \\ M_W^* \Sigma_{-t}(X) M_W & \text{for } t \leq 0. \end{cases} \quad (8.11)$$

Since M_W^* and $\Sigma_t(X)^*$ are bounded operators, they can be represented by matrices. Let $\{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n\}$ be an arbitrary basis in X , where, in general, $n = \infty$. (Such a basis exists, since X is separable.) Then $\Sigma_t(X)\hat{x}_i = \sum_j \alpha_{ij}(t)\hat{x}_j$ for some numbers $\{\alpha_{ij}(t)\}$. Since $\Sigma_t(X)$ is a strongly continuous semigroup, it has a representation $\Sigma_t(X) = e^{At}$, where A is the infinitesimal generator [39,41,45]. Therefore we shall write e^{Ft} to denote the $n \times n$ -matrix $\{\alpha_{ij}(t)\}$; i.e.

$$\Sigma_t(X)\hat{x}_i = \sum_{j=1}^n (e^{Ft})_{ij} \hat{x}_j. \quad (8.12)$$

Note that, unless $n < \infty$, F should not be interpreted as a matrix, since, in general, A is not a bounded operator defined everywhere on X . Similarly, if $\{e_1, e_2, \dots, e_m\}$ is the canonical (orthonormal) basis in R^m , we define the $m \times n$ -matrix H through the relation

$$M_W e_i = \sum_{j=1}^n H_{ij} \hat{x}_j. \quad (8.13)$$

Applying the operator Q_u^{-1} to this relation, we obtain

$$y_i(0) = \sum_{j=1}^n H_{ij} x_j, \quad (8.14)$$

where $\{x_1, x_2, \dots, x_n\} := \{Q_u^{-1}\hat{x}_1, Q_u^{-1}\hat{x}_2, \dots, Q_u^{-1}\hat{x}_n\}$ is a basis in X . Relation (8.14) illustrates the fact that H is a matrix of the observation operator. Define P to be the $n \times n$ covariance matrix with components

$$P_{ij} = E\{x_i x_j\}. \quad (8.15)$$

Then it is clear from (6.5b) that

$$P_{ij} = \langle \hat{x}_i, \hat{x}_j \rangle_X, \quad (8.16)$$

where $\langle \cdot, \cdot \rangle_X$ denotes inner product in the Hilbert space X .

We are now in a position to formulate the results of this section in matrix form. For $t \geq 0$, Proposition 8.4 yields

$$\begin{aligned}
\Lambda_{ij}(t) &= \langle e_i, M_W^* \Sigma_t(X) M_W e_j \rangle_{R^m} \\
&= \langle \Sigma_t(X) M_W e_i, M_W e_j \rangle_X
\end{aligned} \tag{8.17}$$

But, combining (8.12) and (8.13), we have

$$\Sigma_t(X) M_W e_i = \sum_{j=1}^n (He^{Ft})_{ij} \hat{x}_j, \tag{8.18}$$

which inserted into (8.17) together with (8.13) yields

$$\Lambda(t) = He^{Ft} PH' \quad \text{for } t \geq 0 \tag{8.19a}$$

after applying (8.16). In the same way, letting $e^{F't}$ denote the transpose of e^{Ft} , we have

$$\Lambda(t) = HPe^{-F't} H' \quad \text{for } t \leq 0. \tag{8.19b}$$

To obtain matrix representations for the backward operators, first note that $\{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n\}$, where $\hat{x}_i = M_K^{-1} \hat{x}_i$ ($i=1,2,\dots,n$), is a basis in $\bar{X}$. By applying M_K^{-1} to (8.13), we see that M_W has the matrix H with respect to this basis. However, for the semigroup the situation is a bit more complicated. For $t \leq 0$, define the $n \times n$ -matrix $e^{\bar{F}t}$ by

$$\bar{\Sigma}_{-t}(X) \hat{x}_i = \sum_{j=1}^n (e^{\bar{F}t})_{ij} \hat{x}_j, \tag{8.20}$$

which, by Proposition 8.3, can be written

$$\Sigma_t(X) \hat{x}_i = \sum_{j=1}^n (e^{-\bar{F}t})_{ij} \hat{x}_j \tag{8.21}$$

for all $t \geq 0$. Then, taking (8.16) into account, inserting (8.12) and (8.21) in the defining relation

$$\langle \hat{x}_i, \Sigma_t(X) \hat{x}_j \rangle = \langle \Sigma_t(X) \hat{x}_i, \hat{x}_j \rangle$$

yields

$$e^{\bar{F}t} = Pe^{-Ft}P^{-1} \quad \text{for } t \geq 0, \tag{8.22}$$

which should be compared with the corresponding result in [8]. (Also see [46] for the original wide sense result.)

Finally, it is not hard to see that the observability criterion (8.6) has the matrix formulation

$$\bigcap_{t \geq 0} \ker(\text{He}^{Ft}) = 0 \quad (8.23)$$

and that the matrix version of the constructibility criterion (8.7) reads

$$\bigcap_{t \leq 0} \ker(\text{He}^{\bar{F}t}) = 0. \quad (8.24)$$

9. DYNAMICAL REPRESENTATIONS OF MARKOVIAN SPLITTING SUBSPACES

At this point the abstract realization problem has been solved. All proper Markovian splitting subspaces have been determined, and the corresponding semigroups and observation operators have been characterized. The problem is now to what extent these abstract realizations can be represented by stochastic differential equations.

Let X be a proper Markovian splitting subspace with forward generating process u . We shall say that y admits a regular (forward) realization with respect to X if, for some Hilbert space S , there are a strongly continuous semigroup $\{e^{At}; t \geq 0\}$ on S and a bounded operator $B : R^P \rightarrow S$ such that

$$X = \left\{ \sum_{k=1}^P \int_{-\infty}^0 \langle c, e^{-A\sigma} B e_k \rangle_S du_k(\sigma) \mid c \in S \right\}, \quad (9.1)$$

where $\langle \cdot, \cdot \rangle_S$ denotes the inner product in S and $\{e_1, e_2, \dots, e_P\}$ is the canonical basis in R^P . [Of course, in order that (9.1) be well-defined, the function $t \mapsto \langle c, e^{At} B e_k \rangle$ must belong to $L_2(0, \infty)$ for all $c \in S$ and $k = 1, 2, \dots, m$, so this is implicitly assumed in the definition.] Then, since $y_i(0) \in X$ for each $i = 1, 2, \dots, m$, there are $c_1, c_2, \dots, c_m \in S$ such that

$$y_i(0) = \sum_{k=1}^P \int_{-\infty}^0 \langle c_i, e^{-A\sigma} B e_k \rangle_S du_k(\sigma). \quad (9.2)$$

Define the operator $C : S \rightarrow R^m$ in the following way (since we shall have no further use of the constructibility operator, we shall take the liberty to give a new meaning to C): Let Cf be the m -dimensional vector with components $\langle c_i, f \rangle_S$, $i = 1, 2, \dots, m$. Then C is a bounded operator, and (9.2) can be written

$$y(0) = \int_{-\infty}^0 Ce^{-A\sigma} Bdu(\sigma) . \quad (9.3)$$

We shall call such a representation a (*forward*) *regular realization of y with respect to X*, the word "regular" referring to the boundedness of the operators e^{At} , B and C and "with respect to X" referring to property (9.1). The Hilbert space S will be called the *range space* of the realization, X the *state space*. Applying the group $\{U_t; t \in R\}$ of shift operators to (9.3) we obtain

$$y(t) = \int_{-\infty}^t Ce^{A(t-\sigma)} Bdu(\sigma) \quad (9.4)$$

for each $t \in R$, as can be seen by a simple change of variables.

Formally we can write (9.4) as

$$\begin{cases} x(t) = \int_{-\infty}^t e^{A(t-\sigma)} Bdu(\sigma) \\ y(t) = Cx(t) , \end{cases} \quad (9.5)$$

where the *state* $x(t)$ takes values in S. However, in doing so, we shall have to be careful. Unless $\dim S < \infty$, $x(t)$ cannot in general be defined as a Hilbert space-valued random variable in the usual sense [45,47]. For this to be possible the covariance operator must be nuclear, and, as we shall see below, this is usually not the case. However, $\{x(t); t \in R\}$ can be interpreted as a *weak* Markov process, by using the theory for weak random variables developed by Balakrishnan [45]; for a discussion of this, see [16]. In any case, we may formally write the realization in the differential-equation form

$$\begin{cases} dx = Axdt + Bdu \\ y = Cx , \end{cases} \quad (9.6)$$

to be interpreted either by (9.4) or (9.5), the latter requiring the theory of [45].

From (6.4) and (6.5) we have

$$y(0) = \int_{-\infty}^0 w(-\sigma) du(\sigma) , \quad (9.7)$$

where $w = F^{-1}W$ is the inverse Fourier transform of the spectral

factor W . Comparing (9.3) and (9.7) it is seen that, in general, there is no regular realization with respect to an arbitrary X , since w must be continuous for this to be the case. Nevertheless, as we shall see below, the regular realizations are, in a sense, dense so that we can always find one that is an arbitrarily good approximation. Let us assume, for the moment, that y does admit a regular realization with respect to X . Then we may ask the question whether any deterministic realization $w(t) = Ce^{At}B$ which is regular (i.e. B and C are bounded [48,49]) would do the job. To answer this question, note that, for $t \geq 0$,

$$E^S y_i(t) = \sum_{k=1}^n \int_{-\infty}^0 \langle e^{A^*t} c_i, e^{-A\sigma} B e_k \rangle_S du_k(\sigma), \quad (9.8)$$

where e^{A^*t} denotes the adjoint of e^{At} , and that, by the splitting property (2.4), $E^S y_i(t) = E^X y_i(t)$. Therefore, forming the closed span of (9.8) over all $t \geq 0$ and $i = 1, 2, \dots, m$, it is seen that (9.1) is satisfied if $E^X H^+ = X$ and $\overline{\text{sp}}\{e^{A^*t} c_i; t \geq 0, i = 1, 2, \dots, m\} = S$. Hence, if X is observable and the pair (C, A) is observable in the deterministic sense [45], the answer to our question is "yes." For an arbitrary proper Markovian splitting subspace X , however, things are more complicated. This should be expected, since in general two spectral factors, namely W and $\bar{W}$, are needed to characterize X , whereas in the observable case $\bar{W}$ is uniquely determined from W via (4.5). Also it is reasonable to require that the range space is as small as possible. This is achieved by requiring that (A, B) is controllable [45]. Note, however, that the controllability condition has nothing to do with the splitting subspace X , but is desired merely in order to obtain a range space S "of the same size" as X .

In view of the results of Section 8, this suggests that we try to use the spectral image X of X as a range space. To this end, we first need to introduce the concept of *regular splitting subspace*. Let $\hat{C}$ be the class of all functions in $L_2^P(I)$ having a continuous inverse Fourier transform. (Note that $L_1^P(I) \cap L_2(I)$ is a proper subset of $\hat{C}$.) Then, in view of (6.5), $X \subset \hat{C}$ if the regularity condition (9.1) holds, and therefore the operator $V_0 : X \rightarrow R^p$ given by $V_0 f = (F^{-1}f)(0)$ is defined on all of X . We shall say that X is a *regular splitting subspace* if V_0 is a bounded operator.

LEMMA 9.1. [16]. *Let X be a proper Markovian splitting subspace and let $X := Q_u X$ be its spectral image. Then X is regular if and only if $X \subset \hat{C}$.*

Clearly all finite dimensional X , if there are any, are regular. The following lemma is a corollary of Proposition 8.1.

LEMMA 9.2. [16]. Let $\hat{\xi} \in X$ and let $\xi^V := F^{-1}\hat{\xi}$. Then, for each $t \geq 0$,

$$\xi(t + \sigma) = (F^{-1}\Sigma_t(X)\hat{\xi})(\sigma) \quad (9.9)$$

for almost all σ on $[0, \infty)$.

Now, let X be a regular splitting subspace, and let $a \in R^m$. Then, since $aW \in X \subset \hat{C}$ (Lemma 9.1), $aw(t)$ is continuous and Lemma 9.2 yields

$$aw(t) = V_0 \Sigma_t(X) M_W^* a. \quad (9.10)$$

[It can be seen [16] that (9.10) corresponds to the factorization (5.4a).] This leads to the factorization described by the commutative diagram

$$\begin{array}{ccc} R^m & \xrightarrow{M_W(t)} & R^p \\ & \searrow M_W & \nearrow V_0 \\ & X & \xrightarrow{\Sigma_t(X)} X \end{array} \quad (9.11)$$

where $M_W(t)$ denotes multiplication from the right by the matrix $w(t)$. Since matrix multiplication from the left is the adjoint of multiplication by the same matrix from the right,

$$w(t)b = (V_0 \Sigma_t(X) M_W^*)^* b. \quad (9.12)$$

Now, $\Sigma_t(X)$ and M_W are bounded operators, and regularity of X insures that V_0 is bounded also. Therefore (9.7) and (9.12) yield after applying the shift

$$y(t) = \int_{-\infty}^t M_W^* \Sigma_{t-\sigma}(X)^* V_0^* du(\sigma). \quad (9.13)$$

This is a regular realization with respect to X , as is seen from the following theorem.

THEOREM 9.1. [16]. The process y has a regular realization with respect to X if and only if X is a regular splitting subspace. In this case, the range space can be taken to be $X := Q_0 X$ and e^{At} , B and C to be $\Sigma_t(X)^*$, V_0^* and M_W^* respectively.

We shall call (9.13) the standard (forward) realization corresponding to X . We have already encountered the observation operator

M_W^* and the Markov semigroup $\Sigma_t(X)^*$ in Section 8, where conditions for observability and constructibility were given. It can be shown [16] that the standard realization is *spectrally minimal* [39], and that the pair $(\Sigma_t(X)^*, V_0^*)$ is exactly controllable. The covariance matrix of $x(0)$ is the identity I , and therefore the corresponding representation (9.5) must be interpreted in the weak sense.

Obviously, if y admits a forward regular realization with respect to X , it also admits a backward one. In particular, the analysis above can be carried out in the backward setting also to yield the *standard backward realization*

$$y(t) = \int_t^\infty M_W^* \Sigma_{\sigma-t}(X)^* \bar{V}_0^* d\bar{u}(\sigma) \quad (9.14)$$

corresponding to X . The relationship between the operators in (9.13) and (9.14) is described by Proposition 8.3 and

$$V_0 = M_K \bar{V}_0. \quad (9.15)$$

Also it is not hard to see that $\bar{x}(0) := M_K^{-1}x(0)$ is the state corresponding to the standard backward realization.

However, an arbitrary proper Markovian splitting subspace will in general not be regular, since $F^{-1}Q_U X$ may contain functions which are not continuous (Lemma 9.1), and in this case y will have no regular realization with respect to X (Theorem 9.1). (Note that this situation is unique to *continuous-time* processes; in the discrete-time setting all X are regular, since the evaluation operator V_0 is always bounded.) We shall show, however, that in each X there is a *dense* subset of random variables ξ such that each process $\xi(t) := U_t \xi$ admits a regular realization with respect to a subspace of X .

Our basic strategy will be to convolute each function in $F^{-1}Q_U X$ by a scalar L_2 function ϕ . The resulting functions will be continuous [50; p.398] and consequently the techniques described above can be applied. Hence define the subspace $X_\phi \subset H$ to be

$$X_\phi = Q_U^{-1} F[\phi * (F^{-1}Q_U X)]. \quad (9.16)$$

Also let $B_\phi : R^P \rightarrow X$ be the bounded operator defined by $B_\phi b = P_X(F\phi_- b)$, where $X = Q_U X$ and $\phi_-(t) := \phi(-t)$.

PROPOSITION 9.1. [16]. Let $\phi \in L_2(0, \infty)$. Then $X_\phi \subset X$. Moreover,

$$X_\phi = \left\{ \sum_{k=1}^P \int_{-\infty}^0 \langle \xi, \Sigma_{-\sigma}(X) * B_\phi e_k \rangle_X du_k(\sigma) \mid \xi \in X \right\} \quad (9.17)$$

where $\{e_1, e_2, \dots, e_p\}$ is the canonical basis in R^P .

Here (9.17) should be compared with (9.1). Even if y does not admit a regular realization with respect to X , it can be approximated uniformly closely by another process which has a regular realization with respect to $X_\phi \subset X$.

PROPOSITION 9.2. [16]. For any $\varepsilon > 0$, there is a $\phi \in L_2(0, \infty)$ such that, for all $t \in R$ and $k = 1, 2, \dots, m$,

$$\|y_k(t) - y_k(t; \phi)\| < \varepsilon$$

where

$$y(t; \phi) = \int_{-\infty}^t M_W^* \Sigma_{t-\sigma}(X) * B_\phi du(\sigma). \quad (9.18)$$

Note that the only difference between (9.13) and (9.18) is the B -matrix. For the moment, let B denote either V_0^* or B_ϕ , and let $\{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_p\}$ be the canonical basis in R^P , $\{e_1, e_2, \dots, e_m\}$ being reserved for R^m as in Section 8. Let $h(t) := M_W^* \Sigma_t(X) * B$. Then

$$\begin{aligned} h_{ij}(t) &= \langle e_i, M_W^* \Sigma_t(X) * B \tilde{e}_j \rangle_{R^m} \\ &= \langle B^* \Sigma_t(X) M_W e_i, \tilde{e}_j \rangle_{R^P}. \end{aligned} \quad (9.19)$$

Now, let n be the (usually infinite) dimension of X , and let $\{\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n\}$ be the basis in X introduced in Section 8. Let e^{Ft} and H be the matrices of $\Sigma_t(X)^*$ and M_W^* respectively as defined by (8.12) and (8.13), and let G be the corresponding matrix of B , i.e.

$$B^* \hat{x}_i = \sum_{j=1}^n G_{ij} \tilde{e}_j. \quad (9.20)$$

Then it follows from (9.19) that $h(t) = H e^{Ft} G$, i.e. (9.13) or (9.18) has the matrix representation

$$y(t) = \int_{-\infty}^t H e^{F(t-\sigma)} B du(\sigma), \quad (9.21)$$

which, taking due care to properly define the possibly infinite-dimensional state, can be written in differential-equation form.

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A comprehensive theory of stochastic realization for multivariate stationary Gaussian processes is presented. It is coordinate-free in nature, starting out with an abstract state space theory in Hilbert space, based on the concept of splitting subspace. These results are then carried over to the spectral domain and described in terms of Hardy functions. Each state space is uniquely characterized by its structural function, an inner function which contains all the systems theoretical characteristics of the corresponding realizations. Finally coordinated are introduced and concrete differential		

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equation-type representations are obtained. This paper, which in turn summarizes and considerably extends results which have previously been presented in a series of preliminary conference papers .

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