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Research Report CCS 386

AN INFINITE CONSTRAINED GAME
DUALITY CHARACTERIZING
ECONOMIC EQUILIBRIUM

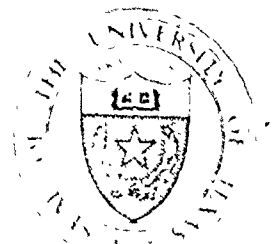
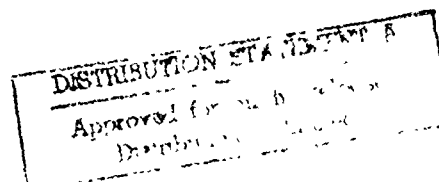
by

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ABSTRACT

The principal economic assumptions of this paper are neoclassical behavior assumptions on a consumer group which owns the resources and a collection of producers employing these resources. A saddle value problem is formulated to characterize equilibrium in the economy in the sense that at equilibrium prices producers determine production plans to maximize profits and that these outputs and inputs are exactly those demanded and supplied respectively by the consumer group.

The saddle value problem is shown to be equivalent to a dual pair of uniextremizations termed the consumer group's problem and the producers' problem. The neoclassical economic assumptions yield sufficient conditions which are among the most general ones for guaranteeing a saddle point and simultaneously a perfect duality for the dual programming pair. Economic interpretations are given for all the variables of the consumer group's problem and for all the variables of the producers' problem even at non-optimal stages in each problem. The approach is an infinite dimensional extension of the Charnes' constrained game linear programming equivalents in finite dimensions.

Key Words: Neoclassical Economics, Saddle Value Problems, Semi-Infinite Programming, Economic Equilibrium, Two Person Constrained Games, Infinite Games.

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1. Introduction

In economic theory there is a large class of benefit maximizing models which are actually mathematical programming models. They embrace the competitive equilibrium problem, the marginal cost pricing problem, and the market surplus maximization problem, see Carey [5], Hotelling [13], Negishi [15], Pressman [16], Samuelson [19], Takayama and Judge [22], and Thore [24], [25]. The development of these extremal principle models for interdependent economic systems has enhanced our understanding of the market mechanisms, while providing access to solutions by the methods and algorithms of mathematical programming.

Conceptually, extremal models of the market system can be developed at any of the following three levels, depending upon the nature of the a priori information which is postulated concerning consumer behavior.

(i) The utility function of each consumer is known, and the consumers maximize utility subject to budget constraints. Essentially, this is the approach taken in the pioneering contribution by Negishi [15], who showed that a certain weighted sum of utilities, subject to the condition that all consumption plans and all production plans be feasible, assumes its maximum at a point of competitive equilibrium.

(ii) All demand functions of the consumers are known and assumed to be integrable and invertible. This approach has been explored in models of spatial equilibrium for some sector of an economy by Takayama and Judge [22]. Their work has recently been extended by Thore ([24],[25]) to the analysis of a general equilibrium system with endogeneous determination of income, and with general convex production sets. The key feature here is that the indirect

demand functions (displaying demand prices as functions of all quantities) are integrated into so-called "quasi-welfare", which, subject to the condition that all consumption plans and all production plans be feasible, assumes its maximum at a point of competitive equilibrium.

The extremal principle for a "resource value-transfer economy", developed by Charnes and Cooper [7], assumes given and known spending proportions of expenditure on the various consumer goods, and given and known proportions of income earned from the sale of the various resources. A certain economic potential function is specified which assumes an unconstrained minimum at the point of "accounting balance", where the markets for all consumer goods, and the markets for all resources are cleared.

(iii) In the present study we shall assume that the expenditure function is the given and known entity. In the common manner, the expenditure function displays the minimal expenditure which has to be incurred (at given prices) in order to reach some given utility level.

Already Arrow and Debreu in their original paper [1] discussed the role of the expenditure function in the neoclassical theory of equilibrium. They wrote:

"From the viewpoint of welfare economics, it is the principle that the consumption vector chosen should be the one which achieves the given utility at least cost which is primary, and the principle of maximizing utility at a given cost only relevant when the two give identical results."

(op. cit., p. 286.)

For a descriptive theory of behavior under perfect competition, on the other hand, they held the concept of utility maximization to be primary.

Assume that all consumers are amalgamated into one composite unit. Let the expenditure function for this unit be given. We shall write down a certain economic potential function, which depends upon the expenditure function. The potential function, subjected to the condition that all production plans are feasible, has a saddle point. At the saddle point there is competitive equilibrium.

The representation of a competitive equilibrium, marginal cost pricing, or market surplus problem as a saddle value problem is not new. In fact, the first author suggested a biextremal problem for extending overly constrained resource-value transfer economies. It is inherent in Negishi's [15] formulation upon applying the Kuhn-Tucker theorem to the original uniextremal formulation; see also Carey [5] and Thore [24], where Charnes' suggestions applied to other economic contexts.

However, the biextremal problems of this paper are different from those obtained by merely assigning Kuhn-Tucker variables to equilibrium constraints and constructing the standard Lagrangian saddle function. While useful in establishing important properties of optimal solutions, these kinds of saddle problems generally admit no useful interpretation for their variables other than when they are at their optimal values.

In contrast, under the usual neoclassical theoretic assumptions we construct a saddle value problem whose saddle point is still guaranteed by standard convexity and which is of the particular bilinear format with separable constraint sets studied by Charnes, Cribik, and Kortanek [9]. Following their approach the polyextremal

problem can be characterized by a dual pair of uniextremal problems. The primal problem (I) involves the production variables plus additional variables stemming from the expenditure minimization problem. The dual problem (II) involves as variables all prices together with additional variables stemming from the maximization of profits, and the production sets.

Thus, each problem (I) and (II) employs additional variables stemming from the constraints and objective functions of the other dual problem. However, neither problem requires any information whatsoever on the other problem's optimal solution. In this sense the problems are completely separable, and each problem has its own distinct set of variables with its own economic interpretation.

Our principal equilibrium duality results are an infinite dimensional extension of the highly unified and complete Von Neumann minimax and Charnes constrained game linear programming equivalents in finite dimensions, where now, as then, meaningful economic interpretations are given for both primal and dual variables even at non-optimal stages of each problem.

We now describe a neoclassical economy.

2. The Economy

Consider an economy which is specified in the following manner:

2.1. Consumers

The consumers are amalgamated into one single composite unit. The unit buys consumer goods, and sells resources (labor, mineral resources, etc.). It controls the producers. It receives income from selling resources, and in the form of dividends (the profits of the producers).

Let $x = [x_1, x_2, \dots, x_n, x_{n+1}]$ be the vector of quantities of consumer goods bought by the unit of consumers or resources sold. Consumer goods are entered as positive elements; resources as negative elements.

The $(n + 1)$ th good is supposed to be a resource good, used as numeraire. The quantity supplied and sold of this resource during the period is x_{n+1} . It is entered with a negative sign.

Let $p = [p_1, p_2, \dots, p_n, 1]$ be the corresponding vector of (non-negative) prices. The price of the numeraire good is unity.

Let $w = [w_1, w_2, \dots, w_n, w_{n+1}]$ denote the initial stocks of the various goods (including resources, and the numeraire) held by the consumer unit. All elements of this vector are nonnegative.

The utility function U is assumed to be twice differentiable, strictly quasi-concave, and possess positive partial derivatives[†] with respect to consumer goods over its domain. The latter assumption is a convenient form of a non-satiation hypothesis, see Arrow and Enthoven [2], Baumol [4], Chapter 14, and Katz [14], Chapter 9. In this setting the consumption set X is given by:

[†]It will be convenient to assume each partial derivative of U is bounded away from zero.

$$X = \{x \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for each consumer good } i \text{ and} \\ 0 \geq x_j \geq -w_j \text{ for each resource good } j\}.$$

Thus, analogous to Debreu [10] our consumption set is also bounded below.

The (neoclassical) indirect utility function V is defined by

$$(2.1) \quad V(p, I) = \max_{x \in X} \{U(x) \mid p^T x \leq I\},$$

where I denotes the income received from all sources including the sale of resources owned and refunded profits of the producing sector.

The expenditure function E is defined by

$$(2.2) \quad E(p, u) = \min_{x \in X} \{p^T x \mid U(x) \geq u\}$$

where u is a fixed level of utility. The optimal solution to program (2.2) will be written $x_i = x_i(p, u)$, $i = 1, 2, \dots, n, n+1$. The functions $x_i(p, u)$ are referred to as the compensated demand functions (compensated supply functions for resources).

We summarize some well-known facts about indirect and expenditure functions and their derivatives, see Baumol [4], Chapter 14 or Silberberg [17], Chapter 8.

Theorem 1. Under the assumptions above on the utility function, let u and I be positive numbers. Then

(1A) E is strictly monotonically increasing with utility level u , concave and differentiable in positive prices p , with

$$\left. \frac{\partial E}{\partial p_i} \right|_{(p, u)} = x_i(p, u), \quad i = 1, \dots, n$$

and

$$(1B) \quad E(p, V(p, I)) = I \quad \text{and} \quad V(p, E(p, u)) = u.$$

2.2. Producers

The producers buy resources and manufacture consumer goods. (They may also buy intermediary goods; but for simplicity of notation such goods will not here be distinguished separately. The required generalizations are immediate.)

Suppose that there are $k = 1, 2, \dots, r$ producers in the economy. The production plan of the producer is specified by the quantities of all his inputs and outputs. The production plan of producer k is represented by the vector $y_k = [y_{ki}, i = 1, 2, \dots, n, n+1]$ where outputs are entered as positive elements, and inputs as negative elements.

In the common manner, the set of all production plans $y_k \in \mathbb{R}^{n+1}$ possible for the k th producer is called his production set.

Assume that each production set Y_k is closed and convex. Assume $0 \in Y_k$ (possibility of inaction).

Let $Y = \sum_{k=1}^r Y_k$ denote the aggregate production set. The following regularity assumptions are introduced for the aggregate production set.

- (i) $Y \cap \mathbb{R}_{(\geq 0)}^{n+1} = \{0\}$, (no free production, outputs require inputs).
- (ii) $Y \supset Y - \mathbb{R}_{(\geq 0)}^{n+1}$, (if a production plan is possible so is one with output no larger and input no smaller).
- (iii) $Y \supseteq \mathbb{R}_{(\leq 0)}^{n+1}$ (free disposal of goods).
- (iv) Y itself is a closed set.
- (v) Production plans having unbounded outputs require unbounded inputs.

(vi) 0 is an extreme point of the sum of the producer asymptotic cones O^+Y_k ,[†] i.e. $y_k \in O^+Y_k$, each k and $\sum_{k=1}^r y_k = 0 \Rightarrow$ each $y_k = 0$.

(vii) The numeraire is an input for each firm, i.e., $y_{k,n+1} \leq 0$ for $k = 1, \dots, r$ and unbounded resource inputs to any firm can yield an unbounded output for any consumption good. Further, there are no vertical supporting hyperplanes with respect to the variable $y_{k,n+1}$ of the closed convex production set Y_k , for each k .

Each producer k maximizes profit on his production set

$$(2.3) \quad \begin{aligned} & \max_{y_k} p^T y_k \\ & \text{subject to } y_k \in Y_k. \end{aligned}$$

We now formulate a saddle value problem to determine an equilibrium for the economy.

[†] See Debreu [10] and Rockafellar [18] and Stoer-Witzgall [21] for the definition of asymptotic or recession cones. (vi) is sufficient to insure that $\sum_k Y_k$ is a closed set.

3. A Polyextremal Principle for Equilibrium

Consider the constrained polyextremal problem: Find

$$(3.1) \quad M(u) \equiv \sup_{y_k, k=1,2,\dots,r} \left[\inf_p p^T \left(\sum_{k=1}^r y_k + w \right) - E(p,u) \right]$$

in $\mathbb{R}^{n+1}$

subject to $y_k \in Y_k, \quad k = 1, 2, \dots, r$
 $p \geq 0.$

In plain words, the "pay-off function", or objective function, is the difference between all income collected by the consumer (obtained in the form of dividends and in the form of the value of initial stock brought to the market-place) and the (minimal) amount of expenditure which is required (at prices p) to sustain the particular utility level u . We term the problem "polyextremal" because it depicts a game-like situation where the $k = 1, 2, \dots, r$ producers (controlled by the consuming unit) are looking for (feasible) production plans y_k which would make the pay-off as large as possible, and where a fictitious "market player" setting prices p would be looking for (nonnegative) prices which would make the same payoff as small as possible.

The saddle function of problem (3.1) can be written in the form

$$(3.2) \quad K_u(y, p) = y^T A p + h_u(p)$$

where $y^T = (y_1^T, \dots, y_r^T) \in \mathbb{R}^{(n+1)r}$, u is fixed positive,

$h_u(p) = p^T w - E(p, u)$, and where A is the $(n+1)$ by $(n+1)r$ matrix consisting of r copies of the $(n+1)$ by $(n+1)$ identity

matrix I_{n+1} , i.e., $A = [I_{n+1} \cdots I_{n+1}]$, r times. Thus, $K_u(\cdot, \cdot)$ is concave in y and convex in p (of course, linear in y , here). Even though both constraint sets for y and p respectively are unbounded, the consumer and producer assumptions of Sections 2 and 3 guarantee existence of a saddle point (y^*, p^*) . This is so because certain recession-like sufficient conditions on the individual functions $K_u(\cdot, p)$ and $K_u(y, \cdot)$ are satisfied and Theorem 37.3 of Rockafellar [18] applies. We shall verify these conditions in an Appendix.

As shown in Section 5 of Charnes, Gribik and Kortanek [9] these particular sufficient conditions for a saddle point also guarantee a perfect duality for the dual pair of uniextremal problems associated with biextremal problems of the form (3.1) above.

Our plan for the remainder and main part of the paper is as follows.

We shall first develop the economic equilibrium implications of the existence of a saddle point (y^*, p^*) for any fixed utility level u . Next, we shall set forth the dual pair of uniextremal problems and prove that the consumer and producer assumptions guarantee a perfect duality. The linearity of the uniextremal problems shall make the transition to the recession conditions stated in the Appendix fairly straightforward.

4. Existence of an Equilibrium for the Economy

Theorem 2. Let the consumer and producer assumptions of Section 2 prevail, and let u be a non-negative a priori specified utility level. Then (3.1) has a saddle point depending on u and denoted $(y^* = (y_1^*, \dots, y_r^*); p^*)$. Moreover, $M(u)$ is continuous for $u \geq 0$.

Proof. Appendix.

The following corollaries shall be proved next for purposes of establishing equilibrium.

Corollary 1. At the saddle point each producer will, given prices $p = p^*$, maximize profits on his production set.

Proof. From the definition of a maxmin it follows immediately that

$$(4.1) \quad p^{*T} y_k^* \geq p^{*T} y_k$$

for all y_k which satisfy

$$y_k \in Y_k.$$

Hence y_k^* is an optimal solution to problem (2.3) with $p = p^*$.

Corollary 2. At the saddle point all markets for consumer goods, and all markets for resources (except possibly the market for the numeraire) will be in equilibrium, provided prices are positive.

Proof. Consider the min part of (3.1), for $y_k = y_k^*$, $k = 1, 2, \dots, r$. It reads

$$(4.2) \quad \min_p p^T \left(\sum_{k=1}^r y_k^* + w \right) - E(p, u)$$

subject to $p \geq 0$.

The Kuhn-Tucker conditions give

$$(4.3) \quad \sum_{k=1}^r y_{ki}^* + w_i - \frac{\partial E(p^*, u)}{\partial p_i} \geq 0$$

$$p_i^* \left(\sum_{k=1}^r y_{ki}^* + w_i - \frac{\partial E(p^*, u)}{\partial p_i} \right) = 0, \quad i = 1, 2, \dots, n$$

or, from (1A) of Theorem 1 on expenditure functions

$$(4.4) \quad \sum_{k=1}^r y_{ki}^* + w_i - x_i(p^*, u) \geq 0$$

$$p_i^* \left(\sum_{k=1}^r y_{ki}^* + w_i - x_i(p^*, u) \right) = 0, \quad i = 1, 2, \dots, n$$

which, per definition, states that all markets $i = 1, 2, \dots, n$ are in equilibrium. It does not state, however, whether the market for the numeraire ($i = n + 1$) is in equilibrium or not.

Corollary 3. The value of the program (3.1), denoted $M(u)$, is equal to the excess demand for numeraire, provided prices are positive.

Proof. The value of the program can be written, using Theorem 1 on the expenditure functions as follows:

$$(4.5) \quad \begin{aligned} M(u) &= p^{*T} \left(\sum_{k=1}^r y_k^* + w \right) - E(p^*, u) \\ &= p^{*T} \left(\sum_{k=1}^r y_k^* + w \right) - \sum_{i=1}^n p_i^* \frac{\partial E(p^*, u)}{\partial p_i} - x_{n+1}(p^*, u) \end{aligned}$$

or, taking note of (4.3)

$$(4.6) \quad M(u) = \sum_{k=1}^r y_{k,n+1}^* + w_{n+1} - x_{n+1}(p^*, u).$$

Since we have assumed that the numeraire is a resource good, the two first terms are negative. If there is a positive demand for numeraire in the economy, the term x_{n+1} is negative, and hence the expression $-x_{n+1}$ is positive. $V(u)$ then is the excess demand for numeraire.

We next establish existence of a positive u^* such that $M(u^*) = 0$.

Lemma 1. $M(0) = w_{n+1}$.

Proof. Observe that $E(p, 0) = 0$ for any $p \geq 0$. Hence (3.1) becomes $M(0) = \sup_{\{y_k\}} \inf_p p^T \left(\sum_{k=1}^r y_k + w \right)$ subject to each $y_k \in Y_k$ and $p \geq 0$. Let $\{y_k\}$ be any saddle value feasible point, in particular $\inf_p p^T \left(\sum_{k=1}^r y_k + w \right) > -\infty$. This implies $\sum_{k=1}^r (y_k + w) \geq 0$ and since $p_j \geq 0$, $j = 1, \dots, n$ it follows that

$$\inf_{p \geq 0} p^T \left(\sum_{k=1}^r y_k + w \right) = \sum_{k=1}^r (y_k)_{n+1} + w_{n+1}.$$

Hence $M(0) = \sup_{\{y_k\}} \left\{ \sum_{k=1}^r (y_k)_{n+1} + w_{n+1} \right\} = w_{n+1}$, the maximum taken on for example with $0 \in Y_k$ for all k .

Lemma 2. There exists $u > 0$ such that $M(u) < 0$.

Proof. Let u increase indefinitely. Since the consumption set X is bounded below, it follows that increasing levels of consumption goods will be required. Moreover, from non-satiation increasing consumption will yield increasing utility levels. Hence

from (4.3) production plans having unbounded outputs will be required.

Now it also follows from (4.3) that all production inputs, except the numeraire are equal to the resources available from the consumer, i.e. they are bounded below. But by assumption (v), Section 2.2, some unbounded inputs are necessarily required and hence it necessarily follows that $\sum_{k=1}^r y_{k,n+1}^* \downarrow -\infty$ as $u \uparrow +\infty$. In particular, it follows from (4.6) that $M(u) < 0$ for some $u > 0$.

Theorem 3. There exists a uniquely determined u^* such that $M(u^*) = 0$. Moreover, the optimal consumption vector can alternatively be determined by letting the consumer maximize utility subject to an income constraint, and u^* is the maximum utility level achievable. The income constraint states that the total expenditure of the consumer must not exceed the value of the initial endowments (valued at optimal prices) plus the value of all profits (occurring at optimum).

Proof. Using the continuity of $M(u)$, Theorem 2, it follows immediately from Lemmas 1 and 2 that there exists u^* such that $M(u^*) = 0$. The value of u^* will now be computed by a formula, proving uniqueness, for the case of positive prices.

Let $I_* = p^{*T} \left(\sum_{k=1}^r y_k^* + w \right)$. We will show that in fact $u^* = V(p^*, I_*)$, see (2.1) in Section 2. Of course, from

$$0 = M(u^*) = p^{*T} \left(\sum_{k=1}^r y_k^* + w \right) - E(p^*, u^*),$$

it follows immediately that $E(p^*, u^*) = I_*$. Applying (1B) of neoclassical Theorem 1, we also obtain

$$V(p^*, I_*) = V(p^*, E(p^*, u^*)) = u^*,$$

proving that u^* is the maximum utility level achievable from total income I_* .

On the other hand, differentiating the right-most equality of (1B) with respect to p_i yields

$$\frac{\partial V}{\partial p_i}(p^*, I_*) + \frac{\partial V}{\partial I}(p^*, I_*) \frac{\partial E}{\partial p_j}(p^*, u^*) = 0.$$

By the non-satiation hypothesis $\frac{\partial V}{\partial I} > 0$ and hence using also (1A) gives

$$x_i(p^*, u^*) = -(\partial V / \partial p_i) / (\partial V / \partial I) \big|_{(p^*, I_*)}.$$

But by Roy's Identity, the right-most term is the optimal consumption vector of problem (2.1) given optimal prices p^* and total income I_* .

The remaining case occurs when some p_i^* , $i \in C$ is zero. From non-satiation of $U(\cdot)$, it follows that $M(u^*) = w_{n+1}$ [with a saddle point termed the trivial saddle point determined by $p_j^* = 0$, $j = 1, \dots, n$; $p_{n+1}^* \equiv 1$ and $\hat{y}_k$ satisfying

$$\sum_k \hat{y}_{kj} = \begin{cases} 0, & j \in C \text{ or } j = n+1 \\ -w_j, & j \notin C, j \neq n+1 \end{cases}.$$

This contradicts $M(u^*) = 0$. Therefore, $p_i^* > 0$, all $i \in C$, and hence no resource price can be zero either because producer profits would then be $+\infty$ by assumption (vii). QED

The final step in our plan is to set forth the dual pair of uniextremal problems and prove perfect duality under our consumer and producer assumptions.

5. An Equivalent Uniextremal Duality for the Economy

The basic construction will use supporting hyperplanes of closed convex sets and generalized finite sequences.

For the convex function $h_u(p) = p^T w - E(p, u)$ introduced in (3.1) of Section 3, we characterize its epigraph

$$\{(p, q_0) \mid h_u(p) \leq q_0\}$$

with the tangential supporting hyperplane system:

$$(5.1) \quad q_0 - \sum_{i=1}^n p_i \left(w_i - \frac{\partial E(\gamma, u)}{\partial p_i} \right) \geq \gamma^T w - E(\gamma, u) - \sum_{i=1}^n \gamma_i \left(w_i - \frac{\partial E(\gamma, u)}{\partial p_i} \right)$$

for all γ in an index set U . Since from (1A) of Theorem 1 we have

$$\sum_{i=1}^n p_i \frac{\partial E(p, u)}{\partial p_i} + x_{n+1}(p, u) = E(p, u),$$

it follows that (5.1) simplifies to

$$(5.2) \quad q_0 - \sum_{i=1}^n p_i (w_i - x_i(\gamma, u)) \geq w_{n+1} - x_{n+1}(\gamma, u)$$

for all γ in an index set U .

By assumption (iv) of Section 2.2 each production set Y_k is a closed convex set in $\mathbb{R}^{n+1}$. There will therefore exist a supporting hyperplane representation, and for notational purposes it will be convenient to introduce y_k as the first n components of $y_k \in Y_k$ so that $y_k = (y_k, y_{k,n+1})$. The same convenient convention will apply to other $(n+1)$ -vectors when necessary.

There will exist a supporting hyperplane representation:

$$Y_k = \left\{ (y_k, y_{k,n+1}) \in \mathbb{R}^{n+1} \mid y_k^T d_k(\alpha) + y_{k,n+1} d_{k,n+1}(\alpha) \leq d_{k,n+2}(\alpha), \right.$$

for all α in an index set $\mathcal{R}$ with $d(\alpha) \in \mathbb{R}^n$,

$$\left. d_{k,n+1}(\alpha), d_{k,n+2}(\alpha) \in \mathbb{R} \right\}$$

such that the moment cone in $\mathbb{R}^{n+2}$:

$$M_k \equiv \left\{ \begin{pmatrix} d_k(\alpha) \\ d_{k,n+1}(\alpha) \\ -d_{k,n+2}(\alpha) \end{pmatrix}_\alpha, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\} \text{ is closed in } \mathbb{R}^{n+2}.^\dagger$$

The production set assumptions of Section 2.2 imply that $d_{k,n+2}(\alpha) \geq 0$ for all k and α , and that there is no loss in generality in taking $d_{k,n+1}(\alpha) = 1$. To see this, observe that $0 \in Y_k$ implies $d_{k,n+2}(\alpha) \geq 0$. On the other hand, upon setting $y_k^0 = 0$ in $\mathbb{R}^n$ it follows from assumption (iii) that $(y_k^0, -\tau) \in Y_k$ for $\tau = 1, 2, \dots$. Hence, $-\tau d_{k,n+1}(\alpha) \leq d_{k,n+2}(\alpha)$ for $\tau = 1, 2, \dots$, implying $-d_{k,n+1}(\alpha) \leq 0$. Finally, by assumption (vii) on inadmissibility of vertical hyperplanes, it follows that $d_{k,n+1}(\alpha) \neq 0$ for all k and α , and there is therefore no loss in generality in taking $d_{k,n+1}(\alpha) = 1$ for all k and α .

Following Charnes, Gribik, and Kortanek [9], we now construct the producer's problem I.

[†]Canonically closed representations are sufficient for closure of the moment cone, see Glashoff [11].

Find $M_I(u) =$

$$(5.4) \quad \sup \sum_{k=1}^r y_{k,n+1} + \sum_{\gamma} \{w_{n+1} - x_{n+1}(\gamma, u)\} \lambda(\gamma)$$

from among $p_0 \in \mathbb{R}$, $y_k = (y_k, y_{k,n+1}) \in \mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^n$ and the
generalized finite sequences $\lambda \in \mathbb{R}^{(U)}$, subject to

$$y_k^T d_k(\alpha) + y_{k,n+1} \leq d_{k,n+2}(\alpha),$$

$$\text{all } \alpha \in \mathbb{R}, \quad k = 1, 2, \dots, r$$

$$\xi_i + \sum_{\gamma} \{x_i(\gamma, u) - w_i\} \lambda(\gamma) - \sum_{k=1}^r y_{ki} = 0$$

$$i = 1, 2, \dots, n$$

$$\sum_{\gamma} \lambda(\gamma) = 1$$

and

$$\xi \geq 0, \quad \lambda(\cdot) \geq 0.$$

The consumer's problem II is the following:

Find $M_{II}(u) =$

$$(5.5) \quad \inf q_0 + \sum_{k=1}^r \sum_{\alpha} d_{k,n+2}(\alpha) \tau_k(\alpha)$$

from among $(q_0, p) \in \mathbb{R} \times \mathbb{R}^n$, $\tau_k \in \mathbb{R}^k$, $k = 1, 2, \dots, r$, where
 $p = (p_1, \dots, p_n)$ subject to

$$p \geq 0$$

$$q_0 + \sum_{i=1}^n p_i (x_i(\gamma, u) - w_i) \geq w_{n+1} - x_{n+1}(\gamma, u)$$

for all γ in u

$$-p + \sum_{\alpha} d_k(\alpha) \tau_k(\alpha) = 0, \quad k = 1, 2, \dots, r$$

$$\sum_{\alpha} \tau_k(\alpha) = 1, \quad k = 1, \dots, r$$

and

$$\tau_k \geq 0, \quad k = 1, 2, \dots, r.$$

We now come to our perfect duality theorem for Problems (I) and (II).

Theorem 4 (Perfect Duality). Assume that the consumer and producer assumptions of Section 2 prevail.

Then for any $u > 0$,

$$(5.6) \quad M_I(u) = M_{II}(u) = M(u), \text{ the saddle value of (3.1).}$$

$$(5.7) \quad \text{There exists an optimal solution } \{p_0^*, \{y_k^*\}, \xi^*, \lambda^*\} \text{ for I} \\ \text{and an optimal solution } \{q_0^*, p^*, \{\tau_k^*\}\} \text{ for II and } \{y_k^*\}, p^* \\ \text{is a saddle point for } M(u).$$

Proof. We first apply Theorem 3, (i) of Charnes, Gribik and Kortanek [9] to (I). For the application at hand the following statement must be verified:

$$(5.8) \quad \xi_i - \sum_{k=1}^r (y_k)_i = 0, \quad i = 1, \dots, n$$

and

$$(5.9) \quad y_k \in O^+ Y_k, \quad k = 1, \dots, r$$

$O^+ Y_k$ is the asymptotic or recession cone mentioned in the list of producer assumptions, see [18].

and

$$(5.10) \quad \xi \geq 0$$

IMPLY THAT

$$(5.11) \quad y_k = 0, \quad k = 1, \dots, r.$$

We shall use the producer set assumptions given in Section 2.2.

Let $\{y_k\}$ and ξ satisfy (5.8), (5.9), and (5.10). Since $0 \in Y_k$, it follows that $0^+ Y_k \subseteq Y_k$ and hence $\sum_{k=1}^r 0^+ Y_k \subseteq Y$. This means that $\sum_{k=1}^r y_k$ also is in Y . Since $\xi \geq 0$, (5.8) gives that $\sum_{k=1}^r y_k \geq 0$ also. But by (i) of Section 2.2 we have therefore $\sum_{k=1}^r y_k = 0$. Application now of the recession condition (vi) implies that each $y_k = 0$, namely (5.11) holds.

Therefore, by [9] Theorem 3(i) it follows that $M_I(u) = M_{II}(u) = M(u)$ and that $M_I(u)$ is a maximum.

The next step in the proof is to apply [9] Theorem 3(ii) to prove that $M_{II}(u)$ is a minimum. The key observation is that inequalities involving a recession function can be stated in terms of homogenizations of the supporting hyperplane system of the epigraph of the given function.

It is sufficient for (II) to assume a minimum that:

$$(5.12) \quad \bar{q}_0 + \sum_{i=1}^n \bar{p}_i (x_i(\gamma, u) - w_i) \geq 0 \quad \text{for all } \gamma \in u$$

and

$$(5.13) \quad \bar{p}_k = \sum_{\alpha} d_k(\alpha) \bar{\tau}_k(\alpha), \quad k = 1, \dots, r$$

and

$$(5.14) \quad \sum_{\alpha} \bar{\tau}_k(\alpha) = 0, \quad k = 1, \dots, r,$$

(where each $\bar{\tau}_k(\cdot) \geq 0$, see (5.5)) and

$$(5.15) \quad \bar{q}_0 \leq - \sum_{\alpha} \sum_{k=1}^r d_{k,n+2}(\alpha) \bar{\tau}_k(\alpha)$$

IMPLIES THAT

$$(5.16) \quad \bar{p} = 0 \quad \text{and} \quad \bar{q}_0 = 0.$$

From (5.14) we have immediately that each $\bar{\tau}_k(\cdot) \equiv 0$ and hence $\bar{p} = 0$ by (5.13), and $\bar{q}_0 \geq 0$ by (5.12). As we have already observed in Section 5, $d_{k,n+2}(\alpha) \geq 0$ and hence by (5.15), $\bar{q}_0 \leq 0$. Since $\bar{q}_0 \geq 0$ has been established it therefore follows that $\bar{q}_0 = 0$, proving that (5.16) holds.

Finally by [9] Theorem 3 we have

$$M(u) = x^{*T} A p^* = p^{*T} \left(\sum_{k=1}^r y_k^* + w \right) - E(p^*, u),$$

using the notation in (3.2) of Section 3, proving the final saddle point statement of the theorem.

We turn now to economic interpretations of the dual programs I and II.

6. Economic Interpretation of the Dual Uniextremizations

Production Sector I

The production plan variables $\{y_k\}_{k=1}^r$ are now augmented by generalized finite sequences stemming from the consumer's expenditure function, which is a function of the price vector for a given, fixed utility level. In this sense the input to the producers' problem I is the expenditure function per se. However, no knowledge is required whatever about the consumer's problem II decisions, or even ranges of decision.

In addition to the finite sequences $\lambda(\cdot)$ there is also a finite list of variables $\{\xi_i\}_{i=1}^n$. These new additions to Program I can be interpreted with the help of the first set of constraining equations of (I) rewritten, slightly as follows:

$$(6.1) \quad -\xi_i + \sum_{k=1}^r y_{ki} = \sum_{\gamma} (x_i(\gamma, u) - w_i) \lambda(\gamma), \quad i = 1, \dots, n.$$

Since $\xi \geq 0$ and $y_k \in Y_k$, we see that the vector on the left in (6.1) lies in the aggregate production set $Y = \sum_{k=1}^r Y_k$ because of (ii) in Section 2.2. We shall therefore term ξ a translation vector which translates a given production plan to a production plan having outputs no larger and inputs no smaller.

Now, observing in (I) that $\sum_{\gamma} \lambda(\gamma) = 1$ where γ is an index vector ranging over prices, we render $\lambda(\cdot)$ the following interpretation.

$\lambda(\cdot)$ is a system of non-negative weighting variables, indexed with prices and summing to 1, applied to demands or supplies of all goods except numeraire less their initial endowments. Then, (6.1)

states that a weighted combination of the demands for goods other than numeraire less initial endowments lie in the aggregated production set.

Consumer's Problem II

The initial price variables of the consumer p are now augmented by generalized finite sequences determined by each of the production sets Y_k , $k = 1, \dots, r$. Thus, the additional data required by the consumer are the production sets themselves or equivalently supporting hyperplane representations of them. Nevertheless, Program II requires no knowledge of the choices of production plans which the producers will make. [The q_0 variable is merely the value of the initial endowments less the expenditure function for any p , and hence can be ignored.]

Now the first set of equations in the consumer's program II states that the price vector should be a convex combination of vectors normal to the frontier of the production set Y_k for $k = 1, \dots, r$. Thus, we term the $\tau_k(\cdot)$ functions as production frontier weights indexed by frontier production plans of producers k , where these themselves are points on the boundary of the closed convex production set Y_k .

In summary, each variable in the producers' problem I and the consumer's problem II has its own interpretation for any value it assumes, in particular non-optimal ones.

Objective Function for Each Player

At dual optimal solutions each player's objective function equals $M(u)$ given in (3.1). Basically, each player takes this joint expression as objective function, but from differing viewpoints according to his own problem constraints. In a generalized sense, with appropriate names yet to be determined, $M(u)$ is the sum of producer's surplus and consumer's surplus and the excess demand for the numeraire good.

In higher dimensions consumer surplus is defined with respect to those prices which vary, which in our case is simply the commodity price vector $p = (p_1, p_2, \dots, p_n)$. When prices are uniquely determined as functions of quantities and vice versa, then one can use the higher dimensional line integral calculus in a neoclassical setting to demonstrate the decomposition of $M(u)$ into the sum of surpluses and excess numeraire demand.

Under this interpretation the producer wishes to maximize this joint sum provided that what he actually produces will be demanded (as discussed more precisely above). On the other hand, the consumer wishes to minimize this joint sum provided that his price vector is normal to the production frontier. It appears that our general formulation thus embraces the important case in economic theory where integrability and invertibility assumptions prevail.

APPENDIX: PROOF OF THEOREM 2, SECTION 4

Most of the work has already been done because of the implications that have been proved in our perfect duality Theorem 4. We shall refer to (a) and (b) of Theorem 37.3 of Rockafellar, specialized to our saddle function given in (3.2) namely for fixed u

$$K_u(x, p) = x^T A p + h_u(p)$$

where

$$x^T = (x_1^T, \dots, x_n^T) \in (Y_1, \dots, Y_r) \equiv U$$

$$h_u(p) = p^T w - E(p, u), \quad p \geq 0$$

and

$$A = [I_{n+1}, \dots, I_{n+1}], \quad r \text{ times,}$$

$$I_{n+1} = (n+1) \text{ by } (n+1) \text{ identity matrix.}$$

For this special case we use the equivalence established in Charnes, Gribik, and Kortanek [9] between condition (i) of Theorem 3 there with condition (b) of Theorem 37.3 [18], and the equivalence between condition (ii) of Theorem 3 [9] with condition (a) of Theorem 37.3 [18]. Since Theorem 4 itself in Section 5 has established that conditions (i) and (ii) [9] are true, it therefore follows that the Rockafellar [18] conditions (a) and (b) hold, establishing existence of a saddle point for all $u \geq 0$.

In relation to other minimax theorems Rockafellar [18] points out on page 431, that the sharpest results in Sion [20] require less

than concavity-convexity of the saddle function but require a compactness assumption. In contrast Theorem 37.3 and the hypotheses of our uniextremal duality require concavity-convexity but no compactness assumption.

The remaining task is to establish continuity of $M(u)$, $u \geq 0$. We shall use the existence of a saddle point for every $u \geq 0$, saddle value inequalities, and some of the production set assumptions of Section 2.2. We have been aided by Gol'stein's book [12].

Continuity of M on $\{u | u \geq 0\}$

Let $\bar{u}$ be fixed, $\bar{u} \geq 0$.

Part I. (Establishing $\lim_{u \rightarrow \bar{u}} M(u) \geq M(\bar{u})$).

Suppose to the contrary that there exists a sequence $\{u_\ell\}$, $\lim_{\ell} u_\ell = \bar{u}$ such that

$$(A1) \quad \sup_y \inf_p K_{u_\ell}(y, p) < M(u) - \delta \quad \text{for all } \ell$$

for some $\delta > 0$. Let $P = \{p = (p, 1) | p \in \mathbb{R}^n, p \geq 0\}$ and $\mathcal{Y} = \{y = (y, \dots, y_r) | \text{each } y_k \in Y_k\}$. Recall that $Y = \sum_k Y_k$, see Section 2.2.

Let $(y^\ell, p_\ell) \in \mathcal{Y} \times P$ denote a saddle point associated with u_ℓ for all ℓ . Then the standard string of saddle value inequalities is:

$$(A.2) \quad K_{u_\ell}(y, p_\ell) \leq K_{u_\ell}(y^\ell, p_\ell) \leq K_{u_\ell}(y^\ell, p)$$

for all $y \in \mathcal{Y}$ and all $p \in P$.

We first establish that the sequence $\{p_\ell\}$ must be bounded solely because of (A.2) and our consumer non-satiation hypothesis.[†] There is

[†]This fact will also be used in Part II of the proof.

nothing to prove if all prices are not positive because trivial saddle points may then be used in (A.2). Suppose to the contrary that $\|p_\ell\| = \|p_\ell\| + 1 \rightarrow \infty$ as $\ell \rightarrow \infty$ and all prices are positive. Let $\theta_\ell = \|p_\ell\| + 1$. Examining (2.2) establishes the well-known homogeneity property of the expenditure function, in particular,

$$(A.3) \quad E(\theta_\ell^{-1} p_\ell, u_\ell) = \min\{\theta_\ell^{-1} p_\ell x \mid u(x) \geq u_\ell\}$$

has the same optimal solution (2.2), and therefore $x_\ell(p_\ell, u_\ell) = \nabla_p E(p_\ell, u_\ell)$ together with $x_{n+1}(p_\ell, u_\ell)$ also solves (A.3).

Because of the second order differentiability conditions on the utility function U , (A.3) is a stable program in the sense of continuity, namely if $\theta_\ell^{-1} p_\ell \rightarrow \bar{p}$, then there is a limit point $\bar{x}$ of $\{x_\ell(p_\ell, u_\ell)\}$ and $\bar{x}$ is an optimal solution to $E(\bar{p}, \bar{u})$. Without loss of generality, we assume that $\{\theta_\ell^{-1} p_\ell\}$ converges to a point, say $\bar{p}$ in $\mathbb{R}^{n+1}$.

Observe that since $p_\ell = (p_\ell, 1)$ it follows that $\bar{p} = (\bar{p}, 0)$ and $\|\bar{p}\| = 1$.

Let $y \in Y$. Rewriting the first inequality of (A.2) in detail gives

$$(A.4) \quad p_\ell^T \sum_k y_k + p_\ell^T w - E(p_\ell, u_\ell) \leq p_\ell^T \sum_k y_k^\ell + p_\ell^T w - E(p_\ell, w_\ell).$$

Multiplying (A.4) by θ_ℓ^{-1} yields

$$(A.5) \quad \theta_\ell^{-1} p_\ell^T \sum_k y_k + \theta_\ell^{-1} p_\ell^T w - E(\theta_\ell^{-1} p_\ell, u_\ell) \leq \theta_\ell^{-1} p_\ell^T \sum_k y_k^\ell - E(\theta_\ell^{-1} p_\ell, u_\ell),$$

using also the homogeneity of E with respect to prices. Since each (y^ℓ, p_ℓ) is a saddle point, Corollary 3 applies to simplify the right-hand side of (A.5), namely (A.5) becomes

$$(A.6) \quad \theta_\ell^{-1} p_\ell^T \sum_k y_k + \theta_\ell^{-1} p_\ell^T w - E(\theta_\ell^{-1} p_\ell, u_\ell) \leq \theta_\ell^{-1} \sum_k y_{k,n+1}^\ell + \theta_\ell^{-1} w_{n+1} - x_{n+1}(\theta_\ell^{-1} p_\ell, u_\ell).$$

Applying the "lim sup" to both sides of (A.6) and inserting limits where possible and observing that $\sum_k y_{k,n+1}^l \leq 0$ each l , gives the limiting inequality,

$$(A.7) \quad \bar{p}^T \sum_k x_k + \bar{p}^T w - E(\bar{p}, \bar{u}) \leq -x_{n+1}(\bar{p}, \bar{u}).$$

(A.7) has several implications.

(A.8.1) The case $\bar{p}_i = 0$ for some consumption good i is impossible. Otherwise, by non-satiation and (1B) of Theorem 1, $u_l = V(p_l, E(p_l, u_l))$ for each l implies

$$\bar{u} = V(\bar{p}, E(\bar{p}, \bar{u})) = +\infty,$$

which is absurd. Therefore, for each consumption good i , $\bar{p}_i > 0$.

(A.8.2) Since, as we have observed, $\bar{p}_{n+1} = 0$, the numeraire can be input without bound at no cost to the producers. Hence by assumption (vi) of Section 2.2 any producer can obtain unbounded outputs of any consumption good i with $\bar{p}_i > 0$. This clearly contradicts (A.7) for now the left-hand side grows without bound.

We have therefore established that $\{p_l\}$ is bounded because of existence of saddle values and our consumer-producer assumptions. Without loss of generality we may assume that the price vector sequence converges in general, namely $\lim_l p_l = \bar{p}$.

However, in this situation, it follows from (A.2) that for

each $y \in Y$ †

$$K_{u_\ell}(y, p_\ell) < M(\bar{u}) - \delta \quad \text{for all } \ell.$$

Upon taking limits we obtain: $K_{\bar{u}}(y, \bar{p}) \leq M(\bar{u}) - \delta$, and hence
 $\sup_y K_{\bar{u}}(y, \bar{p}) \leq M(\bar{u}) - \delta$. We now use the saddle value property in combination with this last inequality, namely,

$$\begin{aligned} M(\bar{u}) &= \sup_y \inf_p K_{\bar{u}}(y, p) = \inf_p \sup_y K_{\bar{u}}(y, p) \\ &\leq \sup_y K_{\bar{u}}(y, \bar{p}) \leq M(\bar{u}) - \delta, \end{aligned}$$

which is an absurdity, since $\delta > 0$.

In summary, then assumption (A.1) has led to a contradiction and therefore we have

$$(A.9) \quad \lim_{u \rightarrow \bar{u}} M(u) \geq M(\bar{u}).$$

Part II. (Establishing $\lim_{u \rightarrow \bar{u}} M(u) \leq M(\bar{u})$).

Suppose to the contrary that there exists a sequence $\{u_\ell\}$, $u_\ell \rightarrow \bar{u}$ such that

$$(A.10) \quad \inf_p \sup_y K_{u_\ell}(y, p) > M(\bar{u}) + \delta$$

for some $\delta > 0$.

Again, we refer to the standard string of saddle value inequalities (A.2) and consider two cases.

†We are following Gol'stein's argument [12], page 48-49.

Case 1. $\sum_k y_k^\ell \rightarrow \bar{z}$.

Since $K_u(y, p)$ depends on y only through $\sum_k y_k$, let us abuse notation for convenience by writing $\bar{z}$ in place of y .

Then for each $p \in P$,

$$M(\bar{u}) + \delta < K_{u_\ell}(y^\ell, p_\ell) \leq K_{u_\ell}(y^\ell, p)$$

implying

$$M(\bar{u}) + \delta \leq K_{\bar{u}}(\bar{z}, p) \text{ for all } p$$

which in turn implies that

$$M(\bar{u}) + \delta \leq \inf_p K_{\bar{u}}(\bar{z}, p).$$

Arguing as before, we obtain a contradiction, namely,

$$M(\bar{u}) = \inf_p \sup_y K_{\bar{u}}(y, p) = \sup_y \inf_p K_{\bar{u}}(y, p) \geq \inf_p K_{\bar{u}}(\bar{y}, p) \geq M(\bar{u}) + \delta.$$

Therefore $\{\sum_k y_k^\ell\}_\ell$ cannot be bounded.

Case 2. $c^\ell \equiv \sum_k y_k^\ell$ and $\|\sigma^\ell\| \rightarrow +\infty$.

Upon dividing (A.2) by σ^ℓ and examining the right-most inequality we obtain:

$$(A.11) \quad \frac{p_\ell^T \sum_k y_k^\ell}{\|\sigma^\ell\|} + \frac{p_{\ell w}^T}{\|\sigma^\ell\|} - \frac{E(p_\ell, u_\ell)}{\|\sigma^\ell\|} \leq \frac{p^T \sum_k y_k^\ell}{\|\sigma^\ell\|} + \frac{p_{\ell w}^T}{\|\sigma^\ell\|} - \frac{E(p, u_\ell)}{\|\sigma^\ell\|}$$

for each $p \in P$.

It has already been established in Part I (in general) that $\{p_\ell\}$ must be bounded, and so without loss of generality we assume that $\lim_\ell p_\ell = \bar{p} \in P$. Similarly, we may assume

$$\lim_{\ell} \frac{\sum_k y_k^\ell}{\|\sigma^\ell\|} = \hat{z} \quad \text{with } \|\hat{z}\| = 1.$$

We are now prepared to examine the limiting inequality of (A.11) as $\ell \rightarrow \infty$.

For each $p \in P$ it follows that

$$(A.12) \quad \overline{p}^T \hat{z} \leq p^T \hat{z},$$

using the facts that $E(0, \bar{u}) = 0$, $\{p_\ell\}$, and $\{u_\ell\}$ converge.

It follows for a net resource j that $\hat{z}_j = 0$. Otherwise, if $\hat{z}_j < 0$ we may take $p_j \uparrow +\infty$ contradicting (A.12).

Now at each saddle point (y^ℓ, p_ℓ) in (U, P) we have (as usual):

$$\nabla_{\bar{p}} E(p_\ell, u_\ell) = \sum_k y_k^\ell,$$

and hence

$$\frac{1}{\|\sigma^\ell\|} \nabla_{\bar{p}} E(p_\ell, u_\ell) = \frac{\sum_k y_k^\ell}{\|\sigma^\ell\|}.$$

Taking limits of both sides yields

$$0 = \hat{z}.$$

Therefore, $\hat{z} \equiv (\hat{z}, z_{n+1}) = (0, 0)$, since in particular $n+1$ is also a resource. This contradicts $\|\hat{z}\| = 1$, and therefore Case 2 is impossible. Hence (A.10) leads to a contradiction, and therefore

$$(A.13) \quad \overline{\lim}_{u \rightarrow \bar{u}} M(u) \leq M(\bar{u}).$$

Combining (A.9) and (A.13) gives

$$M(\bar{u}) \leq \liminf_{u \rightarrow \bar{u}} M(u) \leq \limsup_{u \rightarrow \bar{u}} M(u) \leq M(\bar{u}).$$

Hence, $\lim_{u \rightarrow \bar{u}} M(u) = M(\bar{u})$ establishing continuity.

Remarks. (1) Much shorter and elegant proofs are immediate if one assumes that the solution sets

$$\{y \mid \inf_p K_u(y, p) = M(u)\}$$

and

$$\{p \mid \sup_y K_u(y, p) = M(u)\}$$

are bounded. (In particular, one could obtain differentiability of $M(u)$.) See Gol'stein [12], Section 7.

We could have used these assumptions together with regularization or bounding procedures for the production set U and the price set P . This route, however, would require more complicated dual programs than those employed in our approach.

(2) Professor Jonathan Borwein has indicated how the hypotheses of the main theorems in Charnes, Gribik, and Kortanek [9] may be weakened by using the sharper properties of proper separation of convex sets rather than solely separation alone.

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The saddle value problem is shown to be equivalent to a dual pair of uniextremizations termed the consumer group's problem and the producers' problem. The neoclassical economic assumptions yield sufficient conditions which are among the most general ones for guaranteeing a saddle point and simultaneously a perfect duality for the dual programming pair. Economic interpretations are given for all the variables of the consumer group's problem and for all the variables of the producers' problem even at non-optimal stages in each problem. The approach is an infinite dimensional extension of the Charnes' constrained game linear programming equivalents in finite dimensions.

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