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AN EFFECTIVE DISPERSION THEORY FOR HORIZONTALLY-POLARIZED WAVES--ETC(U)

JAN 81 T J DELPH, G HERRMANN N00014-76-C-0054

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AN EFFECTIVE DISPERSION THEORY FOR HORIZONTALLY-POLARIZED WAVES IN LAYERED COMPOSITES

by
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and
G. Herrmann

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ABSTRACT

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INTRODUCTION

Delph and Herrmann [1] have recently proposed a conceptually simple, yet accurate, approximate theory for one-dimensional wave propagation normal to the layering in a periodically layered elastic composite. This theory, called the effective dispersion theory, has been shown to model quite well various features of the exact dispersion spectrum for wave propagation normal to the layering in a periodically layered unbounded body. In particular, the effective dispersion theory contains the stopping band between the first two Brillouin zones, a feature not present in other approximate theories.

This theory is extended in the present work to two dimensions for antiplane strain motions characterized by two independent wave numbers. Thus we are concerned here only with S H waves. When the direction of propagation is normal to the layering, the extended theory reduces to the original, one-dimensional theory.

The extension is carried out in a simple manner by including additional terms in the strain energy density function. The spectral matching procedure utilized in [1] is then applied. A notable feature of the matching procedure is the retention of the stopping band between the first and second Brillouin zones which was present in the one-dimensional theory.

On the basis of calculations made using a set of representative material parameters, it appears that the extended two-dimensional theory yields excellent results for values of wave numbers measured parallel and normal to the layering of up to the width of one Brillouin zone.

The theory also appears to give acceptable results over a smaller range of wave numbers in the second Brillouin zone.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The layered body, B, is assumed to be composed of periodically alternating layers of homogeneous, isotropic, linearly elastic material (Fig. 1). The layers have thicknesses $2h$; $2h'$, densities ρ ; ρ' and elastic constants λ,μ ; λ',μ' . Any two adjacent layers comprise a *unit cell*, and this unit cell is repeated periodically along the y-axis. The body is taken to be of unbounded extent along the z-axis, but bounded in planes parallel to the x-y plane by the curve $C(x,y)$, which we assume to be sufficiently smooth.

For antiplane strain motion, the displacement component parallel to the z-axis is taken to be the only non-zero displacement component and is assumed to be a function only of time and the x and y coordinates. We now assume that the displacements in the N-th unit cell are given by

$$\begin{aligned} w(x,y,t) &= w_0(x,y,t)|_{y_N=0} + y_N\psi(x,y,t)|_{y_N=0} \\ w'(x,y,t) &= w_0(x,y,t)|_{y'_N=0} \end{aligned} \tag{1}$$

Here w and w' are the displacements in the layers with unprimed and primed constants, respectively. y_N and y'_N are local coordinates having their origin at the midplane of the layers with unprimed and primed constants, respectively. As noted in [1], the validity of equation (1) depends partly on the assumption that the layers with unprimed constants are sufficiently stiff compared to the layers with primed constants so that all the deformation is essentially confined to the softer layers.

Following [1], we now postulate a specific form for the strain and kinetic energy density functions. In doing so, it will be convenient, because of the increased complications introduced by the extra spatial dimension, to adopt the formalism utilized by Achenbach, Sun and Herrmann [2] in dealing with a similar problem. We thus assume that the strain energy density function is given by

$$
\begin{aligned}
W = {} & A_{ijk\ell} e_{ij} e_{k\ell}/2 + B_{ijk\ell} e_{ij} \gamma_{k\ell}/2 + C_{ijk\ell} \gamma_{ij} \gamma_{k\ell}/2 \\
& + D_{ijk\ell mn} \kappa_{ijk} \kappa_{\ell mn}/2 + E_{ijk\ell mn} \kappa_{ijk} \partial_{\ell mn}/2 \\
& + F_{ijk\ell mn} \theta_{ijk} \theta_{\ell mn}/2 \; .
\end{aligned}
\tag{2}
$$

Here $A_{ijk\ell}, \ldots, F_{ijk\ell mn}$ are constants to be determined and e_{ij} , γ_{ij} , κ_{ijk} , and $\theta_{\ell mn}$ are various microstructural strain measures introduced by Mindlin [3]. In our case, only certain of these are taken to be non-zero. These are, following the notation of [2]

$$
\begin{aligned}
& e_{32} = e_{23} = \partial_y w_o/2 \\
& e_{13} = e_{31} = \partial_x w_o/2 \\
& \gamma_{23} = \partial_y w_o - \psi \\
& \theta_{223} = \partial_{yy} w_o - \partial_y \psi \\
& \kappa_{223} = \partial_y \psi \\
& \kappa_{123} = \kappa_{321} = \partial_x \psi/2
\end{aligned}
\tag{3}
$$

where

$$\partial_y(\;) = \frac{\partial}{\partial y}(\;)\,.$$

All other microstructural strain measures not defined explicitly in (3) are assumed to vanish. Furthermore, we assume for the sake of simplicity that

$$\begin{aligned}
&A_{2323} = A_{2332} = A_{3223} = A_{3232} \\
&A_{1313} = A_{3131} = A_{1331} = A_{3131} \\
&A_{1323} = A_{2313} = A_{1332} = A_{3213} = A_{3123} = A_{2331} = A_{3132} = A_{3231} = 0 \\
&B_{2323} = B_{3223} \\
&B_{1323} = B_{3123} = 0 \\
&D_{223123} = D_{123223} = D_{223321} = D_{321223} = 0 \\
&D_{123123} = D_{321321} = D_{123321} = D_{321123} \\
&E_{123223} = E_{321223} = 0
\end{aligned} \tag{4}$$

The kinetic energy density is

$$T = b_1\dot{w}_o^2/2 + b_2\dot{\psi}^2/2 \tag{5}$$

where the superposed dot indicates partial differentiation with respect to time. The assumed forms for the strain and kinetic energy densities are now substituted into Hamilton's principle, which we write as

$$\delta \int_B \int_{t_o}^{t_1} (T - W)\, dt\, dV + \int_{t_o}^{t_1} W_e\, dt = 0 \tag{6}$$

As usual, the variations δw_o and $\delta\psi$ are assumed to vanish at $t = t_o, t_1$. W_e is the work done by external tractions on the bounding surface of the body. Again following [2], we assume this to be given by

$$\delta W_e = \int_S [P(x,y)\ \delta w_o + M(x,y)\ \delta\psi + Q(x,y)\ D\delta w_o]\ dS \qquad (7)$$

Here D is the normal derivative operator on the bounding curve $C(x,y)$ given by $D = n_x \partial/\partial_x + n_y \partial/\partial_y$ and n_x and n_y are, respectively, the x and y components of the unit normal vector to $C(x,y)$. Body forces are assumed to vanish.

The derivation of the governing equations and the associated boundary conditions now follows that presented in [2], which we will not repeat in detail. Briefly, we make use of equations (2), (3), and (4) to define the following generalized stress quantities

$$
\begin{aligned}
\tau_{23} &\equiv \partial W/\partial e_{23} = (A_{2323} + B_{2323}/2)\ \partial_y w_o - B_{2323}\psi/2 \\
\tau_{13} &\equiv \partial W/\partial e_{13} = A_{1313}\partial_x w_o \\
\sigma_{23} &\equiv \partial W/\partial \gamma_{23} = (B_{2323}/2 + C_{2323})\ \partial_y w_o - C_{2323}\psi \\
\mu_{123} &\equiv \partial W/\partial \kappa_{123} = D_{123123}\partial_x \psi \\
\mu_{223} &\equiv \partial W/\partial \kappa_{223} = (D_{223223} - E_{223223}/2)\ \partial_y \psi + E_{223223}\partial_{yy} w_o/2 \\
\chi_{223} &\equiv \partial W/\partial \theta_{223} = (E_{223223}/2 - F_{223223})\ \partial_y \psi + F_{223223}\partial_{yy} w_o
\end{aligned}
\qquad (8)
$$

Equations (7) and (8), after several applications of the divergence theorem, then yield the equations of motion

$$
\begin{aligned}
&-b_1\ddot{w}_0 + \partial_y\tau_{23} + \partial_x\tau_{13} - \partial_{yy}\chi_{223} + \partial_y\sigma_{23} = 0 \\
&-b_2\ddot{\psi} + \partial_y\mu_{223} + \partial_x\mu_{123} - \partial_y\chi_{223} + \sigma_{23} = 0
\end{aligned}
\tag{9}
$$

with the associated natural boundary conditions on the surface bounded by $C(x,y)$

$$
\begin{aligned}
&\{-n_x\tau_{13} - n_y(\tau_{23} + \sigma_{23} - \partial_y\chi_{223}) - [(1 - n_x^2)\,\partial_x n_x - n_x n_y\,(\partial_y n_x + \partial_x n_y) \\
&+ (1 - n_y^2)\,\partial_y n_y]\, n_y^2\chi_{223} + n_y\,[-n_x n_y\partial_x\chi_{223} + (1 - n_y^2)\,\partial_y\chi_{223}] \\
&+ [-n_x n_y\partial_x n_y + (1 - n_y^2)\,\partial_y n_y]\,\chi_{223} + P(x,y)\}\,\delta w_0 = 0 \\
&[-n_x\mu_{123} - n_y\,(\mu_{223} - \chi_{223}) + M(x,y)]\,\delta\psi = 0 \\
&[-n_y^2\chi_{223} + Q(x,y)]\,D\delta u_0 = 0
\end{aligned}
\tag{10}
$$

Note that the last of equations (10) implies that $Q(x,y)$ must vanish on surfaces where $n_y = 0$. This is somewhat analogous to a restriction present in the effective stiffness theory [2].

DETERMINATION OF COEFFICIENTS

We have now to determine the various constants in the expressions for the strain and kinetic energy densities given by equations (2) and (5). This is accomplished, as in [1], by matching the approximate dispersion spectrum for antiplane strain wave propagation in an unbounded body to the exact result given in [4]. To this end, we rewrite equations (9) as

$$-b_1\ddot{w}_o + a_1\partial_{yy}w_o + a_2\partial_{yyyy}w_o - a_3\partial_y\psi - a_4\partial_{yyy}\psi + a_7\partial_{xx}w_o = 0$$
$$-b_2\ddot{\psi} + a_3\partial_y w_o + a_4\partial_{yyy}w_o - a_5\psi - a_6\partial_{yy}\psi + a_8\partial_{xx}\psi = 0 \qquad (11)$$

where, with the aid of equation (4), we have defined

$$\begin{aligned} a_1 &= A_{2323} + B_{2323} + C_{2323} & a_5 &= C_{2323} \\ a_2 &= -F_{223223} & a_6 &= E_{223223} - F_{223223} - D_{223223} \\ a_3 &= B_{2323}/2 + C_{2323} & a_7 &= A_{1313} \\ a_4 &= E_{223223}/2 - F_{223223} & a_8 &= D_{123123} \end{aligned} \qquad (12)$$

Equation (11) is similar to the governing equations derived for the one-dimensional case in [1], but contains in addition terms proportional to $\partial_{xx}w_o$ and $\partial_{xx}\psi$.

In order to examine the case of harmonic wave propagation in an unbounded body, we assume solutions of the form

$$w_o = Ae^{i(k_x x + k_y y - \omega t)} \quad ; \quad \psi = Be^{i(k_x x + k_y y - \omega t)} \qquad (13)$$

Substitution of (13) into (11) yields a set of two linear, homogeneous equations in A and B ; the requirement that the determinant of the matrix of coefficients vanishes gives the dispersion equation in the form

$$\alpha\omega^4 + (\beta k_y^4 + \gamma k_y^2 + \delta + \phi k_x^2)\,\omega^2 + \xi k_y^2 + \sigma k_x^2 + \theta k_y^4 + \rho k_x^2\, k_y^2$$
$$+ \tau k_x^4 + k_y^6 + \epsilon k_x^2\, k_y^4 = 0 \qquad (14)$$

The parameters α , β , γ , ∂ , ξ and θ are various combinations of the constants $a_1, \ldots, b_2$ as defined in [1]. Additionally we have

$$\phi = -(a_7 b_2 + a_8 b_1)/T \; ; \quad \sigma = a_5 a_7/T \; ; \quad \rho = (a_1 a_8 - a_6 a_7)/T \; ;$$
$$\tau = a_7 a_8/T \; ; \quad \varepsilon = -a_2 a_8/T \; ; \quad T = a_2 a_6 - a_4^2 \tag{15}$$

In [1], expressions for the parameters $a_1, a_2, a_3, a_4, a_5, b_1$, and b_2 were determined by matching the approximate dispersion spectrum for propagation normal to the layering to the exact solution, [4], by matching the approximate mode slope at the end of the first Brillouin zone to the exact result, and by imposing the requirement that the frequencies resulting from the approximate dispersion relation be real-valued for all values of the wave number. Since the governing equations are homogeneous, the value of a_6 was arbitrarily set to unity. We retain these previously-developed values in the present work.

In order to determine the values of the constants a_7 and a_8 , we again adopt a matching procedure. We choose to match first the quantity $\left.\frac{\partial \omega}{\partial k_x}\right|_{\omega=k_x=k_y=0}$ to the exact value obtained from [4], which we denote as s_g . From (14), we find that

$$\left.\frac{\partial \omega}{\partial k_x}\right|_{\omega=k_x=k_y=0} = \sqrt{-\sigma/\delta} = s_g \tag{16}$$

The other relation is obtained by requiring the imaginary branch between the first two Brillouin zones, which was present in the one-dimensional theory, to be preserved for values of k_x other than zero.

This is accomplished, as in the one-dimensional theory, by requiring the quantity $\partial\omega/\partial k_y$ to vanish along the end of the first Brillouin zone, where k_y has the value $k_y = k_B$. From (14), we have that

$$\left.\frac{d\omega}{dk_y}\right|_{k_y=k_B} = \frac{-k_B[(2\beta k_B^2 + \gamma)\ \omega^2 + (2\epsilon k_B^2 + \rho)\ k_x^2 + 3k_B^4 + 2\theta k_B^2 + \xi]}{\omega[2\alpha\omega^2 + \beta k_B^4 + \gamma k_B^2 + \delta + \phi k_x^2]} \tag{17}$$

Now from the definitions of β, γ, θ and ξ given in [1], we find

$$2\beta k_B^2 + \gamma = 0 \quad ; \quad 3k_B^4 + 2\theta k_B^2 + \xi = 0 \tag{18}$$

so that the desired result may be achieved by setting

$$\rho = -2\epsilon k_B^2 \tag{19}$$

Equations (16) and (19) suffice for the determination of a_7 and a_8. Using these and the definitions contained in eqn. (15) and in [1], we have

$$a_7 = s_g^2 b_1 \quad ; \quad a_8 = \frac{s_g^2 b_1}{a_1 - 2a_2 k_B^2} \tag{20}$$

where the parameters b_1, a_1 and a_2 are determined as in [1] and, as previously explained, we have set $a_6 = 1$.

The approximate dispersion spectrum is now completely defined and, from (14), the dispersion relation is found to be

$$\omega^2 = [-(\beta k_y^4 + \gamma k_y^2 + \delta + \phi k_x^2) \pm \sqrt{D}]/2\alpha \tag{21}$$

where

$$D = (\beta k_y^4 + \gamma k_y^2 + \delta + \phi k_x^2)^2 - 4\alpha(\xi k_y^2 + \sigma k_x^2 + \theta k_y^4 + \rho k_x^2 k_y^2 + \tau k_x^4 + k_y^6 + \varepsilon k_x^2 k_y^4) \tag{22}$$

The form of D may be considerably simplified, however. Making use of the definitions set forth above and in [1], we may show after some algebra that

$$\phi^2 - 4\alpha\tau = 0 \ ; \quad \beta\phi - 2\alpha\varepsilon = 0 \ ; \quad \gamma\phi - 2\alpha\rho = 0 \ ; \quad \delta\phi - 2\alpha\sigma = 0 \tag{23}$$

Thus the expression for the discriminant D becomes

$$D = (\beta k_y^4 + \gamma k_y^2 + \delta)^2 - 4\alpha(\xi k_y^2 + \theta k_y^4 + k_y^6) \tag{24}$$

and we see that the discriminant is not a function of the wave number k_x; indeed, it reduces to a form identical to that obtained in the one-dimensional theory [1]. We now note from (16) and the last of (23) that

$$\frac{\phi}{2\alpha} = \frac{\sigma}{\delta} = -s_g^2 \tag{25}$$

so that (21) may be rewritten as

$$\omega^2 = s_g^2 k_x^2 + f(k_y) \tag{26}$$

where $f(k_y)$ is the double-valued function

$$f(k_y) = [-(\beta k_y^4 + \gamma k_y^2 + \delta) \pm \sqrt{D}]/2\alpha \qquad (27)$$

and D is given by (24).

Necessary and sufficient conditions were determined in [1] such that $f(k_y) \geq 0$ for all real values of k_y. This requirement was necessary to insure real frequencies and, indeed, supplied the rationale for a part of the matching procedure. It can be seen from (26) that, for $f(k_y) \geq 0$, the frequency ω will always be real-valued in the present case. For constant values of k_y, the spectral lines given by (26) describe hyperbolae in the ω - k_x plane. In the special case $k_y = 0$, the lowest branch degenerates to the straight line $\omega = s_g k_x$.

NUMERICAL RESULTS AND DISCUSSION

As a numerical example, the parameters corresponding to the exact solution presented in [4] were used to evaluate the approximate dispersion equation (26). Here we have introduced the nondimensional quantities $\Omega = 2h\omega/\pi c_T$, $\zeta = 2hk_x/\pi$, and $\eta = 2hk_y/\pi$, where h is the half-thickness of the layers with unprimed constants and $c_T = \sqrt{\mu/\rho}$. Figure 2 shows both exact and approximate lines of constant ζ on the dispersion surface over the first two Brillouin zones. In the first Brillouin zone, the agreement is quite good through $\zeta = 0.20$. In the second Brillouin zone, the agreement is quite good as well for $\zeta = 0$, but worsens as ζ increases. These tendencies are also evidenced in Figure 3, which shows the lines of constant η bounding the first two Brillouin zones.

Figures 4a, b and c show the real and imaginary part of η on the imaginary branch connecting the first two Brillouin zones. Note that the exact value of $\mathrm{Re}(\eta)$ is constant on those branches. The agreement for $\zeta = 0$ is quite good, but becomes worse with increasing ζ, though still adequate through $\zeta = .10$.

On the basis of these results, it appears that the approximate theory outlined in this paper yields excellent results for values of ζ and η up to about one Brillouin zone width, with somewhat less satisfactory agreement over the second Brillouin zone. This degree of agreement, however, is equal or better than the results of other existing approximate theories. In addition, the present theory contains the stopping band between the first two Brillouin zones, a feature present in few other approximate theories. For these reasons, the two-dimensional effective dispersion theory discussed herein is thought to represent a significant advance in the construction of approximate theories for wave propagation in periodically layered bodies.

It should be noted, however, that the two-dimensional effective dispersion theory is still subject to the shortcomings and limitations of the one-dimensional theory. Principally, these are a somewhat restricted validity due to the requirement that the frequencies resulting from the approximate dispersion equation be real-valued, and the lack of a uniqueness proof. Both these points are discussed at some length in [1].

We close by briefly discussing the prospects for improving and extending the present theory. It seems likely that considerably better agreement between exact and approximate solutions can be obtained by including additional terms in the strain energy density function proportional to higher-order derivatives of w_o and ψ, and by introducing a more elaborate matching scheme. This improvement, however, will come at the expense of greatly increased algebraic complication. Additionally, it will become increasingly difficult to satisfy the requirement that the approximate dispersion equation yield only real-valued frequencies. Another possible improvement is the extension of the theory to the plane strain case. Here the difficulties are due to the nature of the exact spectrum for an unbounded body, which has been shown, [5], [6], to be not only double-valued, but extremely complicated as well. It is far from clear as to how a successful modeling of this very complex spectrum ought to proceed.

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FIGURE CAPTIONS

1. Geometry of layered solid

2. Curves of constant ζ

3. Curves of constant η at Brillouin zone ends

4. Imaginary branch between first two Brillouin zones

   a) $\zeta = 0$

   b) $\zeta = .10$

   c) $\zeta = .20$

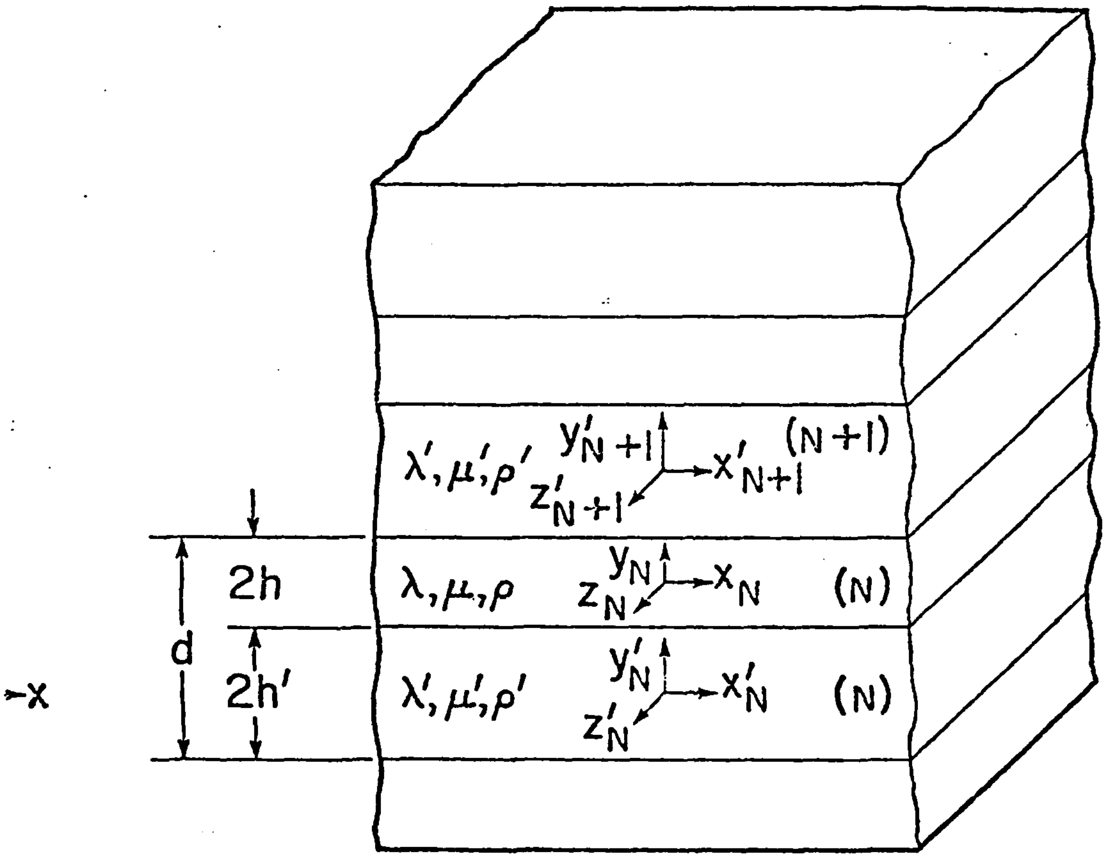


Figure 1

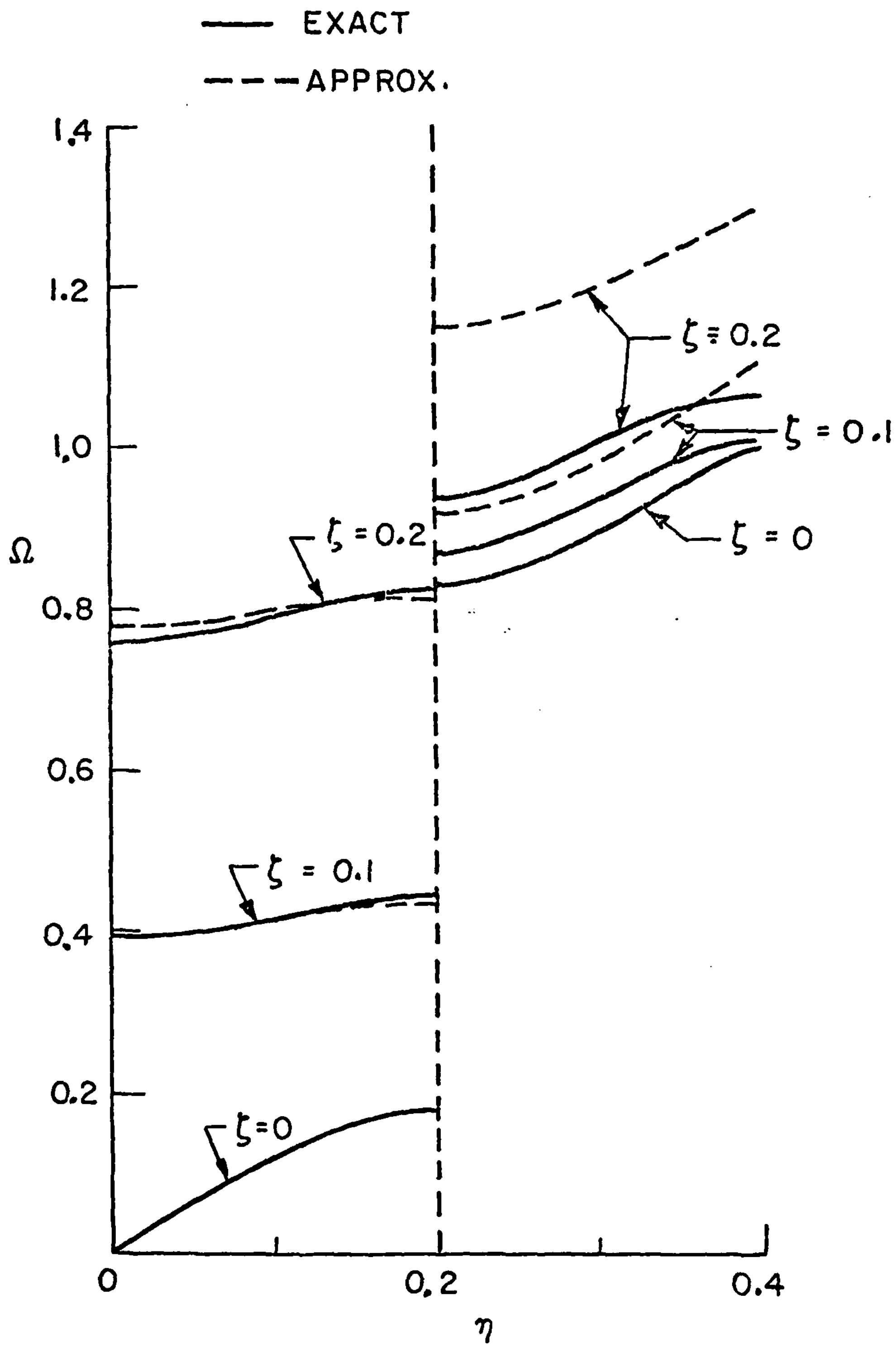


Figure 2

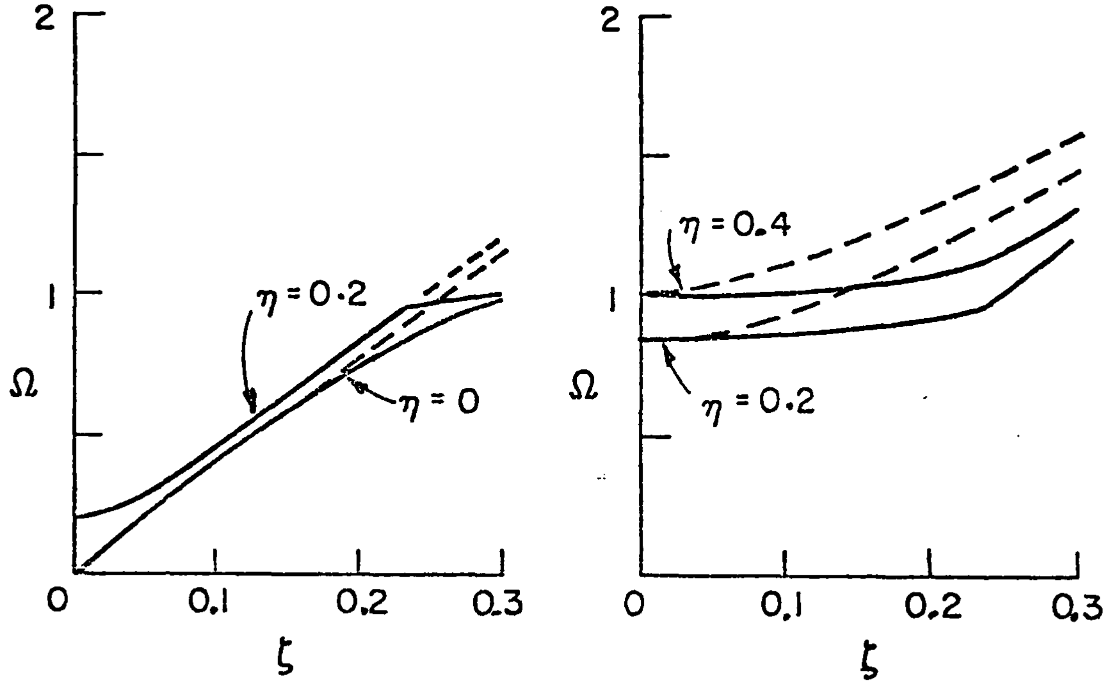


Figure 3

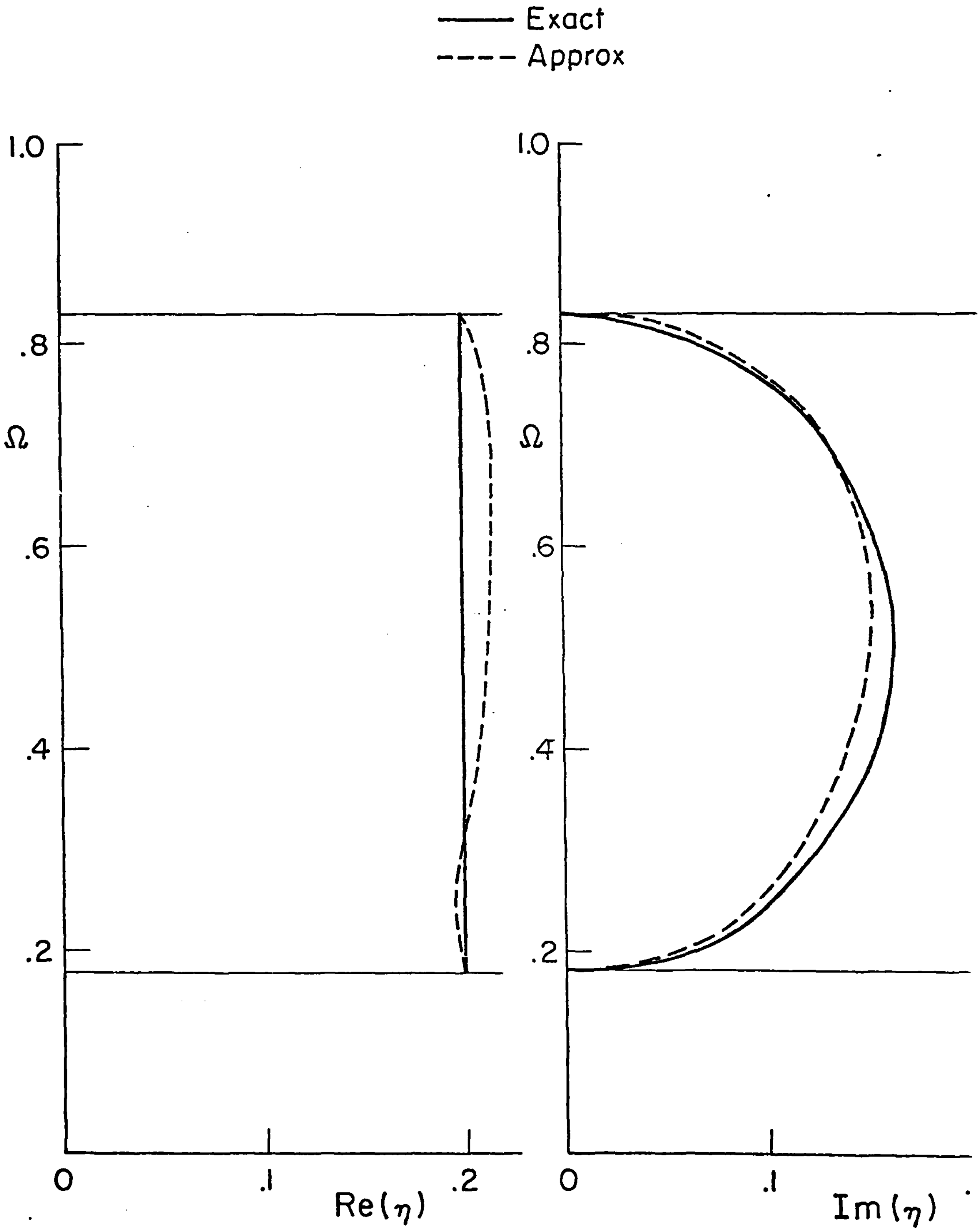
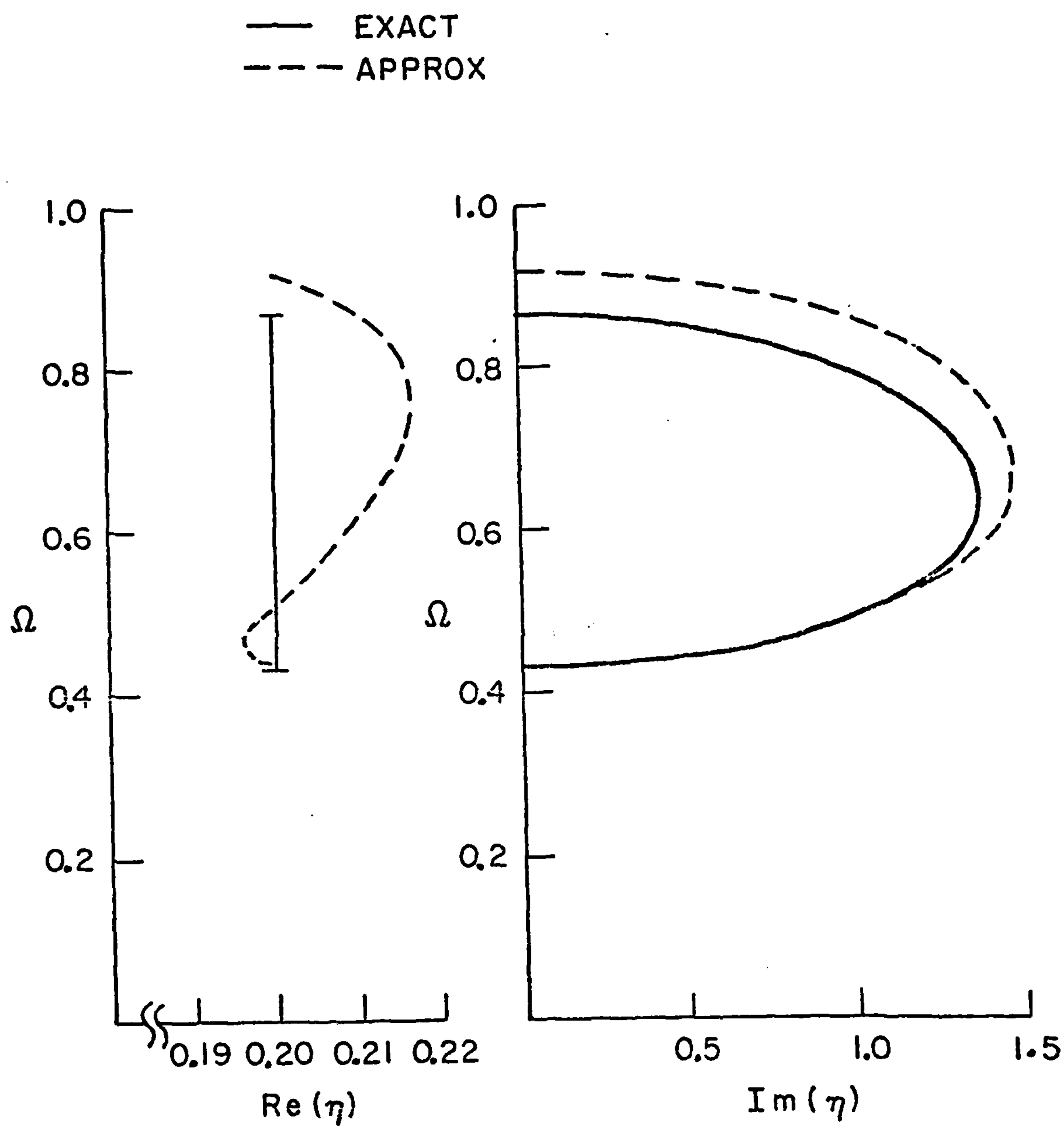


Figure 4a



EXACT
APPROX
1.0
0.8
0.6
0.4
0.2
Ω
0.19 0.20 0.21 0.22
Re (η)
1.0
0.8
0.6
0.4
0.2
Ω
0.5 1.0 1.5
Im (η)

Figure 4b

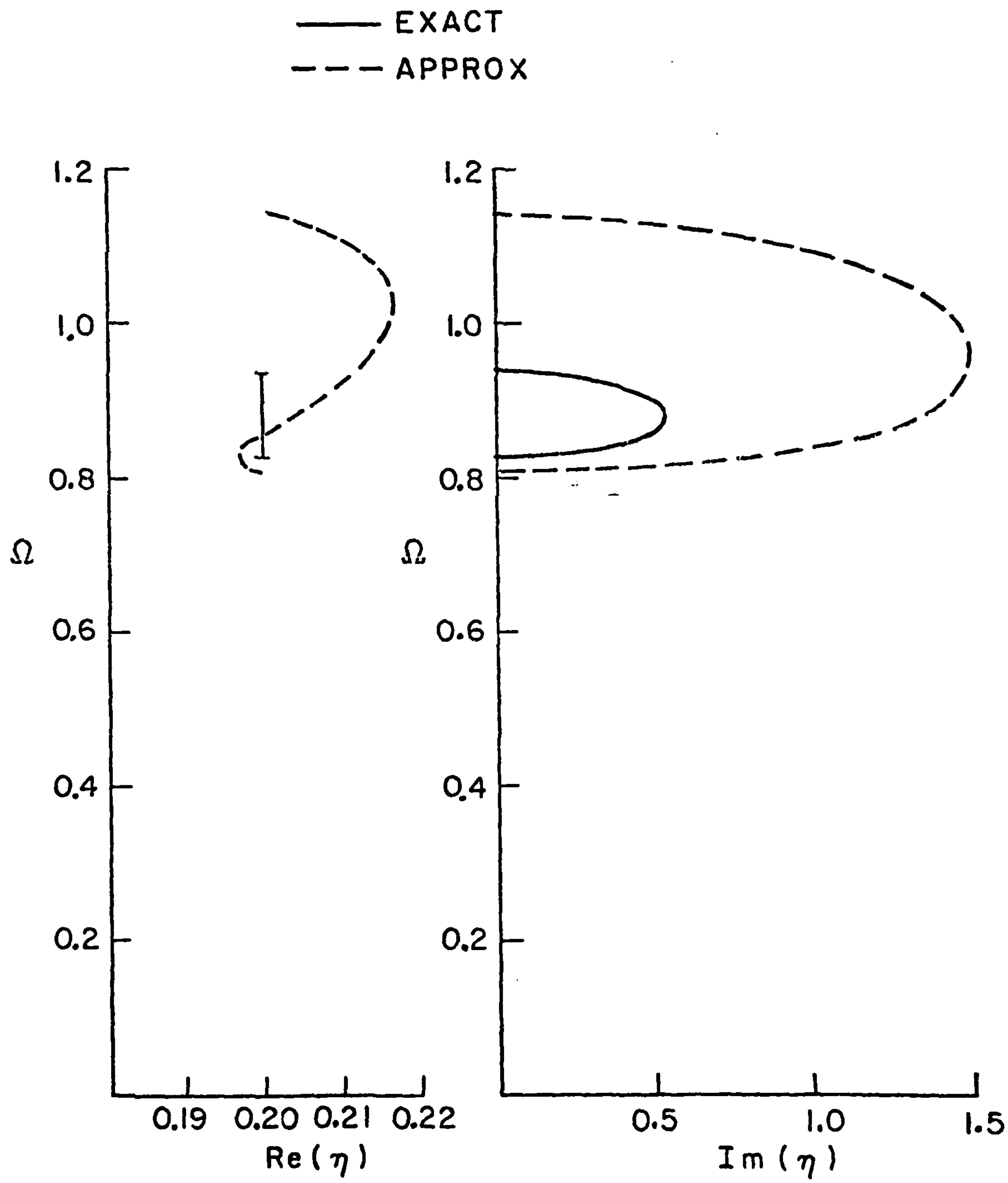

EXACT
APPROX
1.2
1.0
0.8
0.6
0.4
0.2
Ω
0.19 0.20 0.21 0.22
Re(η)
1.2
1.0
0.8
0.6
0.4
0.2
Ω
0.5 1.0 1.5
Im(η)

Figure 4c

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