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This paper presents a hand-held calculator algorithm to compute scale factors for state plane coordinate systems. A second algorithm is presented for digital computers. The algorithms are applicable to any figure of the earth, but this paper addresses Clarke 1866 (NAD 1927) and NAD 1983.

SIMPLE ALGORITHMS FOR CALCULATION OF SCALE FACTORS FOR PLANE COORDINATE SYSTEMS (1927 NAD AND 1983 NAD)

by

[redacted]

Dr. Herbert W. Stoughton, P.E., L.S.

INTRODUCTION

In March 1972, Berry presented a paper by the above title at the Annual Meeting of ACSM. The paper discussed algorithms for the state plane coordinate systems for 1927 NAD. Various plane coordinate systems (state coordinate systems, city systems, Corps of Engineers project systems, the Great Lakes system, special engineering projects, etc.) have been designed for the purpose of simplifying survey calculations from the spheroidal earth to a plane with an orthogonal grid. The controlling design criteria in many instances has been to restrict the maximum difference between the geodetic length and the grid length to one part in ten thousand throughout the area included within the system. This was done on the assumption that if the reduction of measured distances to geodetic lengths to grid lengths was omitted, the surveys would meet Third-Order accuracy (one part in five thousand). This assumption was valid when all the distances were measured by taping, which was normally performed to 0.01 or 0.02 feet per one hundred feet, and thus the scale factor could be ignored.

At the time these systems were designed, this assumption was probably valid, but supplementary tables were required to obtain the scale factors for higher order surveys. This meant that a scale factor was extracted from a table as a function of latitude (for Lambert systems) or the perpendicular distance from the central meridian (transverse Mercator systems). With the advent of the hand-held computer/calculator and digital computers, then it is not practical or efficient to store this data in the computer. Furthermore, the values listed in these tables are calculated to seven decimal places, which is insufficient precision for electronic distance measurements. Also, for longer lines it is recommended that the scale factor for a line should be:

$$k = \frac{1}{6}(k_1 + 4k_m + k_2)$$

Where:
k_1 = the scale factor at one end of the line (point 1).
k_2 = the scale factor at the other end of the line (point 2).
k_m = the scale factor at the midpoint of the line.

PART I

HAND-HELD CALCULATORS VERSION

The formulas to calculate the scale factors require the geodetic position (latitude and longitude) for each point. If the surveyor is employing plane coordinates, this requires a transformation of the plane coordinates to geodetic positions before calculating the scale factors. The formulas employed to calculate scale factors appear in Part II of this paper, and will be employed to derive the algorithms for use on digital computers.

DERIVATION OF THE ALGORITHM

Figure 1 is a section depicting the relationship between a spherical earth and the projection plane. This section is taken *perpendicular* to the principal axis of the coordinate system, which for various projections is:

Lambert: The parallel of latitude approximately midway between the standard parallels (designated ϕ_0), where the scale factor attains its minimum value. Then, Figure 1 is a meridional section.

Transverse Mercator: The central meridian of the zone is a "normal" section, i.e. it is taken perpendicular to the central meridian at the "approximate center" of the north-south range of the zone [see the Appendix], and designated ϕ_0. (This does *not* trace out a parallel of latitude.)

Hotine Skew Orthomorphic (Oblique Mercator): An approximate geodesic line, which passes through a specified point, and has a specified azimuth (non-cardinal at that point). Figure 1 is a section perpendicular

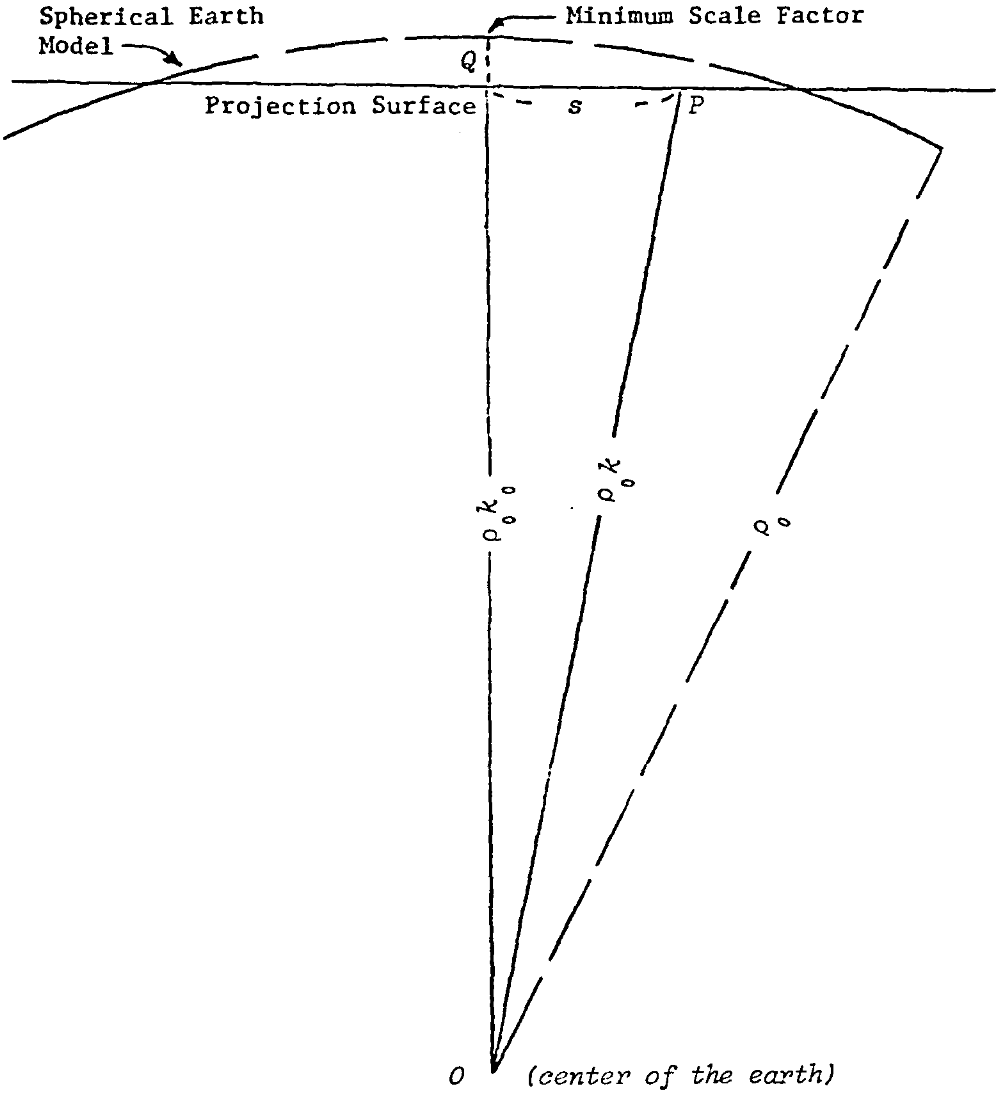

Figure 1

to this line, preferably through the specified point ("center point"), designated ϕ_0.

In all three cases the spherical radius in this section is designated ρ_0, and is calculated by:

$$\rho_0 = \frac{a\sqrt{1. - e^2}}{(1. - e^2 \sin^2\phi_0)} \tag{1}$$

Where:

a = the length of the semi-major axis of the reference ellipsoid (in meters).

e^2 = the square of the eccentricity of the reference ellipsoid.

Table 1 contains the numerical values of a and e^2 for 1927 NAD and 1983 NAD.

Table 1

Ellipsoid	a (meters)	e^2
1927 NAD	6,378,206.400	0.006768657997
1983 NAD	6,378,137.000	0.006694380025

The scale factor of the projection along the principal axis is designated k_0. At any point P located at a distance s, perpendicular to the projection's principal axis, the scale factor is k. Disregarding conformality and assuming spherical shape of the earth, the distance from O (the center of the earth) to point Q (the point on the projected principal axis where it is intersected by the depicted section) is equal to the radius, ρ_0, multiplied by the central meridian scale factor, k_0. At point P, the distance from the center of the earth, O, is $\rho_0 k$. Then, employing the Pythagorean relationship to the right triangle QPO yields:

$$(\rho_0 k)^2 = (\rho_0 k_0)^2 + s^2$$

And:

$$\rho_0 k = \left[(\rho_0 k_0)^2 + s^2\right]^{1/2} \tag{2}$$

Recall the binomial series:

$$(x + y)^n = x^n + nx^{n-1} y + \frac{n(n - 1)}{2} x^{n-2} y^2 + \frac{n(n - 1)(n - 2)}{6} x^{n-3} y^3 + \ldots$$

Let:

$$x = (\rho_0 k_0)^2$$

$$y = s^2$$
$$n = \frac{1}{2}$$

Then, solving for k, equation (2) can be written:

$$k = \frac{1}{\rho_0}\left[(\rho_0 k_0) + \frac{1}{2}(\rho_0 k_0)^{-1}s^2 - \frac{1}{8}(\rho_0 k_0)^{-3}s^4 + \frac{1}{16}(\rho_0 k_0)^{-5}s^6 - \ldots\right]$$

Rearranging:

$$k = k_0 + \frac{s^2}{2\rho_0^2 k_0} - \frac{s^4}{8\rho_0^4 k_0^3} + \frac{s^6}{16\rho_0^6 k_0^5} - \ldots$$

Finally:

$$k = k_0\left[1. + \left(\frac{1}{2\rho_0^2 k_0^2}\right)s^2 - \left(\frac{1}{2\rho_0^2 k_0^2}\right)^2 \frac{s^4}{2} + \left(\frac{1}{2\rho_0^2 k_0^2}\right)^3 \frac{s^6}{2} - \ldots\right] \tag{3}$$

Let:

$$\Gamma^2 = 10^{-15}\, s^2$$
$$K = \frac{10^{15}}{2\rho_0^2 k_0^2}$$

Where, K is a constant for each system.

Then, equation (3) is:

$$k = k_0\left[1. + K\Gamma^2(1. - \frac{K}{2}\Gamma^2(1. - K\Gamma^2 \ldots))\right] \tag{4}$$

In practice, the fourth term is insignificant and can be disregarded. The third term is required only when the scale factor, k, is required at the *extreme* width of the system (greater than seventy miles from the principal axis).

It remains to develop formulae for the computation of s, thus Γ (the distance from the principal axis) for any point P defined by its plane coordinates (x - east; y - north).

Transverse Mercator Systems.

$$\Gamma^2 = (x - x_0)^2 * 10^{-15} \tag{5}$$

Where:
x_0 = the x-coordinate of the system's central meridian (listed in Table 2).

Lambert Conic Systems.

$$\Gamma^2 = [R_0 - ((x - x_0)^2 + (R_b - y)^2)^{1/2}]^2 * 10^{-15} \tag{6}$$

Where:
R_b = the constant for each system [listed in Table 2].
R_0 = the constant for each system [listed in Table 2].

Hotine Skew Orthomorphic (Oblique Mercator) Systems.

$$\Gamma^2 = [(x - x_0)\cos\gamma_0 - (y - y_0)\sin\gamma_0]^2 * 10^{-15} \tag{7}$$

Where:
x_0 = x-translation constant for each system [listed in Table 3].
y_0 = y-translation constant for each system [listed in Table 3].
γ_0 = azimuth of the principal axis at the center point [listed in Table 3].

The data in Table 2 refers to the North American Datum of 1927. A similar set of numerical values of K must be calculated for 1983 NAD. At this time (1981) it is inappropriate to prepare similar tables, as it is unknown whether the states will retain the current systems and defining parameters. For the transverse Mercator systems these parameters (other than the ellipsoid and datum) the central meridian, λ_0, and the scale factor along the central meridian, k_0, and the x-coordinate of the central meridian, x_0, need to be defined. Historically, ϕ_0 was arbitrarily selected at the midgeodetic latitude of the zone, rounded to the nearest five minutes of arc [except for Alaska and Delaware]. If the basic design of the existing transverse Mercator system is identical to the present design, ϕ_0 listed in Table 2 is acceptable for calculating K. Then, for 1983 NAD the equation to calculate K is:

Table 2

Constants For Lambert And Transverse Mercator Systems (1927 NAD)

State	type	zone	ϕ_0	K	k_0	x_0 (feet)	R_0 (feet)	R_b (feet)
Alabama	T	E	32°55'00".000	1.145120	0.9999600000	500,000.00	-	-
	T	W	32°40'00".000	1.145243	0.9999333333	500,000.00	-	-
Alaska	T	2	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	3	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	4	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	5	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	6	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	7	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	8	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	T	9	62°12'30".000	1.137700	0.9999000000	500,000.00	-	-
	L	10	52°50'13".954	1.140103	0.9998480641	3,000,000.00	15,893,950.36	16,564,628.77
Arizona	T	E	34°10'00".000	1.144945	0.9999000000	500,000.00	-	-
	T	C	34°10'00".000	1.144945	0.9999000000	500,000.00	-	-
	T	W	34°30'00".000	1.144785	0.9999333333	500,000.00	-	-
Arkansas	L	N	35°35'03".225	1.144502	0.9999359370	2,000,000.00	29,277,593.61	29,732,882.87
	L	S	34°02'03".874	1.144936	0.9999184698	2,000,000.00	31,014,039.23	31,511,724.20
California	L	1	40°50'06".386	1.143216	0.9998946358	2,000,000.00	24,245,358.05	24,792,436.23
	L	2	39°05'04".766	1.143637	0.9999146793	2,000,000.00	25,795,850.31	26,312,257.65
	L	3	37°45'03".851	1.143956	0.9999291792	2,000,000.00	27,057,475.85	27,512,992.04
	L	4	36°37'33".092	1.144223	0.9999407628	2,000,000.00	28,182,405.33	28,652,931.96
	L	5	34°45'03".806	1.144747	0.9999221277	2,000,000.00	30,194,145.54	30,649,424.27
	L	6	33°20'02".128	1.145030	0.9999541438	2,000,000.00	31,846,570.92	32,271,267.72
	L	7	34°08'30".543	1.144749	0.9999885350	4,186,692.58	30,891,382.10	35,055,396.31
Colorado	L	N	40°15'02".568	1.143230	0.9999568475	2,000,000.00	24,751,897.68	25,086,068.20
	L	C	39°06'03".660	1.143584	0.9999359117	2,000,000.00	25,781,376.91	26,243,052.74
	L	S	37°50'02".980	1.143897	0.9999453995	2,000,000.00	26,977,133.89	27,402,231.82
Connecticut	L	-	41°32'01".034	1.142826	0.9999831405	600,000.00	23,659,233.56	23,914,389.02
Delaware	T	-	39°09'00".000	1.143436	0.9999950281	500,000.00	-	-
Florida	T	E	28°00'00".000	1.146327	0.9999411765	500,000.00	-	-

Table 2

State	type	zone	ϕ_0	K	k_0	x_0 (feet)	R_0 (feet)	R_b (feet)
Florida	T	W	28°00'00".000	1.146327	0.9999411765	500,000.00	-	-
	L	N	30°10'02".112	1.145811	0.9999484343	2,000,000.00	36,030,443.05	36,454,924.53
Georgia	T	E	32°50'00".000	1.145278	0.9999000000	500,000.00	-	-
	T	W	32°50'00".000	1.145278	0.9999000000	500,000.00	-	-
Hawaii	T	1	19°30'00".000	1.147963	0.9999666667	500,000.00	-	-
	T	2	20°55'00".000	1.147713	0.9999666667	500,000.00	-	-
	T	3	21°30'00".000	1.147553	0.9999900000	500,000.00	-	-
	T	4	22°05'00".000	1.147444	0.9999900000	500,000.00	-	-
	T	5	21°55'00".000	1.147452	1.0000000000	500,000.00	-	-
Idaho	T	E	43°50'00".000	1.142287	0.9999473684	500,000.00	-	-
	T	C	43°50'00".000	1.142287	0.9999473684	500,000.00	-	-
	T	W	45°30'00".000	1.141868	0.9999333333	500,000.00	-	-
Illinois	T	E	39°45'00".000	1.143322	0.9999750000	500,000.00	-	-
	T	W	39°45'00".000	1.143399	0.9999411765	500,000.00	-	-
Indiana	T	E	39°50'00".000	1.143319	0.9999666667	500,000.00	-	-
	T	W	39°50'00".000	1.143319	0.9999666667	500,000.00	-	-
Iowa	L	N	42°40'03".521	1.142607	0.9999453686	2,000,000.00	22,736,950.34	23,162,461.59
	L	S	41°12'03".169	1.142995	0.9999483705	2,000,000.00	23,936,585.11	24,374,096.67
Kansas	L	N	39°15'02".540	1.143496	0.9999568556	2,000,000.00	25,644,959.12	25,979,068.57
	L	S	37°55'03".503	1.143896	0.9999359200	2,000,000.00	26,896,024.48	27,351,521.50
Kentucky	L	N	38°28'02".118	1.143692	0.9999620817	2,000,000.00	26,371,820.68	26,724,051.82
	L	S	37°20'02".928	1.144028	0.9999453808	2,000,000.00	27,467,860.75	27,832,235.64
Louisiana	L	N	31°55'03".744	1.145469	0.9999147417	2,000,000.00	33,624,568.36	34,079,629.33
	L	S	30°00'03".024	1.145902	0.9999257458	2,000,000.00	36,271,389.35	36,756,553.45
	L	OS	27°00'03".785	1.146655	0.9998947956	2,000,000.00	41,091,749.54	41,576,762.39
Maine	T	E	45°50'00".000	1.141854	0.9999000000	500,000.00	-	-
	T	W	44°50'00".000	1.141973	0.9999666667	500,000.00	-	-
Maryland	L	-	38°52'32".833	1.143612	0.9999498485	800,000.00	25,989,474.99	26,369,112.76
Massachusetts	L	M	42°12'02".246	1.142689	0.9999645506	600,000.00	23,111,975.14	23,549,477.32
	L	I	41°23'00".484	1.142831	0.9999984844	200,000.00	23,784,678.44	23,924,398.02
Michigan	T	E	44°15'00".000	1.142185	0.9999428571	500,000.00	-	-

Table 2

State	type	zone	ϕ_0	K	k_0	x_0 (feet)	R_0 (feet)	R_b (feet)
Michigan	T	C	44°15'00".000	1.142262	0.9999090909	500,000.00	-	-
	T	W	46°40'00".000	1.141608	0.9999090909	500,000.00	-	-
	L	N	46°17'07".101	1.141638	0.9999410344	2,000,000.00	20,041,716.18	20,589,420.09
	L	C	44°56'36".092	1.141979	0.9999509058	2,000,000.00	21,001,715.22	21,594,768.40
	L	S	42°53'06".055	1.142549	0.9999450783	2,000,000.00	22,564,848.51	23,069,597.22
Minnesota	L	N	47°50'07".498	1.141307	0.9999028166	2,000,000.00	18,984,319.62	19,471,398.75
	L	C	46°20'05".708	1.141668	0.9999220223	2,000,000.00	20,006,679.72	20,493,457.15
	L	S	44°30'05".355	1.142164	0.9999220448	2,000,000.00	21,327,006.06	21,874,349.14
Mississippi	T	E	32°40'00".000	1.145182	0.9999600000	500,000.00	-	-
	T	W	33°10'00".000	1.145101	0.9999411765	500,000.00	-	-
Missouri	T	E	38°20'00".000	1.143793	0.9999333333	500,000.00	-	-
	T	C	38°35'00".000	1.143727	0.9999333333	500,000.00	-	-
	T	W	38°35'00".000	1.143709	0.9999411765	500,000.00	-	-
Montana	L	N	48°17'02".221	1.141029	0.9999714855	2,000,000.00	18,689,498.40	19,157,874.26
	L	C	47°10'05".877	1.141443	0.9999220151	2,000,000.00	19,432,939.76	19,919,806.36
	L	S	45°38'06".383	1.141883	0.9999107701	2,000,000.00	20,500,650.51	21,096,820.93
Nebraska	L	N	42°20'02".258	1.142653	0.9999645501	2,000,000.00	23,004,346.29	23,368,977.46
	L	S	41°00'04".748	1.143109	0.9999220725	2,000,000.00	24,104,561.06	24,590,781.86
Nevada	T	E	39°00'00".000	1.143693	0.9999000000	500,000.00	-	-
	T	C	39°00'00".000	1.143693	0.9999000000	500,000.00	-	-
	T	W	39°00'00".000	1.143693	0.9999000000	500,000.00	-	-
New Hampshire	T	-	44°00'00".000	1.142198	0.9999666667	500,000.00	-	-
New Jersey	T	-	40°00'00".000	1.143255	0.9999750295	2,000,000.00	-	-
New Mexico	T	E	34°30'00".000	1.144840	0.9999090909	500,000.00	-	-
	T	C	34°30'00".000	1.144861	0.9999000000	500,000.00	-	-
	T	W	34°15'00".000	1.144886	0.9999166667	500,000.00	-	-
New York	T	E	42°50'00".000	1.142513	0.9999666667	500,000.00	-	-
	T	C	43°00'00".000	1.142535	0.9999375000	500,000.00	-	-
	T	W	42°40'00".000	1.142625	0.9999375000	500,000.00	-	-
	L	LI	40°51'00".310	1.142983	0.9999949000	2,000,000.00	24,235,000.80	24,462,545.30
North Carolina	L	-	35°15'06".461	1.144733	0.9998725510	2,000,000.00	29,637,059.47	30,183,611.25
North Dakota	L	N	48°05'04".985	1.141164	0.9999358426	2,000,000.00	18,819,849.05	19,215,516.01

Table 2

State	type	zone	ϕ_0	K	k_0	x_0 (feet)	R_0 (feet)	R_b (feet)
North Dakota	L	S	46°50'04".778	1.141501	0.9999358523	2,000,000.00	19,661,027.79	20,086,977.18
Ohio	L	N	41°04'03".718	1.143052	0.9999391411	2,000,000.00	24,048,738.51	24,559,158.47
	L	S	39°23'03".693	1.143509	0.9999359346	2,000,000.00	25,522,875.81	26,027,071.12
Oklahoma	L	N	36°10'02".805	1.144331	0.9999454101	2,000,000.00	28,657,671.66	29,082,831.70
	L	S	34°35'03".107	1.144757	0.9999359432	2,000,000.00	30,382,831.06	30,838,032.96
Oregon	L	N	45°10'07".415	1.142046	0.9998945810	2,000,000.00	20,836,250.94	21,383,852.48
	L	S	43°10'06".926	1.142588	0.9998946058	2,000,000.00	22,341,309.43	22,888,667.15
Pennsylvania	L	N	41°25'02".668	1.142917	0.9999568410	2,000,000.00	23,755,351.27	24,211,050.37
	L	S	40°27'02".425	1.143171	0.9999595012	2,000,000.00	24,577,800.67	24,984,826.43
Rhode Island	T	-	41°40'00".000	1.142766	0.9999937500	500,000.00	-	-
South Carolina	L	N	34°22'02".625	1.144791	0.9999454207	2,000,000.00	30,630,125.53	31,127,724.75
	L	S	33°00'03".083	1.145162	0.9999326284	2,000,000.00	32,252,126.30	32,676,887.65
South Dakota	L	N	45°03'04".264	1.141976	0.9999391116	2,000,000.00	20,922,704.09	21,366,697.03
	L	S	43°37'06".208	1.142433	0.9999068931	2,000,000.00	21,993,575.61	22,461,937.05
Tennessee	L	-	35°50'02".619	1.144410	0.9999484030	2,000,000.00	29,010,231.09	29,535,149.91
Texas	L	N	35°25'04".456	1.144602	0.9999108771	2,000,000.00	29,456,907.29	29,972,959.94
	L	NC	33°03'05".834	1.145287	0.9998726224	2,000,000.00	32,187,609.58	32,691,654.54
	L	C	31°00'05".008	1.145766	0.9998817443	2,000,000.00	34,851,703.46	35,337,121.23
	L	SC	29°20'05".424	1.146201	0.9998632433	2,000,000.00	37,261,509.20	37,807,440.38
	L	S	27°00'03".785	1.146655	0.9998947956	2,000,000.00	41,091,749.54	41,576,762.39
Utah	L	N	41°15'02".655	1.142962	0.9999568422	2,000,000.00	23,894,872.45	24,229,110.29
	L	C	39°50'05".918	1.143474	0.9998988207	2,000,000.00	25,117,176.75	25,664,114.42
	L	S	37°47'02".652	1.143896	0.9999512939	2,000,000.00	27,025,955.35	27,432,812.88
Vermont	T	-	43°55'00".000	1.142226	0.9999642857	500,000.00	-	-
Virginia	L	N	38°37'02".892	1.143683	0.9999483859	2,000,000.00	26,230,200.09	26,576,444.45
	L	S	37°22'02".922	1.144019	0.9999454027	2,000,000.00	27,434,800.06	27,811,312.71
Washington	L	N	48°07'04".498	1.141141	0.9999422551	2,000,000.00	18,798,081.67	19,205,863.43
	L	S	46°35'06".308	1.141617	0.9999145875	2,000,000.00	19,832,653.52	20,289,119.60
West Virginia	L	N	39°37'33".439	1.143433	0.9999407460	2,000,000.00	25,305,029.12	25,715,126.55
	L	S	38°11'04".099	1.143850	0.9999256928	2,000,000.00	26,639,323.45	27,070,620.78
Wisconsin	L	N	46°10'03".978	1.141660	0.9999453461	2,000,000.00	20,124,133.05	20,489,179.67

Table 2

State	type	zone	ϕ_0	K	k_0	x_0 (feet)	R_0 (feet)	R_b (feet)
Wisconsin	L	C	44°52'34".133	1.142020	0.9999407059	2,000,000.00	21,050,746.99	21,430,913.91
	L	S	43°24'04".466	1.142438	0.9999325474	2,000,000.00	22,161,432.25	22,672,134.66
Wyoming	T	E	43°00'00".000	1.142527	0.9999411765	500,000.00	-	-
	T	EC	43°00'00".000	1.142527	0.9999411765	500,000.00	-	-
	T	WC	43°00'00".000	1.142527	0.9999411765	500,000.00	-	-
	T	W	43°00'00".000	1.142527	0.9999411765	500,000.00	-	-
American Samoa	L	-	14°16'00".000*	1.148675	1.0000000000	500,000.00	-82,312,234.65*	-80,000,000.00*
Puerto Rico	L	-	18°14'00".150	1.148110	0.9999939449	500,000.00	63,542,221.66	63,687,479.44
St. Croix Is.	L	-	18°14'00".150	1.148110	0.9999939449	500,000.00	63,542,221.66	63,787,479.44[1]
Virgin Islands	L	-	18°14'00".150	1.148110	0.9999939449	500,000.00	63,542,221.66	63,687,479.44

* American Samoa is in southern latitude, and therefore the values of R_0 and R_b are negative for mathematical reasons in order that the "north coordinate" system be used. Attention to algebraic signs in equation (6), will result in the correct answer.

[1] The numerical value of R_b is 100,000.00 feet larger than the numerical values of R_b for Puerto Rico and the Virgin Islands.

Table 3

Constants For Hotine Skew Orthomorphic Systems (1927 NAD)

State (Area)	ϕ_0	$\sin \gamma_0$	$\cos \gamma_0$	k_0	K	x_0 (meters)	y_0 (meters)
Alaska, Zone 1	57°00'00".000	-0.6000000000	+0.8000000000	0.9999000000	1.138927	+5,000,000.*	-5,000,000.*
Great Lakes, Zone 1	44°00'00".000	+0.8257703085	+0.5640066580	0.9999000000	1.142350	-3,950,000.	-3,430,000.
Great Lakes, Zone 2	43°00'00".000	-0.1630389731	-0.9866196295	0.9999000000	1.142621	+1,200,000.	-3,500,000.
Great Lakes, Zone 3	44°00'00".000	+0.2588190451	+0.9659258263	0.9999000000	1.142350	-1,000,000.	-4,300,000.
Great Lakes, Zone 4	47°12'21".554	-0.9627145794	+0.2705192020	0.9999000000	1.141433	+9,000,000.	-1,600,000.

*the units of the values of x_0 and y_0 for this zone only are *in feet*.

In meters:

$$K = \frac{12.3736955\ (1. - 0.006694380025\ \sin^2\phi_0)^2}{k_0^2} \qquad (8\text{-a})$$

In feet:

$$K = \frac{1.14955853\ (1. - 0.006694380025\ \sin^2\phi_0)^2}{k_0^2} \qquad (8\text{-b})$$

Note: the numerical values of K in Table 2 are for coordinates in feet.

The procedure to calculate K for Lambert systems is identical, but simpler. One of the zone constants is l, which (by definition) is:

$$l = \sin\phi_0$$

Then, equations (8-a) and (8-b) are:

In meters:

$$K = \frac{12.3736955\ (1. - 0.006694380025\ l^2)^2}{k_0^2} \qquad (8\text{-c})$$

In feet:

$$K = \frac{1.14955853\ (1. - 0.006694380025\ l^2)^2}{k_0^2} \qquad (8\text{-d})$$

ACCURACY OF ALGORITHM

The question to be answered is: "How good is the algorithm?" A comparison was made between the scale factors published in the "projection tables" and computed by the algorithm. Table 4 is a comparison of the scale factors calculated by equation (4) and published in the projection tables. California, Zone 1, was selected for the comparison.

Table 4

Comparison Of Scale Factors - Lambert System
(California, Zone 1)

Γ (feet)	k (algorithm)	k (published)	*differ.* ($*10^{-7}$)	ratio (1./diff.)	Γ (miles)
0	0.9998946	0.9998946	0	-	0.0
646	0.9998946	0.9998946	0	-	0.1
31,004	0.9998957	0.9998957	0	-	5.9
61,361	0.9998989	0.9998989	0	-	11.6
91,713	0.9999043	0.9999042	1	1:10,000,000	17.4
122,075	0.9999117	0.9999116	1	1:10,000,000	22.1
152,431	0.9999212	0.9999211	1	1:10.000,000	28.9
182,785	0.9999328	0.9999327	1	1:10,000,000	34.6
213,144	0.9999466	0.9999464	2	1:5,000,000	40.4
243,500	0.9999624	0.9999622	2	1:5,000,000	46.1
273,857	0.9999804	0.9999801	3	1:3,333,000	51.9
304,213	1.0000004	1.0000000	4	1:2,500,000	57.6
334,570	1.0000226	1.0000220	6	1:1,667,000	63.4
364,927	1.0000469	1.0000461	8	1:1,250,000	69.1
395,284	1.0000732	1.0000723	9	1:1,111,000	74.9
425,642	1.0001017	1.0001005	12	1:833,000	80.6
456,000	1.0001323	1.0001308	15	1:667,000	86.4
486,359	1.0001650	1.0001632	18	1:555,000	92.1
516,718	1.0001998	1.0001977	21	1:476,000	97.9
547,078	1.0002367	1.0002341	26	1:384,000	103.6

The version of equation (4) employed was:

$$k = k_0 \left[1. + K\Gamma^2 (1. - 0.5 K\Gamma^2) \right] \tag{4-a}$$

Where:

$$k_0 = 0.9998946358$$
$$K = 1.143216$$
$$x = 0.$$

$$y = y' \text{ (from projection tables)}$$

At the extreme limits of the values in the projection tables (over one hundred miles from the principal axis) the ratio of the difference in the two scale factors exceeded one part in 380,000. At the geographical limits of the zone, the ratio of the difference in the scale factors is one part in 600,000. Incorporation of additional (higher order) terms in equation (4) did not alter the values listed in Table 4.

New Mexico, East Zone, was selected, and equation (4-a) was employed. For the range of values listed in the projection tables the scale factor was calculated, and the results listed in Table 5. There is no difference between the numerical values of the scale factors. The observed difference in the seventh decimal place can be attributed to numerical roundoff, and not to any systematic factors.

Table 5

Comparison Of Scale Factors - Transverse Mercator System (New Mexico, East Zone)

Γ (feet)	*k* (algorithm)	*k* (published)	*differ.* ($*10^{-7}$)	Γ (miles)
0	0.9999091	0.9999091	0	0.0
25,000	0.9999098	0.9999098	0	4.7
50,000	0.9999120	0.9999121	1	9.5
75,000	0.9999155	0.9999156	1	14.2
100,000	0.9999205	0.9999205	0	18.9
125,000	0.9999270	0.9999270	0	23.7
150,000	0.9999348	0.9999348	0	28.4
175,000	0.9999441	0.9999441	0	33.1
200,000	0.9999549	0.9999549	0	37.9
225,000	0.9999670	0.9999671	1	42.6
250,000	0.9999806	0.9999806	0	47.3
275,000	0.9999957	0.9999956	1	52.1
300,000	1.0000121	1.0000121	0	56.8
325,000	1.0000300	1.0000300	0	61.6
350,000	1.0000493	1.0000493	0	66.3
375,000	1.0000701	1.0000701	0	71.0
400,000	1.0000922	1.0000922	0	75.8
425,000	1.0001158	1.0001158	0	80.5
450,000	1.0001408	1.0001409	1	85.2
475,000	1.0001673	1.0001674	1	90.0
500,000	1.0001952	1.0001953	1	94.7
525,000	1.0002246	1.0002246	0	99.4
530,000	1.0002306	1.0002306	0	100.4

Application To UTM

In the previous discussion, the algorithm was applied to systems with relatively narrow north-south extent. Excluding Alaska, the maximum change in latitude is about three degrees of arc. The UTM systems extend over the range of 80°N to 80°S. Therefore, a single numerical value of K may not be appropriate. Table 6 lists the numerical values of K for every 2.5 degrees of latitude for the existing systems for metric coordinates [Clarke 1866 ellipsoid and k_0 = 0.9996000000].

Table 6

Values Of K For Intervals Of 2.5 Degrees Of Latitude

(Clarke 1866 Ellipsoid)

[metric coordinates]

ϕ_0	K	ϕ_0	K	ϕ_0	K
0.0	12.3842570	32.5	12.3359054	65.0	12.2469335
2.5	12.3839380	35.0	12.3291634	67.5	12.2415725
5.0	12.3829635	37.5	12.3222056	70.0	12.2366611
7.5	12.3814009	40.0	12.3150851	72.5	12.2322363
10.0	12.3792023	42.5	12.3078562	75.0	12.2283317
12.5	12.3764045	45.0	12.3005740	77.5	12.2249766
15.0	12.3730291	47.5	12.2932940	80.0	12.2221963
17.5	12.3691021	50.0	12.2860716	82.5	12.2200119
20.0	12.3646535	52.5	12.2789615	85.0	12.2184397
22.5	12.3597174	55.0	12.2720179	87.5	12.2174916
25.0	12.3543318	57.5	12.2652934	90.0	12.2171748
27.5	12.3485379	60.0	12.2588389		
30.0	12.3423801	62.5	12.2527035		

The value of K changes with latitude in a nonlinear manner, but is relatively insensitive to small changes (a few tens of minutes of arc) in latitude.

Let:

$$K_0 = \frac{10^{15}}{2a^2(1. - e^2)k_0^2} \qquad (9)$$

And:

$$\theta = \frac{y}{\mu k_0} \quad \text{(in radians)} \qquad (10)$$

Where:

$$\mu = a(1. - e^2)\left[1. + \frac{3}{4}e^2 + \frac{45}{64}e^4 + \frac{175}{256}e^6\right] \tag{11}$$

Then:

$$K = K_0\ (1. - e^2\sin^2\theta)^2 \tag{12}$$

Where:

a = the semimajor axis of the ellipsoid (in meters).
e^2 = the square of the eccentricity of the ellipsoid.
k_0 = the scale factor along the central meridian of the projection system

For the Clarke 1866 ellipsoid (1927 NAD), equation (12) [for metric coordinates] (k_0 = 0.9996000000) is:

$$\theta'' = 0.03239383\ \frac{y}{k_0}$$

[Note: θ'' is the angle θ in *seconds of arc*.]

$$K = 12.384257\ (1. - 0.006768657997\ \sin^2\theta)^2 \tag{12-a}$$

For the 1983 NAD ellipsoid, equation (12) [for metric coordinates] (k_0 = 0.9996000000) is:

$$\theta'' = 0.03239363\ \frac{y}{k_0}$$

$$K = 12.383600\ (1. - 0.006694380025\ \sin^2\theta)^2 \tag{12-b}$$

If the project does not extend more than about two degrees in latitude, select a mean y-coordinate for the project to calculate θ. There will be *no loss in precision*.

Furthermore, to retain a precision of one in the ninth decimal place of the computed scale factor, the last term given in equation (4) [$-K\Gamma^2$] must be incorporated for distances greater than 250 kilometers from the central meridian. For shorter distances from the central meridian, equation (4-a) is sufficient.

Part 2

Digital Computer Version

The previous material discussed an algorithm for computer/calculator with limited data storage capability. With programmable computers, an algorithm can be developed to employ the plane coordinates and the published equations that require the geodetic positions. These algorithms will convert the given coordinates to requisite geodetic values for computing the scale factors.

Transverse Mercator Systems

The formula to calculate scale factors from a given set of plane coordinates, x and y, requires the *footpoint* geodetic latitude, ϕ_1. The footpoint latitude at the point is calculated from the y-coordinate. To calculate ϕ_1 requires the meridional arc length from the equator. Except for UTM systems, the origin of the y-coordinates ($y = 0$) is a parallel of latitude between the equator and the "working" area of the system. This requires a constant, y_0, for each zone. To obtain the numerical value of y_0 (for the precision required), let:

$$A_0 = 1. + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6$$

$$A_1 = \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6$$

$$A_2 = \frac{15}{128} e^4 + \frac{105}{512} e^6$$

$$A_3 = \frac{35}{1536} e^6$$

Then (in meters):

$$y_0 = a(1. - e^2) \Big[A_0 \bar{\phi}_r + A_1 \sin 2\bar{\phi} + A_2 \sin 4\bar{\phi} + A_3 \sin 6\bar{\phi} \Big] \qquad (13)$$

Where:
a = the semimajor axis of the ellipsoid (in meters).
e^2 = the square of the eccentricity of the ellipsoid.
$\overline{\phi}$ = the geodetic latitude of the origin of the coordinate system. Note: the subscript r designates the angle in radians.

The value of y_0 is a constant for a zone. The next step is to calculate the footpoint latitude. This is performed in two steps.

$$\theta'' = \frac{\mu(y + y_0)}{k_0} \tag{14}$$

Where: y and y_0 are in meters, and μ is a constant for an ellipsoid (listed in Table 7). θ'' is the approximate value of the latitude in *seconds of arc*. Then, the footpoint geodetic latitude, ϕ_1, is:

$$\phi_1 = \theta + \beta \sin 2\theta \tag{15}$$

Table 7

Ellipsoid	μ	β
Clarke 1866 (1927 NAD)	0.03239388	525".3
1983 NAD	0.03239363	519".5

The units of the term "$\beta \sin 2\theta$" in equation (15) is *seconds of arc*.

Then, let:

$$t_1 = \tan \phi_1$$

$$\eta_1^2 = \frac{e^2}{1. - e^2} \cos^2 \phi_1$$

$$N_1 = \frac{a}{\sqrt{1. - e^2 \sin^2 \phi_1}}$$

$$B = \frac{(x - x_0)^2}{k_0^2 N_1^2}$$

Then, the scale factor is:

$$k = k_0 \left[1. + \frac{B}{2}\left[(1. + \eta_1^2) + \frac{B}{12}\,(1. + \eta_1^2(6. + 9\eta_1^2 + 4\eta_1^4 - 24 t_1^2\eta_1^2(1. + \eta_1^2)))\right]\right] \tag{16}$$

The precision of equation (16) is sufficient for the UTM systems as well as the state plane coordinate systems.

Lambert Systems

The procedure for calculating the scale factor utilizes a different set of formulas, but employs an analogous procedure. For each Lambert system, the following constants are provided: R_b, x_0, l ($\sin \phi_0$), and K. Note: the K just mentioned is not the constant, K, that was employed earlier to calculate the scale factors in Part 1 of this paper.

Then:

$$\theta = \arctan\left(\frac{|x - x_0|}{R_b - y}\right)$$

$$\mu = 2 \arctan\left(\frac{|x - x_0|}{K \sin \theta}\right)^{1/l}$$

$$\phi = \mu + \frac{10^6 e^2}{9.6962736}\left[1. + \frac{e^2}{6}\left(\frac{5}{2} + e^2\right)\right]\sin 2\mu$$

The units of the second term in this equation is *seconds of arc*. Then, the scale factor is:

$$k = \frac{lK\sqrt{1. - e^2\sin^2\phi}}{a \cos \phi}\,\tan^l\left(\frac{\mu}{2}\right) \tag{17}$$

The geodetic latitude, ϕ, employed in equation (17) will be accurate to better than five seconds of arc. This is sufficient for computing k to eight decimal places.

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APPENDIX

CALCULATION OF ϕ_0 FOR THE STATE PLANE COORDINATES - TRANSVERSE MERCATOR SYSTEMS

by

Dr. Herbert W. Stoughton, P.E., L.S.

In the "Table Of Constants" [either page 7 or page 10 of the "projection tables" for the state] is listed the numerical value "$\log\left[\frac{1}{6\rho_0^2}\right]$". This number is a function of the geodetic latitude ϕ_0. Research fails to locate the values of ϕ_0 for all the transverse Mercator systems in the United States. This Appendix contains a derivation to calculate ϕ_0 based upon the numerical values listed in the aforementioned "Table Of Constants".

The symbol ρ_0 is the "mean radius" of the ellipsoid for geodetic latitude ϕ_0.

$$\rho_0 = \sqrt{R_{\phi_0} N_0} \qquad \text{(A-1)}$$

Where:

$$R_{\phi_0} = \frac{a(1. - e^2)}{(1. - e^2\sin^2\phi_0)^{3/2}} \qquad \text{(A-2)}$$

$$N_0 = \frac{a}{(1. - e^2\sin^2\phi_0)^{1/2}} \qquad \text{(A-3)}$$

Where:

a = equatorial radius (semimajor axis) of the ellipsoid (in meters).

e^2 = the square of the eccentricity of the ellipsoid.

Substituting equation (A-2) and (A-3) into equation (A-1) yields:

$$\rho_0^2 = \frac{a^2(1. - e^2)}{(1. - e^2\sin^2\phi_0)^2} \qquad \text{(A-1a)}$$

The formula to calculate "$\log\left[\frac{1}{6\rho_0^2}\right]$" [based upon U.S.C.&G.S. Special Publication Nos. 193 and 195] is:

$$\log\left[\frac{1}{6\rho_0^2}\right] = \log\left[\frac{(1. - e^2\sin^2\phi_0)^2}{k_0^2\ a^2(1. - e^2)}\left(\frac{1200}{3937}\right)^2\right] + 20. \tag{A-4}$$

Where:
k_0 = the scale factor along the central meridian of the system.

The constant 20. is added to obtain a positive characteristic.

For the Clarke 1866 ellipsoid (1927 NAD):

$$a = 6{,}378{,}206.400 \text{ meters}$$

$$e^2 = 0.0067686580$$

Then, equation (A-4) is:

$$\log\left[\frac{1}{6\rho_0^2}\right] = \log\left[3.83206*10^{-16}\ \frac{(1. - e^2\sin^2\phi_0)^2}{k_0^2}\right] + 20. \tag{A-4a}$$

Let:

$$\psi = \log\left[\frac{1}{6\rho_0^2}\right]$$

The equation (A-4a) can be written:

$$\psi = \log 3.83206*10^{-16} + 2\log(1. - e^2\sin^2\phi_0)$$
$$- 2\log k_0 + 20. \tag{A-5}$$

The numerical value of ψ is given in the "Table Of Constants". Rearranging equation (A-5) yields:

$$\log(1. - e^2\sin^2\phi_0) = \frac{1}{2}\left[\psi - \log 3.83206*10^{-16}\right.$$
$$\left. + 2\log k_0 - 20\right] \qquad \text{(A-[illegible])}$$

Equation (A-6) contains the known quantities [ψ and k_0] on the right side of the equation, and the unknown quantity, ϕ_0, is on the left side of the equation.

Let:

$$\log \mu = \frac{1}{2}\left[\psi - \log 3.83206*10^{-16} + 2 \log k_0 - 20\right]$$

Then, equation (A-6) can be written:

$$\log (1. - e^2\sin^2\phi_0) = \log \mu \tag{A-7}$$

Or:

$$1. - e^2\sin^2\phi_0 = \mu$$

And:

$$e^2\sin^2\phi_0 = 1. - \mu$$

$$\sin \phi_0 = \sqrt{\frac{1. - \mu}{e^2}}$$

Finally:

$$\phi_0 = \arcsin \sqrt{\frac{1. - \mu}{e^2}} \tag{A-8}$$

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1. Ralph Moore Berry, personal communications.
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