

AD-A092 659

PRINCETON UNIV NJ DEPT OF STATISTICS

F/G 12/1

COVERING THE CIRCLE WITH RANDOM ARCS OF RANDOM SIZES.(U)

SEP 80 A F SIEGEL, L HOLST

DAA629-79-C-0205

UNCLASSIFIED

TR-171-SER-2

ARO-16669.1-M

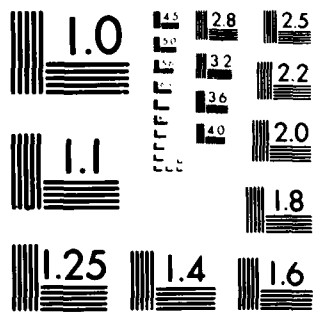
NL

1 OF 1
ALVA
7/26/80



END
DATE
FILMED
1-8
DTIC





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LEVEL II

(13)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS
BEFORE COMPLETING FORM

1. REPORT NUMBER 11-16669.1-M	2. GOVT ACCESSION NO. A.D-A092659	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Covering the Circle with Random Arcs of Random Sizes.		5. TYPE OF REPORT & PERIOD COVERED Technical
7. AUTHOR(s) Andrew F. Siegel Lars Holst		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Princeton University Princeton, NJ 08540		8. CONTRACT OR GRANT NUMBER(s) DAAG29 79-C-0205
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		12. REPORT DATE Sep 80
		13. NUMBER OF PAGES 14
		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) NA		
18. SUPPLEMENTARY NOTES The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) probability arcs spacings circles coverage asymptotic series geometry		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Consider the random uniform placement of a finite number of arcs on the circle, where the arc lengths are sampled from a distribution on (0,1). We provide exact formulae for the probability that the circle is completely covered and for the distribution of the number of uncovered gaps, extending Stevens' (1939) formulae for the case of fixed equal arc lengths. A special class of arc length distributions is considered, and exact probabilities of coverage are tabulated for the uniform distribution on (0,1). Some asymptotic results for the number of gaps are also given.		

AD A092659

DDC FILE COPY

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

80 12 01 T41

COVERING THE CIRCLE WITH RANDOM ARCS
OF RANDOM SIZES

by

Andrew F. Siegel
Princeton University

and

Lars Holst
Stanford University and Uppsala University

Accession For	
NTIS GEA&I	☒
DDC TAB	☐
Unannounced	☐
Justification	
Date	
Author	
Title	
Subject	
Notes	
A	

key words: Geometrical Probability, Coverage Probability, Spacings.

Research was supported by U.S. Army Research Office Grant DAAG29-79-C-0205, the U.S. Office of Naval Research Contract N00014-76-C-0475, a fellowship from the American-Scandinavian Foundation, and a Grant from the Swedish National Science Foundation.

Technical Report No. 171, Series 2
Princeton University
Department of Statistics

September 1980

COVERING THE CIRCLE WITH RANDOM ARCS
OF RANDOM SIZES

by

Andrew F. Siegel

and

Lars Holst

ABSTRACT

Consider the random uniform placement of a finite number of arcs on the circle, where the arc lengths are sampled from a distribution on $(0,1)$. We provide exact formulae for the probability that the circle is completely covered and for the distribution of the number of uncovered gaps, extending Stevens' (1939) formulae for the case of fixed equal arc lengths. A special class of arc length distributions is considered, and exact probabilities of coverage are tabulated for the uniform distribution on $(0,1)$. Some asymptotic results for the number of gaps are also given.

1. Introduction

Many authors have considered problems relating to the coverage of a circle by random arcs of equal sizes, including Whitworth (1897), Stevens (1939), Fisher (1940), Votaw (1946), Domb (1947), Flatto and Konheim (1962), Solomon (1978), Holst (1980 a,b,c and 1981), and Siegel (1978b and 1979). When the arc lengths are not equal, few results are known. Conditions under which a given infinite sequence of arc lengths will almost surely cover the circle were studied by Dvoretzki (1956), Mandelbrot (1972), Shepp (1972), and others. However, when the number of arcs is finite, there is no known simple formula for the coverage probability (Shepp, 1972), although some results are available concerning the amount of the circle covered in this case (Yadin and Zachs, 1980). Some approximations to the coverage probability using moments and simulations were provided by Siegel (1978a).

We provide an exact formula for the probability of covering the circle with n random arcs of random independent sizes in Section 2. The formula is derived by considering the number of uncovered gaps, thereby extending Stevens' (1939) formula to the case of random arc length. Feasibility of computation is demonstrated in Section 3 for the special case of arcs of uniform lengths and a related class of distributions. Some asymptotics concerning the number of uncovered gaps are obtained in Section 4.

2. Distribution of the Number of Gaps and the Probability of Coverage

Consider n arcs placed uniformly and independently at random on a circle of circumference one. Let the arc lengths be independent and identically distributed random variables $L_1, \dots, L_n$, chosen from the cumulative distribution function F on $(0,1)$. The number of uncovered gaps will be denoted N_n . Note that $N_n = 0$ is the event of complete coverage. An example with $N_4 = 2$ is illustrated in the figure.

Theorem 2.1. The distribution of N_n is given by

$$P(N_n = m) = \binom{n}{m} \sum_{k=m}^n \binom{n-m}{k-m} (-1)^{k-m} \xi_k \quad (2.1)$$

where

$$\xi_k = \int_{(\sum_{i=1}^k u_i = 1)} \left[\prod_{i=1}^k F(u_i) \right] \left[\sum_{j=1}^k \int_0^{u_j} F(v) dv \right]^{n-k} du \quad (2.2)$$

$$= E \left\{ \left[\prod_{i=1}^k F(S_{ik}) \right] \left[\sum_{j=1}^k E_L(S_{jk-L})_+ \right]^{n-k} \right\} \quad (2.3)$$

and $S_{1k}, \dots, S_{kk}$ denotes the spacings between k independent uniform points on the circle. The positive part function is $(t)_+ = \max(t, 0)$.

The first terms are

$$\xi_0 = 1$$

$$\xi_1 = [1 - E(L)]^{n-1}$$

$$\xi_2 = \int_0^1 F(u)F(1-u) \left[\int_0^u F(v)dv + \int_0^{1-u} F(v)dv \right]^{n-2} du$$

and the first moment is $E(N_n) = n[1 - E(L)]^{n-1}$.

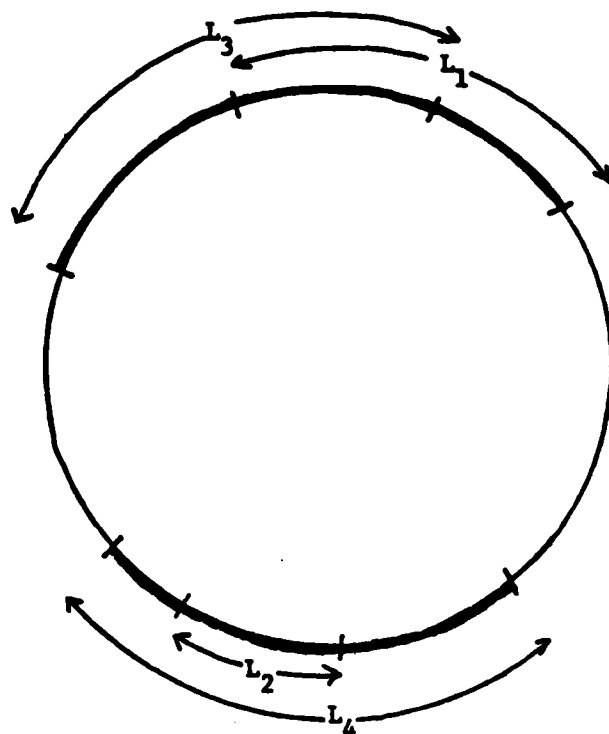


Figure. The event of $N_4 = 2$ uncovered gaps when $n=4$ random arcs are placed on the circle.

Corollary. The probability of completely covering the circle is

$$P(N_n=0) = \sum_{k=0}^n (-1)^k \binom{n}{k} \int_{(\sum_{i=1}^k u_i=1)} \left[\prod_{i=1}^k F(u_i) \right] \left[\sum_{j=1}^k \int_0^{u_j} F(v) dv \right]^{n-k} du. \quad (2.4)$$

The first n terms suffice whenever arcs must have length at least $1/n$ because the product term will be zero for the higher terms.

Proof. Consider the events $A_i =$ "clockwise endpoint of arc i is not covered". The figure shows the case in which A_1 and A_4 occur. Define $\xi_k = P(\bigcap_{i=1}^k A_i)$. Using reasoning analogous to Stevens (1939, formula 4.62) we obtain (2.1). To calculate ξ_k , observe that

$$\xi_k = P\left(\left[\bigcap_{i=1}^k \{L_i \leq S_{ik}\}\right] \cap B_{n-k}\right) \quad (2.5)$$

where $S_{1k}, \dots, S_{kk}$ are the lengths of the gaps between clockwise endpoints of the first k gaps and B_{n-k} is the event "none of the arcs $k+1, \dots, n$ cover the clockwise endpoints of arcs $1, \dots, k$." Conditioning on $S_{1k}, \dots, S_{kk}$, independence of the arc lengths gives us

$$\xi_k = E\left\{P(B_{n-k} | S_{1k}, \dots, S_{kk}) \prod_{i=1}^k P(S_{ik})\right\}. \quad (2.6)$$

An additional arc will fall within one of the first k gaps with conditional probability

$$E\left\{\sum_{j=1}^k (S_{jk} - L)_+ | S_{1k}, \dots, S_{kk}\right\}.$$

Again using independence of arc lengths, we have

$$P(B_{n-k} | S_{1k}, \dots, S_{kk}) = \left[\sum_{j=1}^k E_L(S_{jk} - L)_+\right]^{n-k} \quad (2.7)$$

which, together with (2.6) establishes (2.3). Writing expectations as integrals and integrating by parts we obtain (2.2). Because N_n is the sum of indicator functions $N_n = \sum_{i=1}^n I(A_i)$, its first moment is

$$E(N_n) = nP(A_1) = n\xi_1 = n[1 - E(L)]^{n-1}. \quad \square$$

The formula (2.4) for coverage probability does reduce to Stevens' (1939) formula when arcs have fixed length a . In this case $F(x) = I(x \geq a)$ and from (2.3) we have

$$\xi_k = E \left\{ \left[\prod_{i=1}^k I(S_{ik} \geq a) \right] \left[\sum_{j=1}^k E(S_{jk} - a)_+ \right]^{n-k} \right\}.$$

The product term allows us to drop the positive part to obtain

$$\xi_k = [P(\text{all } S_{ik} \geq a)] \cdot (1-ka)^{n-k}$$

which reduces to

$$\xi_k = (1-ka)_+^{n-1}. \quad (2.8)$$

Substituting this into formula (2.4) we obtain Stevens' formula for equal arcs:

$$P(\text{cover}) = \sum_{k=0}^n (-1)^k \binom{n}{k} (1-ka)_+^{n-1}.$$

3. Uniform Arc Length and Related Distributions

If the cumulative distribution function of the arc length has the special form

$$F(x) = x^\alpha, \quad 0 < x < 1 \quad (3.1)$$

for some $\alpha > 0$, then the probabilities in Section 2 can be simplified by performing the integrations.

Theorem 3.1. If $F(x) = x^\alpha$, then ξ_k from Theorem 2.1 is

$$\xi_k = \frac{(k-1)!}{(\alpha+1)^{n-k} \Gamma[(\alpha+1)n]} \sum_{\left(\sum_{i=1}^k m_i = n-k \right)} \binom{n-k}{m_1, \dots, m_k} \prod_{i=1}^k \Gamma[(\alpha+1)(m_i+1)] \quad (3.2)$$

where the sum is over nonnegative integers m_i . An equivalent formula with fewer terms to sum is obtained if we define $v_j = \#(m_i = j)$:

$$\epsilon_k = \frac{(k-1)!}{(\alpha+1)^{n-k} \Gamma[(\alpha+1)n]} \sum_{\substack{m_1 \leq \dots \leq m_k \\ \sum m_i = n-k}} \binom{k}{v_0, \dots, v_k} \binom{n-k}{m_1, \dots, m_k} \quad (3.3)$$

$$\cdot \prod_{i=1}^k \Gamma[(\alpha+1)(m_i+1)].$$

Corollary. The probability of covering the circle with n random arcs whose lengths are chosen independently from the uniform distribution on $(0,1)$ is

$$P(\text{cover}) = 1 + \frac{n!}{(2n-1)!} \sum_{k=1}^n \frac{(-1)^k}{(n-k)! k 2^{n-k}} \sum_{\substack{m_1, \dots, m_k \\ \sum m_i = n-k}} \binom{n-k}{m_1, \dots, m_k} \quad (3.4)$$

$$\cdot \prod_{i=1}^k (2m_i+1)! \dots$$

Despite its complicated appearance, it was tabulated on a hand calculator for up to 10 arcs, as shown in the table, second column. The third column allows comparison with the case of fixed arcs of the expected length, $1/2$. We see that the differences in coverage probability are most evident when n is small.

To further assess the differences between placing fixed and random arc lengths, column 4 of the table lists the fixed arc length whose coverage probability is equal to that of the uniform random arc lengths. These equivalent fixed arc length values converge to $.5$, again suggesting the similarity as n increases between the random and fixed arc length coverage probabilities.

Table.

<u>n</u>	<u>Coverage Probabilities</u>		<u>fixed arc lengths needed to achieve coverage probability listed for $\mathcal{U}(0,1)$ arcs</u>
	<u>random arc lengths $\mathcal{U}(0,1)$</u>	<u>fixed arc lengths .5</u>	
1	0	0	-
2	.167	0	.583
3	.383	.250	.547
4	.569	.500	.524
5	.730	.688	.518
6	.834	.813	.512
7	.901	.891	.508
8	.943	.938	.506
9	.967	.965	.505
10	.982	.980	.504

Proof of Theorem 3.1. Substituting $F(x) = x^\alpha$ into (2.3) and performing the inner expectation, we have

$$\xi_k = E \left\{ \left(\prod_{i=1}^k S_{ik}^\alpha \right) \left(\sum_{j=1}^k \frac{S_{jk}^{\alpha+1}}{\alpha+1} \right)^{n-k} \right\}.$$

Distributions are preserved by the representation $S_{ik} = X_i / (\sum_{j=1}^k X_j)$ where $X_1, \dots, X_n$ are independent exponential random variables with mean 1. Moreover, $(S_{1k}, \dots, S_{kk})$ is independent of $\sum_{j=1}^k X_j$, so we can show that

$$\xi_k = \frac{E \left[\left(\prod_{i=1}^k X_i^\alpha \right) \left(\sum_{j=1}^k \frac{X_j^{\alpha+1}}{\alpha+1} \right)^{n-k} \right]}{E \left(\sum_{j=1}^k X_j \right)^{(\alpha+1)n-k}}. \quad (3.5)$$

The distribution of $\sum_{j=1}^k X_j$ is Gamma(k,1) so the denominator of (3.5) is

$$\frac{\Gamma[(\alpha+1)n]}{(k-1)!} \quad (3.6)$$

The numerator of (3.5) can be written as

$$\frac{[\Gamma(\alpha+1)]^k}{(\alpha+1)^{n-k}} E \left(\sum_{i=1}^k Y_i^{\alpha+1} \right)^{n-k} \quad (3.7)$$

where $Y_1, \dots, Y_k$ are independent Gamma($\alpha+1, 1$). We do a multinomial expansion to obtain

$$\frac{[\Gamma(\alpha+1)]^k}{(\alpha+1)^{n-k}} \sum_{\left(\sum_{i=1}^k m_i = n-k \right)} \binom{n-k}{m_1, \dots, m_k} \prod_{i=1}^k E(Y_i^{\alpha+1})^{m_i} \quad (3.8)$$

The expectation can be evaluated:

$$E(Y_i^{\alpha+1})^{m_i} = \Gamma[(\alpha+1)(m_i+1)] / \Gamma(\alpha+1) \quad (3.9)$$

Substituting (3.9) in (3.8), substituting the resulting numerator and the denominator (3.6) for the ratio in (3.5), we have the result (3.2).

Formula (3.3) is obtained by combining summands with the same set of exponents $\{m_1, \dots, m_k\}$ and the same multiplicities. $\square$

REMARK: In a similar way, distributions of the form $F(x) = 1-(1-x)^n$, $0 < x < 1$, $n=1,2,\dots$ can be treated.

4. Asymptotic Results

Consider the situation of Section 2, but let the arc lengths be chosen from the distributions F_n which tend to zero in a suitable way when $n \rightarrow \infty$.

Theorem 4.1. Suppose that the arc lengths $0 \leq L_{1n}, L_{2n}, \dots, L_{nn} \leq 1$ are independent, identically distributed, random variables with the distribution F_n such that

$$(A) \quad E(N_n) = n(1-\mu_n)^{n-1} \rightarrow \lambda, \quad n \rightarrow \infty \quad (4.1)$$

where $\mu_n = E(L_{jn})$, and for each $k=2,3,\dots$ we have

$$(B) \quad \frac{\int_{-\infty}^{\infty} [E(F_n(X/k) \exp(n \int_{X/k}^{\infty} (1-F_n(\ell)) d\ell / (1-k\mu_n) + iu(X-1))]^k du}{2\pi k^k e^{-k} / k!} \rightarrow 1, \quad n \rightarrow \infty, \quad (4.2)$$

where X is exponential with mean 1. Then

$$N_n \xrightarrow{D} \text{Poisson}(\lambda), \quad n \rightarrow \infty. \quad (4.3)$$

Proof. By the Lemma below it follows that we need only establish that for all $k=1,2,3,\dots$,

$$n^k \xi_k \rightarrow \lambda^k, \quad n \rightarrow \infty, \quad (4.4)$$

where ξ_k was defined in Theorem 2.1. By the condition (A) this is true for $k=1$. Using the identity $(t)_+ = t + (-t)_+$, we see that

$$\begin{aligned} n^k \xi_k &= n^k E_S \left\{ (1 - k\mu_n + \sum_{j=1}^n E_L(L_{jn} - S_{jk})_+)^{n-k} \prod_{j=1}^k F_n(S_{jk}) \right\} \\ &= n^k (1 - k\mu_n)^{n-k} . \end{aligned} \quad (4.5)$$

$$E_S \left\{ \left(1 + \sum_{j=1}^k E_L(L_{jn} - S_{jk})_+ / (1 - k\mu_n) \right)^{n-k} \prod_{j=1}^k F_n(S_{jk}) \right\} .$$

By (4.1) it follows that

$$n^k (1 - k\mu_n)^{n-k} \rightarrow \lambda^k, \quad n \rightarrow \infty. \quad (4.6)$$

Hence (4.4) is true if for each $k=2,3,\dots$

$$E_S \left\{ (1 + Y_n)^{n-k} \prod_{j=1}^k F_n(S_{jk}) \right\} \rightarrow 1, \quad n \rightarrow \infty, \quad (4.7)$$

where we have introduced the random variable

$$Y_n = \sum_{j=1}^k E_L(L_{jn} - S_{jk})_+ / (1 - k\mu_n) = \sum_{j=1}^k \int_{S_{jk}}^1 (1 - F_n(\ell)) d\ell / (1 - k\mu_n). \quad (4.8)$$

As $1+x \leq e^x$ for $x \geq 0$ it follows that (4.7) holds if

$$E_S \left\{ e^{nY_n} \prod_{j=1}^k F_n(S_{jk}) \right\} \rightarrow 1, \quad n \rightarrow \infty. \quad (4.9)$$

In a similar way as in Holst (1981) Lemmas 2.1 and 3.1 one can prove that

$$\begin{aligned} E_S \left\{ e^{nY_n} \prod_{j=1}^k F_n(S_{jk}) \right\} &= (2\pi k^k e^{-k}/k!)^{-1} \cdot \\ &\cdot \int_{-\infty}^{\infty} \left[E \left\{ F_n(X/k) \exp \left(n \int_{X/k}^{\infty} (1 - F_n(\ell)) d\ell / (1 - k\mu_n) + iu(X-1) \right) \right\} \right]^k du \end{aligned} \quad (4.10)$$

Combining (4.9) and (4.10) with the Assumption (B) proves the assertion. $\square$

Remark. From the proof it actually follows that the probability generating function of N_n converges to that of a Poisson(λ), provided (4.1) and (4.2) are fulfilled.

Corollary. If the arcs have equal lengths μ_n and $E(N_n) \rightarrow \lambda$, then

$$N_n \xrightarrow{D} \text{Poisson}(\lambda), n \rightarrow \infty.$$

Proof. In this case the integral in (4.2) simplifies to

$$\begin{aligned} & \int_{-\infty}^{\infty} \left[\int_{k\mu_n}^{\infty} \exp(iu(x-1)-x) dx \right]^k du = \\ &= \int_{-\infty}^{\infty} [\exp(-iu - k\mu_n + iuk\mu_n) / (1-iu)]^k du \\ &\rightarrow \int_{-\infty}^{\infty} [e^{-iu} / (1-iu)]^k du = 2\pi k^k e^{-k} / k!, n \rightarrow \infty, \end{aligned}$$

As $(1-iu)^{-k}$ is the characteristic function of a Gamma ($k,1$) distribution. $\square$

In a similar way by explicit calculation of the integrand in (4.2) one obtains:

Corollary. Let the arc lengths take the values $0 < a_1\mu_n < \dots < a_r\mu_n$ with probabilities $p_1, \dots, p_r$ where $p_1 + \dots + p_r = 1$ and $a_1p_1 + \dots + a_rp_r = 1$.

If $\mu_n \rightarrow 0$ such that $E(N_n) \rightarrow \lambda$, then

$$N_n \xrightarrow{D} \text{Poisson}(\lambda), n \rightarrow \infty.$$

Lemma. Let $(I_{1n}, \dots, I_{nn})$, $n=1, 2, \dots$ be a triangular array of exchangeable random variables that take only the values 0 or 1. Suppose there exists a $\lambda > 0$ such that for every fixed $k=1, 2, \dots$ we have

$$n^k E \left(\prod_{i=1}^k I_{in} \right) \rightarrow \lambda^k \text{ as } n \rightarrow \infty. \quad (4.11)$$

Then

$$\sum_{i=1}^n I_{in} \xrightarrow{D} \text{Poisson } (\lambda) \text{ as } n \rightarrow \infty. \quad (4.12)$$

Proof. Consider the Stirling number of the second kind $S(m, k) =$ "number of partitions of m distinct elements into k sets." By exchangeability it follows that the m^{th} moment of the left hand side of (4.11) is

$$E \left(\sum_{i=1}^n I_{in} \right)^m = \sum_{k=1}^m S(m, k) \frac{n!}{(n-k)!} E \left[\prod_{i=1}^k I_{in} \right]$$

which tends to the m^{th} moment

$$\sum_{k=1}^m S(m, k) \lambda^k \quad (4.13)$$

of a Poisson (λ) random variable as $n \rightarrow \infty$, establishing (4.12) by convergence of moments. $\square$

Acknowledgement.

This work was completed while the authors were visiting Stanford University. We wish to thank C. M. Oliveri of Princeton University for helpful conversations during the beginning stages of this work.

REFERENCES

- Domb, C. (1947). The Problem of Random Intervals on a Line. Proceedings of the Cambridge Philosophical Society, 43, 329-341.
- Dvoretzki, A. (1956). On Covering a Circle by Randomly Placed Arcs. Proceedings of the National Academy of Sciences U.S.A., 42, 199-203.
- Fisher, R. A. (1940). On the Similarity of the Distributions Found for the Test of Significance in Harmonic Analysis, and in Stevens's Problem in Geometrical Probability. Annals of Eugenics, 10, 14-17.
- Flatto, L., and Konheim, A. G. (1962). The Random Division of an Interval and the Random Covering of a Circle. SIAM Review, 4, 211-222.
- Holst, L. (1980a). On Multiple Covering of a Circle with Random Arcs. Journal of Applied Probability, 17, 284-290.
- _____. (1980b). On the Lengths of the Pieces of a Stick Broken at Random. Journal of Applied Probability, 17, (to appear September 1980).
- _____. (1980c). A Note on Random Arcs on the Circle. Technical Report, Department of Statistics, Stanford University.
- _____. (1981). On Convergence of the Coverage by Random Arcs on a Circle and the Largest Spacing. Annals of Probability, to appear.
- Mandelbrot, M. (1972). On Dvoretzki Coverings for the Circle. Z. Wahrscheinlichkeitstheorie, 22, 158-160.
- Shepp, L. A. (1972). Covering the Circle with Random Arcs. Israel Journal of Mathematics, 11, 328-345.
- Siegel, A. F. (1978a). Random Space Filling and Moments of Coverage in Geometrical Probability. Journal of Applied Probability, 15, 340-355.
- _____. (1978b). Random Arcs on the Circle. Journal of Applied Probability, 15, 774-789.
- _____. (1979). Asymptotic Coverage Distributions on the Circle. Annals of Probability, 7, 651-661.
- Solomon, H. (1978). Geometric Probability. SIAM: Philadelphia.
- Stevens, W. L. (1939). Solution to a Geometrical Problem in Probability. Annals of Eugenics, 9, 315-320.
- Votaw, D. F. (1946). The Probability Distribution of the Measure of a Random Linear Set. Annals of Mathematical Statistics, 17, 240-244.

Whitworth, W.A. (1897, 1959). DCC Exercises on Choice and Chance.
Deighton Bell and Co.: Cambridge. Republished by Hafner: New
York.

Yadin, M., and Zachs, S. (1980). Random Coverage of a Circle with
Application to a Shadowing Problem. Technical Report, Department
of Statistics, Virginia Polytechnic Institute.