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MASSACHUSETTS UNIV AMHERST DEPT OF MATHEMATICS AND S--ETC F/6 12/1  
A CHARACTERIZATION OF A POLYA-EGGENBERGER AND OTHER DISCRETE DI--ETC(U)  
SEP 80 R M KORWAR AFOSR-80-0219

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| 1. REPORT NUMBER                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | 2. AUTHOR                                                      | 3. RECIPIENT'S CATALOG NUMBER               |
| AFOSR-TR- 30 - 1180                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         | AD-A092536                                                     | 536                                         |
| 4. TITLE (and Subtitle)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 5. TYPE OF REPORT & PERIOD COVERED                             |                                             |
| A CHARACTERIZATION OF A POLYA EGGENBERGER<br>AND OTHER DISCRETE DISTRIBUTIONS BY RECORD<br>VALUES                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           | Interim                                                        |                                             |
| 7. AUTHOR(s)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                | 6. PERFORMING ORG. REPORT NUMBER                               |                                             |
| Ramesh M. Korwar                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            |                                                                |                                             |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | 8. CONTRACT OR GRANT NUMBER(s)                                 |                                             |
| The University of Massachusetts<br>Department of Mathematics and Statistics<br>Amherst, Massachusetts 01003                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | AFOSR 80-0219 <i>new</i>                                       |                                             |
| 11. CONTROLLING OFFICE NAME AND ADDRESS                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS |                                             |
| Air Force Office of Scientific Research /NM<br>Bolling AFB, Washington, DC 20332                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                            | 61102F 2304/A5                                                 |                                             |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 | 12. REPORT DATE                                                |                                             |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | September 1980                                                 |                                             |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 13. NUMBER OF PAGES                                            |                                             |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 8                                                              |                                             |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 15. SECURITY CLASS. (of this report)                           |                                             |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | UNCLASSIFIED                                                   |                                             |
|                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             | 15a. DECLASSIFICATION DOWNGRADING<br>SCHEDULE                  |                                             |
| 16. DISTRIBUTION STATEMENT (of this Report)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |                                                                |                                             |
| Approved for public release; distribution unlimited                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                |                                             |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                  |                                                                |                                             |
| DTIC<br>ELECTE<br>DEC 05 1980                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                               |                                                                |                                             |
| 18. SUPPLEMENTARY NOTES                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                     |                                                                |                                             |
| E                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                |                                             |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                          |                                                                |                                             |
| Characterization                      Linear regression<br>Discrete distributions              Polya Eggenberger<br>Generalized hypergeometric        Record Values                                                                                                                                                                                                                                                                                                                                                                                                                                                         |                                                                |                                             |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                           |                                                                |                                             |
| <p><i>note: all n - print "sub"</i></p> <p>Let <math>X_1, X_2, \dots</math>, be a sequence of independent and identically distributed discrete random variables. Define the sequence <math>N(n)</math> by <math>N(1)=1, N(n)=\min \{j \mid j N(n-1), X_j X_N(n-1)\}, n=2,3, \dots</math>. Let <math>R_n = X_{N(n)}</math>. Then <math>R_n</math> is the sequence of record values. By convention <math>R_1 = X_1</math>. Here a characterization of a Polya Eggenberger and other discrete distributions including the geometric, is made by the linearity of regression of <math>R_2 - R_1</math> on <math>R_1</math>.</p> |                                                                |                                             |

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(18) AFOSR-TR-80-1180

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by

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(16) 2384

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AFOSR Technical Report No. 8

(17) A51

(17) 11

(15) ✓ AFOSR-TR-80-1180

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| Accession For      |                                            |
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| Unannounced        | <input type="checkbox"/>                   |
| Justification      |                                            |
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September, 1980

The University of Massachusetts  
Department of Mathematics and Statistics  
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Research sponsored by the Air Force Office of Scientific Research, AFOSR, USAF,  
under Grant AFOSR-80-0219.

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4-2-377

A Characterization of a Polya-Eggerberger and Other  
Discrete Distributions by Record Values

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Abstract

Let  $X_1, X_2, \dots$ , be a sequence of independent and identically distributed discrete random variables. Define the sequence  $W(n)$  by  $W(1)=1$ ,  $W(n)=\min\{j \mid X_j > W(n-1), X_j \leq X_{j+1}(n-1)\}$ ,  $n=2, \dots$ . Let  $R_n = X_{W(n)}$ . Then  $R_n$  is the sequence of record values. By convention  $R_1 = X_1$ . Here a characterization of a Polya-Eggerberger and other discrete distributions including the geometric, is made by the linearity of regression of  $R_2 - R_1$  on  $R_1$ .

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A CHARACTERIZATION OF A  
POLYA-EGGENBERGER AND OTHER DISCRETE  
DISTRIBUTIONS BY RECORD VALUES <sup>1</sup>

by

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SUMMARY. Let  $X_1, X_2, \dots$ , be a sequence of independent and identically distributed discrete random variables. Define the sequence  $\{N(n)\}$  by  $N(1)=1, N(n)=\min\{j \mid j > N(n-1), X_j > X_{N(n-1)}\}$ ,  $n=2, 3, \dots$ . Let  $R_n = X_{N(n)}$ . Then  $\{R_n\}$  is the sequence of record values. By convention  $R_1 = X_1$ . Here a characterization of a Polya-Eggenberger and other discrete distributions including the geometric, is made by the linearity of regression of  $R_2 - R_1$  on  $R_1$ .

Consider a sequence  $X, X_1, X_2, \dots$  of independent and identically distributed (i.i.d.) discrete random variables (r.v.) with

$$P(X=j) = p_j, j=0, \dots, m \quad (1)$$

Here  $m$  is either a positive integer or  $\infty$ . Define

$$N(1)=1, N(n)=\min\{j \mid j > N(n-1), X_j > X_{N(n-1)}\}, n=2, 3, \dots,$$

$R_1 = X_1$  and  $R_n = X_{N(n)}$ ,  $n=2, 3, \dots$ . Then  $\{R_n\}$  is the sequence of record values and  $\{N(n)\}$  the sequence of times at which record values occur.

In this note we characterize the geometric, a Polya-Eggenberger and a generalized hypergeometric distribution by linearity of regression of  $R_2 - R_1$  on  $R_1$ . Thus the

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<sup>1</sup>Research sponsored by the Air Force Office of Scientific Research, AFOSR, USAF, under Grant AFOSR-60-0219.

characterization given here extends Theorem 2 of Srivastava (1979) characterizing the geometric by the constancy of regression of  $R_2 - R_1$  on  $R_1$ .

Consider for  $X$  one of the following distributions.

$$p_j = (1-p_0) c \{1-c\}^{j-1}, \quad j=1,2,\dots, \quad (2)$$

$0 < c < 1.$

$$p_j = (1-p_0) \binom{a}{j-1} \binom{b}{n-j+1} / \binom{a+b}{n}, \quad j=1,\dots, n+1, \quad (3)$$

$a < 0; n > 0; n \text{ integral}; n-b-1 > 0; b < 0, b \neq -1.$

and

$$p_j = (1-p_0) \binom{a}{j-1} \binom{b}{n-j+1} / \binom{a+b}{n}, \quad j=1,2,\dots, \quad (3a)$$

$a < 0; n < 0; n-b-1 < 0; a+b+1 > 0.$

Distribution (2) is a geometric distribution, (3) and (3a) are versions of generalized hypergeometric distribution; Type 2A and Type IV, as described by Kemp and Kemp (1956). For  $a=-1$ , (3) can be recast as

$$p_j = (1-p_0) \binom{n}{j-1} \frac{A(A+s), \dots, (A+j-2s) \quad B(B+s), \dots, (B+n-(j-1)-1s)}{(A+B) (A+B+s) \dots (A+B+n-1s)} \quad (3b)$$

$j=1,\dots, n+1$

with  $A=1$ ,  $B=-b$ ,  $s=1$ . An example of a Polya-Eggenberger distribution is (3b).

We are now ready to prove the

Theorem: Suppose  $X$  takes on only the values  $0, \dots, n$  with positive probabilities and assume that  $X$  has a finite expectation. Then

$$E(R_2 - R_1 \mid R_1 = i) = \alpha + i\beta \quad \text{a.s.}$$

$$\alpha, \beta \text{ Constants} \quad (4)$$

$$i = 0, \dots, m-1$$

if, and only if,  $X$  has either the geometric (2), or a Polya-Eggenberger (3b) with  $A=1$  and  $s=1$ , or a generalized hypergeometric distribution (3a) with  $a=-1$ . Furthermore,  $\beta=0$  iff  $X$  is geometric (2),  $\beta < 0$ ,  $\beta \neq 1$  iff  $X$  is (3b) with  $A=1$ ,  $s=1$  and  $\beta > 0$  iff  $X$  is (3a) with  $a=-1$ .

Proof: First consider the 'only if' part. Suppose (4) holds. Now it is shown in Srivastava (1979) that

$$E(R_2 - R_1 \mid R_1 = i) = \frac{\sum_{j=1}^{m-i} p_{i+j}}{\sum_{j=i+1}^m p_j}, \quad i=0, \dots, m-1$$

Thus it follows that

$$\sum_{j=1}^{m-i} j p_{i+j} = (\alpha + i\beta) \sum_{j=i+1}^m p_j \quad (5)$$

and

$$\sum_{j=1}^{m-i-1} j p_{i+j+1} = (\alpha + \overline{i+1} \beta) \sum_{j=i+2}^m p_j \quad (6)$$

Subtracting (6) from (5) we have

$$(1+\beta) \sum_{j=i+1}^m p_j = (\alpha + \overline{i+1} \beta) p_{i+1} \quad (7)$$

Replacing  $i$  by  $(i+1)$  in (7) and subtracting the result from (7), we finally obtain

$$p_{i+2} / p_{i+1} = (\alpha + \beta i - 1) / \{ \alpha + \beta(i+2) \}, \quad i=0, 1, \dots, m-2.$$

This last result and (7) yield

$$p_j = p_1 \frac{(\alpha-1) \dots (\alpha+j-2\beta-1)}{(\alpha+2\beta) \dots (\alpha+j\beta)} \quad j=1, \dots, m. \quad (8)$$

and

$$p_1 = (1+\beta) (1-p_0) / (\alpha+\beta). \quad (9)$$

From (4) it follows that

$$\alpha \geq 1 \quad (10)$$

We consider three cases (i)  $\beta=0$ , (ii)  $\beta<0$  and (iii)  $\beta>0$ . The case  $\beta=0$  is already considered by Srivastava (1979). His result is included in (8) and specializing for  $\beta=0$ , (8) and (9) give (2) with  $c=1/\alpha$ .

Next consider the cases  $\beta \neq 0$ . Now, from (4), it follows that

$$\beta < 0 \quad \text{if, and only if, } X \text{ is bounded} \quad (11)$$

$$\text{If } \beta < 0 \quad \text{then } \alpha + \beta(m-1) = 1 \quad (12)$$

We obtain (12) from (4) by specializing  $i=m-1$ . From (8) and (9) it is clear that  $\beta=-1$  leads to the degeneracy of  $X$  at 0, a contradiction of the assumption of  $m$  being a positive integer. For the cases  $\beta \neq 0$  (8) can be recast as

$$p_j = p_1 \left\{ \frac{A^{[j-1]} B^{[j-1]}}{C^{[j-1]} (j-1)!} \right\}, \quad j=1, \dots, m \quad (13)$$

where  $x$  is the ascending factorial,

$$x^{[j]} = x(x+1) \dots (x+j-1), \quad j=1, 2, \dots, \quad x^{[0]} = 1;$$

$p_1$  is given by (9) and

$$A = (\alpha-1)/\beta \quad B=1, \quad \text{and } C = \alpha/\beta+2. \quad (14)$$

The product in the curled brackets in (13) is the coefficient of  $z^{j-1}$  in the hypergeometric function

$$F(A;B;C;z) = \sum_{j=0}^{\infty} \left\{ \frac{[j]_A [j]_B}{[j]_C j!} \right\} z^j.$$

Thus (13) is the generalized hypergeometric distribution (3) ( or 3(a) ) with

$$a=-1, b=1+1/\beta, n=-(\alpha-1)/\beta \quad (15)$$

First suppose  $\beta < 0$ . Then, from (12) and (15) we have that  $n=m-1$ , and if  $m \geq 2$ , from (9), that  $b < 0$ . Obviously,  $n-b-1 > 0$  for  $m \geq 2$ . Thus, for the case  $\beta < 0$ ,  $\beta \neq 1$  and  $m \geq 2$  we have a generalized hypergeometric distribution (3) with parameters  $a=-1$ ,  $b=1+1/\beta$  and  $n=m-1$ . This distribution is also a Polya-Eggenberge distribution (3b) with parameter  $A=1$ ,  $B=-1/\beta$ ,  $n=m-1$  and  $s=1$ . However, (4) will not lead to any particular distribution if  $m=1$ .

Finally assume  $\beta > 0$ . From (15), we have that  $X$  has the generalized hypergeometric distribution (3a) with the parameters given by (15).

Now for the "if" part. Let  $X$  have (3) or (3a) with  $a=-1$ . Then we have

$$p_{j+1}/p_j = (j-1-n) / (b-n+j), \quad j=1,2,\dots,m-1.$$

from which we get

$$\sum_{j=k+1}^m j p_j / \sum_{j=k+1}^m p_j = (b-n-1+bk) / (b-1), \quad k=0,\dots,m-1$$

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which in turn yields (4) with  $\alpha=(b-n-1)/(b-1)$  and  $\beta=b/(b-1)$ . Hence the 'if' part is proved.

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