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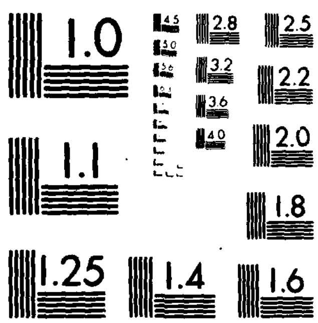

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On Characterizing Discrete Signals In Additive Noise - A Unified Treatment

Russell A. Boyles and Francisco J. Samaniego

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A characterization and classification result is established which applies to Binomial, Negative Binomial or Poisson signals in additive noise. The result unifies and generalizes three separate characterization results appearing in the recent literature.

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A characterization and classification result is established which applies to Binomial, Negative Binomial or Poisson signals in additive noise. The result unifies and generalizes three separate characterization results appearing in the recent literature.

I. INTRODUCTION.

The distributions of discrete signals in additive noise have been characterized via systems of differential equations satisfied by their probability mass functions in a series of recent papers, ([2], [3] and [4]). These papers have dealt with signal distributions belonging to various discrete exponential families, and each characterization result has roughly been in terms of equations of the form

$$\frac{\partial}{\partial\theta} f(x|\theta) = c[f(x-1|\theta') - f(x|\theta')].$$

While the main results in each of these works bear a definite resemblance, the proofs have differed substantially, and the regularity required of the signal distribution has varied among the specific models studied. For example, only the characterizing result for

Pascal signals in noise [4] require the existence of moments of all orders. We have sought here to present a single theorem which identifies the commonality of these earlier results.

The present result relies on a new and general parametrization of a discrete family of distributions which includes all discrete convolutions of Binomial, Negative Binomial (Pascal) and Poisson distributions as special cases. The proof of our characterization and classification theorem differs radically from the individual proofs of the characterization results in the papers cited. Moreover, the theorem requires somewhat weaker assumptions than cummulatively contained in previous results. In particular, no moment conditions are required in the present result.

II. TWO THEOREMS.

The following notation is used in the results of this section: $\mathbb{Z}$ for the set of all integers, and $\mathcal{B}$, $\mathcal{NB}$, and $\mathcal{P}$ for the binomial, negative binomial and Poisson distributions respectively.

Theorem 1. Let Z, $\{X_{n,\mu} : n \in \mathbb{Z} \cup \{\infty\}, \mu \in [0, +\infty)\}$ be random variables on the nonnegative integers. Assume the distribution of Z is independent of (n,μ), and assume moreover that

$$Z = X_{0,\mu} = X_{n,0} \quad \forall\, n,\mu.$$

Then the differential equation

$$(1) \quad \frac{\partial}{\partial\mu} P(X=x|n,\mu) = P(X=x-1|n-1,\mu-\frac{\mu}{n})$$

$$- P(X=x|n-1,\mu-\frac{\mu}{n}) \qquad (n \neq 0)$$

is equivalent to

(2) $X_{n,\mu} = Y_{n,\mu} + Z$

where Z and $\{Y_{n,\mu}\}$ are independent, and

$$(3) \quad Y_{n,\mu} \sim \begin{cases} \mathcal{B}(n, \frac{\mu}{n}) & \text{if } 0 < n \in \mathbb{Z} \text{ and } \mu \le n \\ \mathcal{NB}(-n, \frac{\mu}{\mu - n}) & \text{if } 0 > n \in \mathbb{Z} \\ \mathcal{P}(\mu) & \text{if } n = \infty \end{cases}$$

Remark: While the case $0 < n \in \mathbb{Z}$ and $\mu > n$ appears to be unclassified in (3) above, we note that the differential equations in (1) do not apply for this case since the parameter $\mu - \frac{\mu}{n}$ is negative and the probability functions on the right side of (1) are therefore undefined.

<u>Proof</u>. First we will show that the variables $Y_{n,\mu}$ defined in (3) satisfy (1). If $n = \infty$ we have

$$\begin{aligned} \frac{\partial}{\partial\mu} P(Y = x|\infty,\mu) &= \frac{\partial}{\partial\mu}\left\{e^{-\mu}\mu^x/x!\right\} \\ &= (-e^{-\mu}\mu^x/x!) + (e^{-\mu}x\mu^{x-1}/x!) \\ &= (e^{-\mu}\mu^{x-1}/(x-1)!) - (e^{-\mu}\mu^x/x!) \\ &= P(Y = x-1|\infty,\mu) - P(Y = x|\infty,\mu) \end{aligned}$$

which is (1), since $\mu = \mu - 0 = \mu - \frac{\mu}{\infty}$ and $\infty = \infty - 1$.

For $0 < n \in \mathbb{Z}$ we have

$$\frac{\partial}{\partial\mu}P(Y=x|n,\mu) = \frac{\partial}{\partial\mu}\left\{\binom{n}{x}\left(\frac{\mu}{n}\right)^{x}\left(\frac{n-\mu}{n}\right)^{n-x}\right\}$$

$$= \binom{n}{x}x\left(\frac{\mu}{n}\right)^{x-1}\left(\frac{1}{n}\right)\left(\frac{n-\mu}{n}\right)^{n-x} + \binom{n}{x}\left(\frac{\mu}{n}\right)^{x}(n-x)\left(\frac{n-\mu}{n}\right)^{n-x-1}\left(\frac{-1}{n}\right)$$

$$= \binom{n-1}{x-1}\left(\frac{\mu}{n}\right)^{x-1}\left(\frac{n-\mu}{n}\right)^{n-x} - \binom{n-1}{x}\left(\frac{\mu}{n}\right)^{x}\left(\frac{n-\mu}{n}\right)^{n-x-1}$$

$$= \binom{n-1}{x-1}\left[\frac{\left(\frac{n-1}{n}\right)\mu}{(n-1)}\right]^{x-1}\left[\frac{(n-1)-\left(\frac{n-1}{n}\right)\mu}{(n-1)}\right]^{(n-1)-(x-1)}$$

$$- \binom{n-1}{x}\left[\frac{\left(\frac{n-1}{n}\right)\mu}{(n-1)}\right]^{x}\left[\frac{(n-1)-\left(\frac{n-1}{n}\right)\mu}{(n-1)}\right]^{(n-1)-x}$$

$$= P\left(Y=x-1\middle|n-1,\left(\frac{n-1}{n}\right)\mu\right) - P\left(Y=x\middle|n-1,\left(\frac{n-1}{n}\right)\mu\right),$$

which is (1).

For $0 > n \in \mathbb{Z}$, we have

$$\frac{\partial}{\partial\mu}P(Y=x|n,\mu) = \frac{\partial}{\partial\mu}\left\{\binom{-n+x-1}{x}\left(\frac{-n}{-n+\mu}\right)^{-n}\left(\frac{\mu}{-n+\mu}\right)^{x}\right\}$$

$$= \frac{\partial}{\partial\mu}\left\{\binom{-n+x-1}{x}\left(\frac{\mu}{-n}\right)^{x}\left(\frac{\mu-n}{-n}\right)^{n-x}\right\}$$

$$= \binom{-n+x-1}{x}x\left(\frac{\mu}{-n}\right)^{x-1}\left(\frac{-1}{n}\right)\left(\frac{\mu-n}{-n}\right)^{n-x} + \binom{-n+x-1}{x}\left(\frac{\mu}{-n}\right)^{x}(n-x)\left(\frac{\mu-n}{-n}\right)^{n-x-1}\left(-\frac{1}{n}\right)$$

$$= \binom{-n+x-1}{x-1}\left(\frac{\mu}{-n}\right)^{x-1}\left(\frac{\mu-n}{-n}\right)^{n-x} - \binom{-n+x}{x}\left(\frac{\mu}{-n}\right)^{x}\left(\frac{\mu-n}{-n}\right)^{n-x-1}$$

$$= \binom{-(n-1)+(x-1)-1}{x-1}\left[\frac{(\frac{n-1}{n})\mu}{-(n-1)}\right]^{x-1}\left[\frac{(\frac{n-1}{n})\mu-(n-1)}{-(n-1)}\right]^{(n-1)-(x-1)}$$

$$- \binom{-(n-1)+x-1}{x}\left[\frac{(\frac{n-1}{n})\mu}{-(n-1)}\right]^{x}\left[\frac{(\frac{n-1}{n})\mu-(n-1)}{-(n-1)}\right]^{(n-1)-x}$$

$$= P(Y=x-1|n-1,(\tfrac{n-1}{n})\mu) - P(Y=x|n-1,(\tfrac{n-1}{n})\mu).$$

It is now easy to show that the variables $X_{n,\mu}$ defined by (2) satisfy (1), since

$$P(X=x|n,\mu) = \sum_{k=0}^{x} P(Z=k)P(Y=x-k|n,\mu).$$

Now assume equation (1) holds. To prove that (2) and (3) hold, we use the following result:

Lemma 1. If equation (1) holds, then we have

$$(4) \quad \frac{\partial^k}{\partial\mu^k} P(X=x|n,\mu)$$

$$= \frac{n(n-1)\ldots(n-k+1)}{n^k} \sum_{\ell=0}^{\infty} (-1)^{k-\ell} \binom{k}{\ell} P(X=x-\ell|n-k,(\tfrac{n-k}{n})\mu)$$

for $\mu \in [0,+\infty)$, $0 \neq n \in \mathbb{Z} \cup \{\infty\}$, $x=0,1,2,\ldots$, and $k=1,2,3,\ldots$.

<u>Remarks</u>. The sum on the right is actually a finite sum, since the summand is 0 whenever $\ell > \min\{x,k\}$. This follows from the definition of $\binom{k}{\ell}$ and the assumption that X has support on the non-negative integers. Also, for $n = \infty$ the correct expression is obtained by letting $n \to \infty$.

<u>Proof of Lemma 1</u>. The proof is by induction on k. The case $k=1$ is just equation (1). Now assume (4) holds. Then

$$\frac{\partial^{k+1}}{\partial\mu^{k+1}} P(X=x|n,\mu)$$

$$= \frac{n(n-1)\ldots(n-k+1)}{n^k} \sum_{\ell=0}^{\infty} (-1)^{k-\ell}\binom{k}{\ell}\frac{\partial}{\partial\mu} P(X=x-\ell|n-k,(\tfrac{n-k}{n})\mu)$$

which, by (1) and the chain rule,

$$= \frac{n(n-1)\ldots(n-k+1)}{n^k}\Big\{\sum_{\ell=0}^{\infty}(-1)^{k-\ell}\binom{k}{\ell}P(X=x-\ell-1|n-k-1,(\tfrac{n-k-1}{n})\mu)$$

$$- \sum_{\ell=0}^{\infty}(-1)^{k-\ell}\binom{k}{\ell}P(X=x-\ell|n-k-1,(\tfrac{n-k-1}{n})\mu)\Big\}(\tfrac{n-k}{n})$$

$$= \frac{n(n-1)\ldots(n-k+1)(n-k)}{n^{k+1}}\Big\{\sum_{\ell=1}^{\infty}(-1)^{k-\ell+1}\binom{k}{\ell-1}P(X=x-\ell|n-k-1,(\tfrac{n-k-1}{n})\mu)$$

$$+ \sum_{\ell=0}^{\infty}(-1)^{k-\ell+1}\binom{k}{\ell}P(X=x-\ell|n-k-1,(\tfrac{n-k-1}{n})\mu)\Big\}$$

$$= \frac{n(n-1)\ldots(n-[k+1]+1)}{n^{k+1}} \sum_{\ell=0}^{\infty}(-1)^{(k+1)-\ell}\binom{k+1}{\ell}P(X=x-\ell|n-(k+1),(\tfrac{n-(k+1)}{n})\mu)$$

which completes the proof of (4).

Recall that for any real number r, and any $k \in \mathbb{Z}$, the binomial coefficient $\binom{r}{k}$ is defined by (see [1], p. 50)

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\ldots(r-k+1)}{k!} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

Lemma 2. If equation (1) holds, and $\mu < |n|$, then

$$(5) \quad P(X) = x|n,\mu) = \sum_{\ell=0}^{x} P(Z = x-\ell)\binom{n}{\ell}\left(\frac{\mu}{n}\right)^{\ell}\left(1-\frac{\mu}{n}\right)^{n-\ell} \qquad \text{if } 0 \neq n \in \mathbb{Z}$$

and

$$(5') \quad P(X = x|\infty,\mu) = \sum_{\ell=0}^{x} P(Z = x-\ell)\frac{\mu^{\ell}}{\ell!}e^{-\mu},$$

for $x=0,1,2,\ldots$.

Proof. Use Lemma 1 in the Taylor expansion for $P(X = x|n,\mu)$ about $\mu = 0$:

$$(6) \quad P(X = x|n,\mu) = \sum_{k=0}^{m-1} \frac{\partial^k}{\partial\mu^k} P(X = x|n,\mu)\Big|_{\mu=0} \frac{\mu^k}{k!}$$

$$+ \frac{\partial^m}{\partial\mu^m} P(X = x|n,\mu)\Big|_{\mu=\mu_m^*} \frac{\mu^m}{m!}$$

where $0 \leq \mu_m^* \leq \mu$. When $n = \infty$, the remainder term is equal to

$$(7) \qquad \left\{\sum_{\ell=0}^{x} (-1)^{m-\ell}\binom{m}{\ell}P(X = x-\ell|\infty,\mu)\right\}\frac{\mu^m}{m!}.$$

Since the quantity in braces is a polynomial of degree $\leq x$ in m, expression (7) tends to 0 as $m \to \infty$. When $0 < n \in \mathbb{Z}$, the remainder term is equal to 0 whenever $m > n$. When $0 > n \in \mathbb{Z}$, we have

$$(8)\quad \left|\frac{\partial^m}{\partial\mu^m} P(X=x|n,\mu)\Big|_{\mu=\mu_m^*} \frac{\mu^m}{m!}\right| \le \frac{|n(n-1)\ldots(n-m+1)|}{m!}\left(\frac{\mu}{|n|}\right)^m \sum_{\ell=0}^{x}\binom{m}{\ell}$$

$$= \frac{(-n+m-1)!}{(-n-1)!\ m!}\left(\frac{\mu}{|n|}\right)^m \sum_{\ell=0}^{x}\binom{m}{\ell}.$$

By Stirling's approximation,

$$\frac{(-n+m-1)!}{m!} \sim \frac{(-n+m-1)^{-n+m-1+(1/2)}e^{n-m+1}}{m^{m+1/2}\,e^{-m}}$$

$$= (-n+m-1)^{-n-1}\left(\frac{-n+m+(1/2)}{m}\right)^{m+(1/2)} e^{n+1}$$

$$= (-n+m-1)^{-n-1}\left(1+\frac{-n+(1/2)}{m}\right)^{1/2}\left(1+\frac{-n+(1/2)}{m}\right)^{m} e^{n+1}$$

As $m\to\infty$ the product of the last three factors converges to $1\cdot e^{-n+(1/2)}\cdot e^{n+1} = e^{3/2}$. Thus, for m sufficiently large, $(-n+m-1)!/m!$ *is bounded* by a polynomial in m. Since $\sum_{\ell=0}^{x}\binom{m}{\ell}$ is a polynomial in m, the assumption $\frac{\mu}{|n|} < 1$ forces the right-hand expression in (8) to tend to 0 as $m\to\infty$. (This is the only time we use $\mu < |n|$.)

For any nonzero $n \in Z$ and any nonnegative $\mu < |n|$, we may now write, by virtue of (4),

$$P(X=x|n,\mu) = \sum_{k=0}^{\infty} \frac{\partial^k}{\partial\mu^k} P(X=x|n,\mu)\Big|_{\mu=0} \cdot \frac{\mu^k}{k!}$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{x} (-1)^{k-\ell} \binom{n}{k}\binom{k}{\ell}\left(\frac{\mu}{n}\right)^k P(Z=x-\ell)$$

$$= \sum_{\ell=0}^{x} P(Z=x-\ell)(-1)^{\ell} \sum_{k=0}^{\infty} \binom{n}{k}\binom{k}{\ell}\left(\frac{\mu}{-n}\right)^k .$$

Now,

$$\sum_{k=0}^{\infty} \binom{n}{k}\binom{k}{\ell}\left(\frac{\mu}{-n}\right)^k = \sum_{k=\ell}^{\infty} \binom{n}{k}\binom{k}{\ell}\left(\frac{\mu}{-n}\right)^k$$

$$= \sum_{k=0}^{\infty} \binom{n}{k+\ell}\binom{k+\ell}{\ell}\left(\frac{\mu}{-n}\right)^{k+\ell}$$

$$= \sum_{k=0}^{\infty} \binom{n}{\ell}\binom{n-\ell}{k}\left(\frac{\mu}{-n}\right)^{k+\ell}$$

$$= \binom{n}{\ell}\left(\frac{\mu}{-n}\right)^{\ell} \sum_{k=0}^{\infty} \binom{n-\ell}{k}\left(\frac{\mu}{-n}\right)^k .$$

Application of equation (8.7), page 51 of [1], to this latter expression yields (5). Moreover,

$$P(X=x|\infty,\mu) = \sum_{k=0}^{\infty} \sum_{\ell=0}^{x} P(Z=x-\ell)(-1)^{k-\ell}\binom{k}{\ell} \frac{\mu^k}{k!}$$

$$= \sum_{\ell=0}^{x} P(Z=x-\ell)\frac{1}{\ell!} \sum_{k=\ell}^{\infty} \frac{(-1)^{k-\ell}\mu^k}{(k-\ell)!}$$

$$= \sum_{\ell=0}^{x} P(Z=x-\ell)\frac{1}{\ell!} \sum_{k=0}^{\infty} \frac{(-1)^k \mu^{k+\ell}}{k!}$$

$$= \sum_{\ell=0}^{x} P(Z=x-\ell) \frac{\mu^{\ell}}{\ell!} e^{-\mu},$$

which is (5'). This completes the proof of Lemma 2.

Equation (5') identifies $X_{\infty,\mu}$ as a sum $Y + Z$ where Y and Z are independent and $Y \sim \mathcal{P}(\mu)$. Assume next that $0 < n \in \mathbb{Z}$. Then (5) becomes

$$P(X = x | n,\mu) = \sum_{\ell=0}^{x} P(Z = x-\ell) \binom{n}{\ell} \left(\frac{\mu}{n}\right)^{\ell} \left(1 - \frac{\mu}{n}\right)^{n-\ell}.$$

Although this equation was derived without using the assumption $\mu \leq n$, the right-hand side is not a probability mass function if $\mu > n$. With this restriction, the above equation identifies $X_{n,\mu}$ as a sum $Y + Z$ where Y and Z are independent and $Y \sim \mathcal{B}(n, \frac{\mu}{n})$.

Finally, let $0 > n \in \mathbb{Z}$. For a fixed value of μ, consider first any n such that $\mu < |n|$. Then by (5)

$$P(X = x | n,\mu) = \sum_{\ell=0}^{x} P(Z = x-\ell) \binom{n}{\ell} \left(\frac{\mu}{-n}\right)^{\ell} \left(1 - \frac{\mu}{n}\right)^{n-\ell}$$

$$= \sum_{\ell=0}^{x} P(Z = x-\ell) \binom{-n+\ell-1}{\ell} \left(\frac{-n}{-n+\mu}\right)^{-n} \left(\frac{\mu}{-n+\mu}\right)^{\ell}.$$

Thus $X_{n,\mu} = Y + Z$, where Y and Z are independent and $Y \sim \mathcal{NB}(-n, \frac{\mu}{-n+\mu})$.

To finish this case, we need to show that $X_{n,\mu}$ has the indicated form even when $\mu \geq |n|$. This is accomplished by the following result:

<u>Lemma 3</u>. Assume that equation (1) holds. If $X_{n,\mu} = Y + Z$ where Y and Z are independent and $Y \sim \mathcal{NB}(-n, \frac{\mu}{-n+\mu})$, and

$n \leq -2$, then $X_{n+1,\mu} = Y' + Z$ where Y' and Z are independent and $Y' \sim \mathcal{NB}(-(n+1), \frac{\mu}{-(n+1)+\mu})$.

<u>Proof</u>. By (1) we have

$$P(X=x|n+1,\mu) = C + \int \left\{ P(X=x-1|n,(\tfrac{n}{n+1})\mu) - P(X=x|n,(\tfrac{n}{n+1})\mu) \right\} d\mu$$

$$= C + \sum_{\ell=0}^{x} P(Z=x-\ell)\binom{-(n+1)+\ell-1}{\ell}\left(\frac{-(n+1)}{-(n+1)+\mu}\right)^{-(n+1)}\left(\frac{\mu}{-(n+1)+\mu}\right)$$

by hypothesis and the fact that the mass function for $Z+Y$, where $Y \sim \mathcal{NB}(-(n+1), \frac{\mu}{-(n+1)+\mu})$, obeys equation (1). The value of the constant C may be found by evaluating that last equation at $\mu = 0$:

$$P(Z=x) = C + P(Z=x)$$

$\Rightarrow C = 0$. This completes the proof of the lemma and the theorem.

An alternative unifying result, containing nonstochastic as well as stochastic solutions, is as follows:

<u>Theorem 2</u>. Let $\{f(\cdot|n,r,\mu) : (n,r,\mu) \in \Gamma\}$ be a family of functions on the nonnegative integers, where $\Gamma = \{(n,r,\mu) : n \in \mathbb{Z},\ 0 < r < +\infty,\ 0 \leq \mu < +\infty\}$. Define $f(x|n,r,\mu) = 0$ for $x < 0$. Assume that

$$(9) \qquad \frac{\partial}{\partial\mu} f(x|n,r,\mu) = \frac{|n|}{r}\Big[f(x-1|n-1,r,\mu) - f(x|n-1,r,\mu)\Big]$$

for $n \neq 0$. If we assume $f(x|0,r,\mu)$ and $f(x|n,r,0)$ are functions of x alone, and assume moreover that

$$\hat{f}(x) \triangleq f(x|0,r,\mu) = f(x|n,r,0) \qquad \forall\ x,r,n,\mu,$$

then

$$(10) \quad f(x|n,r,\mu) = \sum_{\ell=0}^{x} \hat{f}(x-\ell)\binom{n}{\ell}\left(\frac{\mu}{\sigma_n r}\right)^{\ell}\left(1-\frac{\mu}{\sigma_n r}\right)^{n-\ell}$$

for $n \neq 0$, where $\sigma_n = \begin{cases} 1 & \text{if } n > 0 \\ -1 & \text{if } n < 0 \end{cases}$. Of course, (conversely) any function of the form (10) satisfies (9).

<u>Remarks</u>. The function

$$g(x|n,r,\mu) = \binom{n}{x}\left(\frac{\mu}{\sigma_n r}\right)^{x}\left(1-\frac{\mu}{\sigma_n r}\right)^{n-x}$$

will be a probability mass function on $\{0,1,2,\ldots\}$ for a certain subset of Γ. When $n > 0$ and $0 \leq \mu \leq n$ we have

$$g(x|n,n,\mu = \binom{n}{x}\left(\frac{\mu}{n}\right)^{x}\left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \text{mass function of } \mathcal{B}(n,\tfrac{\mu}{n}).$$

When $n < 0$, we have

$$g(x|n,-n,\mu) = \binom{-n+x-1}{x}\left(\frac{-n}{-n+\mu}\right)^{-n}\left(\frac{\mu}{-n+\mu}\right)^{x}$$

$$= \text{mass function of } \mathcal{NB}\left(-n,\frac{\mu}{-n+\mu}\right).$$

As $n \to \pm\infty$, in both of the above cases, we have

$$g(x|n, |n|, \mu) \to g(x|\pm\infty, +\infty, \mu) \stackrel{\Delta}{=} \text{mass function of } \mathcal{P}(\mu).$$

For the special case $r = |n|$, equation (9) becomes

$$(9') \qquad \frac{\partial}{\partial\mu} f(x|n, |n|, \mu) = f(x-1|n-1, |n|, \mu) - f(x|n-1, |n|, \mu).$$

Thus, if we enlarge the parameter space to

$$\Gamma' \stackrel{\Delta}{=} \Gamma \cup \{(\pm\infty, +\infty, \mu) : 0 \le \mu < +\infty\},$$

Theorem 2 has the following

<u>Corollary 1</u>. The family $\{f(\cdot|n,r,\mu) : (n,r,\mu) \in \Gamma'\}$ satisfies (9') iff the following conditions hold:

(a) $f(x|n, |n|, \mu)$ is not a probability mass function if $n > 0$ and $\mu > n$;

(b) when $\hat{f}(x)$ is the mass function of a r.v. Z on the non-negative integers, $f(x|n, |n|, \mu)$ is the mass function of the r.v. $Y_{n,\mu} + Z$, where $Y_{n,\mu}$ and Z are independent and (3) holds.

The proof of Theorem 2 is obtained from the corresponding portion of the proof of Theorem 1 by replacing "n" by "$\sigma_n r$" in certain appropriate places.

__Sketch of Proof__. Use induction on k to show that

$$\frac{\partial^k}{\partial\mu^k} f(x|n,r,\mu) = \frac{n(n-1)\ldots(n-k+1)}{(\sigma_n r)^k} \sum_{\ell=0}^{\infty} (-1)^{k-\ell}\binom{k}{\ell} f(x-\ell|n-k,r,\mu).$$

Next, show that $f(x|n,r,\mu)$ is represented by its Taylor series at $\mu = 0$, provided $n > 0$, or $n < 0$ and $\mu < -n$ (as in Lemma 2):

$$f(x|n,r,\mu) = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \frac{n(n-1)\ldots(n-k+1)}{(\sigma_n r)^k} \sum_{\ell=0}^{\infty} (-1)^{k-\ell}\binom{k}{\ell}\hat{f}(x-\ell)$$

$$= \ldots = \sum_{\ell=0}^{x} \hat{f}(x-\ell)\binom{n}{\ell}\left(\frac{\mu}{\sigma_n r}\right)^{\ell}\left(1 - \frac{\mu}{\sigma_n r}\right)^{n-\ell}.$$

Now remove the restriction $\mu < -n$ when $n < 0$ by an argument similar to the proof of Lemma 3.

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