1.0

1.1

1.25 1.4 1.6

2.8 2.5

3.2 2.2

3.6

4.0 2.0

1.8

MICROCOPY RESOLUTION TEST CHART

AFOSR-TR. 80 - 1 3 4 0

AASE
80-222

A Method for the Generation
of
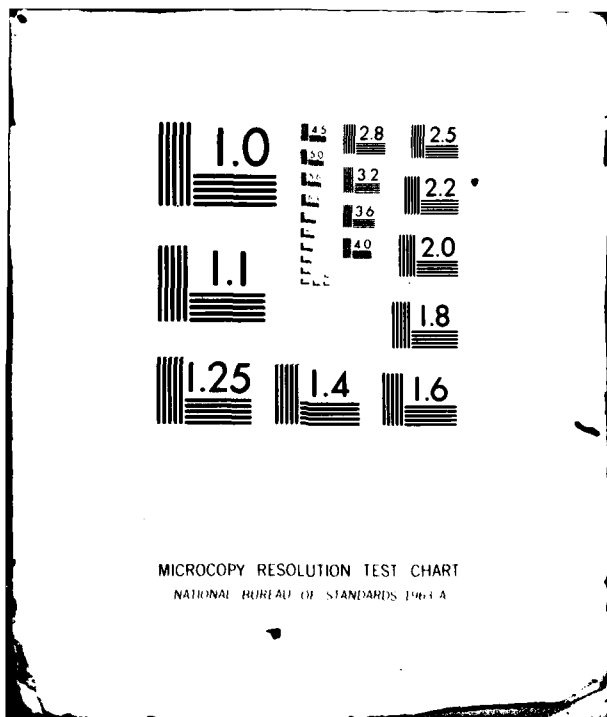# General Three-Dimensional Coordinates
Between
Bodies of Arbitrary Shapes

LEVEL Ⅱ ③

AD A093420

eirs
ENGINEERING & INDUSTRIAL RESEARCH STATION
Department of Aerospace Engineering

by

Z. U. A. Warsi

DTIC
ELECTE
S DEC 30 1980 D
D

Mississippi State University
Mississippi State, Miss. 39762

MSSU-EIRS-80-7

COLLEGE OF ENGINEERING ADMINISTRATION

WILLIE L. MCDANIEL, PH.D.
DEAN, COLLEGE OF ENGINEERING

WALTER R. CARNES, PH.D.
ASSOCIATE DEAN

RALPH E. POWE, PH.D.
ASSOCIATE DEAN

LAWRENCE J. HILL, M.S.
DIRECTOR, ENGINEERING EXTENSION

CHARLES B. CLIETT, M.S.
AEROSPACE ENGINEERING

WILLIAM R. FOX, PH.D.
AGRICULTURAL & BIOLOGICAL ENGINEERING

JOHN L. WEEKS, JR., PH.D.
CHEMICAL ENGINEERING

ROBERT M. SCHOLTES, PH.D.
CIVIL ENGINEERING

B. J. BALL, PH.D.
ELECTRICAL ENGINEERING

W. H. EUBANKS, M.ED.
ENGINEERING GRAPHICS

FRANK E. COTTON, JR., PH.D.
INDUSTRIAL ENGINEERING

C. T. CARLEY, PH.D.
MECHANICAL ENGINEERING

JOHN I. PAULK, PH.D.
NUCLEAR ENGINEERING

ELDRED W. HOUGH, PH.D.
PETROLEUM ENGINEERING

For additional copies or information
address correspondence to

ENGINEERING AND INDUSTRIAL RESEARCH STATION
DRAWER DE
MISSISSIPPI STATE UNIVERSITY
MISSISSIPPI STATE, MISSISSIPPI 39762

TELEPHONE (601) 325-2266

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS
BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
AFOSR-TR-80-1340	AD-AC93420	

4. TITLE (and Subtitle)	5. TYPE OF REPORT & PERIOD COVERED
A METHOD FOR THE GENRATION OF GENERAL THREE-DIMENSIONAL COORDINATES BETWEEN BODIES OF ARBITRARY SHAPES.	Interim
	6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Z. U. A. Warsi	AFOSR-80-0185

9. PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
Mississippi State University Department of Aerospace Engineering Mississippi State, MS 39762	61102F 2304/A3

11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
Air Force Office of Scientific Research /NM Bolling AFB, Washington, DC 20332	October 1980
	13. NUMBER OF PAGES
	25

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Grid Generation, Curvilinear Coordinates, Numerical Methods, Computational Fluid Dynamics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Analytical development of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies is presented. The resulting equations have only two independent variables and therefore require an order of magnitude less working core capacity than when equations depending on all three independent variables are considered. The method also allows, in a straight forward manner the possibility of coordinate contraction in the desired regions.

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE

AFOSR-TR-80-1340

A Method for the Generation of General Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes*

by

Z. U. A. Warsi[†]
Department of Aerospace Engineering
Mississippi State University
Mississippi State, MS   39762

Abstract

Analytical development of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies is presented. The resulting equations have only two independent variables and therefore require an order of magnitude less working core capacity than when equations depending on all three independent variables are considered. The method also allows, in a straight forward manner, the possibility of coordinate contraction in the desired regions.

An exact solution of the proposed equations for the case of an inner prolate ellipsoid and an outer sphere with coordinate contraction is presented to demonstrate that by using these equations it is possible to generate three-dimensional coordinates between analytically specified surfaces of simple forms by analytical means.

The fundamental constraining equations which have been adopted for the generation of coordinates are $\Delta_2\xi = 0$ and $\Delta_2\eta = 0$, where Δ_2 is the surface Beltrami operator of the second order.

[†]Professor

80 12 29 109

1. Introduction

At present a number of techniques are under active development
for the generation of three-dimensional body-oriented coordinate systems
for use in the numerical solution of the Navier-Stokes equations and
other field equations where the exact specification of the boundary
conditions is of prime importance. Among these efforts two easily dis-
cernable groups can be formed, (i) algebraic methods, and (ii) the
elliptic equations method. In the first group the grid points in space
are obtained by some interpolation or blending functions scheme which
depends on the given boundary data. The choice of the interpolation
scheme or of the blending functions is crucial in achieving a desired
order of smoothness and distribution of the grid points in space. This
line of effort has actively been considered by Eiseman [1,2], Smith and
Weigel [3], and Eriksson [4]. In the second group of efforts, a set of
three poisson equations in the curvilinear coordinates are first inverted
and then solved for the Cartesian coordinates under the prescribed values
at the given boundaries. Thus in essence all the methods of the second
group are a straight forward extension of the work of Thompson et al [5]
in two dimensions. Research in this area has been conducted by Mastin
et al [6], Yu [7], Ghia et al [8], and Graves [9].

At this stage of research it is premature to compare the two groups
since neither of them have been fully investigated for their inherent
potentials. However, based on the success of the differential equations
approach in two dimensions, e.g. [5], it is desirable to further investi-
gate the elliptic equations approach for the generation of coordinates.

The elliptic equations approach presented in this paper is different
from the approaches adopted in the previously cited works, i.e., References

2

[6] - [9]. The proposed method depends heavily on the formulae of Gauss and on the concept of principal curvatures of a surface. It has been shown that a fruitful arrangement of the classical differential-geometric results can yield a method which is easily programmable on a computing machine, and which at any time solves a two-dimensional partial differential equation of the form used in Ref. [5]. In this paper only the theoretical development of the method along with a technique to redistribute the coordinate surfaces near the inner boundary surface has been considered. The developed equations have been solved for the generation of three-dimensional coordinates between an inner prolate ellipsoid and an outer spherical surface in an exact analytic form.

2. Notation and Collection of Formulas

In what follows, the general coordinates are denoted as x^i ($i = 1$, 2,3). However when an expression has been expanded out in full and there is no use for an index notation, we have set

$$x^1 = \xi \ , \ x^2 = \eta \ , \ x^3 = \zeta \ .$$

The derivatives of the position vector $\underline{r} = (x,y,z)$ are denoted as

$$\underline{r}_i = \frac{\partial \underline{r}}{\partial x^i} \ , \ \underline{r}_{ij} = \frac{\partial^2 \underline{r}}{\partial x^i \partial x^j} \ .$$

The covariant components of the metric tensor are

$$g_{ij} = \underline{r}_i \cdot \underline{r}_j \tag{2.1}$$

while the contravariant components are given by

$$g^{ij} g_{kj} = \delta_k^i \tag{2.2}$$

Thus in three dimensions

$$g = \det(g_{ij})$$

$$= g_{11} g_{22} g_{33} + 2 g_{12} g_{13} g_{23} - (g_{23})^2 g_{11} - (g_{13})^2 g_{22} - (g_{12})^2 g_{33} \tag{2.3}$$

Writing

$$G_1 = g_{22} g_{33} - (g_{23})^2$$

$$G_2 = g_{11} g_{33} - (g_{13})^2$$

$$G_3 = g_{11}g_{22} - (g_{12})^2$$

$$G_4 = g_{13}g_{23} - g_{12}g_{33}$$

$$G_5 = g_{12}g_{23} - g_{13}g_{22}$$

$$G_6 = g_{12}g_{13} - g_{11}g_{23} \qquad (2.4)$$

we have

$$g^{11} = G_1/g \ , \ g^{22} = G_2/g \ , \ g^{33} = G_3/g$$

$$\qquad (2.5)$$

$$g^{12} = G_4/g \ , \ g^{13} = G_5/g \ , \ g^{23} = G_6/g$$

The Christoffel symbols based on the metric g_{ij} are

$$\Gamma^i_{jk} = g^{i\ell}[jk,\ell]$$

where

$$[jk,\ell] = \underline{r}_{jk} \cdot \underline{r}_\ell \qquad (2.6)$$

$$= \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^k} + \frac{\partial g_{k\ell}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right)$$

and repeated lower and upper indices imply summation. In the sequel we have also used the surface Christoffel symbols which have been denoted as $T^\alpha_{\beta\gamma}$, where the Greek indices range over (1,2) or (3,1) or (2,3).

The coordinate which is held fixed to account for the surface geometry is denoted by a superscript in parentheses. Thus the unit normal vector on the surface $\nu = $ const. is given by

5

$$n^{(\nu)} = (r_\alpha \times r_\beta)/|r_\alpha \times r_\beta| \qquad (2.7)$$

where

$$\nu = 1 : \alpha = 2 , \beta = 3 \quad (\text{surface } x^1 = \text{const.})$$

$$\nu = 2 : \alpha = 3 , \beta = 1 \quad (\text{surface } x^2 = \text{const.}) \qquad (2.8)$$

$$\nu = 3 : \alpha = 1 , \beta = 2 \quad (\text{surface } x^3 = \text{const.})$$

The rectangular Cartesian components of $n^{(\nu)}$ are denoted as

$$n^{(\nu)} = (X^{(\nu)} , Y^{(\nu)} , Z^{(\nu)}) \qquad (2.9)$$

The coefficients of the second fundamental form are denoted by $S^{(\nu)}$, $T^{(\nu)}$ and $U^{(\nu)}$ defined as

$$S^{(\nu)} = n^{(\nu)} \cdot r_{\alpha\alpha} \quad (\text{no sum on } \alpha)$$

$$T^{(\nu)} = n^{(\nu)} \cdot r_{\alpha\beta} \qquad (2.10)$$

$$U^{(\nu)} = n^{(\nu)} \cdot r_{\beta\beta} \quad (\text{no sum on } \beta)$$

where (ν,α,β) are in the cyclic permutations of $(1,2,3)$, in this order.

The partial derivatives of the second order are expressible in terms of the first order as

$$r_{ij} = \Gamma^k_{ij} r_k \qquad (2.11)$$

For a surface on which one of the coordinates is fixed, the Gauss' equations are

6

$$\mathbf{r}_{\alpha\alpha} = T^{\gamma}_{\alpha\alpha}\mathbf{r}_{\gamma} + S^{(\nu)}\mathbf{n}^{(\nu)}$$

$$\mathbf{r}_{\alpha\beta} = T^{\gamma}_{\alpha\beta}\mathbf{r}_{\gamma} + T^{(\nu)}\mathbf{n}^{(\nu)} \qquad (2.12)$$

$$\mathbf{r}_{\beta\beta} = T^{\gamma}_{\beta\beta}\mathbf{r}_{\gamma} + U^{(\nu)}\mathbf{n}^{(\nu)}$$

Where (ν,α,β) are in the permutational sequences of $(1,2,3)$ as shown in (2.8), and the repeated index γ implies summation on the two indices of a surface.

The sum of the principal curvatures of the surface ν = const. is, [10],

$$k_1^{(\nu)} + k_2^{(\nu)} = (g_{\alpha\alpha}U^{(\nu)} - 2g_{\alpha\beta}T^{(\nu)} + g_{\beta\beta}S^{(\nu)})/G_{\nu} \qquad (2.13)$$

where in writing equation (2.13) for a particular value of ν, use must be made of Eqs. (2.8) and (2.10). We now introduce two second order surface differential operators by using (2.8), which for ν = const. are

$$D^{(\nu)} \equiv g_{\beta\beta}\partial_{\alpha\alpha} - 2g_{\alpha\beta}\partial_{\alpha\beta} + g_{\alpha\alpha}\partial_{\beta\beta} \qquad (2.14)$$

$$\Delta_2^{(\nu)} \equiv \frac{1}{\sqrt{G_{\nu}}}[\partial_{\alpha}\{\frac{1}{\sqrt{G_{\nu}}}(g_{\beta\beta}\partial_{\alpha} - g_{\alpha\beta}\partial_{\beta})\}$$

$$+ \partial_{\beta}\{\frac{1}{\sqrt{G_{\nu}}}(g_{\alpha\alpha}\partial_{\beta} - g_{\alpha\beta}\partial_{\alpha})\}] \qquad (2.15)$$

As is well known, the operator Δ_2 is the Beltrami differential operator of the second order [10].

The three space Christoffel symbols which have been referred to in the next section are given below.

$$\Gamma^3_{11} = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \xi} + G_6(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta})]$$

$$+ \frac{G_3}{g}(x_{\xi\xi}x_\zeta + y_{\xi\xi}y_\zeta + z_{\xi\xi}z_\zeta) \qquad (2.16)$$

$$\Gamma^3_{12} = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \eta} + G_6 \frac{\partial g_{22}}{\partial \xi}]$$

$$+ \frac{G_3}{g}(x_{\xi\eta}x_\zeta + y_{\xi\eta}y_\zeta + z_{\xi\eta}z_\zeta) \qquad (2.17)$$

$$\Gamma^3_{22} = \frac{1}{2g}[G_5(2 \frac{\partial g_{11}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_6 \frac{\partial g_{22}}{\partial \eta}]$$

$$+ \frac{G_3}{g}(x_{\eta\eta}x_\zeta + y_{\eta\eta}y_\zeta + z_{\eta\eta}z_\zeta) \qquad (2.18)$$

3. Formulation of the Problem

The principal idea of the method to be presented is to generate a series of surfaces on each of which a certain a'priori chosen variable or coordinate is kept fixed. Each surface to be generated starts from a given curve of the inner body and ends on the corresponding curve of the outer boundary, cf. Fig. 1. A routine, preferably a spline fit, can then be used to join the successive generated surfaces so as to have a smooth three-dimensional computational net for solving other physical field equations.

To illustrate the method, we take $\nu = 3$ or $x^3 = \zeta = $ constant on each surface to be generated. Thus $\alpha = 1$ and $\beta = 2$, viz., α and β respectively correspond to the coordinates $x^1 = \xi$ and $x^2 = \eta$. For the sake of brevity of notation we will not use the superscript (3) unless it becomes necessary. Thus from (2.10)

$$S = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} \; , \; T = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} \; , \; U = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} \tag{3.1}$$

where

$$\underset{\sim}{n} = \underset{\sim}{i}X + \underset{\sim}{j}Y + \underset{\sim}{k}Z \tag{3.2}$$

Equations (2.12) are

$$\underset{\sim}{r}_{\xi\xi} = T^{\gamma}_{11}\underset{\sim}{r}_{\gamma} + S\underset{\sim}{n} \tag{3.3}$$

$$\underset{\sim}{r}_{\xi\eta} = T^{\gamma}_{12}\underset{\sim}{r}_{\gamma} + T\underset{\sim}{n} \tag{3.4}$$

$$\underset{\sim}{r}_{\eta\eta} = T^{\gamma}_{22}\underset{\sim}{r}_{\gamma} + U\underset{\sim}{n} \tag{3.5}$$

From (2.14) and (2.15) the operators $D^{(3)}$ and $\Delta_2^{(3)}$ are

$$D \equiv g_{22} \frac{\partial^2}{\partial \xi^2} - 2g_{12} \frac{\partial^2}{\partial \xi \partial \eta} + g_{11} \frac{\partial^2}{\partial \eta^2} \tag{3.6}$$

$$\Delta_2 \equiv \frac{1}{\sqrt{G_3}} [\frac{\partial}{\partial \xi} \{ \frac{1}{\sqrt{G_3}} (g_{22} \frac{\partial}{\partial \xi} - g_{12} \frac{\partial}{\partial \eta}) \}$$

$$+ \frac{\partial}{\partial \eta} \{ \frac{1}{\sqrt{G_3}} (g_{11} \frac{\partial}{\partial \eta} - g_{12} \frac{\partial}{\partial \xi}) \}] \tag{3.7}$$

We now multiply equations (3.3) – (3.5) respectively by g_{22}, $-2g_{12}$, g_{11} adding and using equations (2.13) and (2.15) to have

$$D\underset{\sim}{r} + G_3(\underset{\sim}{r}_\xi \Delta_2 \xi + \underset{\sim}{r}_\eta \Delta_2 \eta) = G_3 \underset{\sim}{n}(k_1 + k_2) \tag{3.8}$$

where

$$\Delta_2 \xi = \frac{1}{G_3}(2g_{12}T_{12}^1 - g_{22}T_{11}^1 - g_{11}T_{22}^1)$$

$$\Delta_2 \eta = \frac{1}{G_3}(2g_{12}T_{12}^2 - g_{22}T_{11}^2 - g_{11}T_{22}^2) \left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} \tag{3.9}$$

$$G_3 = g_{11}g_{22} - (g_{12})^2$$

To obtain an expression for $k_1 + k_2$ consider equation (2.11) and utilize the property that $\underset{\sim}{n}$ is orthogonal to $\underset{\sim}{r}_\xi$ and $\underset{\sim}{r}_\eta$, so that

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} = \Gamma_{11}^3 (\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} = \Gamma_{12}^3 (\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

10

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} = \Gamma^3_{22}(\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)$$

$$(3.10)$$

where all the derivatives with respect to ζ are evaluated at $\zeta =$ constant. Multiplying Eq. (3.8) scalarly by $\underset{\sim}{n}$ and using (3.10), we get

$$G_3(k_1 + k_2) = (\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11})$$

$$(3.11)$$

We now propose the following deterministic problem: Let ξ and η be the surface coordinates on the surface $\zeta =$ constant, subject to the constraints

$$\Delta_2 \xi = 0$$

$$(3.12)$$

$$\Delta_2 \eta = 0$$

Then the Cartesian coordinates x,y,z of the surface satisfy the differential equations

$$D\underset{\sim}{r} = G_3(k_1 + k_2)\underset{\sim}{n}$$

$$(3.13)$$

The three scalar differential equations for the generation of the Cartesian coordinates are then

$$g_{22}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{11}x_{\eta\eta} = XR$$

$$(3.14)$$

$$g_{22}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{11}y_{\eta\eta} = YR$$

$$(3.15)$$

$$g_{22}z_{\xi\xi} - 2g_{12}z_{\xi\eta} + g_{11}z_{\eta\eta} = ZR$$

$$(3.16)$$

where

$$R = (Xx_\zeta + Yy_\zeta + Zz_\zeta)(g_{11}\Gamma_{22}^3 - 2g_{12}\Gamma_{12}^3 + g_{22}\Gamma_{11}^3) \qquad (3.17)$$

and

$$
\left.
\begin{aligned}
X &= (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3} \\[2ex]
Y &= (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3} \\[2ex]
Z &= (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3}
\end{aligned}
\right\} \qquad (3.18)
$$

Equations (3.14) - (3.16) form a quasilinear system of partial differential equations in which the components of $\underset{\sim}{r}_\zeta$ are assumed to be known. Since the values of x,y,z are known on the basic inner and outer boundaries (denoted at B and ∞ respectively in Fig. 1), a suitable way of prescribing $\underset{\sim}{r}_\zeta$ can be to take

$$\underset{\sim}{r}_\zeta = f_1(\eta)(\underset{\sim}{r}_\zeta)_B + f_2(\eta)(\underset{\sim}{r}_\zeta)_\infty \qquad (3.19)$$

where $f_1(\eta)$ and $f_2(\eta)$ are suitable weights having the properties

$$f_1(\eta_B) = 1 \ , \ f_2(\eta_B) = 0$$

$$f_1(\eta_\infty) = 0 \ , \ f_2(\eta_\infty) = 1$$

For exposing the essential nonlinear terms in the factor R we refer to Eqs. (2.16) - (2.18) in which the $\underset{\sim}{r}_\zeta$ terms have been collected separately.

Referring to Figure 2, we now solve Eqs. (3.14) - (3.16) for each ζ = const., on a rectangular plane by prescribing the values of x, y and z on the lower side (C_1) and upper side (C_2) which represent the curves

12

on B and ∞ respectively. The sides C_3 and C_4 are the cut lines on which periodic boundary conditions are to be imposed. The preceding analysis thus completes the formulation of the problem.

4. Coordinate Transformation (Contraction)

For the purpose of generating coordinates between the space of the inner and outer boundary which can be distributed in a desired manner, we consider a coordinate transformation from $\xi \rightarrow \chi$ and $\eta \rightarrow \sigma$. Let

$$\xi = \xi(\chi) + \xi_o$$

$$\eta = \eta(\sigma) + \eta_B \qquad (4.1)$$

then

$$\xi = \xi_o \text{ at } \chi = \chi_o \ , \ \xi(\chi_o) = 0$$

$$\eta = \eta_B \text{ at } \sigma = \sigma_B \ , \ \eta(\sigma_B) = 0 \qquad (4.2)$$

Writing

$$\lambda(\chi) = \frac{d\xi}{d\chi} \ , \ \theta(\sigma) = \frac{d\eta}{d\sigma}$$

and denoting the transformed metric tensor as $\bar{g}_{ij}$, we have

$$g_{11} = \bar{g}_{11}/\lambda^2 \ , \ \bar{g}_{11} = x_\chi^2 + y_\chi^2 + z_\chi^2$$

$$g_{12} = \bar{g}_{12}/\theta\lambda \ , \ \bar{g}_{12} = x_\chi x_\sigma + y_\chi y_\sigma + z_\chi z_\sigma$$

$$g_{22} = \bar{g}_{22}/\theta^2 \ , \ \bar{g}_{22} = x_\sigma^2 + y_\sigma^2 + z_\sigma^2$$

$$G_3 = \bar{G}_3/\theta^2\lambda^2 \ , \ \bar{G}_3 = \bar{g}_{11}\bar{g}_{22} - (\bar{g}_{12})^2 \qquad (4.3)$$

$$X = \bar{X} \ , \ Y = \bar{Y} \ , \ Z = \bar{Z}$$

14

$$k_1 + k_2 = \bar{k}_1 + \bar{k}_2$$

$$R = \bar{R}/\theta^2 \lambda^2$$

Further noting that

$$\underline{r}_{\xi\xi} = (\underline{r}_{\chi\chi} - \frac{\underline{r}_\chi \lambda_\chi}{\lambda})/\lambda^2$$

$$\underline{r}_{\xi\eta} = \underline{r}_{\chi\sigma}/\theta\lambda \qquad \Bigg\} \qquad (4.4)$$

$$\underline{r}_{\eta\eta} = (\underline{r}_{\sigma\sigma} - \frac{\underline{r}_\sigma \theta_\sigma}{\sigma})/\theta^2$$

Using (4.3) and (4.4) in Eqs. (3.14) – (3.16), we have

$$\bar{g}_{22}x_{\chi\chi} - 2\bar{g}_{12}x_{\chi\sigma} + \bar{g}_{11}x_{\sigma\sigma} = Px_\chi + Qx_\sigma + \bar{X}\bar{R} \qquad (4.5)$$

$$\bar{g}_{22}y_{\chi\chi} - 2\bar{g}_{12}y_{\chi\sigma} + \bar{g}_{11}y_{\sigma\sigma} = Py_\chi + Qy_\sigma + \bar{Y}\bar{R} \qquad (4.6)$$

$$\bar{g}_{22}z_{\chi\chi} - 2\bar{g}_{12}z_{\chi\sigma} + \bar{g}_{11}z_{\sigma\sigma} = Pz_\chi + Qz_\sigma + \bar{Z}\bar{R} \qquad (4.7)$$

where

$$P = \frac{\bar{g}_{22}}{\lambda}\lambda_\chi$$

$$\qquad (4.8)$$

$$Q = \frac{\bar{g}_{11}}{\theta}\theta_\sigma$$

Thus, by choosing λ and θ arbitrarily we can redistribute the coordinates in the desired manner. An example of this choice is given in the next section.

5. An Analytical Example of Coordinate Generation

In this section we shall consider the problem of coordinate generation between a prolate ellipsoid and a sphere with coordinate contraction near the inner surface. This problem yields an exact solution of the equations $(4.5) - (4.7)$.

Let $\eta = \eta_B$ and $\eta = \eta_\infty$ be the inner prolate ellipsoid and the outer sphere respectively. The coordinates which vary on these two surfaces are ξ and ζ. We now envisage a net of lines $\xi = $ const. and $\zeta = $ const. on these two surfaces. A curve C_1 on the inner surface designated as $\zeta = \zeta_0$ is

$$
\left.
\begin{aligned}
x &= \cosh\eta_B \cos\zeta_0 \\[2mm]
y &= \sinh\eta_B \sin\zeta_0 \cos\xi \\[2mm]
z &= \sinh\eta_B \sin\zeta_0 \sin\xi
\end{aligned}
\right\}
\tag{5.1}
$$

Similarly, the curve C_2 corresponding to $\zeta = \zeta_0$ on the outer surface is

$$
\left.
\begin{aligned}
x &= e^{\eta_\infty} \cos\zeta_0 \\[2mm]
y &= e^{\eta_\infty} \sin\zeta_0 \cos\xi \\[2mm]
z &= e^{\eta_\infty} \sin\zeta_0 \sin\xi
\end{aligned}
\right\}
\tag{5.2}
$$

Based on the forms of the functions x,y,z in (5.1) and (5.2), we assume the following forms of x,y,z for the surface $\zeta = \zeta_0$:

16

$$x = f(\sigma)\cos\zeta_o$$

$$y = \phi(\sigma)\sin\zeta_o\cos\xi \qquad (5.3)$$

$$z = \phi(\sigma)\sin\zeta_o\sin\xi$$

The boundary conditions for f and ϕ are

$$f(\sigma_B) = \cosh\eta_B$$

$$f(\sigma_\infty) = e^{\eta_\infty}$$

$$\phi(\sigma_B) = \sinh\eta_B \qquad (5.4)$$

$$\phi(\sigma_\infty) = e^{\eta_\infty}$$

Calculating the various derivatives, metric coefficients, and all other data needed in the equations (4.5) - (4.7), we get on substitution an equation which has $\sin^2\zeta_o$ and $\cos^2\zeta_o$. Equating to zero the coefficients of $\sin^2\zeta_o$ and $\cos^2\zeta_o$, we obtain

$$\frac{f''}{f'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \qquad (5.5)$$

$$\frac{\phi''}{\phi'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \qquad (5.6)$$

where a prime denotes differentiation with respect to σ. On direct integration of Eqs. (5.5) and (5.6) under the boundary conditions (5.4), we get

$$f(\sigma) = Ae^{Bn(\sigma)} + C \qquad (5.7)$$

$$\phi(\sigma) = De^{Bn(\sigma)} \qquad (5.8)$$

where

$$A = \frac{(e^{n_\infty} - \cosh n_B)\sinh n_B}{e^{n_\infty} - \sinh n_B} \qquad (5.9a)$$

$$B = \ln[\frac{e^{n_\infty}}{\sinh n_B}]^{1/(n_\infty - n_B)} \qquad (5.9b)$$

$$C = \frac{e^{n_\infty}(\cosh n_B - \sinh n_B)}{e^{n_\infty} - \sinh n_B} \qquad (5.9c)$$

$$D = \sinh n_B \qquad (5.9d)$$

As an application we may take [11]

$$\xi(\chi) = a\chi$$

$$n(\sigma) = b(\sigma - \sigma_B)K^\sigma$$

where a and b are constants. Since at n_∞,

$$n(\sigma_\infty) = n_\infty - n_B$$

hence

$$n(\sigma) = \frac{(n_\infty - n_B)(\sigma - \sigma_B)}{(\sigma_\infty - \sigma_B)}K^{(\sigma - \sigma_\infty)}$$

By taking a value of K slightly greater than one (K = 1.05 or 1.1), we can have sufficient contraction of coordinates near the inner surface.

For the chosen problem, since the dependence on ζ is simple, we find that the coordinates between a prolate ellipsoid and a sphere are

$$x = [Ae^{B\eta(\sigma)} + C]\cos\zeta$$

$$y = De^{B\eta(\sigma)}\sin\zeta\cos\xi$$

$$z = De^{B\eta(\sigma)}\sin\zeta\sin\xi$$

where A, B, C, and D are given in equation (5.9).

6. Conclusions

A new method for the generation of three-dimensional coordinates between two arbitrary shaped bodies has been presented. The method is based on some simple differential-geometric concepts such as the equations of Gauss and the expressions for the principal curvatures of a surface. The simplicity of the method lies in solving, at one time, only three partial differential equations of the two-dimensional type. This aspect is bound to reduce the working core requirements for a given problem on a computing machine. Finally the method allows, in a very direct fashion, the possibility of coordinate redistribution in the desired regions (cf. Eqs. (4.5) − (4.7)).

An analytic solution of the proposed equations for the case of an inner prolate ellipsoid and an outer sphere has been presented. This example shows that one can generate coordinates between two analytically specified surfaces of simple forms by exact solutions of the proposed equations.

In this paper the fundamental equations which form a set of constraints for the generation of coordinates in the surface are

$$\Delta_2 \xi = 0$$

$$\Delta_2 \eta = 0 \ ,$$

where Δ_2 is the surface Beltrami operator of the second order. It must be noted that Δ_2 is neither a Laplace operator in the Cartesian plane (x,y), nor in the Cartesian space (x,y,z). However, in the case of a Cartesian plane (x,y), when there is no dependence on z, Δ_2 reduces to the Laplace operator ∇^2.

References

1. P. R. Eisemann, "A Multi-Surface Method of Coordinate Generation," *Journal of Computational Physics*, 33 (1979), 118.

2. P. R. Eisemann and R. E. Smith, "Mesh Generation Using Algebraic Techniques," *Workshop on Numerical Generation Techniques for Partial Differential Equations*, NASA Langley Research Center, Hampton, Virginia, Oct. 6-7, (1980).

3. R. E. Smith and B. L. Weigel, "Analytical and Approximate Boundary-Fitted Coordinate Systems," AIAA paper 80-0192 (1980).

4. L.-E. Eriksson, "Three-Dimensional Spline-Generated Coordinate Transformations for Grids Around Wing-Body Configurations," *Workshop on Numerical Generation Techniques for Partial Differential Equations*, NASA Langley Research Center, Hampton, Virginia, Oct. 6-7, (1980).

5. J. F. Thompson, F. C. Thames, and C. W. Mastin, "Automatic Numerical Generation of Body-Fitted Curvilinear Coordinate System for Field Containing any Number of Arbitrary Two-Dimensional Bodies," *Journal of Computational Physics*, 15 (1974), 299.

6. C. W. Mastin and J. F. Thompson, "Transformation of Three-Dimensional Regions onto Rectangular Regions by Elliptic Systems," *Numerische Mathematik*, 29 (1978), 397.

7. N. J. Yu, "Grid Generation and Transonic Flow Calculations for Three-Dimensional Configurations," AIAA paper 80-1391 (1980).

8. U. Ghia and K. N. Ghia, "Numerical Generation of a System of Curvilinear Coordinates for Turbine Cascade Flow Analysis," Rep. No. AFL 75-4-17, University of Cincinnati (1975).

9. R. A. Graves, "Application of a Numerical Orthogonal Coordinate Generator to Axisymmetric Blunt Bodies," NASA Tech. Memo. 80131 (1980).

10. L. P. Eisenhart, "A Treatise on the Differential Geometry of Curves and Surfaces," Ginn and Company, N.Y. (1937).

11. Z. U. A. Warsi and J. F. Thompson, "Machine Solutions of Partial Differential Equations in the Numerically Generated Coordinate Systems," Engineering and Experimental Research Station, Mississippi State University, Rep. No. MSSU-EIRS-ASE-77-1 (1976).
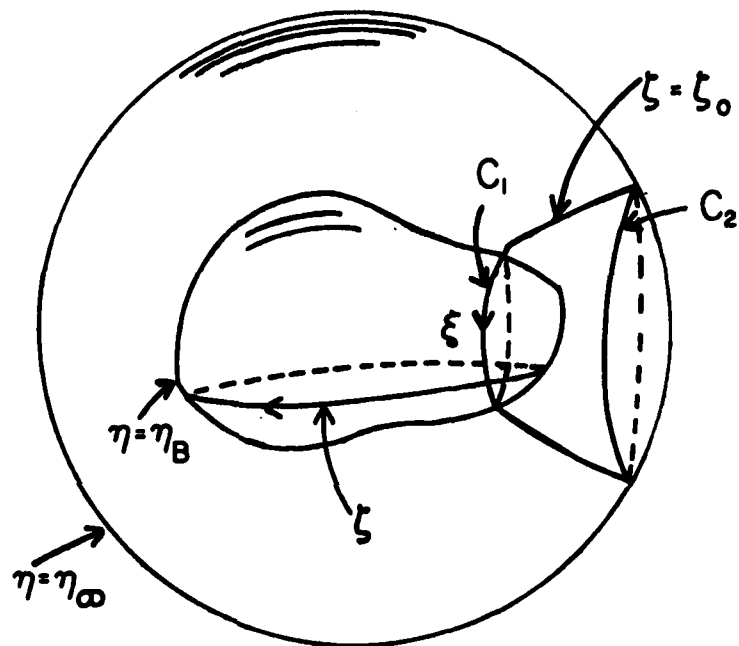
(a)

(b)
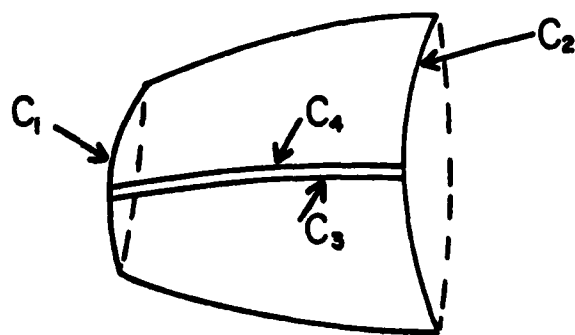
Figure 1: (a) Topoloty of the given surfaces. Inner $\eta = \eta_B$, outer $\eta = \eta_\infty$, current variables ξ, ζ. (b) Surface to be generated for each ζ = const., current variables ξ, η.
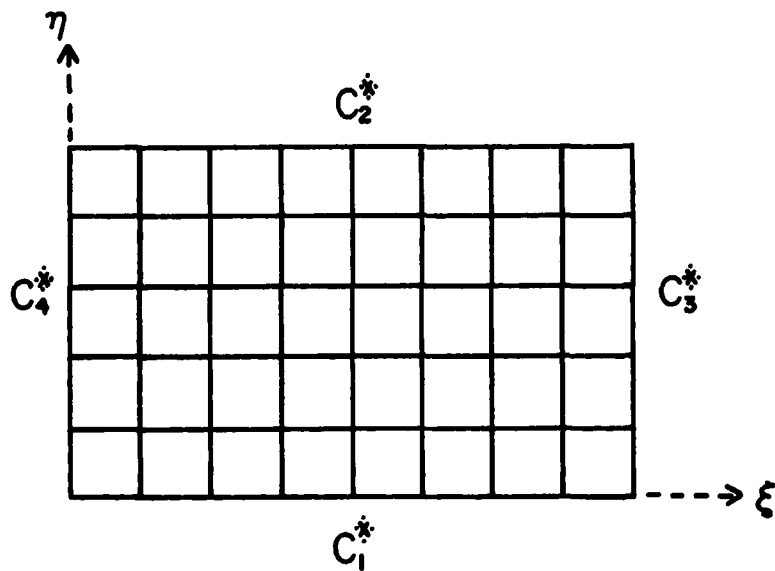
Figure 2: Figure 1(b) opened in a rectangular plane by imagining a cut.

DATE
FILMED

— 8