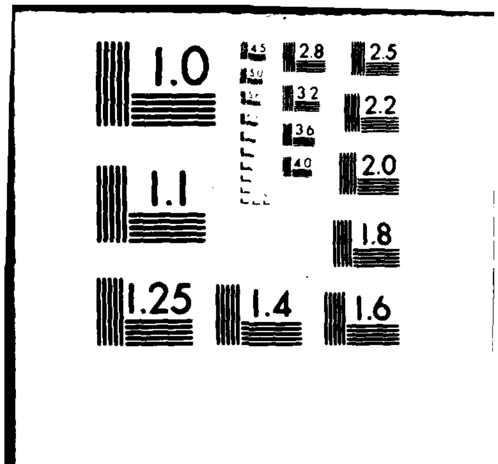

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A CONSTRUCTIVE PROOF OF THE BORSUK-ULAM ANTIPODAL POINT THEOREM

by

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TECHNICAL REPORT SOL 80-9
May 1980

Research and reproduction of this report were partially supported by Department of Energy Contract DE-AC03-76-SF00326, PA #DE-AT-03-76ER72018; the National Science Foundation Grants MCS-77-05623 and SOC-78-16811; and the Army Research Office-Durham Contract DAAG-29-78-G-0026.

I. Introduction

In this paper, a proof of the Borsuk-Ulam Antipodal Point Theorem is presented by means of a constructive algorithm that computes an approximate solution by means of a simplicial subdivision and integer labels.

II. Main Results

For $x \in \mathbb{R}^n$, let $\|x\|$ denote the L^∞ norm. Let $S^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$. By an odd function, we mean a function such that $f(-x) = -f(x)$. The Borsuk-Ulam Antipodal Point Theorem [1] can be stated as follows:

Theorem: Let $f : S^n \to \mathbb{R}^{n-1}$ be an odd continuous function. Then there exists a point $x^* \in S^n$ such that $f(x^*) = 0$. ⊠

Let T be any symmetric triangulation of S^n such that its restriction to $S^n \cap \{x | x_i = 0 \ i \in U\}$ for any U is also a triangulation, with grid size δ. For example, consider a scaling of J^1 [2] restricted to S^n. Let T^0 denote the vertices of the triangulation. Consider the following labeling function on T^0:

$$\ell(x) = \begin{cases} i & \text{if } i \text{ is the smallest index such that } \|f(x)\| = f_i(x) \text{ and } f_i(x) > 0 \\ -i & \text{if } i \text{ is the smallest index such that } \|f(x)\| = f_i(x) \text{ and } f_i(x) \le 0 \end{cases}$$

Note $\ell(x) = -\ell(-x)$.

Fix $\varepsilon > 0$ and choose δ such that $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \varepsilon$.

<u>Lemma 1</u>: Suppose $\ell(x) = i > 0$ and $\ell(y) = -i$ and $\|x - y\| < \delta$. Then $\|f(x)\| < 3\varepsilon$.

<u>Proof</u>:

$$f_i(x) > 0$$

$$f_i(y) \leq 0$$

$$f_i(x) - f_i(y) < \varepsilon$$

$$f_i(x) \leq \varepsilon + f_i(y) < \varepsilon$$

$$f_i(y) > f_i(x) - \varepsilon \geq -\varepsilon \quad .$$

Therefore

$$|f_i(x)| < \varepsilon \quad \text{and} \quad |f_i(y)| < \varepsilon \ ,$$

also

$$f_j(x) \leq f_i(x) \quad \text{for any} \quad j = 1, \ldots, n$$

$$f_j(y) \geq f_i(y) \quad \text{for any} \quad j = 1, \ldots, n \quad .$$

Therefore

$$f_i(x) - 2\varepsilon < f_i(y) - \varepsilon \leq f_j(y) - \varepsilon < f_j(x) \leq f_i(x) \ .$$

Therefore

$$|f_j(x) - f_i(x)| < 2\varepsilon \ ,$$

hence

$$\|f(x)\| < 3\varepsilon \ . \qquad \boxtimes$$

In the next section, we will prove constructively:

Lemma 2: For a given T and induced labeling $\ell(\cdot)$ as above, there exists a pair of adjacent vertices x and $y \in S^n$ such that $\ell(x) = i$ and $\ell(y) = -i$. ⊠

Combining Lemmas 1 and 2 and taking a limiting subsequence of x's as $\varepsilon \to 0$, we obtain the main theorem.

3. An Algorithm for Computing Oppositely Labeled Adjacent Vertices

The algorithm of this section is a modification of that of Reiser [3]. Let T^i denote the collection of i-dimensional simplices of T. We shall define a simplex $\sigma \in T$ to be oppositely labeled if there are two vertices x, y of σ such that $\ell(x) = i$ and $\ell(y) = -i$. The algorithm will terminate with an oppositely labeled simplex. Let $R \subset \{1, \ldots, n-1, -1, \ldots, -n+1\}$ such that $i \in R$ implies $-i \notin R$. Define

$$A(R) = \{x \in S^n : \begin{array}{ll} x_i \geq 0 & \text{for } 0 < i \in R, \\ x_i \leq 0 & \text{for } 0 < -i \in R, \\ x_i = 0 & \text{otherwise} \end{array} \} .$$

The following algorithm, analogous to that of Reiser, will produce an oppositely labeled simplex. d is the dimension of the simplex under question. q is the index of the newly added vertex. R is the index set of the orthant of $\mathbb{R}^{n-1}$ under consideration, and X is the set of vertices of the simplex under question.

<u>Step 0</u>: $R \leftarrow \emptyset$. $v' \leftarrow e^n$. $X \leftarrow \{v'\}$. $d \leftarrow 0$. $q \leftarrow 1$.

<u>Step 1</u>: Let $\ell = \ell(v^q)$. If there is a vertex $v \in X$ with $\ell(v) = -\ell$, stop. If there is a vertex $v^k \in X$, $k \neq q$ with $\ell(v^k) = \ell$, go to Step 2, otherwise go to Step 3.

<u>Step 2</u>: v^k is replaced by the unique vertex $\bar{v}^k$ in $A(R)$ for which we have a d-dimensional simplex of T in $A(R)$, if such a $\bar{v}^k$ exists. In this case set $v^k \leftarrow \bar{v}^k$, set $q \leftarrow k$ and go to Step 1. Otherwise, go to Step 4.

<u>Step 3</u>: $R \leftarrow R \cup \{\ell\}$, $d \leftarrow d + 1$. Define v^{d+2} to be the unique vertex $v \in A(R)$ such that $\langle v^1, \ldots, v^{d+1}, v\rangle \in T^d$. $X \leftarrow X \cup \{v^{d+2}\}$, $q \leftarrow d + 2$. Go to Step 1.

<u>Step 4</u>: $X \leftarrow X \setminus \{v^k\}$. There now is a unique index $i \in R$ such that $v_i = 0$ for all $v \in X$. Set $R = R \setminus \{i\}$. $d \leftarrow d - 1$. $q \leftarrow$ that index s.t. $v^q \in X$ and $\ell(v^q) = i$. Set $k \leftarrow q$ and go to Step 2.

Note that upon returning to Step 1, we have $X = \{v^1, \ldots, v^{d+1}\}$, each $v^i \in A(R)$, and R has d elements. The algorithm cannot cycle, and must terminate with either an oppositely-labeled subsimplex σ, or the simplex $\{-e^n\}$. The fact that $\ell(x) = -\ell(-x)$ quarantees that $\{-e^n\}$ cannot be the terminal simplex.

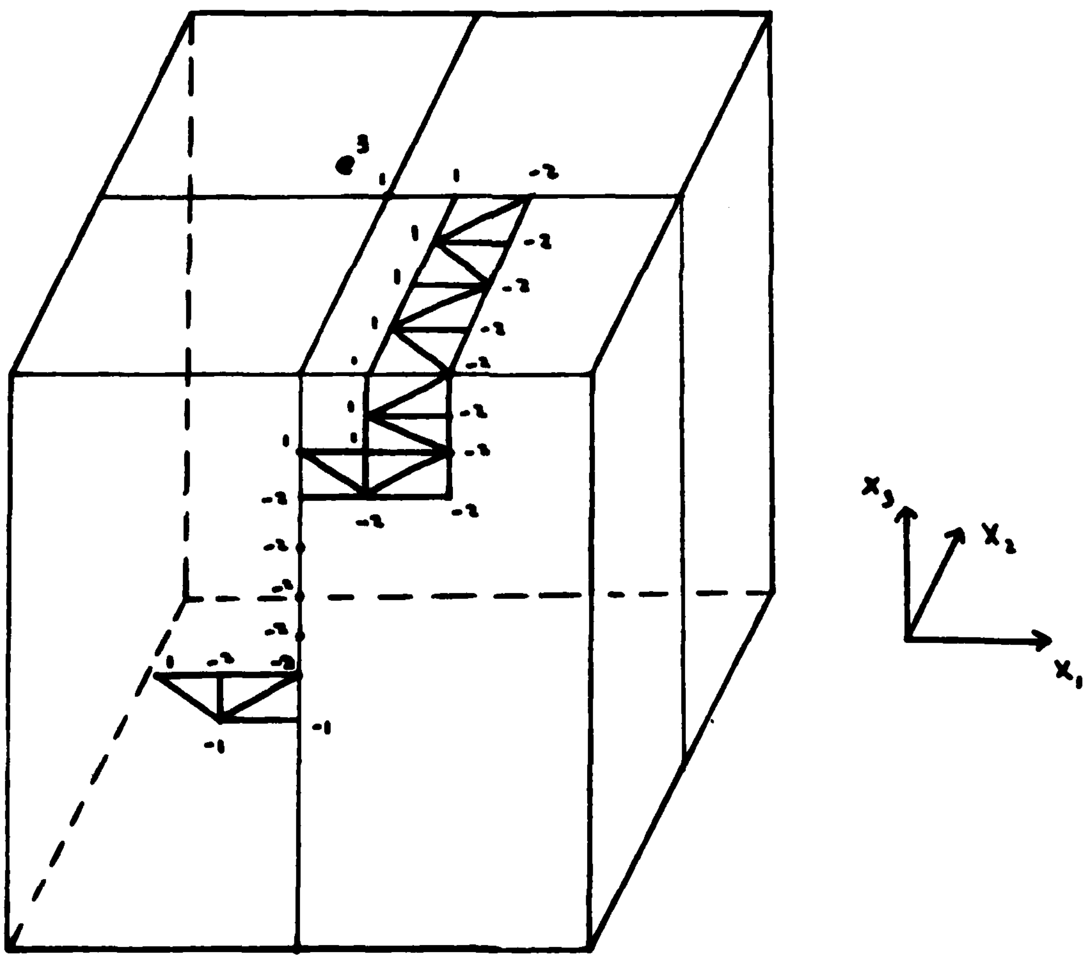

Figure 1. Sample path of algorithm

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[3] Reiser, P. M., "A Modified Integer Labelling for Complementarity Algorithms," Institut für Operations Research, Universität Zürich, June 1978.

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REPORT DOCUMENTATION PAGE	READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER SOL-80-9	2. GOVT ACCESSION NO. AD-A092 679
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) A CONSTRUCTIVE PROOF OF THE BORSUK-ULAM ANTIPODAL POINT THEOREM.	5. TYPE OF REPORT & PERIOD COVERED Technical
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) R. M. Freund	8. CONTRACT OR GRANT NUMBER(s) DAAG-29-78-G-0026
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research, Stanford University, Stanford, California 94305	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematics Division, U.S. Army Research Office, Box CM, Duke Station, Durham, N.C. 27706	12. REPORT DATE May 1980
	13. NUMBER OF PAGES 7
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE NA

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

NA

18. SUPPLEMENTARY NOTES

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Antipodal points — Algorithm

Simplicial methods — Integer labels

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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