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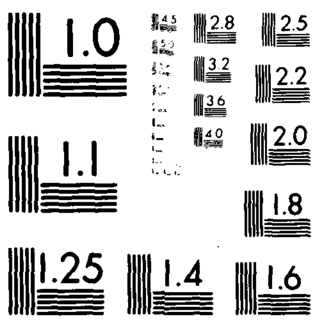

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DIGITAL AND CELLULAR CONVEXITY

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ABSTRACT

A new definition of cellular convexity is introduced. We then show that given a complex, it is cellularly convex if and only if the digital region determined by the complex is digitally convex. It is also shown that a digital region (a complex) is digitally (cellularly) convex if and only if the MPP of the digital region (the half-cell expansion of the complex) contains only the digital region (the centers of the cells of complex). An algorithm is presented for determining the concavity tree of a digital region or a complex.

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Special Characters

κ	k	IBM symbol element (lower case characters)
σ	s	IBM symbol element (lower case characters)
δ	w	IBM symbol element (lower case characters)
γ	q	IBM symbol element (lower case characters)
ε	e	
∂	M	IBM symbol element (upper case)
$\mathcal{H}$	script element H	
$\mathcal{K}$	script element K	

1. Introduction

In several earlier papers Sklansky et al. derived techniques and an associated theory for determining whether or not a digitized blob represented by a finite subset of a mosaic of squares (a "cellular complex") can be the digitization of a convex region (a convex "preimage") [4,5,6]. Sklansky defined a complex to be "cellularly convex" if it has a convex preimage [4]. He showed that if a complex is "regular", its polygonal preimage of minimum perimeter reveals the concavities that every preimage of the complex must have. The restriction of regularity requires that a complex have no peninsular protrusion that is one cell wide. Although this is a weak restriction it is still significant, particularly in noisy images.

The Minsky-Papert definition of convexity may be specialized to define digital convexity of digital regions (finite 8-connected subsets of lattice points on the Euclidean plane) [2]. However, no useful results concerning digital convexity have been obtained in terms of this definition.

Recently in [1] Kim proposed a new definition of cellular convexity, and showed that a complex is convex under the new definition if and only if its corresponding digital region is convex by the definition of Minsky-Papert. This leads us to two distinct but equivalent concepts: digital convexity and cellular convexity. (Digital blobs and digital convexity refer to arrays of points; cellular blobs and cellular convexity refer to arrays of cells.)

In this paper we eliminate the restriction of regularity by using Kim's revised definition. To do so, we use Sklansky's work on the half-cell expansion of a complex [6]: a complex is defined to be cellularly convex if there exists a convex region whose digitized image is the half-cell expansion of the complex.

A complex determines a unique digital region: the array of center points of the cells of the complex. We show that a complex is cellularly convex if and only if the unique digital region determined by the complex is digitally convex.

We define the minimum-perimeter polygon (MPP) of digital regions and cellular complexes. Then we show that given a complex, the minimum-perimeter polygon of its half-cell expansion and the minimum-perimeter polygon of the digital region determined by the complex are identical. We prove that a simply 8-connected digital region is digitally convex if and only if its MPP contains only the points of the digital region. A corollary to this is an equivalent result for tight complexes that are simply 8-connected. These are the principal results of this paper.

We also show that the convex hull of the set of center points of the cells of a complex determines the cellular convex hull of the complex, that is, the smallest cellularly convex complex containing the complex. In fact, the convex hull of the set of center points of the cells of a complex is the MPP of the half-cell expansion of the cellular convex hull of the complex. This result leads to an algorithm for determining the concavity tree of any digital region or any complex.

In the next section, we introduce definitions and notation. In Section 3, we discuss digital and cellular concavities and the relation between them. Section 4 is concerned with the MPPs of digital regions and complexes. The relation between the MPP and convexity is also discussed. The concavity tree of a complex is considered in Section 5.

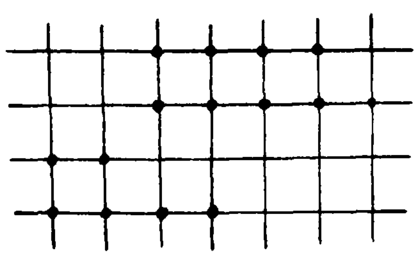
2. Definitions

In this section we introduce definitions and notations that are used in this paper.

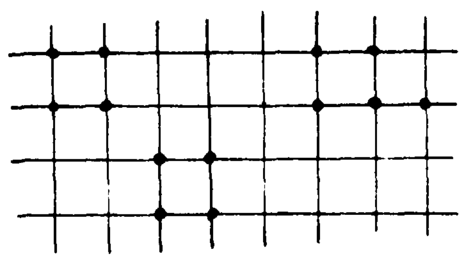
Digital region

Consider the set of all the lattice points--i.e., the integer-valued points--in the plane. Let S be a set of lattice points. Then $\overline{S}$ denotes the complement of S, that is, the set of all lattice points not in S. A point of S is an interior point if all of its four 4-neighbors [3] are points of S. A point of S is a boundary point if it is not an interior point. A point of S is a corner point if two of its 4-neighbors are mutually 8-neighbors and in $\overline{S}$. A <u>digital 8-chain</u> (<u>4-chain</u>) is a sequence of digital points such that every element of the sequence except the first is an 8-neighbor (4-neighbor) of its predecessor. A set S of lattice points is <u>8-connected</u> (<u>4-connected</u>) if for any two points c_1, c_2 of S, there is a digital 8-chain (4-chain) between them. A set of S is <u>simply 8-connected</u> if it is 8-connected and $\overline{S}$ is 4-connected.

A <u>digital region</u> D is any finite set of lattice points (see Figure 1). The digital region of Figure 1(a) is 8-connected;



(a)



(b)

Figure 1. Two digital regions.

the digital region of Figure 1(b) is not. We denote the (Euclidean) convex hull of the points of D by κD. The line segment between two points c_1 and c_2 is denoted by $\overline{c_1 c_2}$ and the directed line segment from c_1 to c_2 by $\overrightarrow{c_1 c_2}$.

Digital convexity (Minsky-Papert [2])

A digital region D is said to be digitally convex if there exists no triplet (c_1, c_2, c_3) of collinear points such that c_1 and c_3 are points of D and c_2 is a point of $\overline{D}$.

Cellular complex

A square cellular mosaic is the set of squares, called cells, in a grid of equal horizontal and vertical unit distance spacing on the plane. Let R be a finite subset of a square cellular mosaic and σR the set of points of R. ∂R denotes the set of points of the edges of all the cells of R and $\partial\sigma R$ the boundary of σR. A square cellular complex J is a finite set J' of cells together with a boundary ∂J, where ∂J is any subset of the edges of J' including $\partial\sigma J'$. Thus $J = \langle J', \partial J \rangle$, where ∂J is any set of edges satisfying $\partial\sigma J' \subseteq \partial J \subseteq \partial J'$. Hereafter, we use "complex" in place of "square cellular complex."

Given a complex J, J' denotes the set of cells in J and $\delta(J)$ the set of the centers of the cells of J'. Thus, $\delta(J)$ is a digital region uniquely determined by complex J. Given a digital region D, let J' be the set of cells whose centers are the points of D. The J' and $\partial\sigma J'$ determine a complex $J=\langle J', \partial\sigma J'\rangle$. The complex J is uniquely determined by D and denoted by $\gamma(D)$.

The complement $\bar{J}$ of a complex J is the set of cells $\bar{J}'$, the complement of J', with $\partial\bar{J}=\partial\sigma J'$. Thus $\bar{J}=\langle\bar{J}', \partial\sigma J'\rangle$. Two cells e_1, e_2 of J are <u>4-connected in J</u> (or <u>4-connected</u>, for short) if they share an edge which is not a subset of ∂J. Two cells e_1, e_2 of J are <u>8-connected in J</u> (or <u>8-connected</u>, for short) if they share a corner point and no two edges one of whose endpoints is the corner point belong to ∂J. A <u>cellular 8-chain</u> (<u>4-chain</u>) is a sequence of cells such that every two successive elements are 8-connected (4-connected). A complex J is 8-connected if for any two cells e_1 and e_2 of J, there is a cellular 8-chain between them. A complex J is simply 8-connected if it is 8-connected, ∂J is connected and $\bar{J}$ is 4-connected. In Figure 2, complex (a) is simply 8-connected, complex (b) is 8-connected but not simply 8-connected, and complex (c) is not 8-connected. The complex (c) consists of two 8-connected components of which one is simply 8-connected and the other is not.

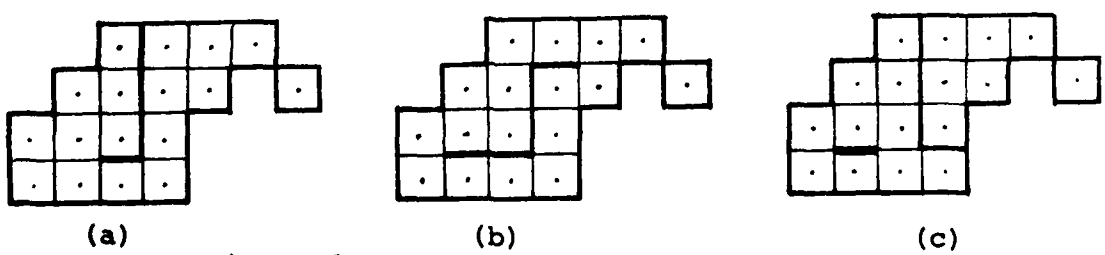


Figure 2. Examples of complexes

We call $\partial\sigma J'$ the outer boundary of complex J and $\partial J-\partial\sigma J'$ the inner boundary of J. A complex J is said to be tight if it has an empty inner boundary. Otherwise it is nontight.

An element of J' is said to be penisolated if three of its sides belong to ∂J. A complex J is said to be regular if J' has no penisolated element. An element of J' is an interior cell if no point of its edges belongs to the boundary of J, namely ∂J. An element of J' is a boundary cell if it is not an interior cell.

A boundary chain of a simply 8-connected complex J is a cellular 8-chain whose elements consist of all boundary cells of J'. The spinal path of a chain is the piecewise linear curve obtained by connecting the centers of the successive cells of the chain by straight line segments.

Cellular image

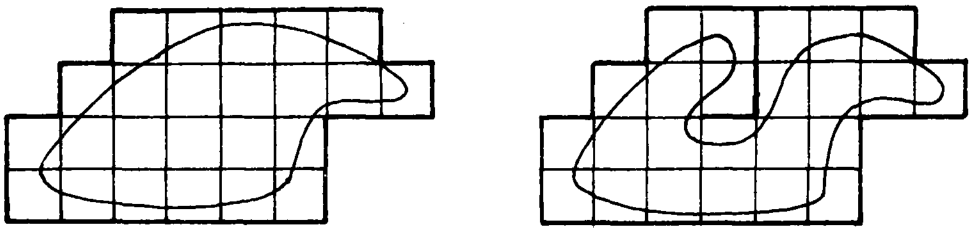
A complex J is said to be the cellular image (or simply, image) of a plane region q, and q a preimage of J, if

(i) $q \subseteq \sigma J'$

(ii) for each element e of J', $e^o \cap q \neq \emptyset$, where e^o is the interior of e, and

(iii) if r is an edge of an element e of J' and $r^o \cap q = \emptyset$, then r is a subset of ∂J, where r^o is the interior of r.

We denote the image of a plane region q by I(q). Figures 3(a) and 3(b) show plane regions together with their images, which are tight and nontight complexes, respectively.

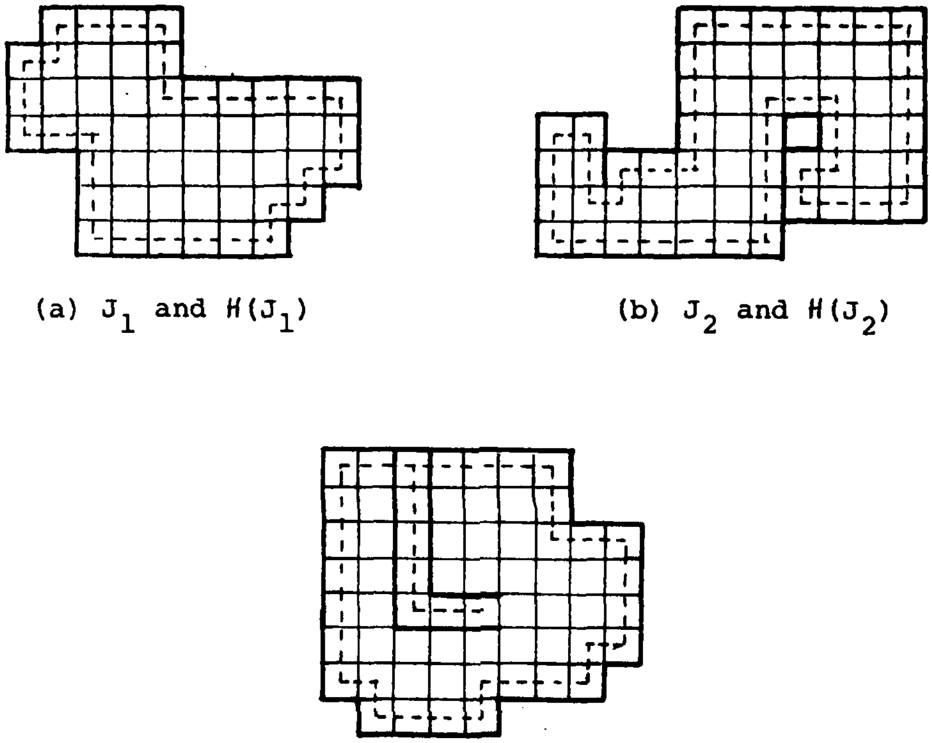


(a) (b)

Figure 3. Examples of the images of plane regions.

Half-cell expansion [6]

Let M denote the given square cellular mosaic. Construct another square cellular mosaic M' by displacing M by one-half of the diagonal of a cell along the direction of a diagonal. Let J be a complex on M and ∂J its boundary. The half-cell expansion of J, denoted $H(J)$, is the complex on M' such that the spinal path of its boundary chain is ∂J, the boundary of J. In Figure 4, the complex J_i is shown by the dashed lines and its half-cell expansion $H(J_i)$ by the solid lines. Figure 4(a) shows the complex J_1 and $H(J_1)$ both of which are tight. In Figure 4(b) J_2 is tight and $H(J_2)$ is not. Neither J_3 nor $H(J_3)$ in Figure 4(c) is tight.



(a) J_1 and $H(J_1)$

(b) J_2 and $H(J_2)$

(c) J_3 and $H(J_3)$

Figure 4. Complexes and their half-cell expansions.

Cellular convexity

A complex J is said to be cellularly convex if there exists a convex plane region q such that the half-cell expansion of J is the image of q, that is, $H(J)=I(q)$.

This is a modification of the original definition of cellular convexity of Sklansky in [4]. In the original definition, a complex J is cellularly convex if there exists a convex plane region q such that $J=I(q)$.

Cellular concavity (Kim [1])

Given a complex J, let c_1 and c_2 be the centers of elements e_1 and e_2 of J', respectively. (From now on, the center of an element e will always be denoted by a corresponding c.) Then $P(J;c_1,c_2)$ denotes the set of polygons each of whose boundaries consists of a segment of $\overline{c_1c_2}$ and the outer boundary of J and whose interiors are subsets of $\overline{\sigma J'}$, the complement of $\sigma J'$. In Figure 5, $P(J;c_1,c_2)$ is the set of four polygons shown as shaded regions.

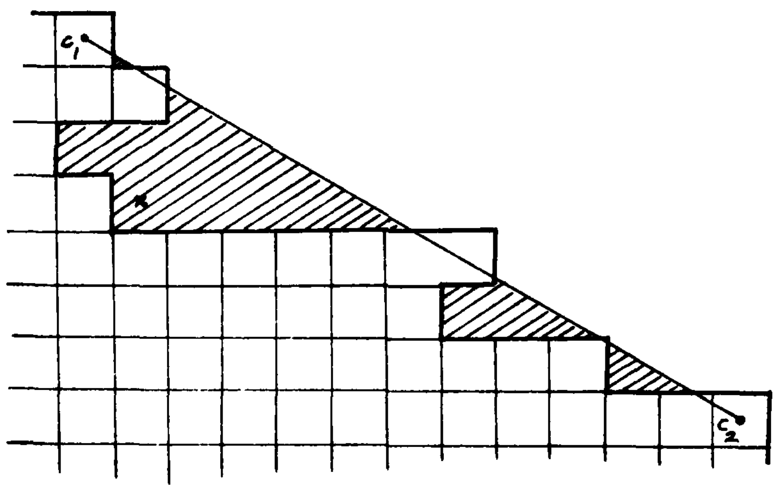


Figure 5. $P(J;c_1,c_2)$

A complex J is said to have a concavity between e_1 and e_2 if $P(J;c_1,c_2)$ contains a point of $\overline{\delta(J)}$. A complex J is said to have a concavity if there are two cells e_1 and e_2 of J' between which there is a concavity.

In the remainder of the paper we assume that complexes and digital regions are simply 8-connected unless stated otherwise.

3. Digital and cellular convexity

In this section we derive a relationship between digital convexity and cellular convexity. The main result is that a tight complex J is cellularly convex if and only if the digital region $\delta(J)$ is digitally convex.

First we state as lemmas three known results that will be used to prove our result.

Lemma 1 (Lemma 3 [1])

If a tight complex J has a concavity, then there is no convex plane region q whose image is J.

Lemma 2 (Theorem 4 [1])

A regular tight complex J does not have any concavity if and only if there is a convex plane figure q whose image is J.

Lemma 3 (Theorem 5 [1])

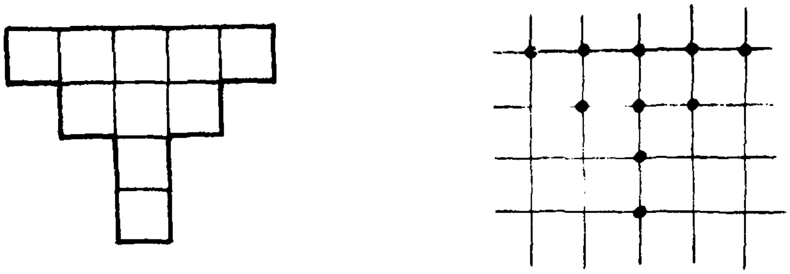
A tight complex J does not have any concavity if and only if $\delta(J)$ is digitally convex.

Theorem 4

Given a tight complex J, $\delta(J)$ is digitally convex if there is a convex plane region q whose image is J.

Proof: By Lemma 1, J has no concavity. Thus, $\delta(J)$ is digitally convex by Lemma 3. □

Now consider the tight complex T and $\delta(T)$ shown in Figure 6.



(a) Complex T (b) Digital region $\delta(T)$

Figure 6.

It is easy to see that $\delta(T)$ is digitally convex but there is no convex region q such that $T=I(q)$.

<u>Lemma 5</u>

If a complex J has a convex preimage, then it is tight.

<u>Proof</u>: Suppose J is not tight. Then it has nonempty inner boundary. Let e_1 and e_2 be two elements of J' sharing an edge which is a part of the inner boundary of J, as shown in Figure 7.

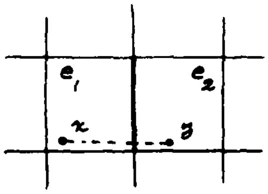


Figure 7.

Let q be any preimage of J. Since e_1 and e_2 are elements of J', there are interior points x of e_1 and y of e_2 that are points of q. Since the points of the edge shared by e_1 and e_2 except possibly the endpoints are not in q, $\overline{xy}$ contains a point not in q. Hence, q is not convex, and so J does not have any convex preimage.□

Lemma 6

If $\mathcal{H}(J)$ is tight, then J is also tight.

Proof: Suppose J is not tight. Then it has a nonempty inner boundary. A nonempty inner boundary of J induces a nonempty inner boundary of $\mathcal{H}(J)$. Thus $\mathcal{H}(J)$ is also not tight.□

Lemma 7

If a complex J is tight, then its half-cell expansion $\mathcal{H}(J)$ is regular.

Proof: Suppose $\mathcal{H}(J)$ is not regular. Then it has a penisolated cell. Consider such a cell shown in Figure 8. The part uvw of the boundary $\partial\mathcal{H}(J)$ is obtained by displacing the part u'v'x' of the boundary ∂J, and the part xwv of $\partial\mathcal{H}(J)$ by displacing the part

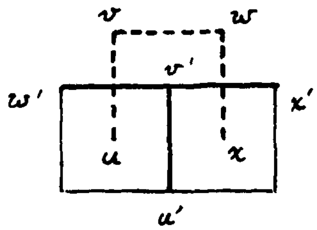


Figure 8. A penisolated cell of $\mathcal{H}(J)$.

u'v'w' of ∂J. The part w'v'x' is a subset of the outer boundary of J and the part v'u' is a subset of the inner boundary of J. Thus, the inner boundary of J is not empty and J is not tight, which proves the lemma.□

Lemma 8

A tight complex J has a concavity if and only if its half-cell expansion is either nontight or has a concavity.

Proof: Suppose J has a concavity. Then by Lemma 3, $\delta(J)$ is not digitally convex and therefore there exists a triplet of collinear points (c_1,c,c_2) such that $c_1,c_2\in\delta(J)$ and $c\in\overline{\delta(J)}$. Let e be the cell of which c is the center. If a pair of opposite edges of e belongs to ∂J, the line segment which is parallel to the edges and passes through c becomes a part of the inner boundary of $H(J)$. Hence, $H(J)$ is not tight. Now suppose that neither pair of opposite edges of e belongs to ∂J. Then there are two adjacent edges of e that do not belong to ∂J. Without loss of generality assume that these two edges are the top and right side edges. Let c_1', c' and c_2' be the points one half diagonal of a cell away in the direction of diagonals upward to the right of c_1, c and c_2, respectively. Then they are collinear, c_1' and c_2' are points of $\delta(H(J))$, and c is a point of $\overline{\delta(H(J))}$. Hence, $H(J)$ has a concavity.

Next suppose that $H(J)$ is either nontight or has a concavity. First consider the case when $H(J)$ is not tight. There exists a portion of the inner boundary of $H(J)$ that starts from the outer boundary of $H(J)$ and has unit length as shown in Figure 9.

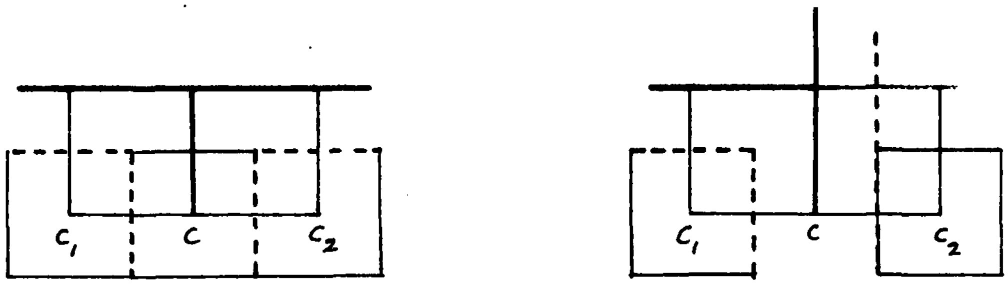
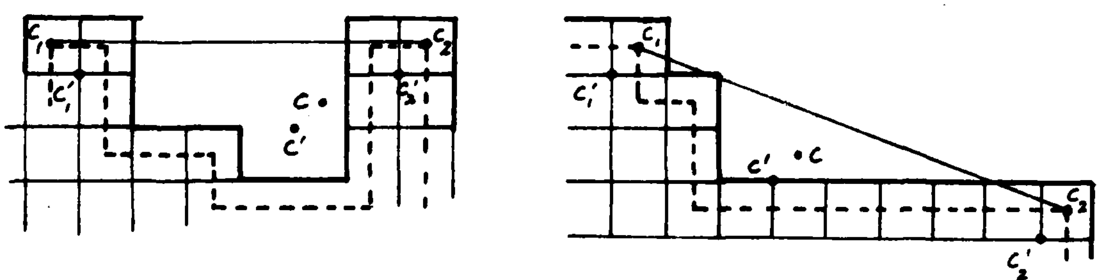


Figure 9.

Then the portion is formed by the half-cell expansion from the edges of two elements e_1 and e_2 of J' separated by a cell e of $\overline{J'}$. Then $P(J;c_1,c_2)$ contains c and so J has a concavity. Now consider the case when $\mathcal{H}(J)$ has a concavity. Then there are two vertices c_1 and c_2 of $\kappa\delta(\mathcal{H}(J))$ such that $P(\mathcal{H}(J);c_1,c_2)$ contains a point c of $\overline{(\mathcal{H}(J))'}$. The points c_1 and c_2 are corner points of cells e_1' and e_2' of J', respectively. Let e' be a cell of $\overline{J'}$ such that c is one of its corner points and c' is farther away from $\overline{c_1c_2}$ than c is. (See Figure 10 for an example.) Then $P(J;c_1',c_2')$ contains c' which is a point of $\overline{J'}$. Thus J has a concavity.□



(a) (b)

Figure 10.

Theorem 9

A tight complex J is cellularly convex if and only if the digital region $\delta(J)$ is digitally convex.

Proof: Suppose that J is cellularly convex. Then $H(J)$ has a convex preimage, and therefore is tight by Lemma 5. By Lemma 6, J is also tight. Hence, by Lemma 7, $H(J)$ is regular. So $H(J)$ does not have any concavity because of Lemma 3, and J also has no concavity by Lemma 8. Therefore, $\delta(J)$ is digitally convex by Lemma 4.

Now suppose that $\delta(J)$ is digitally convex. Then J does not have any concavity because of Lemma 3. By Lemma 8, $H(J)$ is tight and has no concavity. $H(J)$ is also regular because of Lemma 7. Therefore, by Lemma 2, $H(J)$ has a convex preimage and thus J is cellularly convex.□

4. The minimum-perimeter polygon (MPP)

In this section we define the minimum-perimeter polygons (MPPs) of digital regions and complexes. Then we show that given a tight simply 8-connected complex J, the MPP of its half-cell expansion $H(J)$ and the MPP of $\delta(J)$ are identical. The main results are: A digital region D is digitally convex if and only if its MPP does not contain any point of $\bar{D}$, and a complex J is cellularly convex if and only if the MPP of $H(J)$ does not contain the center of any cell of $\overline{J'}$.

MPPs of digital regions

Let D be a digital region. A polygon q is proper with respect to D if it contains every point of D and its interior does not contain any point of $\bar{D}$. Consider $J=\gamma(D)$, the tight cellular complex uniquely determined by D. Since $\gamma(D)$ is tight, ∂J is a connected closed curve. Hence ∂J can be traversed covering every edge of ∂J exactly once. The cell whose center is a corner point of D (or $\bar{D}$) is called a corner cell of J (or $\bar{J}$). Whenever the edges of a corner cell of either J or $\bar{J}$ are traversed, ∂J changes its direction. We construct a sequence P of corner points of D and $\bar{D}$ as follows: let c_0 denote the leftmost point of the top row of D and let e_0 denote the left top cell of J. Note that c_0 is a corner point of D, that c_0 is the center point of e_0, and that e_0 is a corner cell of J. Starting from the left top point of e_0, traverse ∂J clockwise.

Whenever ∂J changes its direction, a pair of adjacent edges of a corner cell is traversed. Then the corner point of D, the center of the corner cell, becomes the next element of P. The sequence P is completed when ∂J is traversed once. For the case of the digital region shown in Figure 11, $P=(c_0,c_1,...,c_{23})$.

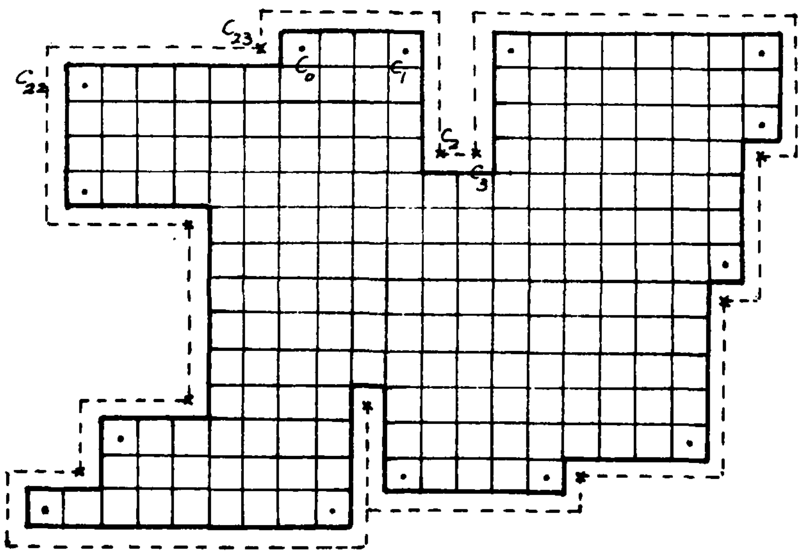


Figure 11. A digital region D and the corner points of D and $\overline{D}$.

Let $P=(c_0,c_1,\cdots,c_\ell)$ be the sequence of the corner points of D and $\overline{D}$ defined above. Let $P_i=(c_{i_1},c_{i_2},\cdots,c_{i_{k_i}})$ be a subsequence of P and consider it as a polygon whose sequence of vertices is P_i. Since P is proper, for any digital region, there is at least one proper polygon P_i whose sequence of vertices is a subsequence of P.

Lemma 10

If $P_m=(c_{m_1},c_{m_2},\cdots,c_{m_\ell})$ is a proper polygon with the shortest perimeter, then P_m is a subsequence of P and every convex vertex of P_m is a point of D and every concave vertex of P_m is a point of $\bar{D}$.

Proof: If P_m has a vertex which is not a point of D or $\bar{D}$, then we can construct a proper polygon whose perimeter is strictly shorter than P_m. Thus P_m must be a subsequence of P.

Now suppose that c_{m_i} is a convex vertex and a point of $\bar{D}$. Let $(c_{m_{i,1}},c_{m_{i,2}},\ldots,c_{m_{i,k}})$ be the subsequence of P such that

(i) $m_{i-1}<m_{i,1}<,\ldots,<m_{i,k}<m_{i+1}$,

(ii) its elements are points of D and thus points of P, and

(iii) the polygon $P'=(c_{m_1},\ldots,c_{m_{i-1}},c_{m_{i,1}},\ldots,c_{m_{i,k}},c_{m_{i+1}},\ldots,c_{m_\ell})$ is a proper polygon, where vertices $c_{m_{i,j}}$, $1\le j\le k$, are convex.

Then P' has shorter perimeter than P_m, which is a contradiction.

Now suppose that c_{m_i} is a concave vertex and a point of D. Then by a similar argument as above, we can obtain a proper polygon P" whose perimeter is shorter than P_m. □

Lemma 11

Given a digital region, its proper polygon with the shortest perimeter is unique.

Proof: Suppose not, and let $P_1=(c_{i_1},\ldots,c_{i_{k_i}})$ and $P_2=(c_{j_1},\ldots,c_{j_{k_j}})$ be two proper polygons with the shortest perimeter. We note that $c_{i_1}=c_{j_1}=c_0$. Let s be the smallest integer such that $c_{i_s}\neq c_{j_s}$.

Without loss of generality assume that $\overrightarrow{c_{i_{s-1}}c_{i_s}}$ lies to the right of $\overrightarrow{c_{j_{s-1}}c_{j_s}}$. Let x be the first point at which the boundaries of P_1 and P_2 intersect after $c_{i_{s-1}}$ $(=c_{j_{s-1}})$. Such a point x exists, since P_1 and P_2 meet again at c_0. Since P_1 and P_2 are proper polygons, between c_{s-1} and x, the vertices of P_1 and P_2 are points of D and $\overline{D}$, and hence convex and concave by Lemma 10, respectively. But this is not possible. □

Given a digital region, its minimum-perimeter polygon (MPP) is the unique proper polygon with the shortest perimeter. As we noted in Lemma 10, the convex vertices of the MPP of a digital region D are corner points of D and the concave vertices are corner points of $\overline{D}$.

MPPs of regular complexes

We first define the distance between two polygons. (Even though more generally we can define the distance between two compact sets of points on the plane, we restrict our discussion to polygons, which are compact sets.) For any two points p and q, let d(p,q) denote the Euclidean distance between them.

The distance between a point q and a polygon P is defined as

$$d(q,P) = \inf\{d(q,p) \mid p \in P\}.$$

We define the ε-neighborhood of a polygon P as

$$N_\varepsilon(P) = \{q \mid d(q,P) < \varepsilon\}.$$

The distance between two polygons P and Q is defined as

$$d(P,Q) = \inf\{\varepsilon \geq 0 \mid P \subseteq N_\varepsilon(Q) \text{ and } Q \subseteq N_\varepsilon(P)\}.$$

Let J be a regular complex. A polygon q is proper with respect to J if $J=I(q)$. Suppose that P_i is a proper polygon. Note that if e is a boundary cell of J, then there is an interior point of e that is also a point of P_i. Let ℓ_i denote the perimeter of a proper polygon P_i. Let ℓ be the greatest lower bound among the perimeters of all proper polygons. Construct a Cauchy sequence of proper polygons $(P_1,P_2,\ldots,P_i,\ldots)$ such that $\ell_i \to \ell$ as $i \to \infty$. It has been shown that there exists a polygon P such that $P_i \to P$, that is, $d(P_i,P) \to 0$, and that the perimeter of P is ℓ (see [7]).

<u>Lemma 12</u>

The polygon P defined above is unique. That is, any Cauchy sequence $(P_1,P_2,\ldots,P_i,\ldots)$ such that $\ell_i \to \ell$ as $i \to \infty$ converges to P.

<u>Proof</u>: Suppose not and let $(P_1,P_2,\ldots,P_i,\ldots)$ and $(P_1',P_2',\ldots,P_i',\ldots)$ be two Cauchy sequences such that $P_i \to P$, $P_i' \to P'$ and $P \neq P'$. Since $P \neq P'$, there exists an $\varepsilon > 0$ such that either $P \not\subseteq N_\varepsilon(P')$ or $P' \not\subseteq N_\varepsilon(P)$.

Thus, there exists a point p on the boundary of P and a point q on the boundary of P' such that $d(p,q) \geq \varepsilon$. Let $\delta>0$ and i be such that $\delta<<\varepsilon$ and $d(P_i-P)<\delta, \ell_i-\ell<\delta, d(P_i',P')<\delta$ and $\ell_i'-\ell<\delta$. Then there exist points p' on the boundary of P_i and q' on the boundary of P_i' such that $d(p',q')>\frac{\varepsilon}{2}$. Then it is not difficult to see that there is a proper polygon P_j such that $\max\{\ell_i-\ell_j, \ell_i'-\ell_j\}>\delta$. That is, the perimeter of P_j is less than ℓ, which is a contradiction. □

Given a regular complex J, its <u>minimum-perimeter polygon (MPP)</u> is the unique polygon P defined above.

<u>Lemma 13</u>

Let J be a regular complex and ℓ be the greatest lower bound of the perimeter of all proper polygons of J. If P is a polygon such that its perimeter is ℓ and there is a proper polygon P_i satisfying $d(P_i,P)<\varepsilon_i$ and $\ell_i-\ell<\varepsilon_i$ for any $\varepsilon_i>0$, then P is the MPP of J.

<u>Proof</u>: Let $\varepsilon_i=1/i$ for each $i=1,2,\cdots$. Then $(P_1,P_2,\cdots, P_i,\cdots)$ is a Cauchy sequence such that $P_i \to P$ as $i\to\infty$. Thus P is the MPP of J. □

<u>Theorem 14</u>

Let J be a complex such that its half-cell expansion $H(J)$ is a tight complex. Then the MPP of the complex $H(J)$ and the MPP of the digital region $\delta(J)$ are identical.

Proof: By Lemma 6, J is tight and so $H(J)$ is regular by Lemma 7. Since $H(J)$ is tight, every boundary cell of $H(J)$ contains a boundary point of $\delta(J)$. For any polygon to be a preimage of $H(J)$, it must contain every point of $\delta(J)$ and its interior must not contain any point of $\overline{\delta(J)}$. By definition the MPP P of $\delta(J)$ contains every point of $\delta(J)$, does not contain any point of $\overline{\delta(J)}$ in its interior and has the shortest perimeter ℓ among all such polygons. We also note that if e is a cell of $H(J)$, then $e \cap P \neq \emptyset$.

We claim that P is also the MPP of the complex $H(J)$. Because of Lemma 13, to prove our claim, we only have to

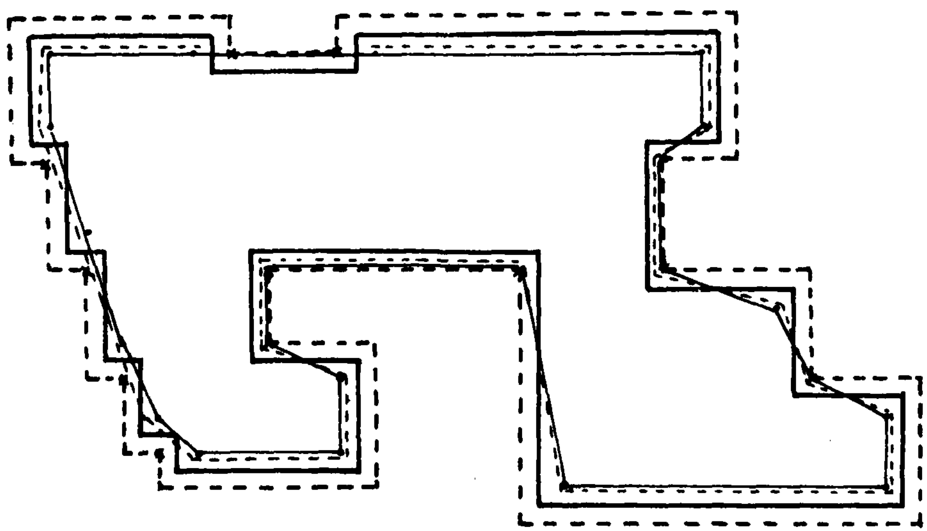


Figure 12. A complex J, $H(J)$, a proper polygon P_i and MPP P of $H(J)$.

show that for every $\varepsilon>0$, there exists a proper polygon P_i of $H(J)$ such that $d(P_i,P)<\varepsilon$ and $\ell_i-\ell<\varepsilon$, where ℓ_i is the perimeter of P_i. Let $P'=(c_0,c_1,\cdots,c_\ell)$ be a sequence of corner points on the boundary of P in clockwise order such that if c_i is a corner point of $\delta(J)$, then it is a convex vertex of P and if c_i is a corner point of $\overline{\delta(J)}$, then it is either a concave vertex of P or a virtual vertex [6] of P. (Considered as polygons, P and P' are identical except that P' may contain virtual vertices.) Let e_i be the corner cell to which a convex vertex c_i of P belongs. Let d_i be the diagonal of e_i with c_i as an endpoint. Given an $\varepsilon>0$, determine $\delta>0$ such that the polygon $P_i=(c'_0,c'_1,\cdots,c'_\ell)$ has perimeter ℓ_i which is less than $\ell+\varepsilon$ and $d(P_i,P)>\varepsilon$, where c'_i is the point on the diagonal d_i δ distance away from c_i if c_i is a convex vertex, and $c'_i=c_i$ otherwise. Also δ is small enough so that the interior of P_i does not contain a point of $\overline{D}$. Then P_i is the proper polygon that was sought. □

<u>Theorem 15</u>

A digital region D is digitally convex if and only if no point of $\overline{D}$ is a point of the MPP of D.

<u>Proof</u>: Suppose that no point of $\overline{D}$ is a point of P, the MPP of D. Then P is a convex polygon and contains all

points of D. Thus, there are no two points c_1, c_2, of D such that a point of $\overline{D}$ is on $\overline{c_1 c_2}$. Therefore, D is digitally convex.

Now suppose that there is a point, say c, of $\overline{D}$ which is also a point of P, the MPP of D. Then c is either a concave vertex or a virtual vertex. Let c_1 and c_2 be the vertices of P such that they are points of D and nearest to c on each side of c. Since there are at least three convex vertices in any polygon and every convex vertex of P is a point of D, the points c_1 and c_2 exist. Then $P(\gamma(D); c_1, c_2)$ contains c and thus $\gamma(D)$ is not cellularly convex. Since $\gamma(D)$ is simple, D is not digitally convex by Lemma 3. □

Theorem 16

A tight complex J is cellularly convex if and only if there is no cell of $\overline{J}'$ such that its center lies in the MPP of $H(J)$, the half-cell expansion of J.

Proof: By Theorem 14, the MPP of $H(J)$ is identical to the MPP of $\delta(J)$. Thus, the proof is immediate from Theorems 9 and 15. □

5. The cellular concavity tree

We use a structure called the cellular concavity tree to describe the shapes of concavities of a complex. In this section we restrict our attention to tight complexes that are simply 8-connected. We define the concavities of a tight complex as the simply 8-connected components of the difference between the complex and its cellular convex hull.

Given a tight complex J, let $\kappa J=(c_1,c_2,\cdots,c_\ell)$ be the convex hull of $\delta(J)$. For each i, $1\le i\le\ell$, let $(c_{i_1},c_{i_2},\cdots,c_{i_{j_i}})$ be the sequence of the centers of cells on $\overline{c_i c_{i+1}}$, where $\ell+1=1$. Then $P_h=(c_1,c_{1_1},\cdots,c_{1_{j_1}},c_2,\cdots,c_\ell,c_{\ell_1},\cdots,c_{\ell_{j_\ell}})$ is the same polygon as κJ except that P_h may have virtual vertices. Let KJ be the unique complex such that the MPP of its half-cell expansion HKJ is P_h and every vertex of P_h is a center of KJ. (See Figure 13 for an example.)

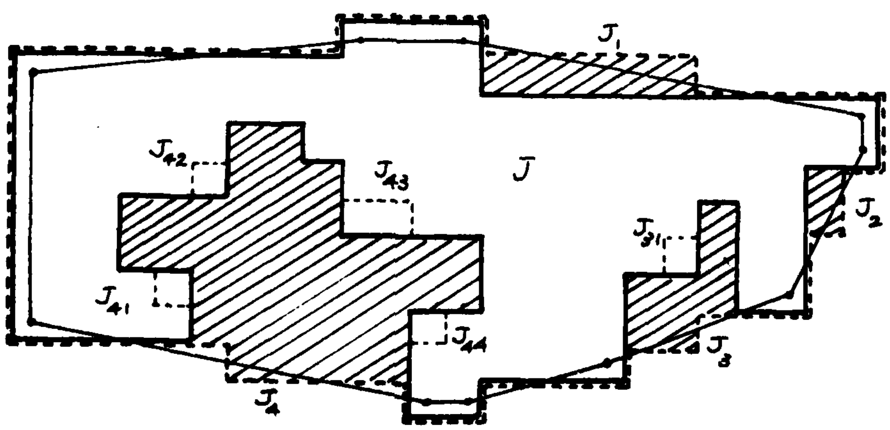


Figure 13. A complex J, κJ and KJ.

Theorem 17

Given a complex J, KJ is the smallest complex such that it is cellularly convex and contains J, that is, $J' \subseteq (KJ)'$.

Proof: By Theorem 16, KJ is cellularly convex. Since κJ has all the points of $\delta(J)$ and κJ and P_h are the same polygons (except possibly for the extra virtual vertices of P_h), J is contained in KJ. Suppose $J \subsetneq KJ$ and I is any complex such that $J \subseteq I \subsetneq KJ$. Then there exists a point c of $\delta(KJ)$ which is not a point of $\delta(I)$. Thus there exists i, $1 \leq i \leq \ell$, such that c_i is a vertex of κJ and $P(J; c_i, c_{i+1})$ contains c. Therefore I is cellularly concave. This shows that KJ is the smallest cellularly convex complex containing J. □

Cellular convex hull

The cellular convex hull of a complex J is the smallest cellularly convex complex which contains J, denoted by KJ.

The above theorem shows that the cellular convex hull of a complex J is well defined and is obtained from the convex hull of $\delta(J)$. The next theorem states a result analogous to that in Euclidean geometry.

Theorem 18

Given a complex J, let $\{J_i\}$ be the set of all cellularly convex complexes that contain J. Then $KJ = \bigcap_i J_i$.

Proof: Since KJ is cellularly convex and contains J, $KJ=J_i$ for some i, and hence $\bigcap_i J_i \subseteq KJ$. Now it remains to show that $KJ \subseteq \bigcap_i J_i$. It is sufficient to show that $\bigcap_i J_i$ is cellularly convex. Suppose not. Then by Theorem 9, there is a triplet (c_1,c_2,c_3) of collinear points such that $c_1,c_3 \in \delta(\bigcap_i J_i)$ and $c_2 \notin \delta(\bigcap_i J_i)$. Then $c_2 \notin \delta(J_i)$ for some i. But $c_1,c_3 \in J_i$ and thus J_i is not cellularly convex, which is a contradiction. □

Cellular concavity tree

The cellular concavity tree of a complex J is defined operationally by the following recursive algorithm [6].

Algorithm CONCAVITY-TREE(J)

1. If J is cellularly convex then return.
2. Find KJ.
3. For each J_i, where J_i denotes the i-th 8-connected component of $KJ-J$

   do Son(i)$\leftarrow J_i$; call CONCAVITY-TREE(J_i) end

In step 1, whether or not a complex is cellularly convex may be determined due to Theorem 16. In step 2 to find KJ, first κJ is found. An algorithm to find κJ is given in [1]. From κJ, KJ may be found.

Figure 14 shows the cellular concavity tree of the complex J shown in Figure 13.

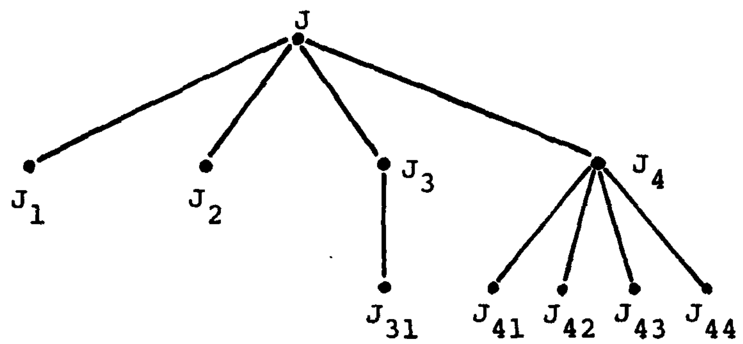


Figure 14. The cellular concavity tree of J in Figure 13.

6. Conclusion

Initially Sklansky defined cellular convexity in order to treat the following question: under what conditions does a complex have a convex preimage? He succeeded in answering the question by showing that a regular complex is cellularly convex (has a convex preimage) if and only if its MPP is convex.

Digital convexity of digital regions may be viewed as a special case of the Minsky-Papert definition of convexity. The relation between the two definitions was not noticed until recently partly because the former definition is in terms of preimages while the latter is in terms of a geometric property. Recently Kim showed that these two are equivalent except for the regularity condition.

In this paper, we unify the concepts of digital convexity and cellular convexity. We accomplish this by revising the definition of cellular convexity, using the concept of the half-cell expansion of a complex. We have shown that with this revised definition, digital convexity and cellular convexity are equivalent. Furthermore, the MPPs and concavity trees of digital regions and complexes are shown to be identical.

These results reinforce the soundness of our definitions of digital convexity, cellular convexity and the associated theory.

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