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by
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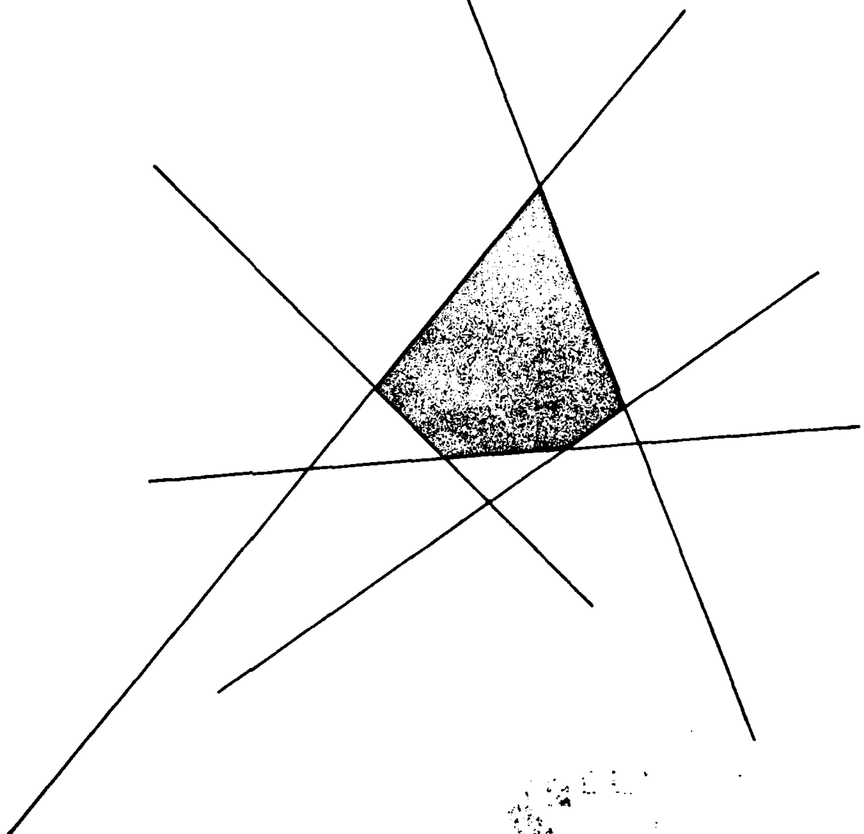

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CONDITIONALLY INCREASING PROCESSES

by

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NOVEMBER 1980 ORC 80-24

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ABSTRACT

The conditional distribution of $X(t)$ given $\sup_{0\le u\le t} X(u) < a$ of certain processes is studied. We use it to prove the IFR property of k-out-of-n systems and certain shock models and other processes.

CONDITIONALLY INCREASING PROCESSES

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0. INTRODUCTION AND SUMMARY

Let $\{X(t) : t \geq 0\}$ denote a stochastic process such that $X(t)$ is increasing stochastically in t. We are interested in studying the conditional distribution of $X(t)$ given that $\left\{\max_{0\leq u\leq t} X(u) < a\right\}$. In particular, we are interested in processes whose conditional distributions $P\left(X(t) < s \mid \max_{0\leq u\leq t} X(u) < a\right)$ are monotone in t. A nonnegative stochastic process $\{X(t) : t \geq 0\}$ is said to be IFR if for any $a > 0$, $\tau_a = \inf\{t : X(t) < a\}$ has an IFR distribution. IFR processes are used in reliability theory to model cumulative damage processes, wear and fatigue of materials. The above definition of IFR process is equivalent to saying that for any $a > 0$, $x > 0$, $P\left(\max_{0\leq u\leq t+x} X(u) < a \mid \max_{0\leq u\leq t} X(u) < a\right)$ is decreasing in t. If the process happens to be of independent increments, then to determine whether the given process is IFR is basically to determine the behavior of $X(t)$ given $\max_{0\leq u\leq t} X(u) < a$.

In this report we study the conditional distribution of $X(t)$ given $\left\{\max_{0\leq u\leq t} X(u) < a\right\}$ of certain processes which are widely used in reliability theory. In Section 1 we show, using this approach, that k-out-of-n system of i.i.d. IFR components has an IFR lifetime. In Section 2 we state conditions under which the homogeneous (nonhomogeneous) compound Poisson process is IFR. It should be mentioned here that most of the above results are not new but all of them were obtained using purely analytical methods. The advantage of our method here is that it provides a stronger intuitive insight.

1. k-out-of-n SYSTEM

In this section we consider a coherent system (c,ϕ) which is composed of n components (for definitions and basic properties consult Barlow-Proschan (1975)), and let $h(\cdot)$ denote its reliability function. The components lifetimes $T_1, T_2, \ldots, T_n$ are nonnegative i.i.d. random variables having distribution F $(\bar{F} = 1 - F)$ and density $F' = f$. Let $X_i(t)$, $i = 1, \ldots, n$ be 0 or 1 according to whether $\{T_i \le t\}$ or $\{T_i > t\}$ respectively and let $\underline{X}(t) = (X_1(t), \ldots, X_n(t))$ be the state vector at time t. Let $N_t = \sum_{i=1}^{n} X_i(t)$ denote the number of functioning components at time t. Then for an arbitrary coherent structure ϕ, we have

Lemma 1:

For any k, $(k = 0,1, \ldots, N)$

$$E(N_t \mid N_t \ge k , \phi(\underline{X}(t)) = 1) \ge E(N_t \mid \phi(\underline{X}(t)) = 1) .$$

Proof:

Check that for $j = 1, \ldots, n$

$$P(N_t \ge j \mid N_t \ge k , \phi(\underline{X}(t)) = 1) \ge P(N_t \ge j \mid \phi(\underline{X}(t)) = 1) . \qquad \text{Q.E.D.}$$

Lemma 2:

$$E(N_t \mid \phi(\underline{X}(t)) = 1) = \frac{\bar{F}(t) \sum_{j=1}^{n} h(1_j, \bar{F}(t))}{h(\bar{F}(t))}$$

where $h(1_j, \bar{F}(t)) = E(\phi(X_1(t), \ldots, X_{j-1}(t), 1, X_{j+1}(t), \ldots, X_n(t))$.

Proof:

$$E(N_t \mid \phi(\underline{X}(t) = 1) = \sum_{i=1}^{n} E[X_i(t) \mid \phi(\underline{X}(t)) = 1]$$

$$= \frac{\sum_{i=1}^{n} P(X_i(t) = 1 \;,\; \phi(\underline{X}(t)) = 1)}{P(\phi(\underline{X}(t)) = 1)}$$

$$= \frac{\sum_{i=1}^{n} P(X_i(t) = 1 \;,\; \phi(X_i(t), \ldots, X_{i-1}(t), 1, X_{i+1}(t), \ldots, X_n(t) = 1)}{h(\bar{F}(t))}$$

and by the independent assumption this equals

$$\frac{\sum_{i=1}^{n} P(X_i(t) = 1) P(\phi(X_1(t), \ldots, X_{i-1}(t), 1, X_{i+1}(t), \ldots, X_n(t)) = 1)}{h(\bar{F}(t))}$$

$$= \frac{\bar{F}(t) \sum_{i=1}^{n} h(1_i, \bar{F}(t))}{h(\bar{F}(t))} \; .$$

Q.E.D.

Lemma 3:

$$h'(p) = \sum_{i=1}^{n} h(1_i, p) - \sum_{i=1}^{n} h(0_i, p) \; .$$

Proof:

See Barlow and Proschan (1975). Q.E.D.

The following is the main theorem of this section.

Theorem 1:

Let ϕ be an arbitrary coherent structure, then for any k $(k = 0, \ldots, n)$:

$P(N_t \geq k \mid \phi(\underline{X}(t)) = 1)$ is nonincreasing in t .

Loosely speaking, given the system is up at t , the number of functioning components is stochastically decreasing in t .

<u>Proof</u>:

$P(N_t \geq k \mid \phi(\underline{X}(t)) = 1)$ can be written in the following way:

$$\frac{\sum_{j=k}^{n} c_j [\bar{F}(t)]^j [F(t)]^{n-j}}{h(\bar{F}(t))}$$

where c_j is the number of distinct ϕ-path sets of size j . Thus:

$$\frac{d}{dt} P(N_t \geq k \mid \phi(\underline{X}(t)) = 1)$$

$$= \frac{d}{dt} \frac{\sum_{j=k}^{n} c_j [\bar{F}(t)]^j [F(t)]^{n-j}}{h(\bar{F}(t))}$$

$$= \frac{1}{h^2(\bar{F}(t))} \left\{ \frac{h(\bar{F}(t)) f(t)}{\bar{F}(t) F(t)} P(N_t \geq k , \phi(\underline{X}(t)) = 1) \times \right.$$

$$(E[N_t] - E[N_t \mid N_t \geq k , \phi(\underline{X}(t)) = 1]$$

$$\left. + f(t) h'(\bar{F}(t)) P(N_t \geq k , \phi(\underline{X}(t)) = 1) \right\} .$$

This is nonpositive for $t \geq 0$ iff

$$(*) \quad \frac{h(\bar{F}(t))}{\bar{F}(t) F(t)} (E[N_t] - E[N_t \mid N_t \geq k , \phi(\underline{X}(t)) = 1]) + h'(\bar{F}(t)) \leq 0 .$$

By Lemma 1,

$$(*) \leq \frac{h(\bar{F}(t))}{\bar{F}(t) F(t)} (E[N_t] - E[N_t \mid \phi(\underline{X}(t)) = 1]) + h'(\bar{F}(t))$$

which by Lemmas 2, 3 is equal to:

$$= \frac{h(\bar{F}(t))}{\bar{F}(t)F(t)} \left[n\bar{F}(t) - \frac{\bar{F}(t) \sum_{i=1}^{n} h(1_i, \bar{F}(t))}{h(\bar{F}(t))} \right]$$

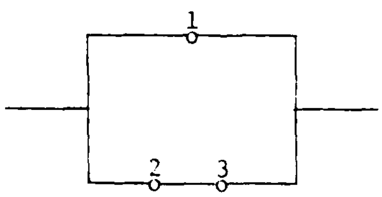
$$+ \sum_{i=1}^{n} h(1_i, \bar{F}(t)) - \sum_{i=1}^{n} h(0_i, \bar{F}(t))$$

$$= \frac{1}{F(t)} \left[nh(\bar{F}(t)) - \bar{F}(t) \sum_{i=1}^{n} h(1_i, \bar{F}(t)) - F(t) \sum_{i=1}^{n} h(0_i, \bar{F}(t)) \right]$$

$$= \frac{1}{F(t)} [nh(\bar{F}(t)) - nh(\bar{F}(t))] = 0 .$$

Thus $\frac{d}{dt} P(N_t \geq k \mid \phi(\underline{X}(t)) = 1) \leq 0$. Q.E.D.

Note that even though the unconditional process N_t is decreasing stochastically (in fact, decreasing w.p.1) it is not always true for the process conditional on $\phi(\underline{X}(t)) = 1$. Indeed, for the nonidentical component case this is not true. The following example illustrates it. Let the system be



Its structure function is $\phi(\underline{X}) = \max \{X_1, X_2, X_3\}$ and suppose the lifetimes are independent and distributed uniformly on (0,1) and (0,2) (0,2) respectively, then check

$$P(N_t \geq 2 \mid \phi(\underline{X}(t)) = 1) < 1 \quad \text{for } 0 < t < 1$$
$$= 1 \quad \text{for } 1 < t < 2 .$$

Definition:

A structure ϕ is called k-out-of-n iff $\phi(\underline{X}) = 1$ or 0 according to whether $\sum_{i=1}^{n} X_i \geq k$ or $< k$ respectively.

Corollary 1:

If the component lifetime is IFR, then the k-out-of-n system has an IFR distribution.

Proof:

We have to show that for $x > 0$

$$P(N_{t+x} \geq k \mid N_t \geq k) \text{ is decreasing in } t .$$

Let

$$S(r,t,x) = \sum_{j=k}^{r} \binom{r}{j} \left[\frac{\bar{F}(t+x)}{\bar{F}(t)}\right]^{j} \left[1 - \frac{\bar{F}(t+x)}{\bar{F}(t)}\right]^{r-j}$$

where $r \geq k$, $x > 0$ and $F(\cdot)$ is the component lifetime distribution function, which is assumed to be IFR. Thus for fixed $x > 0$, $S(r,t,x)$ is increasing in r and decreasing in t and

$$P(N_{t+x} \geq k \mid N_t \geq k) = E(S(N_t,t,x) \mid N_t \geq k) .$$

By Theorem 1, given $N_t \geq k$, N_t is stochastically decreasing in t , which implies

$$E(S(N_t,t,x) \mid N_t \geq k) \text{ is decreasing in } t . \qquad \text{Q.E.D.}$$

Remark:

Corollary 1 has been previously proven but the above argument yields more probabilistic insight.

2. RENEWAL AND OTHER RELATED PROCESSES

In this section we consider two processes, the renewal process and the compound Poisson process. For both processes we state conditions under which each process is stochastically increasing in t, conditional on the event that the process has not exceeded some threshold point by time t.

We start with the renewal process and state the following known lemma (see Barlow and Proschan (1965)).

Lemma 4:

If X_1, X_2 are independent IFR random variables with failure rate functions λ_1, λ_2 respectively, then λ the failure rate function of $X_1 + X_2$ satisfies:

$$\lambda(t) \leq \min\{\lambda_1(t), \lambda_2(t)\} \quad t \geq 0 .$$

Theorem 2:

If the interarrival times $X_1, X_2, \ldots$ are IFR and possess a density (i.e., failure rate function exists) then the correspondent renewal process $\{N_t : t \geq 0\}$ satisfies:

$$P(N_t < n \mid N_t < n+m) \text{ is decreasing in } t \text{ for } n, m$$

positive integers.

Proof:

Denote the failure rate function of $X_1 + \ldots + X_n$ by λ_n. Now since IFR is closed under convolution, by the previous lemma we have that λ_n is decreasing in n. Now

$$P(X_t < n \mid N_t < n+m) = \frac{P(X_1 + \ldots + X_n > t)}{P(X_1 + \ldots + X_{n+m} > t)}$$

$$= e^{-\int_0^t (\lambda_n(u) - \lambda_{n+m}(u))\,du}$$

which is decreasing in t . Q.E.D.

For a more general renewal process, Theorem 2 does not necessarily hold. The following example illustrates it. $X_1, X_2, \ldots$ are the inter-arrival times and suppose

$$P(X_1 = 1) = 1 - P(X_1 = 10) = p .$$

Thus

$$P(N_t \geq 2 \mid N_t < 3) = \begin{cases} p^2 & 2 < t < 3 \\ \dfrac{p^2(1-p)}{1-p^3} & 3 < t < 4 \end{cases}$$

and for $0 < p < 1$, $p^2 > \dfrac{p^2(1-p)}{1-p^3}$.

<u>Corollary 2</u>:

If $\{N_t : t \geq 0\}$ is nonhomogeneous Poisson process, then for n , m positive integers,

$$P(N_t < n \mid N_t < m+n) \text{ is decreasing in } t .$$

<u>Proof</u>:

$\{N_t\}$ can be viewed as a homogeneous Poisson process with a proper change of time-scale.

Next we consider a compound Poisson process. Let $\{N_t : t \geq 0\}$ be a homogeneous Poisson process with intensity $\lambda > 0$ and let $X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables independent of $\{N_t\}$ and distributed according to F. The process

$$Y(t) = \begin{cases} X_1 + \ldots + X_{N_t} & \text{if } N_t > 0 \\ 0 & \text{if } N_t = 0 \end{cases}$$

is called a compound Poisson process. Esary, Marshall and Proschan (1973) studied this process and derived, among other things, conditions which make the process IFR. (A nonnegative process Y is IFR iff for any $a > 0$, $T_a = \inf\{t : Y_t \geq a\}$ has an IFR distribution.) Their approach is purely analytical and they derive the conditions using total positivity theory. Our approach here is more probabilistic and as such, provides more insight. We start with the following theorem:

<u>Theorem 3</u>:

For any $k \geq 0$, $c \geq 0$:

$$P(N_t \geq k \mid Y(t) \leq c) \text{ is increasing in } t .$$

<u>Proof</u>:

$$P(N_t \geq k \mid Y(t) \leq c) = \frac{\sum_{j=k}^{\infty} (\lambda t)^j/j!F^{*j}(c)}{\sum_{j=0}^{\infty} (\lambda t)^j/j!F^{*j}(c)}$$

where F^{*j} denotes the j^{th} fold convolution of F. The proof is by induction on k.

For $k = 1$, check

$$\frac{\partial}{\partial t} \frac{\sum_{j=1}^{\infty} (\lambda t)^j/j!F^{*j}(c)}{1 + \sum_{j=1}^{\infty} (\lambda t)^j/j!F^{*j}(c)} \geq 0 .$$

Now assume it holds for $k = n$, and to show that it holds for $k = n + 1$

$$\frac{\sum_{j=n+1}^{\infty} (\lambda t)^j/j!F^{*j}(c)}{\sum_{j=0}^{\infty} (\lambda t)^j/j!F^{*j}(c)} = \frac{\sum_{j=n}^{\infty}}{\sum_{j=0}^{\infty}} \frac{\sum_{j=n+1}^{\infty}}{\sum_{j=n}^{\infty}} .$$

By the induction hypothesis it suffices to show

$$\frac{\partial}{\partial t} \frac{\left(\sum_{j=n+1}^{\infty}\right)}{\left(\sum_{j=n}^{\infty}\right)} \geq 0$$

but

$$\frac{\partial}{\partial t} \frac{\left(\sum_{j=n+1}^{\infty}\right)}{\left(\sum_{j=n}^{\infty}\right)} = \frac{\left(\frac{\partial}{\partial t} \sum_{j=n+1}^{\infty}\right)\left(\sum_{j=n}^{\infty}\right) - \left(\frac{\partial}{\partial t} \sum_{j=n}^{\infty}\right)\left(\sum_{j=n+1}^{\infty}\right)}{\left(\sum_{j=n}^{\infty}\right)^2} .$$

It is easy to check that the numerator is nonnegative iff:

$$\frac{t}{n}\left(\frac{\partial}{\partial t} \sum_{j=n+1}^{\infty} (\lambda t)^j/j!F^{*j}(c)\right) - \left(\sum_{j=n+1}^{\infty} (\lambda t)^j/j!F^{*j}(c)\right) \geq 0$$

which is true since it is equal to

$$[(\lambda t)^{n+1}/nn!F^{*(n+1)}(c) + (\lambda t)^{n+2}/n(n+1)!F^{*(n+2)}(c) + \ldots]$$

$$- [(\lambda t)^{n+1}/(n+1)!F^{*(n+1)}(c) + (\lambda t)^{n+2}/(n+2)!F^{*(n+2)}(c) + \ldots] \geq 0 \text{ . Q.E.D.}$$

Corollary 3:

Theorem 3 holds even when $Y(t)$ is nonhomogeneous compound Poisson. We also need the following well known lemma.

Lemma 5:

If the distribution function F is PF_2, i.e., if for any $x_1 < x_2$, $y_1 < y_2$:

$$\begin{vmatrix} F(x_1 - y_1) & F(x_1 - y_2) \\ F(x_2 - y_1) & F(x_2 - y_2) \end{vmatrix} \geq 0$$

then for any $x_1 < x_2$ and $n \geq 0$

$$F^{*n}(x_1)F^{*(n+1)}(x_2) \geq F^{*n}(x_2)F^{*(n+1)}(x_1) .$$

Proof:

See Esary-Marshall-Proschan (1973), Theorem 4.9. The above states that if a sequence of i.i.d. random variables $X_1, X_2, \ldots$ have a PF_2 distribution F, then for any $x_1 < x_2$:

$$P(X_1 + \ldots + X_n \leq x_1 \mid X_1 + \ldots + X_n \leq x_2) \text{ is decreasing in } n .$$

Corollary 4:

If F is PF_2, then for $x_1 < x_2$

$$P(Y(t) \leq x_1 \mid Y(t) \leq x_2) \text{ is decreasing in } t .$$

Proof:

$$P(Y(t) \le x_1 \mid Y(t) \le x_2)$$

$$= \sum_{n=0}^{\infty} P(Y(t) \le x_1 \mid Y(t) \le x_2 , N_t = n)P(N_t = n \mid Y(t) \le x_2)$$

$$= \sum_{n=0}^{\infty} P(X_1 + \ldots + X_n \le x_1 \mid X_1 + \ldots + X_n \le x_2)P(N_t = n \mid Y(t) \le x_2)$$

given $Y(t) \le x_2$, N_t is stochastically increasing in t and $P(X_1 + \ldots + X_n \le x_1 \mid X_1 + \ldots + X_n \le x_2)$ is decreasing in n and hence

$$\sum_{n=0}^{\infty} P(X_1 + \ldots + X_n \le x_1 \mid X_1 + \ldots + X_n \le x_2)P(N_t = n \mid Y(t) \le x_2)$$

is decreasing in t .

Corollary 5:

If F is PF_2 then the compound Poisson process Y is IFR.

Corollary 6:

Suppose Y is nonhomogeneous compound Poisson with intensity $\lambda(\cdot)$, then if F is PF_2 and λ is increasing, Y is IFR process.

Proof:

Have to show that for $x > 0$, $c \ge 0$,

$$P(Y(t + x) \le c \mid Y(t) \le c) \text{ is decreasing in } t .$$

Fix $x > 0$ and let $h(t,y) = \sum_{j=0}^{\infty} \frac{\left[\int_t^{t+x} \lambda(u)du\right]^j}{j!} F^{*j}(c - y)$ for $0 \le y \le c$.

Now since λ is increasing in t, $h(t,y)$ is decreasing in t and y. But

$$P(Y(t + x) \leq c \mid Y(t) \leq c) = Eh((t,Y(t)) \mid Y(t) \leq c)$$

which is decreasing in t. A similar result concerning the nonhomogeneous compound Poisson was obtained by A.-Hameed-Proschan (1973) using similar argument as in Esary-Marshall-Proschan (1973).

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