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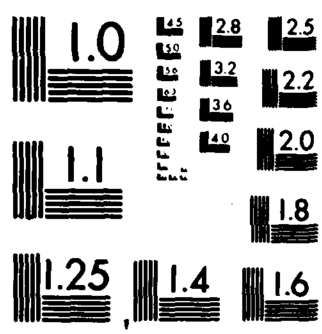


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# ON GENERALIZATIONS OF COCHRAN'S THEOREM AND PROJECTION MATRICES

BY

AKIMICHI TAKEMURA

TECHNICAL REPORT NO. 44
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PREPARED UNDER CONTRACT N00014-75-C-0442
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OFFICE OF NAVAL RESEARCH
THEODORE W. ANDERSON, PROJECT DIRECTOR

DEPARTMENT OF STATISTICS
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# ON GENERALIZATIONS OF COCHRAN'S THEOREM AND PROJECTION MATRICES

by

Akimichi Takemura

## 1. Introduction

In statistical analysis the decomposition of a sum of squares into various independent components is essential in ANOVA-type procedures. In this connection Cochran's theorem gives nice conditions to guarantee the independence of components. There have been many generalizations or refinements of Cochran's theorem, and recently Anderson and Styan (1980) reviewed those results and gave new ones.

In this paper we present a unified approach to those problems by consistent use of "projection" matrices which are generally not orthogonal. This approach enables us to extend various results to much greater generality.

In section 2 we review the properties of projection. In section 3 we prove some generalizations of Cochran's theorem. In section 4 we discuss decomposition of matrices into projections to their eigenspaces and its relation to Cochran-type decomposition.

Matrices may be considered as real or complex. Proofs are given in general vector space terms, and often they cover both cases. In section 4 we use complex matrices explicitly.

## 2. Projection Matrix

In this section properties of projection matrices are summarized. To be self-contained brief proofs will be given.

A matrix $A$ is called a projection matrix if it is idempotent, i.e., $A^2 = A$. If in addition $A' = A$ (or $A^* = A$ for complex matrices) then $A$ is called an orthogonal projection matrix.[1] Orthogonal projections have many extra properties which are not shared by general projections. However, there are quite a few facts which have counterparts in general projections or hold without orthogonality. Moreover, the advantage of developing a theory for general projections is that the theory holds for vector spaces not equipped with an inner product.

We need some definitions about subspaces of a vector space. Let $X$ be a vector space and $U_1,\ldots,U_k$ be subspaces of $X$. $U_1,\ldots,U_k$ are called (linearly) independent if $x_i \in U_i$, $i = 1,\ldots,k$, $\sum_{i=1}^k x_i = 0$ implies $x_i = 0$ $i = 1,\ldots,k$. When $U_1,\ldots,U_k$ are independent, the subspace $U$ spanned by $U_1,\ldots,U_k$ is called the direct sum of $U_i$'s and denoted by $U = U_1 + \ldots + U_k = \sum_{i=1}^k U_i$. An element $y$ of $U$ has a unique representation $y = x_1 + \ldots + x_k$ where $x_i \in U_i$, $i = 1,\ldots,k$. It is easy to see that the union of bases of $U_i$'s form a basis of $U$ and therefore $\dim U = \sum \dim U_i$. For a matrix $A$ the column vector space or the range of $A$ is denoted by $C(A)$. The null space of $A$ is denoted by $N(A)$. $\lambda(A) = \{\lambda_1,\ldots,\lambda_k\}$ is the set of distinct eigenvalues of $A$ and $E(\lambda_i) = N(A - \lambda_i I)$ denotes the eigenspace associated with $\lambda_i$.

[1] Usually "projection" is used to mean orthogonal projection but here we use it for general projection.

Now we state properties of projection matrices.

<u>Proposition 2.1.</u> The following conditions are equivalent:

(i) $A$ is a projection.

(ii) $I - A$ is a projection.

(iii) $C(A) = \{x : Ax = x\}$ .

(iv) $N(I-A) = C(A)$ or $N(A) = C(I-A)$ .

(v) $C(A)$ and $C(I-A)$ are independent.

<u>Proof</u>. We show that (i) is equivalent to each of (ii)-(v) .

(ii): (i) $\Leftrightarrow A(I-A) = 0 \Leftrightarrow$ (ii).

(iii): Let $V = \{x : Ax = x\}$ . Assume $A^2 = A$ . If $x \in C(A)$ then $x = Ay = A^2y = A(Ay) = Ax$ . Therefore $x \in V$ . If $x \in V$ then $x = Ax \in C(A)$ . Therefore $V = C(A)$.

Assume $C(A) = V$. Then $Ax \in C(A) = V$. Hence $A(Ax) = Ax \ \forall x$ . Therefore $A^2 = A$ .

(iv): (iv) $\Leftrightarrow$ (iii).

(v): Necessity is proved in the next proposition. Sufficiency is proved noting

$$(A-A^2)x - (I-A)Ax = 0 , \quad \forall x ,$$

and $(A-A^2)x \in C(A)$ , $(I-A)Ax \in C(I-A)$. Q.E.D.

Properties (iii), (iv), (v) show that $A$ projects a vector $x \notin C(A)$ to $C(A)$ along a line parallel to $C(I-A)$ .

**Proposition 2.2.** Suppose $A_1,\ldots,A_k$ are projections such that $A_iA_j = 0$ for $i \neq j$. Then (i) $A = A_1 + \ldots + A_k$ is a projection, and (ii) $C(A_i)$'s are independent and $C(A) = \sum_{i=1}^k C(A_i)$.

**Proof.** Let $x_i \in = C(A_i)$ then $x_i = A_iy_i$. Suppose $\sum_{i=1}^k x_i = 0$. Then

$$0 = A_j(\sum_{i=1}^k x_i) = A_j^2y_j = A_jy_j = x_j, \quad j = 1,\ldots,k .$$

Therefore $C(A_i)$'s are independent. $Ax = \sum_{i=1}^k A_ix$ , therefore

$$(2.1) \qquad C(A) \subset \sum_{i=1}^k C(A_i) .$$

On the other hand $AA_ix = A_i^2x = A_ix$ implies $C(A_i) \subset C(A)$, $i = 1,\ldots,k$. Therefore

$$(2.2) \qquad C(A) \supset \sum_{i=1}^k C(A_i) .$$

From (2.1) and (2.2) $C(A) = \sum_{i=1}^k C(A_i)$. Q.E.D.

The converse of the above proposition is given by the following proposition.

**Proposition 2.3.** Let $A$ be a projection onto $V = C(A)$. Suppose $V = \sum_{i=1}^k V_i$ where $V_i$'s are given independent subspaces. Then $A = \sum_{i=1}^k A_i$ where $A_i$'s are uniquely determined projections such that $A_iA_j = 0$ for $i \neq j$ and $C(A_i) = V_i$, $i = 1,\ldots,k$.

**Proof.** Let $V_0 = C(I-A)$. Then $V_0,V_1,\ldots,V_k$ are independent and $\sum_{i=0}^k V_i = X$ (whole space). Now we choose vectors $\{x_{01},\ldots,x_{0m_0}, x_{11},\ldots, x_{1m_1},\ldots,x_{km_k}\}$ such that $\{x_{i1},\ldots,x_{im_i}\}$ form a basis for $V_i$.

Then $x_{ij}$'s form a basis for the whole space. Let

$$T = (x_{01},\ldots,x_{km_k}) = (T_0,T_1,\ldots,T_k)$$

be a matrix with column vectors $x_{ij}$ with $T_i = (x_{i1},\ldots,x_{im_i})$, $i = 0, 1,\ldots,k$. Let $T^{-1}$ be partitioned accordingly

$$T^{-1} = \begin{pmatrix} T^0 \\ \vdots \\ T^k \end{pmatrix} .$$

Then

$$I = TT^{-1} = \sum_{i=0}^{k} T_i T^i .$$

Let $A_i = T_i T^i$. Now it is easy to check that $A_i^2 = A_i$, $C(A_i) = V_i$, and $A_i A_j = 0$ for $i \neq j$. To prove uniqueness let $A = \sum_{i=1}^k A_i = \sum_{i=1}^k B_i$. Then $\sum_{i=1}^k (A_i - B_i)\, x = 0$, $\forall x$, implies $A_i x = B_i x$, $i = 1,\ldots,k$, because $(A_i - B_i)\, x \in V_i$. Hence $A_i = B_i$, $i = 1,\ldots,k$. Q.E.D.

In view of the above two propositions it may well be justified to call projection matrices $A_i$, $i = 1,\ldots,k$, <u>(linearly) independent</u> if $A_i A_j = 0$ for $i \neq j$.

We note here the relation between diagonalizable matrices and projections. Let $A$ be a $p \times p$ diagonalizable matrix and let $\lambda(A) = \{\lambda_1,\ldots,\lambda_k\}$. There exists a nonsingular $T$ such that

$$T^{-1} A T = D_\lambda$$

where $D_\lambda$ is a diagonal matrix with $\lambda_i$'s as diagonal elements in an appropriate order, i.e.,

$$D_\lambda = \mathrm{diag}(\lambda_1,\ldots,\lambda_1,\lambda_2,\ldots,\lambda_k) .$$

Now $\underset{\sim}{A} = \underset{\sim}{T}\underset{\sim}{D}_\lambda \underset{\sim}{T}^{-1}$ and with the similar partitioning of $\underset{\sim}{T}\ \underset{\sim}{T}^{-1}$ as in the proof of Proposition 2.3

$$\underset{\sim}{A} = \sum_{i=1}^{k} \lambda_i \underset{\sim}{H}_i \tag{2.3}$$

where $\underset{\sim}{H}_i$'s are independent projections onto eigenspace $E(\lambda_i)$'s of $\lambda_i$'s . It follows that for any polynomial $P$

$$P(\underset{\sim}{A}) = \sum_{i=1}^{k} P(\lambda_i)\ \underset{\sim}{H}_i\ . \tag{2.4}$$

$\underset{\sim}{H}_i$'s are unique by Proposition 2.3 and $\underset{\sim}{I} = \sum_{i=1}^{k} \underset{\sim}{H}_i$ . Conversely if $\sum_{i=1}^{k} E(\lambda_i) = X$ [2] then $\underset{\sim}{A}$ can be diagonalized by $\underset{\sim}{T}$ as in the proof of Proposition 2.3.

When $0 \in \lambda(\underset{\sim}{A})$ then putting $\lambda_k = 0$ we can write (2.3) as

$$\underset{\sim}{A} = \sum_{i=1}^{k-1} \lambda_i \underset{\sim}{H}_i\ .$$

(2.4) holds with $k$ replaced by $k-1$ if $P(0) = 0$ , i.e., $P$ does not have constant term.

Despite its apparent plausibility we need the diagonalizability of a projection matrix to prove:

<u>Proposition 2.4</u>. If $\underset{\sim}{A}^2 = \underset{\sim}{A}$ then rank $\underset{\sim}{A}$ = tr $\underset{\sim}{A}$ .

<u>Proof</u>. $\underset{\sim}{A}^2 = \underset{\sim}{A}$ implies $\lambda(\underset{\sim}{A}) \subset \{0,1\}$ . It will be shown that $\underset{\sim}{A}$ is diagonalizable in section 4 and therefore

$$\begin{aligned} \text{rank } \underset{\sim}{A} &= \text{number of eignevalues } 1 \\ &= \text{tr } \underset{\sim}{A}\ . \end{aligned} \qquad \text{Q.E.D.}$$

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[2] Eigenspaces $E(\lambda_i)$'s are always independent.

## 3. Cochran-Type Theorems

In terms of matrix theory Cochran's theorem essentially asserts the following: Let $A$ be an orthogonal projection and $A = \sum_{i=1}^{k} A_i$ where $A_i$'s are symmetric. If $\text{rank}\, A = \sum_{i=1}^{k} \text{rank}\, A_i$ then $A_i$'s are independent orthogonal projections. Various kinds of extensions and refinements are discussed in Anderson and Styan (1980). In this section we present simpler proofs for a number of them and give a generalization (Theorem 3.3 together with Corollary 4.2).

We begin by showing that Cochran's theorem holds <u>without orthogonality</u>.

<u>Theorem 3.1</u>. (Cochran's theorem) Let $A = \sum_{i=1}^{k} A_i$ . $A_i$'s are independent projections if and only if

(i) $A$ is a projection,

(ii) $\text{rank}\, A = \sum_{i=1}^{k} \text{rank}\, A_i$ .

The key point to prove this is contained in

<u>Lemma 3.1</u>. Let $A = \sum_{i=1}^{k} A_i$ where $A_i$'s are general matrices. Then

(i) $C(A_i)$'s are independent and $C(A) = \sum_{i=1}^{k} C(A_i)$

if and only if

(ii) $\text{rank}\, A = \sum_{i=1}^{k} \text{rank}\, A_i$ .

<u>Proof</u>.

Necessity: $\dim\{C(A)\} = \sum_{i=1}^{k} \dim\{C(A_i)\}$. This is equivalent to $\text{rank}\, A = \sum_{i=1}^{k} \text{rank}\, A_i$ .

Sufficiency: We show the independence of $C(\underset{\sim}{A}_i)$ by contradiction. Suppose that there are vectors $\underset{\sim}{x}_1,\ldots,\underset{\sim}{x}_k$ with at least one nonzero vector such that $\underset{\sim}{x}_i \in C(\underset{\sim}{A}_i)$, $i = 1,\ldots,k$, and $\sum_{i=1}^k \underset{\sim}{x}_i = \underset{\sim}{0}$. Without loss of generality let

$$\underset{\sim}{0} \neq \underset{\sim}{x}_1 = -\sum_{i=2}^k \underset{\sim}{x}_i .$$

Let $\{\underset{\sim}{x}_1, \underset{\sim}{q}_1,\ldots,\underset{\sim}{q}_m\}$ be a basis for $C(\underset{\sim}{A}_1)$ where $m = \text{rank } \underset{\sim}{A}_1 - 1$. Then $C(\underset{\sim}{A}) \subset C(\underset{\sim}{A}_1,\ldots,\underset{\sim}{A}_k) = C(\underset{\sim}{q}_1,\ldots,\underset{\sim}{q}_m,\underset{\sim}{A}_2,\ldots,\underset{\sim}{A}_k)$. But $\dim\{C(\underset{\sim}{A})\} = \text{rank } \underset{\sim}{A}$ and

$$\dim C(\underset{\sim}{q}_1,\ldots,\underset{\sim}{q}_m, \underset{\sim}{A}_2,\ldots,\underset{\sim}{A}_k) \leq m + \sum_{i=2}^k \dim C(\underset{\sim}{A}_i) = \text{rank } \underset{\sim}{A} - 1$$

which is a contradiction. Therefore $C(\underset{\sim}{A}_i)$'s are independent. Now $C(\underset{\sim}{A}) \subset \sum_{i=1}^k C(\underset{\sim}{A}_i)$ but again by the rank condition (ii) we get $C(\underset{\sim}{A}) = \sum_{i=1}^k C(\underset{\sim}{A}_i)$. Q.E.D.

Proof of Theorem 3.1.

Sufficiency: From the lemma we know $C(\underset{\sim}{A}_i) \subset C(\underset{\sim}{A})$, $i = 1,\ldots,k$. Therefore by (iii) of Proposition 2.1 we get $\underset{\sim}{A}\underset{\sim}{A}_i\underset{\sim}{x} = \underset{\sim}{A}_i\underset{\sim}{x}$. Then

$$\sum_{j\neq i} \underset{\sim}{A}_j\underset{\sim}{A}_i\underset{\sim}{x} + (\underset{\sim}{A}_i^2 - \underset{\sim}{A}_i)\underset{\sim}{x} = \underset{\sim}{0} .$$

By independence of $C(\underset{\sim}{A}_i)$'s we get $\underset{\sim}{A}_j\underset{\sim}{A}_i\underset{\sim}{x} = \underset{\sim}{0}$ $i \neq j$ and $(\underset{\sim}{A}_i^2 - \underset{\sim}{A}_i)\underset{\sim}{x} = \underset{\sim}{0}$. This being true for any $\underset{\sim}{x}$ shows $\underset{\sim}{A}_j\underset{\sim}{A}_i = \underset{\sim}{0}$, $i \neq j$, and $\underset{\sim}{A}_i^2 = \underset{\sim}{A}_i$.

Necessity: In view of lemma 3.1 necessity was proved already in Proposition 2.2. Q.E.D.

Chipman and Rao (1964), Khatri (1968) give more algebraic proofs of this theorem.

We proceed to generalize this result to an r-potent matrix. A matrix $\underset{\sim}{A}$ is called r-potent if $\underset{\sim}{A}^r = \underset{\sim}{A}$. A general theorem is given by Anderson and Styan (1980, their theorem 3.3). Here we present a simpler (but equivalent) version.

Theorem 3.2. (r-potent matrix) Let $\underset{\sim}{A} = \sum_{i=1}^{k} \underset{\sim}{A}_i$ and $r \geq 3$.

$$\text{(i)} \quad \underset{\sim}{A}_i^r = \underset{\sim}{A}_i , \quad \underset{\sim}{A}_i \underset{\sim}{A}_j = \underset{\sim}{0} , \quad i \neq j ,$$

if and only if

$$\text{(i')} \quad \underset{\sim}{A}^r = \underset{\sim}{A} , \quad \text{(ii')} \quad \text{rank } \underset{\sim}{A} = \sum_{i=1}^{k} \text{rank } \underset{\sim}{A}_i ,$$

$$\text{(iii')} \quad \underset{\sim}{A}\underset{\sim}{A}_i = \underset{\sim}{A}_i \underset{\sim}{A} , \quad i = 1,\ldots,k .$$

Proof.

Necessity: We need to prove (ii'). By multiplying (i) by $\underset{\sim}{A}_i^{r-2}$ or $\underset{\sim}{A}_j^{r-2}$ we get $\underset{\sim}{A}_i^{2(r-1)} = \underset{\sim}{A}_i^{r-1}$, $\underset{\sim}{A}_i^{r-1} \underset{\sim}{A}_j^{r-1} = \underset{\sim}{0}$, $i \neq j$. Therefore the previous theorem applies to $\underset{\sim}{A}_i^{r-1}$ and we get

$$\text{rank } \underset{\sim}{A}^{r-1} = \sum_{i=1}^{k} \text{rank } \underset{\sim}{A}_i^{r-1} .$$

But (i) and (i') show that rank $\underset{\sim}{A}^{r-1}$ = rank $\underset{\sim}{A}$, rank $\underset{\sim}{A}_i^{r-1}$ = rank $\underset{\sim}{A}_i$ noting rank $\underset{\sim}{A} \geq$ rank $\underset{\sim}{A}^{r-1} \geq$ rank $\underset{\sim}{A}^r$ = rank $\underset{\sim}{A}$, etc. (ii') follows from this.

Sufficiency: From (iii') we get

$$\sum_{j=1}^{k} \underset{\sim}{A}_j \underset{\sim}{A}_i \underset{\sim}{x} = \underset{\sim}{A}_i \underset{\sim}{A} \underset{\sim}{x} .$$

Hence

$$\sum_{j\neq i} \underset{\sim}{A}_j\underset{\sim}{A}_i\underset{\sim}{x} + \underset{\sim}{A}_i(\underset{\sim}{A}_i-\underset{\sim}{A})\underset{\sim}{x} = \underset{\sim}{0} .$$

By Lemma 3.1 $C(A_i)$'s are independent and therefore $\underset{\sim}{A}_j\underset{\sim}{A}_i\underset{\sim}{x} = \underset{\sim}{0}$, $\forall\underset{\sim}{x}$, $j \neq i$ .
Hence

$$\underset{\sim}{A}_j\underset{\sim}{A}_i = \underset{\sim}{0}, \quad j \neq i .$$

Now

$$\underset{\sim}{0} = \underset{\sim}{A}^r - \underset{\sim}{A} = \sum_{i=1}^{k}(\underset{\sim}{A}_i^r-\underset{\sim}{A}_i) .$$

Noting $(\underset{\sim}{A}_i^r-\underset{\sim}{A}_i)\underset{\sim}{x} \in C(\underset{\sim}{A}_i)$, $\forall\underset{\sim}{x}$, we get

$$\underset{\sim}{A}_i^r = \underset{\sim}{A}_i , \quad i = 1,\ldots,k . \quad \text{Q.E.D.}$$

We have assumed $r \geq 3$ above because (iii') is not needed in the case $r = 2$ . Anderson and Styan (1980, their Theorem 3.3) replaces (iii') by $\underset{\sim}{A}_i\underset{\sim}{A}^{r-2} = \underset{\sim}{A}^{r-2}\underset{\sim}{A}_i$ , for example. This condition is nice in the sense that it reduces to the null condition when $r=2$. On the other hand, (iii') is desirable in connection with simultaneous diagonalizability (Theorem 4.3 in section 4) and actually easier to check for $r > 3$ .

Our proof of sufficiency is quite general and it yields the following result almost without changes.

<u>Theorem 3.3</u>. Let $P(x)$ be any polynomial of $\deg P \geq 2$ . Suppose

(i) $P(\underset{\sim}{A}) = \underset{\sim}{0}$ ,

(ii) $\operatorname{rank} \underset{\sim}{A} = \sum_{i=1}^{k} \operatorname{rank} \underset{\sim}{A}_i$ ,

(iii) $\underset{\sim}{A}\underset{\sim}{A}_i = \underset{\sim}{A}_i\underset{\sim}{A}$ .

Then

$$A_i A_j = 0 \quad \text{and} \begin{cases} P(A_i) = 0, & i = 1,\ldots,k, \text{ if } P(0) = 0, \\ A_i\, P(A_i) = 0, & i = 1,\ldots,k, \text{ if } P(0) \neq 0. \end{cases}$$

Proof. As in the proof of Theorem 3.2 we get $A_i A_j = 0$, $i \neq j$. Therefore $0 = P(A) = \sum_{i=1}^k P(A_i)$ if $P(0) = 0$. Then

$$0 = \sum_{i=1}^k A_i\, Q(A_i)$$

where $Q(x) = P(x)/x$. By independence of $C(A_i)$'s we get $A_i\, Q(A_i) = 0$, $i = 1,\ldots,k$. Therefore $P(A_i) = A_i\, Q(A_i) = 0$. If $P(0) \neq 0$, let $R(x) = xP(x)$. Then $R(0) = 0$, $R(A) = 0$, and by the above argument $R(A_i) = A_i\, P(A_i) = 0$, $i = 1,\ldots,k$. Q.E.D.

When $P(0) = 0$ the converse of this is true except for (ii). A partial converse will be given in Corollary 4.2.

## 4. Decomposition to Projections

The decomposition discussed in the previous section may be called the Cochran-type decomposition. Here we want to investigate another type of decomposition which we call the decomposition to projections.[3] For a diagonalizable[4] matrix such decomposition was discussed at the end of

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[3]It is often called "spectral decomposition, as in the case of orthogonal projections.

[4]A more "mathematical" term is "semisimple".

section 1. First we derive a simple criterion which guarantees the diagonalizability of a matrix. After that the relation of two types of the above decompositions will be discussed and the main theorem of this section (Theorem 4.4) asserts that two types of projections commute. Theorem 4.4 will be applied to generalize Cochran's theorem to sum of weighted $\chi^2$ variables.

To discuss the diagonalizability of matrices we rely on the theory of Jordan canonical form and minimal polynomial. For a $p \times p$ matrix $\underset{\sim}{A}$ let $\lambda(\underset{\sim}{A}) = \{\lambda_1,\ldots,\lambda_k\}$ and let $m_i$ be the multiplicity of eigenvalue $\lambda_i$, $i = 1,\ldots,k$ $(\sum_{i=1}^k m_i = p)$. $E(\lambda_i) = N(\underset{\sim}{A}-\lambda_i\underset{\sim}{I})$ is called the eigenspace associated with $\lambda_i$. $E^w(\lambda_i) = N([\underset{\sim}{A}-\lambda_i\underset{\sim}{I}]^{m_i})$ is called the wide-sense eigenspace associated with $\lambda_i$. A polynomial $\phi_{\underset{\sim}{A}}(\cdot)$ is called the minimal polynomial of $\underset{\sim}{A}$ if $\phi_{\underset{\sim}{A}}$ is the lowest degree polynomial such that $\phi_{\underset{\sim}{A}}(\underset{\sim}{A}) = \underset{\sim}{0}$. The theory of Jordan canonical forms tells us the following: $E^w(\lambda_i)$, $i = 1,\ldots,k$, are independent, $\dim E^w(\lambda_i) = m_i$, and $\sum_{i=1}^k E^w(\lambda_i) = X$ (the whole space); there exists a nonsingular $\underset{\sim}{T} = (\underset{\sim}{T}_1, \ldots,\underset{\sim}{T}_k)$ with $T_i$: $p \times m_i$ consisting of a basis of $E^w(\lambda_i)$ such that

$$\underset{\sim}{T}^{-1}\underset{\sim}{A}\underset{\sim}{T} = \begin{pmatrix} \underset{\sim}{C}_1 & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{C}_2 & \cdots & \underset{\sim}{0} \\ \vdots & \vdots & & \vdots \\ \underset{\sim}{0} & \underset{\sim}{0} & \cdots & \underset{\sim}{C}_k \end{pmatrix}$$

where $\underset{\sim}{C}_i$ is an $m_i \times m_i$ upper triangular matrix; furthermore $\underset{\sim}{C}_i$ is of the form

$$\underset{\sim}{C}_i = \begin{pmatrix} \underset{\sim}{B}_{i1} & \cdots & \underset{\sim}{0} \\ \vdots & & \vdots \\ \underset{\sim}{0} & \cdots & \underset{\sim}{B}_{i\ell i} \end{pmatrix}$$

where $\underset{\sim}{B}_{ij}$ is a $\beta_{ij} \times \beta_{ij}$ matrix which has the form

$$\begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix} .$$

The minimal polynomial $\phi_{\underset{\sim}{A}}$ turns out to be

(4.1) $$\phi_{\underset{\sim}{A}}(x) = \Pi_{i=1}^{k}(x-\lambda_i)^{\beta_i}$$

where $\beta_i = \max_j\{\beta_{ij}\}$ . See Shilov (1971, Chapter 6) for example.

If $\beta_i = 1$ then $\underset{\sim}{C}_i = \lambda_i \underset{\sim}{I}_{m_i}$ . Therefore we have

<u>Fact 3.1</u>.

The following conditions are equivalent:

(i) $\underset{\sim}{A}$ is diagonalizable.

(ii) $\beta_i = 1, \ i = 1,\ldots,k$ .

(iii) $E(\lambda_i) = E^W(\lambda_i), \ i = 1,\ldots,k$ .

(iv) $\sum_{i=1}^{k} E(\lambda_i) = X$ .

These conditions are rather hard to check for particular matrices. Therefore it is desirable to have a sufficient condition for the diagonalizability. Here is a strong but fairly general sufficient condition.

**Lemma 4.1.** Let $P(x)$ $(P \not\equiv 0)$ be a polynomial such that $P(x) = 0$ has no multiple root. Suppose $P(\underset{\sim}{A}) = \underset{\sim}{0}$ ; then $\underset{\sim}{A}$ is diagonalizable.

**Proof.** Let

$$P(x) = Q(x)\ \phi_{\underset{\sim}{A}}(x) + R(x)$$

where $\deg R < \deg \phi_{\underset{\sim}{A}}$ . Then $P(\underset{\sim}{A}) = Q(\underset{\sim}{A})\ \phi_{\underset{\sim}{A}}(\underset{\sim}{A}) + R(\underset{\sim}{A})$ and we have $R(\underset{\sim}{A}) = \underset{\sim}{0}$ . But minimality of $\phi_{\underset{\sim}{A}}$ implies $R(x) \equiv 0$ . Therefore $P(x) = Q(x)\ \Pi_{i=1}^{k}(x-\lambda_i)^{\beta_i}$ by (4.1). By assumption about $P$ we get $\beta_i = 1$ , $i = 1,\ldots,k$, and $P$ is diagonalizable. Q.E.D.

Applying this criterion we obtain the following theorem.

**Theorem 4.1.** (Decomposition to Projections) Let $P(x)$ $(P \not\equiv 0)$ be a polynomial such that $P(x) = 0$ has no multiple root. Suppose $P(\underset{\sim}{A}) = \underset{\sim}{0}$ ; then $\underset{\sim}{A}$ is diagonalizable and

$$\underset{\sim}{A} = \lambda_1 \underset{\sim}{H}_1 + \ldots + \lambda_k \underset{\sim}{H}_k \tag{4.2}$$

where $\lambda(\underset{\sim}{A}) = \{\lambda_1,\ldots,\lambda_k\}$ , $\underset{\sim}{H}_i$'s are independent projections and $\sum_{i=1}^{k} \underset{\sim}{H}_i = \underset{\sim}{I}$ . This expression is unique.

**Proof.** By Lemma 4.1 $\underset{\sim}{A}$ is diagonalizable. Decomposition (4.2) is given in (2.3). Q.E.D.

**Corollary 4.1.** Let $\underset{\sim}{A}$ be r-potent, i.e., $\underset{\sim}{A}^r = \underset{\sim}{A}$ . Then

(4.3)
$$A = H_1 + \omega H_2 + \dots + \omega^{r-2} H_{r-1}$$
$$= H_1 + \omega H_2 + \dots + \omega^{r-2} H_{r-1} + 0 \cdot H_r$$

where $\omega = e^{i2\pi/(r-1)}$ and $H_i$'s are independent projections. This expression is unique.

Proof. Let $P(x) = x^r - x = x(x-\omega) \dots (x-\omega^{r-2})(x-1)$. Then $P(x)$ has no multiple root. Noting that $\lambda(A) \subset \{1,\omega,\dots,\omega^{r-2},0\}$ (4.3) follows from (4.2). Q.E.D.

This decomposition has the following intuitive meaning. First note $C(A) = \sum_{i=1}^{r-1} C(\omega^{i-1} H_i) = \sum_{i=1}^{r-1} C(H_i)$ . Now $A^r = A$ implies $C(A) = C(A^r)$ $= C(A^{2r-1}) = \dots$ . On the other hand, $C(A) \supset C(A^2) \supset \dots$ , therefore, $C(A) = C(A^k)$ , $k = 2,3,\dots$ . Let $x \in C(A)$ then $x = Ay$ . Then $A^{r-1} x = A^r y = A y = x$ . Therefore, $A$ can be interpreted as "rotating" $C(A)$ completely in $r - 1$ steps. Actually (4.3) tells us that $A$ divides $C(A)$ into $r - 1$ independent spaces $C(H_i)$ , $i = 1,\dots,r - 1$ , and rotates $C(H_i)$ by the angle $(i-1)2\pi/(r-1)$ .

At this point we have two decompositions: Cochran-type decomposition (Theorem 3.3) and decomposition to projections (Theorem 4.1). It is conceivable that these two decompositions are "independent" and the order of applying them is interchangeable.

Theorem 4.2. (Commutativity)

Let $P(x)$ $(P \not\equiv 0)$ be a polynomial with no multiple root. Let $A = \sum_{i=1}^{k} A_i$ . Suppose

(i) $P(\underset{\sim}{A}) = \underset{\sim}{0}$ ,

(ii) rank $\underset{\sim}{A} = \sum_{i=1}^{k}$ rank $\underset{\sim}{A}_i$ ,

(iii) $\underset{\sim}{A}\underset{\sim}{A}_i = \underset{\sim}{A}_i\underset{\sim}{A} \quad i = 1,\ldots,k$ .

Then there exist independent projections $\underset{\sim}{H}_{11},\ldots,\underset{\sim}{H}_{k1}, \underset{\sim}{H}_{12},\ldots\underset{\sim}{H}_{k2},\ldots,$ $\underset{\sim}{H}_{1\ell},\ldots,\underset{\sim}{H}_{k\ell}$ such that

$$\begin{aligned}
\underset{\sim}{A}_1 &= \lambda_1 \underset{\sim}{H}_{11} + \lambda_2 \underset{\sim}{H}_{12} + \cdots + \lambda_\ell \underset{\sim}{H}_{1\ell} , \\
\underset{\sim}{A}_2 &= \lambda_1 \underset{\sim}{H}_{21} + \lambda_2 \underset{\sim}{H}_{22} + \cdots + \lambda_\ell \underset{\sim}{H}_{2\ell} , \\
&\vdots \\
\underset{\sim}{A}_k &= \lambda_1 \underset{\sim}{H}_{k1} + \lambda_2 \underset{\sim}{H}_{k2} + \cdots + \lambda_\ell \underset{\sim}{H}_{k\ell} , \\
\underset{\sim}{A} &= \lambda_1 \underset{\sim}{H}_1 + \lambda_2 \underset{\sim}{H}_2 + \cdots + \lambda_\ell \underset{\sim}{H}_\ell ,
\end{aligned} \tag{4.4}$$

where $\underset{\sim}{H}_j = \sum_{i=1}^{k} \underset{\sim}{H}_{ij}$ and $\{\lambda_1,\ldots,\lambda_\ell\} = \lambda(\underset{\sim}{A}) - \{0\}$ . These expressions are unique.

We first prove the following essential

<u>Theorem 4.3</u>. (Simultaneous Diagonalization)

Let $\underset{\sim}{A},\underset{\sim}{B}$ be diagonalizable. Then

$$\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{B}\underset{\sim}{A}$$

if and only if $\underset{\sim}{A},\underset{\sim}{B}$ are simultaneously diagonalizable.

<u>Proof</u>.

Sufficiency: Let $A = TD_1T^{-1}$ , $B = TD_2T^{-1}$ , then $AB = TD_1D_2T^{-1}$ $= TD_2D_1T^{-1} = BA$ .

Necessity: Let $\lambda(A) = \{\lambda_1,\ldots,\lambda_k\}$ , $\lambda(B) = \{\mu_1,\ldots,\mu_\ell\}$ and $A = \sum_{i=1}^k \lambda_i H_i^A$ , $B = \sum_{j=1}^\ell \mu_j H_j^B$ . Suppose $x \in E(\lambda_i) = C(H_i^A)$ . Then $ABx = BAx$ $= \lambda_i Bx$ . Therefore $Bx \in E(\lambda_i)$ or $C(BH_i^A) \subset C(H_i^A)$ . Noting $I = \sum_{i=1}^k H_i^A$ we get

$$0 = B - B = \sum_{i=1}^k (H_i^A B - BH_i^A) .$$

Therefore $H_i^A B = BH_i^A$ . By Corollary 4.1 $H_i^A$ is diagonalizable , and repeating the same argument we obtain

$$H_i^A H_j^B = H_j^B H_i^A \quad \text{for any } i \text{ and } j .$$

Let $H_{ij} = H_i^A H_j^B$ . Then

$$H_{ij}^2 = H_i^A H_j^B H_i^A H_j^B = (H_i^A)^2 (H_j^B)^2 = H_i^A H_j^B = H_{ij} ;$$

similarly

$$H_{ij} H_{i'j'} = 0 \quad \text{for } i \neq i' \text{ or } j \neq j' .$$

It follows that $H_{ij}$ , $i = 1,\ldots,k$ , $j = 1,\ldots,\ell$, are independent projections such that $I = \sum_{i,j} H_{ij}$ . By Cochran's theorem $\sum_{i,j} \text{rank } H_{ij} = p$ . Now taking appropriate bases from $C(H_{ij})$'s and letting $T$ as in the proof of Proposition 2.3, it is easy to see that $A,B$ are simultaneously diagonalized by $T$ and $A,B$ can be written as

$$\tilde{A} = \sum_{i=1}^{k} \lambda_i \sum_{j=1}^{\ell} \tilde{H}_{ij} \ ,$$

(4.5)

$$\tilde{B} = \sum_{j=1}^{\ell} \mu_j \sum_{i=1}^{k} \tilde{H}_{ij} \ .$$

Now $C(\tilde{H}_{ij}) \subset C(\tilde{H}_i^A) \cap C(\tilde{H}_j^B) = E(\lambda_i) \cap E(\mu_j)$ . On the other hand if $\tilde{x} \in E(\lambda_i) \cap E(\mu_j)$ then $\tilde{H}_{ij}\tilde{x} = \tilde{H}_i^A \tilde{H}_j^B \tilde{x} = \tilde{H}_i^A \tilde{x} = \tilde{x} \in C(\tilde{H}_{ij})$ . Therefore $C(\tilde{H}_{ij}) = E(\tilde{\lambda}_i) \cap E(\tilde{\mu}_j)$ . Q.E.D.

Remark: It is immediate to generalize the above proof to more than two diagonalizable matrices commuting with each other. For more standard proofs see Satake (1975, Sec. 4.2, ex. 1), Greub (1975, Sec. 13.27).

Now we are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. The case deg P = 1 reduces to Cochran's Theorem. Suppose deg $P \geq 2$ , then by Theorem 3.3 $\tilde{A}_i\tilde{A}_j = \tilde{0}$ , $i \neq j$, and $P(\tilde{A}_i) = \tilde{0}$ or $\tilde{A}_i P(\tilde{A}_i) = \tilde{0}$ . By Lemma 4.1 $\tilde{A}_i$'s can be diagonalized. Let $\{\mu_{i1},\ldots,\mu_{ih_i}\} = \lambda(\tilde{A}_i) - \{0\}$ . We want to show

$$\lambda(\tilde{A}_i) - \{0\} \subset \lambda(\tilde{A}) - \{0\}$$

and

$$E(\mu_{ij}) \subset E(\lambda_\alpha)$$

where

$$\lambda_\alpha = \mu_{ij} \ .$$

Suppose $\tilde{x} \in E(\mu_{ij})$ . Then

$$\mu_{ij} \tilde{A} \tilde{x} = \tilde{A} \tilde{A}_i \tilde{x} = \tilde{A}_i^2 \tilde{x} = (\mu_{ij})^2 \tilde{x} \ .$$

Dividing by $\mu_{ij}$ we get $\tilde{A}\tilde{x} = \mu_{ij} \tilde{x}$ ; hence $\mu_{ij} \in \lambda(\tilde{A}) - \{0\}$ and if $\mu_{ij} = \lambda_\alpha$ then $E(\mu_{ij}) \subset E(\mu_\alpha)$ .

Summarizing this we obtain now from Theorem 4.2

$$\underset{\sim}{A} = \lambda_1 \underset{\sim}{H}_1 + \dots + \lambda_\ell \underset{\sim}{H}_\ell ,$$

$$\underset{\sim}{A}_1 = \lambda_1 \underset{\sim}{H}_{11} + \dots + \lambda_\ell \underset{\sim}{H}_{1\ell} ,$$

$$\vdots$$

$$\underset{\sim}{A}_k = \lambda_1 \underset{\sim}{H}_{k1} + \dots + \lambda_\ell \underset{\sim}{H}_{k\ell} ,$$

where

(i) $\underset{\sim}{H}_1,\dots,\underset{\sim}{H}_\ell$ are independent projections,

(ii) for fixed $i$ $\underset{\sim}{H}_{i1},\dots,\underset{\sim}{H}_{i\ell}$ are independent projections, and

(iii) $E(\mu_{i\alpha}) = C(\underset{\sim}{H}_{ij}) \subset C(\underset{\sim}{H}_j) = E(\lambda_j)$, $i = 1,\dots,k$, where $\mu_{i\alpha} = \lambda_j$ .

Now

$$\underset{\sim}{A}\,\underset{\sim}{x} = \sum_{j=1}^{\ell} \lambda_j \underset{\sim}{H}_j \underset{\sim}{x} ,$$

$$\sum_{i=1}^{k} \underset{\sim}{A}_i \underset{\sim}{x} = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \lambda_j \underset{\sim}{H}_{ij} \underset{\sim}{x} .$$

Therefore

$$(\underset{\sim}{H}_j - \textstyle\sum_{i=1}^{k} \underset{\sim}{H}_{ij})\lambda_j \underset{\sim}{x} = \underset{\sim}{0} , \ \forall \underset{\sim}{x} ,$$

and we get

$$\underset{\sim}{H}_j = \sum_{i=1}^{k} \underset{\sim}{H}_{ij} . \tag{4.6}$$

Let $\underset{\sim}{H} = \sum_{j=1}^{\ell} \underset{\sim}{H}_j = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \underset{\sim}{H}_{ij}$ . Now

$$\text{rank } \underset{\sim}{A} = \sum_{j=1}^{\ell} \text{rank } \underset{\sim}{H}_j = \text{rank } \underset{\sim}{H} ,$$

$$\text{rank } \underset{\sim}{A}_i = \sum_{j=1}^{\ell} \text{rank } \underset{\sim}{H}_{ij} .$$

Hence

$$\sum \text{rank } \underset{\sim}{A}_i = \text{rank } \underset{\sim}{A}$$

implies

$$\sum\sum_{i,j} \text{rank } \underset{\sim}{H}_{ij} = \text{rank } \underset{\sim}{H} \ .$$

Therefore by Cochran's Theorem (Theorem 3.1) $\underset{\sim}{H}_{ij}$'s are independent projections. From the construction the expressions are unique. Q.E.D.

Corollary 4.2. (Converse to Theorem 3.3) Suppose $P(x)$ has no multiple root and $P(0) = 0$ . Let $\underset{\sim}{A} = \sum_{i=1}^{k} \underset{\sim}{A}_i$ . If

(i) $\underset{\sim}{A}_i \underset{\sim}{A}_j = \underset{\sim}{0}$ , $i \neq j$ ,

(ii) $P(\underset{\sim}{A}_i) = \underset{\sim}{0}$ , $i = 1,\ldots,k$ ,

then

(iii) $P(\underset{\sim}{A}) = \underset{\sim}{0}$ ,

(iv) $\underset{\sim}{A}\,\underset{\sim}{A}_i = \underset{\sim}{A}_i \underset{\sim}{A}$ ,

(v) $\text{rank } \underset{\sim}{A} = \sum_{i=1}^{k} \text{rank } \underset{\sim}{A}_i$ .

Proof. (iii), (iv) follow immediately from (i) and (ii). Therefore we assume (i) - (iv) instead. Then the above proof requires no change up to (4.6). Now we want to show that

$$\underset{\sim}{H}_{is} \underset{\sim}{H}_{jt} = \underset{\sim}{0}$$

whenever $i \neq j$ or $s \neq t$ without the rank condition. Fix $s,t$ such that $s \neq t$ . Let $\alpha,\beta$ be such that $\mu_{i\alpha} = \lambda_s$ , $\mu_{j\beta} = \lambda_t$ . If $\lambda_s \notin \lambda(\underset{\sim}{A}_i)-\{0\}$ then let $E(\mu_{i\alpha}) = \emptyset$ instead. In any case $E(\mu_{i\alpha}) \cap E(\mu_{j\beta}) \subset E(\lambda_s)$

$\cap E(\lambda_t) = \emptyset$ . Theorem 4.3 applied to $\underset{\sim}{A}_i, \underset{\sim}{A}_j$ (note $\underset{\sim}{A}_i \underset{\sim}{A}_j = \underset{\sim}{A}_j \underset{\sim}{A}_i = \underset{\sim}{0}$) implies that in (4.5) the second sums are just single terms and now $s \neq t$ implies $\underset{\sim}{H}_{is} \underset{\sim}{H}_{jt} = \underset{\sim}{0}$ . Then for $i \neq j$ , $\underset{\sim}{0} = \underset{\sim}{A}_i \underset{\sim}{A}_j = \sum_{s=1}^{\ell} \lambda_s^2 \underset{\sim}{H}_{is} \underset{\sim}{H}_{js}$ . Noting $C(\underset{\sim}{H}_{is} \underset{\sim}{H}_{js}) \subset C(\underset{\sim}{H}_s)$ we get $\underset{\sim}{H}_{is} \underset{\sim}{H}_{js} = 0$ . Therefore we have shown $\underset{\sim}{H}_{ij}$'s are independent and by Cochran's Theorem used in the other direction we get rank $\underset{\sim}{H} = \sum_{ij}$ rank $\underset{\sim}{H}_{ij}$ which yields rank $\underset{\sim}{A} = \sum_{i=1}^{k}$ rank $\underset{\sim}{A}_i$. Q.E.D.

Khatri (1977, Sec. 3) discusses similar results without assuming the commutativity (iii) which is essential for our development.

Actually we can put Theorem 4.2 into a more natural and general form.

<u>Theorem 4.4</u>. (Commutativity) Let $\underset{\sim}{A} = \sum_{i=1}^{k} \underset{\sim}{A}_i$ . Suppose

(i) $\underset{\sim}{A}$ is diagonalizable,

(ii) rank $\underset{\sim}{A} = \sum_{i=1}^{k}$ rank $\underset{\sim}{A}_i$ ,

(iii) $\underset{\sim}{A} \underset{\sim}{A}_i = \underset{\sim}{A}_i \underset{\sim}{A} \quad i = 1,\ldots,k$ ,

Then $\underset{\sim}{A}_i$'s are simultaneously diagonalizable and (4.4) holds.

<u>Proof</u>. By Fact 4.1, $\underset{\sim}{A}$ is diagonalizable $\Leftrightarrow \phi_{\underset{\sim}{A}}$ has no multiple root. Therefore the result follows if we let $P = \phi_{\underset{\sim}{A}}$ in Theorem 4.2 . Q.E.D.

Now looking at (4.4) we see that row-wise it gives a decomposition to projections and column-wise it gives Cochran-type decompositions. In addition, marginal sums add up correctly. To get $\underset{\sim}{H}_{ij}$ we could either (1) decompose $\underset{\sim}{A}_i$ into projections, or (2) decompose $\underset{\sim}{A}$ into projections

$H_1, \ldots, H_\ell$ and let $H_{ij} = A_i H_j$ . In this sense the two decompositions commute.

Finally we give an application of Theorem 4.4 to weighted $\chi^2$-variables. If $A$ and $A_i$'s are real symmetric the theory developed in this section can be interpreted entirely in real numbers and real matrices. Furthermore, diagonalization can be restricted to the ones by orthogonal matrices, which is always possible. In statistical terms, then, Theorem 4.4 reads:

Theorem 4.5. Let $x_1, \ldots, x_p$ be independent standard normal variables. Let $Q = x'A\,x$ , $Q_1 = x'A_1\,x, \ldots, Q_k = x'A_k\,x$ be quadratic forms in $x_i$'s such that

$$\text{(i)}\quad A = \sum_{i=1}^{k} A_i \;,$$

$$\text{(ii)}\quad \operatorname{rank} A = \sum_{i=1}^{k} \operatorname{rank} A_i \;,$$

$$\text{(iii)}\quad AA_i = A_iA \;, \quad i = 1, \ldots, k \;.$$

Then there exist independent standard normal variables $y_1, \ldots, y_p$ such that $(y_1, \ldots, y_p)' = \Gamma(x_1, \ldots, x_p)'$ for some orthogonal $\Gamma$ and

$$\begin{aligned}
Q &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_m y_m^2 \;, \\
Q_1 &= \lambda_1 y_1^2 + \ldots + \lambda_{m_1} y_{m_1}^2 \;, \\
Q_2 &= \lambda_{m_1+1} y_{m_1+1}^2 + \ldots + \lambda_{m_1+m_2} y_{m_1+m_2}^2 \;, \\
&\vdots \\
Q_k &= \lambda_{m-m_k+1} y_{m-m_k+2}^2 + \ldots + \lambda_m y_m^2
\end{aligned}$$

where $\lambda_i$'s are eigenvalues of $\underset{\sim}{A}$ and $m_i$ = rank $\underset{\sim}{A}_i$ , $m$ = rank $\underset{\sim}{A}$ .

The converse of this theorem is easy to prove. Therefore it actually gives necessary and sufficient conditions. This theorem in its matrix form is essentially proved in Luther (1965, his Theorem 1). See also Khatri (1977) for further discussion.

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cochran's theorem, r-potent matrix, spectral decomposition, projection, diagonalizable matrix, simultaneous diagonalization

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Generalizations of Cochran's theorem including (i) nonsymmetric matrices and (ii) r-potent matrices are proved by consistent use of projection matrices. Decomposition of diagonalizable matrices into projections to eigenspaces (or spectral decomposition) and its relation to Cochran-type decomposition are studied.

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