ACCEPTANCE CONTROL CHARTS WITH
STIPULATED ERROR PROBABILITIES
BASED ON POISSON COUNT DATA

Research Report No. 80-9

by

Suresh Mhatre*
Richard L. Scheaffer**
Richard S. Leavenworth*

RESEARCH
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Industrial & Systems
Engineering Department
University of Florida
Gainesville, FL. 32611

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*Department of Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611

**Department of Statistics
University of Florida
Gainesville, Florida 32611

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TABLE OF CONTENTS

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INTRODUCTION

The application of control chart methods to accept and reject the output of a process has been described in the literature on several occasions. Winterhalter (1945) suggested the use of what he called *reject limits* in conjunction with the usual control limits to control a process average, $\bar{X}$. So long as process dispersion was held in control and the control limits lay within the reject limits, virtually all product would meet specifications.

Hill (1956) expanded this idea by employing the reject limits in place of standard control limits for those cases in which the difference between the upper and lower specification limits (U - L) substantially exceeded the natural tolerances of the process, 6 σ'. In his 1957 article, Richard Freund gave more form and substance to Winterhalter's earlier work by providing an analytical basis for deriving the location of a reject limit. He also coined the phrase *Acceptance Control Chart* and referred to the derived limit as the *Acceptance Control Limit* (ACL). His development closely follows that for variables acceptance sampling plans. Essentially, it requires the specification of two points on an operating characteristic curve in terms of a quality level and probability of acceptance for each. From these inputs are derived a control limit, the ACL, and a subgroup size, n. So long as the plotted values of $\bar{X}$ fall within the ACL's, the process may be assumed to be turning out product that meets specifications, subject to the defined risks.

In this paper, we extend Freund's work to the attributes case of counts of nonconformances or nonconformances per unit which can be shown to follow the Poisson distribution. Traditionally, the c-chart has been used when the area of opportunity for a nonconformity to occur is constant; the u-chart has been used when the area of opportunity varies from subgroup to subgroup. Three methods for finding the optimum subgroup size and acceptance control limit are compared. These are: (1) the exact method, employing the Poisson distribution; (2) the standard normal approximation; and (3) the square-root normal transformation.

PROBLEM FORMULATION

Control charts for nonconformances have found many uses in industry. Examples include counts of surface imperfections on film, flaws in fabric weave and nonconformities in completed units and subassemblies. The particular application developed in this paper relates to maintenance activities. Frequently maintenance shops process similar types of units, such as hydraulic assemblies, but the units vary substantially in size and time required to process them. In such cases, it may be reasonable to assume that the act of committing an error in processing (the occurence of a nonconformity) has a constant probability as a function of time. The area of opportunity for the occurrence of a nonconformity is thus measured in units of time.

We assume that the quality control procedure consists of observing a process for a length of time, H, and counting the number of nonconformances, X, that occur during this time interval. We assume that X has a Poisson distribution with intensity λ. That is, the mean number of nonconformances observed in time H is λH. Formulating the problem requires the specification of two pairs of values:

(1) An Acceptable Process Level, λ_o, and its associated risk level, α. λ_o is the process quality level that is considered acceptable as a process average measured in terms of nonconformances per 100 worker-hours. The probability of accepting the hypothesis that the process is operating at or below λ_o, when it actually is operating at λ_o, is $1 - \alpha$.

(2) A Rejectable Process Level, λ_1, and its associated risk level, β. λ_1 is the process quality level that is considered unacceptable. The risk of accepting the hypothesis that the process is operating at or below λ_o when it actually is operating at or above λ_1 is β.

The two points $(\lambda_o, 1 - \alpha)$ and (λ_1, β) thus define the operating characteristic curve of the acceptance control chart plan. From these two points we will derive the Acceptance Control Limit, K, and the optimal subgroup size, H.

Generally speaking, the quality control procedure will involve looking at a series of time intervals, H_1, H_2, ..., and observing X_1, X_2, ... In this case, we assume X_i has a Poisson distribution with mean λH_i. H_i can be thought of as the size of the i^{th} subgroup.

The intensity of nonconformances at the acceptable process level (APL) will be deonted by λ_0, while the intensity at the rejectable process level (RPL) will be denoted by λ_1. With K denoting the acceptance control limit (ACL), we can make the identifications shown in Figure 1.

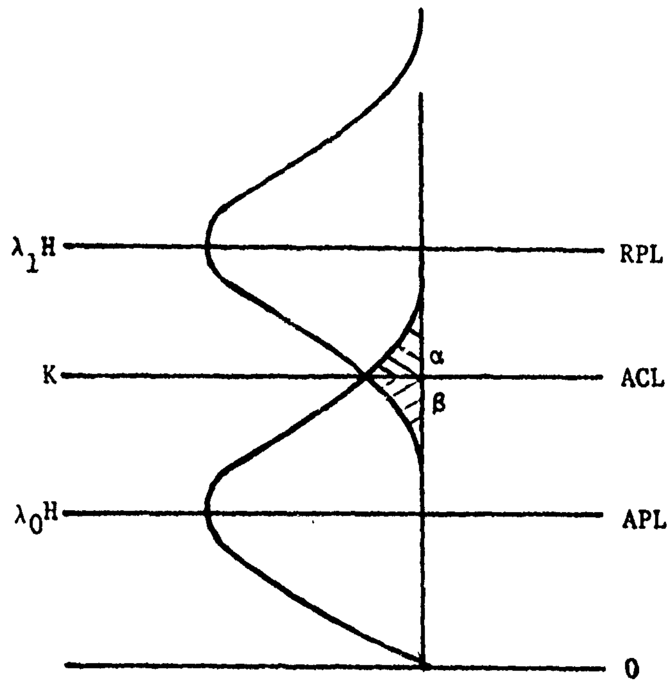

Fig. 1 An Acceptance Control Charting Scheme for Poisson Counts

PROBLEM SOLUTION

Our problem is to determine values of H and K for fixed values of α, β, λ_0 and λ_1. Recall that we want to choose H and K so that the probability of X exceeding K, when λ_0 is the true intensity, is α; and the probability of X being less than or equal to K, when λ_1 is the true intensity, is β.

We shall investigate three methods of calculating H and K, for fixed λ_0, λ_1, α and β. The first will use exact Poisson probabilities; the other two will involve normal approximations to the Poisson. One might ask why approximate procedures are needed when an exact solution is known. The answer lies in the fact that the exact solution, for all possible industrial applications, requires entensive tables of Poisson (or Chi-square) probabilities. Normal approximations have been used in industry and can be worked out quickly and easily with reference to only a table of normal curve areas.

Exact Solution

We can find the exact solution for $\underline{H}$ and $\underline{K}$ based on Poisson probabilities. If $\underline{X}$ has a Poisson distribution with mean θ, then

$$P_{\theta}(X \leq C) = P(X^2_{2(C+1)} > 2\theta)$$

where $\underline{X}^2_r$ denotes a Chi-square random variable with $\underline{r}$ degress of freedom. Thus, a table of Chi-square probabilities can be used in place of Poisson probabilities.

Now $\underline{H}$ and $\underline{K}$ are found by simultaneously solving the equations

$$P_{\lambda_0 H}(X \leq K) = P(X^2_{2(K+1)} > 2\lambda_0 H) = 1 - \alpha$$

and

$$P_{\lambda_1 H}(X \leq K) = P(X^2_{2(K+1)} > 2\lambda_1 H) = \beta.$$

These equations must be solved iteratively.

Standard Normal Approximation

If $\underline{X}$ has a Poisson distribution with mean $\underline{\lambda H}$, then it is well-known that

$$\frac{X - \lambda H}{\sqrt{\lambda H}}$$

has, approximately, a standard normal distribution if $\underline{\lambda H}$ is large. (That is, $(\underline{X} - \underline{\lambda H})/\sqrt{\underline{\lambda H}}$ has a distribution which tends to the standard normal distribution as $\underline{\lambda H}$ tends to infinity; the approximation seems to work well for λH greater than 5.)

If $\underline{z}_{\gamma}$ denotes the value that cuts off an upper tail area of γ under the standard normal curve, then $\underline{H}$ and $\underline{K}$ can be found by solving the equations

$$z_{\alpha} = \frac{K-\lambda_0 H}{\sqrt{\lambda_0 H}}$$

and

$$-z_{\beta} = \frac{K-\lambda_1 H}{\sqrt{\lambda_1 H}}$$

Solving these equations yields

$$H = \left[\frac{z_\alpha \sqrt{\lambda_o} + z_\beta \sqrt{\lambda_1}}{\lambda_1 - \lambda_o} \right]^2$$

and

$$K = \lambda_0 H + z_\alpha \sqrt{\lambda_0 H} = \lambda_1 H - z_\beta \sqrt{\lambda_1 H}$$

Square-Root Normal Transformation

Since $\underline{X}$, suitably standardized, is approximately normally distributed, it can be shown that $\sqrt{\underline{X}}$ also is approximately normally distributed. The variance of $\sqrt{\underline{X}}$ is essentially free of $\underline{\lambda}$, for large $\underline{\lambda H}$, and the distribution of $\sqrt{\underline{X}}$ tends to be more accurately approximated by a normal distribution than does the distribution of $\underline{X}$, for moderate values of $\underline{\lambda H}$.

The theory (see Johnson and Kotz [1969]) actually states the

$$2(\sqrt{X} - \sqrt{\lambda H})$$

is approximately distributed as a standard normal random variable if $\underline{\lambda H}$ is large. Working on the true square-root scale, we find $\underline{H}$ and $\sqrt{\underline{K}}$ by solving the equations

$$z_\alpha = 2(\sqrt{K} - \sqrt{\lambda_0 H})$$

and

$$-z_\beta = 2(\sqrt{K} - \sqrt{\lambda_1 H})$$

which yield

$$H = \frac{0.25\ (z_\alpha + z_\beta)^2}{(\sqrt{\lambda_1} - \sqrt{\lambda_0})^2}$$

and

$$\sqrt{K} = \sqrt{\lambda_0 H} + 1/2\ z_\alpha$$

Transforming back to the original count scale

$$K = (\sqrt{\lambda_0 H} + 1/2\ z_\alpha)^2$$

The equation for H is the same as the one in the standard normal case when $\alpha = \beta$.

NUMERICAL STUDY

The values of H and K were found for various fixed values of λ_0, λ_1, α and β. A representative sample illustrative of our findings is shown in Table 1. The complete set of results is contained in the Appendix. To illustrate these results, we will look at the first row of figures where $\alpha = 0.025$, $\beta = 0.01$, $\lambda_0 = 0.1$, and $\lambda_1 = 0.6$. We say that the process is out of control if there are more than K nonconformances in H time units of observation. K has the value of 5 for both the standard normal approximation and the exact case and the value 6 for the square-root normal case. Note that the standard normal approximation gives a value of H, 23.469 time units, much larger than the true value, 22.000. Thus we would be observing the process longer than we should, for the same K value, and, as a result, have a greater probability of seeing more than K nonconformities than the nominal value of α indicates.

The square-root transformation results in an H approximately equal to the true value, but the K is slightly larger. Thus the probability of seeing more than K nonconformities would be slightly smaller than the nominal value. This pattern prevails throughout most of the cases studied.

TABLE 1

Values of H and K for Specified
λ_0, λ_1, α and β.

$\alpha = 0.025$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE ROOT NORMAL		EXACT (CHI-SQUARE)	
λ_0	λ_1	H	K	H	K	H	K
0.1	0.6	23.469	5	21.858	6	22.000	5
0.2	0.7	31.887	11	30.280	12	31.000	11
0.3	0.8	39.813	18	38.222	19	38.250	18
0.4	0.9	47.534	27	45.950	28	46.500	27
1.0	4.0	4.860	9	4.595	10	4.795	9
2.0	5.0	7.067	21	6.803	22	6.900	21
3.0	6.0	9.191	37	8.927	38	8.967	37
4.0	7.0	11.283	58	11.019	58	11.172	58
		$\alpha = 0.05$		$\beta = 0.01$			
0.1	0.6	21.578	4	18.773	5	19.700	4
0.2	0.7	28.783	9	26.005	10	27.250	9
0.3	0.8	35.575	16	32.812	16	33.500	15
0.4	0.9	42.195	23	39.442	23	39.556	22
1.0	4.0	4.408	7	3.944	8	4.000	7
2.0	5.0	6.299	18	5.839	18	5.860	17
3.0	6.0	8.131	32	7.663	32	7.767	31
4.0	7.0	9.915	50	9.459	49	9.536	48

TABLE 1, CONT. VALUES OF $\underline{H}$ AND $\underline{K}$ FOR SPECIFIED λ_0, λ_1, α and β.

$\alpha = 0.05$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE ROOT NORMAL		EXACT (CHI-SQUARE)	
λ_0	λ_1	H	K	H	K	H	K
0.1	0.6	16.626	3	15.469	4	14.583	3
0.2	0.7	22.580	8	21.429	8	20.570	7
0.3	0.8	28.186	13	27.038	13	28.167	13
0.4	0.9	33.647	19	32.501	20	33.000	19
1.0	4.0	3.442	6	3.250	7	3.285	6
2.0	5.0	5.003	15	4.812	15	4.700	14
3.0	6.0	6.505	26	6.314	27	6.350	26
4.0	7.0	7.985	41	7.794	41	7.988	40

EXAMPLE APPLICATIONS

Example 1. Check of Repaired Items Against a Standard. The data of Table 2 shows the number of maintenance errors, $\underline{X}_i$, observed upon sampling repaired aircraft parts for which the actual repair time was $\underline{H}_i$ hours. In this example, $\underline{H}_i$ is fixed by the practical sampling circumstances, and so no specific λ_1 needs to be determined. It is desired that λ_0 be 0.01 and α be 0.01. Thus $\underline{z}_\alpha$ is 2.33.

TABLE 2 NONCONFORMANCES AMONG REPAIRED ITEMS

SAMPLE	X_i	H_i	STANDARD NORMAL K	SQUARE ROOT NORMAL K
1	1	58.33	2.36	3.72
2	4	80.22	2.89	4.24
3	1	209.24	5.46	6.82
4	2	164.70	4.64	5.99

Table 2 also shows the values of $\underline{K}$ obtained by the standard normal approximation and the square-root normal transformation. For the first sample:

Standard normal approximation

$$K = \lambda_0 H + z_\alpha \sqrt{\lambda_0 H}$$

$$= 0.01(58.33) + 2.33 \sqrt{0.01(58.33)}$$

$$= 2.36$$

Square-root normal transformation

$$K = (\sqrt{\lambda_0 H} + 1/2\ z_\alpha)^2$$

$$= (\sqrt{0.01(58.33)} + 2.33/2)^2$$

$$= 3.72$$

Samples 1, 3 and 4 would be declared "in control" at the standard value of λ_0 under either scheme. However, sample 2, with $\underline{X}_2 = 4$, would be declared "out-of-control" under the standard normal scheme and "in control" if the square-root normal transformation were used; the observed value is very close to the boundary in either case. Whether we declare the process to be "out-of-control" or "in control" at the point that sample 2 was taken depends upon whether we want to think of the true α risk value as being slightly larger than 0.01 or slightly smaller than 0.01. In many cases declaring a process to be out-of-control when, in fact, it is in control is a costly error. Thus a quality control engineer may wish to use the more conservative procedure that lends itself to smaller α value.

Example 2. Establishing a Standard Plan to Check Maintenance Errors in a Paint Shop. It is desired to set up a standard Acceptance Control Chart plan for checking maintenance errors in an aircraft subassembly paint shop. The acceptable process level is 3 errors per 100 worker-hours with a risk level (α) of 0.05. The rejectable process level is to be 15 errors per 100 worker-hours with a risk level of 0.10. Values of $\underline{K}$ and $\underline{H}$ will be found by the three methods.

As previously stated, the Chi-square may be used to solve for Poisson probabilities. Using the Hald Statistical Tables (1952);

$$P(X^2_{2(K+1)} \leq 2\lambda_0 H) = \alpha$$

$$P(X^2_{2(K+1)} \leq 2\lambda_1 H) = 1 - \beta$$

Substituting the values of λ_0, λ_1, α, and β into these equations

$$P(X^2_{2(K+1)} \leq 2(0.03)H) = 0.05$$

$$P(X^2_{2(K+1)} \leq 2(0.15)H) = 0.90$$

A convenient search procedure for solving these equations for K and H is to take the ratios of the values of X_r^2 for even values of r and solve for the value of r that is closest to this ratio. The value of H may then be found from the resulting values of X_r^2 taken from the table.

$$\frac{2\lambda_1 H}{2\lambda_0 H} = \lambda_1/\lambda_0 = 0.15/0.03 = 5.0$$

From the Hald Tables of the Chi-square distribution:

r	$X^2(r,\ 0.90)/X^2(r,\ 0.05)$
6	10.6 /1.64 = 6.46
8	13.4 /2.73 = 4.91
10	16.0 /3.94 = 4.06

Clearly, the ratio of the two Chi-squares is closest to the desired value of 5.0 when r equals 8. The value of K then must be

$$K = (8/2) - 1 = 3$$

H is found by solving the equation

$$2\lambda H = X^2_{r,\ \gamma}$$

for each $(\lambda,\ \gamma)$ pair and selecting the larger (more conservative value. Thus

$$H = X^2_{(8,\ 0.90)} / 2\lambda_1 = 13.4 / 2(0.15) = 44.67$$

or

$$H = X^2_{(8,\ 0.05)} / 2\lambda_0 = 2.73 / 2(0.03) = 45.50$$

Thus our observation time should be 45.50 hours.

In comparison, using the standard normal approximation yields values of H and K of

$$H = \left[\frac{1.645\sqrt{0.03} + 1.282\sqrt{0.15}}{0.15 - 0.03}\right]^2 = 42.41$$

$$K = 0.03(42.41) + 1.645\sqrt{0.03(42.41)} = 3.13$$

By the square-root normal transformation, these values are

$$H = \frac{0.25(1.645 + 1.282)^2}{(\sqrt{0.15} - \sqrt{0.03})^2} = 46.73$$

$$K = (\sqrt{0.03(46.73)} + 1.645/2)^2 = 4.03$$

It should be noted that the actual values of α and β in this case are 0.040 and 0.122 using the standard normal approximation and 0.014 and 0.172 using the square-root transformation. Thus both approximations are more conservative with respect to α error and less conservative with respect to β error than the plan design called for ($\alpha = 0.05$ and $\beta = 0.10$).

As with many cases involving observations on maintenance operations the actual total maintenance time involved in a sample subgroup is likely to differ from the planned, or design, time. Table 3 shows the actual subgroup times and nonconformities found in 21 subgroups. The actual times range from a low of 32.1 hours to a high of 57.8. This results from the fairly wide discrete time variation required to process a unit. As a consequence, it may be necessary to recompute the control limit based upon the actual time in a subgroup as opposed to the value found for the design time. Since the count of nonconformities is integer-valued, recalculation of the control limit is not always required.

Figure 2 shows the Acceptance Control Chart, using three sets of control limits, for the sampling data of Table 3. Since the actual sample hours vary from subgroup to subgroup, it is inappropriate to plot a central line on this chart. (Where the sample hours can be held constant, a central line would be plotted at $\underline{\lambda_0 H}$.) The control limit using the exact Poisson is plotted as a dash line at the value 3.5 for all points except subgroups 7, 9, 12, and 21. Recalculation was necessary

TABLE 3. Calculation of ACLs and Actual α and β Error Probabilities for Aircraft Maintenance Data

				Standard Normal Approximation				Square-root Normal Transformation				
						Actual				Actual		
Subgroup	ACL by Actual Poisson	Actual Hours	Total noncon-formities c	K	ACL	α	β	K	ACL	α	β	Subgroup
1	3.5	51.5	4	3.59	3.5	0.071	0.051	4.26	4.5	0.021	0.116	1
2	3.5	51.5	1	3.59	3.5	0.071	0.051	4.26	4.5	0.021	0.116	2
3	3.5	47.1	0	3.37	3.5	0.055	0.078	4.04	4.5	0.015	0.167	3
4	3.5	47.1	0	3.37	3.5	0.055	0.078	4.04	4.5	0.015	0.167	4
5	3.5	53.1	2	3.67	3.5	0.078	0.043	4.35	4.5	0.023	0.102	5
6	3.5	40.4	3	3.02	3.5	0.035	0.146	3.70	3.5	0.035	0.146	6
7	4.5	57.7	0	3.90	3.5	0.098	0.027	4.57	4.5	0.032	0.068	7
8	3.5	51.5	0	3.59	3.5	0.071	0.051	4.26	4.5	0.021	0.116	8
9	2.5	34.8	0	2.72	2.5	0.089	0.107	3.40	3.5	0.022	0.235	9
10	3.5	51.5	1	3.59	3.5	0.071	0.051	4.26	4.5	0.021	0.116	10
11	3.5	38.8	1	2.94	2.5	0.113	0.070	3.62	3.5	0.031	0.168	11
12	4.5	57.8	3	3.90	3.5	0.098	0.027	4.58	4.5	0.032	0.067	12
13	3.5	39.1	0	2.95	2.5	0.115	0.068	3.63	3.5	0.031	0.164	13
14	3.5	49.1	2	3.47	3.5	0.062	0.065	4.15	4.5	0.017	0.142	14
15	3.5	41.6	0	3.09	3.5	0.038	0.131	3.76	3.5	0.038	0.131	15
16	3.5	51.5	1	3.59	3.5	0.071	0.051	4.26	4.5	0.021	0.116	16
17	3.5	42.1	1	3.11	3.5	0.039	0.125	3.79	3.5	0.039	0.125	17
18	3.5	54.3	2	3.73	3.5	0.083	0.038	4.41	4.5	0.025	0.092	18
19	3.5	36.6	1	2.82	2.5	0.099	0.089	3.50	3.5	0.026	0.203	19
20	3.5	36.6	0	2.82	2.5	0.099	0.089	3.50	3.5	0.026	0.203	20
21	2.5	32.1	2	2.58	2.5	0.074	0.141	3.25	3.5	0.017	0.292	21

for subgroups 7 and 12 (4.5) because of the larger than standard sample hours and for subgroups 9 and 21 because of the smaller than standard sample hours.

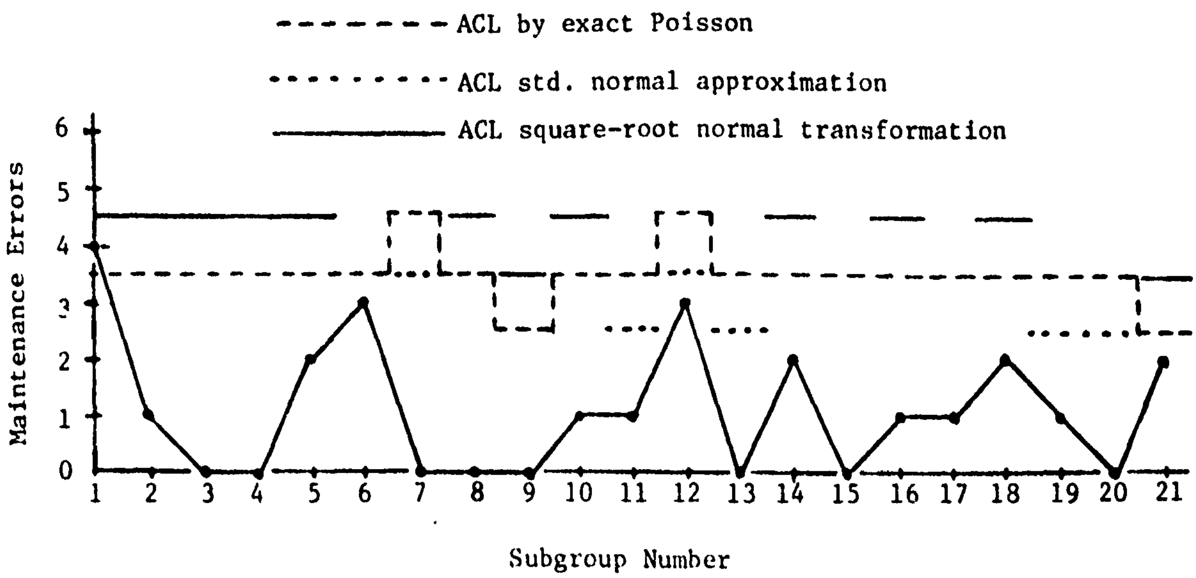

Fig. 2 Acceptance Control Chart for Data of Table 3

Table 3 also shows the values of K calculated from the standard normal approximation and from the square-root normal transformation. Again, since the count of nonconformities, c, is integer-valued, the ACL for each approximation has been set half-way between integerized values of K and K+1. Where these ACL values differ from those found by the exact Poisson, they are plotted on Figure 2 as dotted lines for the standard normal approximation and as solid lines for the square-root normal transformation.

Observing Figure 2, it should be noted that an out-of-control condition is signalled for subgroup 1 by both the exact Poisson and the standard normal approximation but not by the square-root normal transformation. In those instances wherein the ACL by the standard normal approximation differs from the exact Poisson, it tends to be tighter. Thus the standard normal approximation tends to protect more against β error at the sacrifice of α error. The square-root normal transformation tends to act just the opposite. Where it differs from the exact Poisson it tends to be looser affording greater protection against α error at the expense of β error.

This feature is born out by examination of Table 3 in which are tabulated the actual α and β error for each subgroup using each approximation method. Recall that the design level for α was 0.05. By the standard normal approximation, actual α protection ranged from 0.035 to 0.115 with 18 of 21 case above 0.05. In the case of the square-root normal transformation, actual α error ranged from 0.015 to 0.039; all cases were below the design level of 0.05.

The design level for β was 0.10. In the case of the standard normal approximation, the actual β error ranged from 0.027 to 0.146 with 5 of 21 cases above 0.10. For the square-root transformation all but three cases were above the design level with the actual values ranging from 0.067 to 0.292. In four cases the actual risk levels were more than double the design level.

It should be noted that where the ACL found by an approximation method agrees with that found by the exact Poisson, the true values of α and β apply to the exact Poisson as well. Thus when actual sample hours differ from the value of $\underline{H}$ found from applying the Chi-square formulas, the actual levels of protection may change significantly.

CONCLUSIONS

This paper has described an Acceptance Control Charting approach for process control of cases involving the observation of Poisson counts. In addition to deliniating a procedure utilizing the exact Poisson, procedures for use of the popular standard normal approximation and of the not-so-frequently used square-root normal transformation were developed and evaluated.

It was shown that the standard normal approximation tended to favor protection against β error. To the extent that the results of ACL calculations differed from the exact Poisson, the difference was biased in favor of β error protection. On the other hand, usage of the square-root normal transformation leads to ACL calculations offering better protection against α error. To the extent that these calculations differed from the exact Poisson, the bias favored α error protection.

Study of a number of cases, of which Table 1 includes a sample, indicated that the square-root normal transformation gives values of $\underline{H}$ and $\underline{K}$ that oscillate around the true values but that large discrepancies between the approximate and true values are rare. We therefore recommend using the square-root transformation when it is cumbersome or impossible to use exact values and when the cost of α error is high in relation to β error. However, for those cases in which the cost of β error is equal to or greater than that of α error, the standard normal approximation is preferable.

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APPENDIX

COMPLETE RESULTS OF NUMERICAL STUDY

APPENDIX

$\alpha = 0.025$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
0.1	0.3	89.71	14	85.74	15	84.83	14
0.2	0.4	137.86	37	133.91	38	134.80	37
0.3	0.5	184.83	70	180.88	70	181.14	69
0.4	0.6	231.37	111	227.43	111	227.71	110
0.5	0.7	277.71	161	273.77	161	273.72	160
0.6	0.8	323.94	221	320.00	220	319.48	219
0.7	0.9	370.09	290	366.16	289	366.29	288
0.8	1.0	416.20	368	412.27	366	411.59	365
0.9	1.1	462.28	456	458.35	453	457.81	452
1.0	1.2	508.34	552	504.41	549	503.83	548

$\alpha = 0.025$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
0.1	0.6	23.469	5	21.870	6	22.000	5
0.2	0.7	31.887	11	30.296	12	31.000	11
0.3	0.8	39.813	18	38.226	19	38.250	18
0.4	0.9	47.534	27	45.950	28	46.500	27
0.5	1.0	55.145	37	53.563	38	53.800	37
0.6	1.1	62.691	49	61.110	49	61.833	49
0.7	1.2	70.192	62	68.613	63	70.087	63
0.8	1.3	77.665	77	76.086	77	76.785	77
0.9	1.4	85.115	93	83.537	93	84.148	93
1.0	1.5	92.549	111	90.971	111	91.091	111

APPENDIX

$\alpha = 0.025$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	2.0	27.57	37	26.78	38	26.90	37
2.0	3.0	46.27	111	45.49	111	45.54	110
3.0	4.0	64.79	221	64.00	220	63.90	219
4.0	5.0	83.24	368	82.45	366	82.32	365
5.0	6.0	101.67	552	100.88	549	100.77	548
6.0	7.0	120.08	773	119.29	769	119.23	768
7.0	8.0	138.48	1030	137.70	1026	137.57	1024
8.0	9.0	156.88	1324	156.09	1319	156.07	1318
9.0	10.0	175.27	1655	174.49	1649	174.35	1647
10.0	11.0	193.67	2022	192.88	2016	192.77	2014

$\alpha = 0.025$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	3.0	8.972	14	8.574	15	8.483	14
2.0	4.0	13.786	37	13.391	38	13.450	37
3.0	5.0	18.483	70	18.088	70	18.114	69
4.0	6.0	23.137	111	22.743	111	22.771	110
5.0	7.0	27.771	161	27.377	161	27.372	160
6.0	8.0	32.394	221	32.000	220	31.948	219
7.0	9.0	37.009	290	36.616	289	36.629	288
8.0	10.0	41.620	368	41.227	366	41.159	365
9.0	11.0	46.228	456	45.834	453	45.781	452
10.0	12.0	50.834	552	50.440	549	50.383	548

APPENDIX

$\alpha = 0.025$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	4.0	4.860	9	4.595	10	4.795	9
2.0	5.0	7.067	21	6.803	22	6.900	21
3.0	6.0	9.191	37	8.927	38	8.967	37
4.0	7.0	11.283	58	11.019	58	11.172	58
5.0	8.0	13.358	82	13.095	82	13.176	82
6.0	9.0	15.425	111	15.162	111	15.181	110
7.0	10.0	17.485	144	17.223	143	17.189	142
8.0	11.0	19.542	180	19.279	180	19.304	179
9.0	12.0	21.596	221	21.333	220	21.299	219
10.0	13.0	23.648	266	23.385	265	23.382	264

$\alpha = 0.05$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
0.1	0.3	80.52	12	73.6	13	77.00	12
0.2	0.4	121.81	32	114.94	32	116.50	31
0.3	0.5	162.10	60	155.26	58	156.06	58
0.4	0.6	202.04	95	195.22	93	196.25	93
0.5	0.7	241.81	138	235.00	136	234.58	135
0.6	0.8	281.49	190	274.68	187	274.91	186
0.7	0.9	321.10	249	314.30	245	313.88	244
0.8	1.0	360.68	316	353.88	311	354.23	311
0.9	1.1	400.23	391	393.44	386	393.46	385
1.0	1.2	439.76	474	432.97	468	432.85	467

APPENDIX

$\alpha = 0.05$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
0.1	0.6	21.578	4	18.773	5	19.700	4
0.2	0.7	28.783	9	26.005	10	27.250	9
0.3	0.8	35.575	16	32.812	16	33.500	15
0.4	0.9	42.195	23	39.442	23	39.556	22
0.5	1.0	48.724	32	45.977	32	46.600	31
0.6	1.1	55.197	42	52.455	41	53.250	41
0.7	1.2	61.633	53	58.895	52	59.256	52
0.8	1.3	68.045	66	65.310	65	65.246	64
0.9	1.4	74.438	80	71.706	78	72.032	78
1.0	1.5	80.817	95	78.087	93	78.499	93

$\alpha = 0.05$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	2.0	24.36	32	22.99	32	23.30	31
2.0	3.0	40.41	95	39.04	93	39.25	93
3.0	4.0	56.30	190	54.94	187	54.98	186
4.0	5.0	72.14	316	70.78	311	70.85	311
5.0	6.0	87.95	474	86.59	468	86.57	467
6.0	7.0	103.76	663	102.40	656	102.38	657
7.0	8.0	119.55	884	118.20	875	118.11	874
8.0	9.0	135.34	1136	133.99	1126	134.03	1126
9.0	10.0	151.13	1420	149.77	1409	149.96	1410
10.0	11.0	166.92	1736	165.56	1723	165.91	1726

APPENDIX

$\alpha = 0.05$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	3.0	8.052	12	7.360	13	7.700	12
2.0	4.0	12.181	32	11.494	32	12.075	32
3.0	5.0	16.210	60	15.526	58	15.606	58
4.0	6.0	20.204	95	19.522	93	19.625	93
5.0	7.0	24.181	138	23.500	136	23.458	135
6.0	8.0	28.149	190	27.468	187	27.491	186
7.0	9.0	32.110	249	31.430	245	31.388	244
8.0	10.0	36.068	316	35.388	311	35.423	311
9.0	11.0	40.023	391	39.343	386	39.346	385
10.0	12.0	43.976	474	43.296	468	43.285	467

$\alpha = 0.05$ $\beta = 0.01$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	4.0	4.408	7	3.944	8	4.000	7
2.0	5.0	6.299	18	5.839	18	5.860	17
3.0	6.0	8.121	32	7.663	32	7.767	31
4.0	7.0	9.915	50	9.459	49	9.536	48
5.0	8.0	11.696	71	11.241	69	11.174	68
6.0	9.0	13.470	95	13.015	93	13.083	93
7.0	10.0	15.238	123	14.783	121	14.766	120
8.0	11.0	17.003	155	16.549	152	16.523	151
9.0	12.0	18.766	190	18.312	187	18.327	186
10.0	13.0	20.527	228	20.073	225	20.077	224

APPENDIX

$\alpha = 0.05$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
0.1	0.3	63.52	10	60.65	11	61.5	10
0.2	0.4	97.58	26	94.72	27	95.25	26
0.3	0.5	130.80	49	127.94	49	129.83	49
0.4	0.6	163.72	78	160.86	78	159.99	77
0.5	0.7	196.49	114	193.64	114	193.76	113
0.6	0.8	229.19	156	226.64	156	226.49	155
0.7	0.9	261.82	205	258.99	204	258.54	203
0.8	1.0	294.45	260	291.60	259	291.24	258
0.9	1.1	327.04	322	324.20	321	324.41	320
1.0	1.2	359.62	390	356.77	389	356.99	388

$\alpha = 0.05$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
0.1	0.6	16.626	3	15.469	4	14.583	3
0.2	0.7	22.580	8	21.429	8	20.571	7
0.3	0.8	28.186	13	27.038	13	28.167	13
0.4	0.9	33.647	19	32.601	20	33.125	19
0.5	1.0	39.031	26	37.886	27	38.100	26
0.6	1.1	44.368	35	43.224	35	43.182	34
0.7	1.2	49.674	44	48.531	44	48.250	43
0.8	1.3	54.959	54	53.817	55	54.068	54
0.9	1.4	60.229	66	59.087	66	59.002	65
1.0	1.5	65.487	78	64.345	78	63.995	77

APPENDIX

$\alpha = 0.05$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
1.0	2.0	19.515	26	18.94	27	19.05	26
2.0	3.0	32.743	78	32.17	78	32.00	77
3.0	4.0	45.820	156	45.27	156	45.30	155
4.0	5.0	58.890	260	58.32	259	58.25	258
5.0	6.0	71.923	390	71.35	389	71.40	388
6.0	7.0	84.946	546	84.38	544	84.34	543
7.0	8.0	97.960	728	97.40	725	97.31	724
8.0	9.0	110.980	936	110.41	933	110.40	932
9.0	10.0	123.990	1170	123.42	1166	123.36	1165
10.0	11.0	136.990	1430	136.43	1426	136.34	1424

$\alpha = 0.05$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
1.0	3.0	6.352	10	6.063	11	6.150	10
2.0	4.0	9.758	26	9.471	27	9.525	26
3.0	5.0	13.080	49	12.794	49	12.679	48
4.0	6.0	16.372	78	16.086	78	15.999	77
5.0	7.0	19.649	114	19.364	114	19.376	113
6.0	8.0	22.919	156	22.634	156	22.650	155
7.0	9.0	26.184	205	25.899	204	25.854	203
8.0	10.0	29.445	260	29.160	259	29.124	258
9.0	11.0	32.704	322	32.419	321	32.441	320
10.0	12.0	35.962	390	35.677	389	35.699	388

APPENDIX

$\alpha = 0.05$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	4.0	3.442	6	3.250	7	3.285	6
2.0	5.0	5.003	15	4.812	15	4.700	14
3.0	6.0	6.505	26	6.314	27	6.350	26
4.0	7.0	7.985	41	7.794	41	7.779	40
5.0	8.0	9.453	58	9.262	58	9.364	58
6.0	9.0	10.914	78	10.724	78	10.666	77
7.0	10.0	12.372	101	12.182	101	12.138	100
8.0	11.0	13.827	127	13.637	127	13.616	126
9.0	12.0	15.279	156	15.089	156	15.100	155
10.0	13.0	16.731	188	16.541	187	16.500	186

$\alpha = 0.01$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
0.1	0.3	81.86	14	85.74	17	82.50	15
0.2	0.4	130.01	37	133.91	40	133.75	39
0.3	0.5	176.98	70	180.88	73	180.88	72
0.4	0.6	223.52	111	227.43	115	227.97	114
0.5	0.7	269.86	161	273.77	165	274.31	165
0.6	0.8	316.08	221	319.99	226	320.26	225
0.7	0.9	362.24	290	366.16	295	365.96	294
0.8	1.0	408.35	368	412.27	373	411.55	372
0.9	1.1	454.43	456	458.35	461	457.98	460
1.0	1.2	500.48	552	504.41	558	504.17	557

APPENDIX

$\alpha = 0.01$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
0.1	0.6	20.328	5	21.870	7	23.300	6
0.2	0.7	28.746	11	30.296	13	30.500	12
0.3	0.8	36.672	18	38.226	21	37.000	19
0.4	0.9	44.392	27	35.950	30	44.944	28
0.5	1.0	52.004	37	53.563	40	53.500	39
0.6	1.1	59.549	49	61.110	52	60.736	51
0.7	1.2	67.051	62	68.613	66	68.917	65
0.8	1.3	74.523	77	76.086	80	76.501	80
0.9	1.4	81.973	93	83.537	97	83.582	96
1.0	1.5	89.407	111	90.971	115	91.186	114

$\alpha = 0.01$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
$\lambda 0$	$\lambda 1$	H	K	H	K	H	K
1.0	2.0	26.00	37	26.78	40	26.75	39
2.0	3.0	44.70	111	45.49	115	45.59	114
3.0	4.0	63.22	221	64.00	226	64.05	225
4.0	5.0	81.67	368	82.45	373	82.31	372
5.0	6.0	100.10	552	100.88	558	100.83	557
6.0	7.0	118.51	773	119.29	779	119.20	778
7.0	8.0	136.92	1030	137.70	1037	137.60	1036
8.0	9.0	155.31	1324	156.10	1332	156.03	1331
9.0	10.0	173.70	1655	174.48	1664	174.36	1662
10.0	11.0	192.10	2022	192.88	2032	192.73	2030

APPENDIX

$\alpha = 0.01$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
λ0	λ1	H	K	H	K	H	K
1.0	3.0	8.186	14	8.574	17	8.250	15
2.0	4.0	13.001	37	13.391	40	13.375	39
3.0	5.0	17.697	70	18.088	73	18.088	72
4.0	6.0	22.353	111	22.743	115	22.797	114
5.0	7.0	26.986	161	27.377	165	27.431	165
6.0	8.0	31.608	221	32.000	226	32.026	225
7.0	9.0	36.224	290	36.616	295	36.596	294
8.0	10.0	40.835	368	41.227	373	41.155	372
9.0	11.0	45.443	456	45.834	461	45.789	460
10.0	12.0	50.048	552	50.440	558	50.417	557

$\alpha = 0.01$ $\beta = 0.025$

		STANDARD NORMAL		SQUARE-ROOT NORMAL		ACTUAL CHI-SQUARE	
0	λ1	H	K	H	K	H	K
1.0	4.0	4.337	9	4.595	11	4.275	9
2.0	5.0	6.544	21	6.803	24	6.675	22
3.0	6.0	8.667	37	8.927	40	10.550	39
4.0	7.0	10.759	58	11.019	61	10.998	60
5.0	8.0	12.835	82	13.095	86	13.113	85
6.0	9.0	14.901	111	15.162	115	15.198	114
7.0	10.0	1[illegible].962	144	17.223	147	17.261	147
8.0	11.0	19.018	180	19.279	184	19.311	184
9.0	12.0	21.072	221	21.333	226	21.351	225
10.0	13.0	23.124	266	23.385	271	23.383	270