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BREAKING SOLUTIONS OF THE VARIABLE COEFFICIENT KORTEWEG-DEVRIES--ETC(U)
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Strongly shoaling solutions to the variable coefficient Korteweg-deVries equation have been obtained for arbitrary initial or off-shore waveforms and depth variations. These solutions are valid when $\epsilon << \delta << 1$, where ϵ is the ratio of the initial wave amplitude to the depth and δ is the ratio of the initial length of the wave to the length scale associated

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20. Abstract (Continued)

with the depth variation. The conditions at breaking are given in terms of the initial shape of the wave and the depth variation. The actual wave amplitude at breaking was found to be proportional to $\epsilon^{\frac{6\alpha-4}{7\alpha-4}}\delta^{\frac{\alpha}{7\alpha-4}}$, where α is a parameter depending on the specific form of the depth variation; for a beach of constant slope, $\alpha = 1$. Numerical solutions of this equation were also obtained; these were in excellent agreement with the asymptotic results.

CONTENTS

BREAKING SOLUTIONS OF THE VARIABLE COEFFICIENT KORTEWEG-deVRIES EQUATION

1. Introduction

One of the most important problems in physical oceanography is the description of surface gravity waves as they propagate in water of variable depth. An extremely useful tool in the study of this phenomenon is the variable coefficient Korteweg-deVries equation. This equation contains the most important effects of modulation due to the depth variation, nonlinearity and dispersion. It has been shown that this equation can describe the gradual modulation of solitary waves and cnoidal wavetrains, see Johnson (1973b), Ostrovskiy and Pelinovskiy (1970) and Svendsen and Hansen (1978), as well as the shoal-induced fissioning of solitary waves, see, e.g., Johnson (1972). In the appropriate long wave limit, it can also describe waves which steepen and break. The main objective of this report is to describe another class of solutions to this equation. Over most of their propagation, these solutions may be described by the linearized version of this equation. Near the shore, the linear theory breaks down and in this region nonlinear effects predominate and the wave is observed to break by plunging. The theory presented here gives the breaking point and the wave amplitude at breaking explicitly in terms of the initial, i.e., off-shore, wave shape and the bottom topography. Once the wave breaks and a bore forms, the resultant interaction of the bore with the flow behind it will tend to decrease the bore amplitude; the results obtained here are also expected to give reasonable order of magnitude estimates for the maximum observed waveheight.

Although the starting point for our analysis is the Korteweg-deVries equation, the results obtained will also be applicable to actual flow problems. The conditions under which this is true, i.e., under which the Korteweg-deVries equation is valid, are discussed in Section 2.

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An interesting result which comes out of our analysis is the dependence of the wave shape and break point on the initial parameters of the problem. For the cases considered here, these depend on the wave shape and bottom variation through ϵ, μ and δ and, in particular, the ratios $\varepsilon \equiv \frac{\epsilon}{\delta}$ and $\nu \equiv \frac{\mu^2}{\delta}$; see Section 3 for this notation. The results of numerical or experimental studies can therefore be presented in terms of these natural parameters.

It is important to note how the present investigation differs from previous studies of shoaling water waves. While the authors mentioned above calculate the behavior of a single type of wave as it approaches a beach, we have shown how a wave transforms from a linear dispersive wave into one which is basically nonlinear and nondispersive. Furthermore, we do not claim to solve the very difficult problems of run-up and backwash discussed by many authors, see, e.g., Hibberd and Peregrine (1979). The beach slopes are so gradual that the waves break many wavelengths off shore. Thus, in the context of the present theory, run-up occurs in a second inner or near-shore region.

In the next section, we introduce the Korteweg-deVries equation and discuss some of the limitations on its use in water wave problems. In Sections 4 and 5 we derive the main results of the report and compare these to other studies of this equation. In Section 6, we compare the results of this asymptotic theory to numerical solutions of the Korteweg-deVries equation.

2. The Variable Coefficient Korteweg-deVries Equation and its Limitations.

We will consider one-dimensional surface waves of length λ and amplitude a propagating into water of variable depth. The undisturbed depth of the water is

$$H \equiv h_o h\,(\frac{\bar{x}}{L}),$$

where $\bar{x}$ is the distance measured parallel to the free surface in the direction of propagation, h_o is the depth at $\bar{x} \equiv 0$, i.e., $h(0) \equiv 1$, and L is a scale length measuring the rate of change of the bottom topography, see Figure 1. When the parameters

$$\epsilon \equiv \frac{a}{h_o}\ , \ \mu \equiv \frac{h_o}{\lambda} \text{ and } \delta \equiv \frac{\lambda}{L}$$

are all small, Johnson (1973a), Svendsen and Hansen (1978) and Cramer (1980) have all shown that the wave motion is governed by the following equation

$$\eta_x + \frac{3}{2}\ \epsilon h^{-\frac{3}{2}} \eta\eta_\tau + \mu^2\ \frac{h^{\frac{1}{2}}}{6}\ \eta_{\tau\tau\tau} + \frac{\delta}{4}\frac{h'}{h}\ \eta = 0, \qquad (1)$$

where $\eta \equiv h_o \epsilon \bar{\eta}$ and $\bar{\eta}$ is the displacement of the fluid surface from its undisturbed position. Furthermore,

$$x \equiv \frac{\bar{x}}{\lambda}\ ,\ h = h(x\delta) \text{ and } h'(x\delta) \equiv \frac{dh}{dx\delta}\ .$$

This equation has been written in a coordinate system which moves at the speed of a linearized shallow water wave; thus,

$$\tau \equiv \frac{1}{\delta}\int_o^{x\delta} \frac{dp}{\sqrt{h(p)}} - t,$$

where $t \equiv \frac{\sqrt{gh_o}}{\lambda}\ \bar{t}$ and $\bar{t}$ is the time and g is the acceleration of gravity. In the remainder of this section we will discuss possible limitations on the use of this equation.

In using the above equation it is crucial that the ratios represented by ϵ, μ and δ remain small; that is, the amplitude must remain small compared to the depth and the length must be large compared to the depth and small compared to the scale length of the bottom variation. These conditions can fail as the shore is approached. However, we only make use of this equation up to the point at which the wave breaks. We therefore only need to show that these ratios are small at the point of minimum depth, i.e., the depth at which breaking occurs. To provide a simple yet convincing argument that such waves exist, we shall discuss the particular depth variation given by

$$h = (1-x\delta)^p$$

where $p > \frac{4}{7}$. We first consider the small amplitude assumption. The ratio of the amplitude of the wave to the local depth is given by

$$\frac{\bar{\eta}}{h_o h(\frac{\bar{x}}{L})} = \frac{\epsilon\eta(x,\tau)}{h(x\delta)} .$$

In the following sections we will see that $\eta = 0(h^{-\frac{1}{4}})$ at breaking. In order for the small amplitude assumption to be valid at breaking we must therefore require

$$\epsilon << h_b^{\frac{5}{4}} ,$$

where h_b is the depth at which the wave breaks. It is shown later that, for the above depth variation,

$$h_b = 0(\frac{\epsilon}{\delta})^{\frac{4p}{7p-4}} = o(1)$$

Thus, at breaking, the wave amplitude will be small compared to depth provided

$$\epsilon << \delta << \epsilon^{\frac{2}{5}\frac{2-p}{p}} ;$$

the first condition will be seen to be necessary in order to induce breaking rather than fissioning. The above order relation is self-consistant provided $\frac{2}{5}\frac{2-p}{p} < 1$, i.e., $p > \frac{4}{7}$; this is seen to come naturally out of the analysis later. Note that the amplitude to depth ratio is of order

$$\left(\epsilon^{\frac{2}{5}\frac{p-2}{p}}\delta\right)^{\frac{5p}{7p-4}},$$

at breaking. Thus, when $p > 2$, this ratio is always small and we only need to require

$$\epsilon \ll \delta \ll 1 .$$

We now consider our long wave assumption. As the wave approaches shore, the linearized shallow water theory predicts that the length of the wave decreases. However, the rate at which the depth decreases is much faster than this shortening of the wave. Until the wave comes very close to shore, the long wave approximation is actually getting better rather than worse. The main effect may be illustrated using the linearized shallow water theory. If we ignore reflected waves (these will only increase the length of the wave, thereby making our approximation better), this theory may be used to show that the ratio of the depth to the length of the wave is given by

$$\frac{h_o}{\lambda}\sqrt{h} ,$$

provided

$$\delta \ll h^{\frac{2-p}{2p}} .$$

In order to ensure that this long wave assumption is not violated, we therefore require

$$\delta \ll h_b^{\frac{2-p}{2p}} .$$

Because $h_b = O(\frac{\epsilon}{\delta})^{\frac{4p}{7p-4}}$, we have

$$\epsilon \ll \delta \ll \epsilon^{\frac{2}{5}\frac{2-p}{p}} ,$$

which is the same as that obtained for the small amplitude assumption.

We now consider the assumption of a gradually varying bathymetry. If we again use the linearized shallow water theory and neglect reflections, we find that the ratio of the length of the wave to the bathymetry length scale,

$\left|\frac{1}{h}\frac{dh}{dx}\right|^{-1}$, is given by

$$p\,\frac{\delta}{h^{\frac{2-p}{2p}}}$$

provided $\delta \ll h^{\frac{2-p}{2p}}$. Thus, the conditions imposed earlier, viz,

$$\delta \ll h_b^{-\frac{2-p}{2p}} ,$$

i.e., $$\epsilon \ll \delta \ll \epsilon^{\frac{2}{5}\frac{2-p}{p}} ,$$

are sufficient to ensure that this ratio remains small even at the breaking point.

Thus, we see that the wave will always break before the small amplitude, long wave and gradual depth variation assumptions fail provided we require

$$\epsilon \ll \delta \ll \epsilon^{\frac{2}{5}\frac{2-p}{p}} \ll 1.$$

If $p > 2$ these assumptions will always be valid at breaking and we do not need to impose any conditions other than those of Sections 3-5, i.e.,

$$\epsilon \ll \delta \ll 1.$$

It should be noted that these results were obtained for a special form of the depth variation; it is expected that this discussion is easily extended to more general bathymetries. The conditions given here are related to the local conditions given by Varley, Venkataraman and Cumberbatch (1971).

Equation (1) also assumes that we may neglect the effect of reflected waves. The above discussion shows that the length of the wave remains small compared with the length scale associated with the depth variation. It is therefore clear that the reflected waves generated by the depth variation will be small, at least for the time period considered here. For wavetrains, we expect the effect of reflections to be diminished even further by destructive interference, see, e.g., Kajiura (1961). Previous numerical and experimental studies also provide support for this conclusion. Peregrine (1967) has used the Boussinesq equations to describe a solitary wave as it propagates over a gently sloping beach; the resultant reflected wave was observed to be at least an order of magnitude smaller than the incident wave. Waves which break by plunging without significant reflection have also been observed in the laboratory studies of Street and Camfield (1966) and Galvin (1968).

We conclude that the Korteweg-deVries equation (1) will provide a valid model for shoaling water waves under relatively mild restrictions on the initial wave shape and bottom topography. It should be noted that the theory here only attempts to calculate the conditions associated with the breaking of the wave. In the final stages of the approach to shore, most of the assumptions made here fail and one must resort to more complicated theories capable of treating the problems of run-up and backwash.

3. Problem Formulation

In this report we will obtain approximate solutions to equation (1) subject to the initial condition

$$\eta(0,\tau) = F(\tau).$$

If we define

$$\tilde{x} \equiv x\delta \text{ and } u(\tilde{x},\tau) \equiv h^{+\frac{1}{4}}(\tilde{x})\eta(\tilde{x},\tau)$$

the initial value problem reads

$$u_{\tilde{x}} + \frac{\nu}{6} h^{\frac{1}{2}} u_{\tau\tau\tau} + \frac{3}{2}\varepsilon h^{-\frac{7}{4}} u u_{\tau} = 0$$

$$u(0,\tau) = F(\tau), \tag{2}$$

where $h = h(x\delta) \equiv h(\tilde{x})$, $\nu \equiv \frac{\mu^2}{\delta}$ and $\varepsilon \equiv \frac{\epsilon}{\delta}$.

On Figure 1 we have sketched typical bottom topographies. Figure 1a depicts a closed beach; here $h(1) \equiv 0$. In Figure 1b a shelf is sketched; here $h(\tilde{x}) = d$ for $\tilde{x} \geq 1$. In the numerical studies described below we will always take $h \equiv 1$ for $\tilde{x} \leq 0$. It should be noted that the length scales have been distorted for the purposes of illustration.

In this report we will consider the case where ε is relatively small, i.e.,

$$\epsilon << \delta << 1.$$

The parameter ν is taken to be of order one, i.e., at most bounded. Inspection of equations (2) suggests that the wave motion may be described by the linearized version of this equation

$$u_{\tilde{x}} + \frac{\nu}{6} h^{\frac{1}{2}} u_{\tau\tau\tau} = 0$$

$$u(0,\tau) = F(\tau),$$

at least for $\tilde{x} \neq 1$, i.e., $h \neq 0$. Of course, if ν is also small, e.g., $\mu^2 = O(\varepsilon)$, then the wave motion is described by the classical Green's law formula $u(\tilde{x},\tau) \equiv F(\tau)$, see, e.g., Lamb (1932). However, as the shore is approached, that is, as $h \to 0$, the nonlinear term is expected to become larger and larger and the linear solution is therefore expected to break down near the shore. An inspection of the dispersive term in (2) suggests that it will become smaller and smaller as $h \to 0$. We would therefore expect that the approximation to (2) which is valid near the shore is

$$u_{\tilde{x}} + \frac{3}{2}\varepsilon h^{-\frac{7}{4}} u u_{\tau} \simeq 0.$$

Of course, the initial condition for this equation is simply the incoming linear solution. This equation is immediately recognized as a version of the inviscid Burger equation or Airy's equation. Solutions to this equation steepen at the front face of the wave and eventually become triple-valued. Although the theory must fail where the surface slope becomes infinite, the usual interpretation of this phenomena is that the actual wave will be observed to topple over and break in a similar fashion, see, e.g., Stoker (1957). These remarks are expected to hold not only for a closed beach but for a shelf, see Figure 1b, provided d is sufficiently small.

This heuristic description of the wave evolution is shown to be an accurate one in the following sections. In order to illustrate the basic theory, the case of $\nu = O(\varepsilon) = o(1)$ is described in detail in Section 4; this will also enable us to quickly extract results appropriate for solitary waves. Because ϵ and μ^2 are small compared to δ it will be referred to as weakly dispersive and weakly nonlinear. In Section 5, shorter waves having $\nu = O(1)$ are discussed. Because these exhibit some dispersion before entering the near-shore region and breaking, this will be referred to as the dispersive case.

4. Weakly Dispersive and Weakly Nonlinear Waves

In this section it will be assumed that both ε and ν are small. We assume an outer or off-shore expansion of the form

$$u = U_o + O(\varepsilon,\nu),$$

which is assumed to be valid for all $\tilde{x} = O(1)$, $\tilde{x} \neq 1$. If we substitute this in equation (2) and retain only lowest order terms, we find that

$$U_o(\tilde{x},\tau) = F(\tau),$$

which implies,

$$u = F(\tau) + O(\varepsilon,\nu),$$

for $\tilde{x} \neq 1$. This can be recognized as Green's law.

An examination of higher order terms shows that this straightforward expansion scheme breaks down in the vicinity of the shore, i.e., $\tilde{x} \simeq 1$. We therefore need to find a second approximation to the Korteweg-deVries equation in (2) which is valid near the shore. To do this we define a new independent variable

$$\hat{x} \equiv \frac{\tilde{x}-1}{\Delta}, \quad (4)$$

where $\Delta = o(1)$ is a scale factor to be determined later. Note that $\hat{x} \leqslant 0$ when $\tilde{x} \leqslant 1$. In the usual way, we will assume that $\hat{x} = O(1)$ in the inner or near-shore region; the scaling Δ essentially gives the nearness of $\tilde{x}$ to 1. We will also need to write out explicitly the expansion of h in the vicinity of $\tilde{x} = 1$. This will be written

$$h(\tilde{x}) = h(1+\Delta\hat{x}) \sim \Delta^{\alpha}\hat{h}(\hat{x}) + o(\Delta^{\alpha}) \quad (5)$$

for small Δ and $\hat{x} = O(1)$. The exponent $\alpha > 0$ and the form of the function $\hat{h}(\hat{x})$ will depend on the particular behavior of h as $\tilde{x}\to 1$. For example, for a closed beach and

$$h \sim (1-\tilde{x})^n \text{ as } \tilde{x}\to 1,$$

we have $\alpha = n$ and $\hat{h}(\hat{x}) = (-\hat{x})^n$. For the shelf given by

$$h(\tilde{x}) \equiv \begin{cases} 1 & \text{for } \tilde{x} \leqslant 0 \\ \frac{1+d}{2} + \frac{1-d}{2}\cos \Pi\tilde{x} & \text{for } 0 \leqslant \tilde{x} \leqslant 1 \\ d & \text{for } \tilde{x} \geqslant 1 \end{cases}$$

where d is assumed small, we have $\alpha = 2$ and

$$\hat{h}(\hat{x}) = \begin{cases} \hat{d} + \frac{\Pi^2\hat{x}^2}{4} & \text{for } \hat{x} \leqslant 0 \\ \hat{d} & \text{for } \hat{x} \geqslant 0 \end{cases}$$

where $\hat{d} \equiv \Delta^{-2}d$. The most interesting case is when $\hat{d} = O(1)$.

If we now use (4) to change variables in (2), substitute the expansion (5) and the following inner or near-shore expansion

$$u = u_o + o(1)$$

in (2), we then have

$$u_{o_{\hat{x}}} + \nu\Delta^{1+\frac{\alpha}{2}}\hat{h}^{\frac{1}{2}}u_{o\tau\tau\tau} + \frac{3}{2}\varepsilon\Delta^{1-\frac{7}{4}\alpha}\hat{h}^{\frac{-7}{4}}u_o u_{o\tau} = o(1).$$

For $\nu = o(1)$, $\varepsilon = o(1)$ and $\alpha > 0$, the only reasonable balance of terms is obtained when

$$\Delta = \varepsilon^{\frac{4}{7\alpha-4}} \text{ and } \alpha > \frac{4}{7}.$$

In the former, the order symbols have been dropped for convenience. The Korteweg-deVries equation therefore becomes

$$u_{o\hat{x}} + \frac{3}{2}\hat{h}^{\frac{-7}{4}}u_o u_{o\tau} = 0,$$

to lowest order. The general solution to this equation is easily obtained by the method of characteristics; this may be written in the following parametric form

$$u_o(\hat{x},\tau) = G(\tau_\infty)$$

and $$\tau = \tau_\infty + \frac{3}{2} G(\tau_\infty) I(\hat{x}),$$

where $$I(\hat{x}) \equiv \int_{-\infty}^{\hat{x}} \hat{h}^{-\frac{7}{4}}(p)\,dp \ . \qquad (6)$$

Here G is an unknown function to be determined by matching to the linear solution. Alternatively, one could recast this as

$$u_o(\hat{x},\tau) = G(\tau - \frac{3}{2} GI(\hat{x})) \ ,$$

which just follows from substituting τ_∞ from the second parametric equation into the first.

Our description of the near-shore solution will be complete once G is specified. This will be obtained by matching the near-shore solution to the off-shore solution through the formalism of the method of matched asymptotic expansions, see, e.g., Van Dyke (1975). We first consider the near-shore solution

$$u = G(\tau - \frac{3}{2} GI(\hat{x})) + o(1) \ .$$

When this is recast in the off-shore variables and expanded for $\varepsilon \to 0$ and $\tilde{x}$ fixed, we have

$$u \sim G(\tau) + o(1) \ ,$$

because $I(\hat{x}) = I(\frac{x-1}{\Delta}) \to 0$ as $\Delta = \varepsilon^{\frac{4}{7\alpha-4}} \to 0$. The method of matched asymptotic expansions then requires that this match the off-shore solution

$$u = F(\tau) + o(1) \ .$$

As a result $G(p) \equiv F(p)$, for all p, and the near-shore solution therefore becomes

$$u = F(\tau - \frac{3}{2} FI(\hat{x})) + o(1) \ ,$$

or, in terms of the characteristic lines,

$$u(\hat{x},\tau) = F(\tau_\infty) + o(1)$$

on $$\tau = \tau_\infty + \frac{3}{2} F(\tau_\infty) I(\hat{x}) \ . \qquad (7)$$

It is well-known that some of the characteristics will converge on themselves leading to a steepening of the wave at the front face. The break point $\hat{x}_b$ will be defined as the point at which the slope of the wave first becomes infinite. The equation of the characteristic lines (7) may be used to show that this is given by the following implicit formula:

$$I(\hat{x}_b) \equiv \frac{2}{3} \frac{1}{|F'|_{max}} , \tag{8}$$

where $F'(\xi) \equiv \frac{dF}{d\xi}$. Here we note that higher order matchings have been carried out; these produce no changes in the basic lowest order results.

In summary, we have derived approximate solutions to the initial value problem (2) for small ν and small ε. We have found that for all $\tilde{x} \neq 1$ the behavior of the wave is given by Green's law, i.e.,

$$u \simeq F(\tau) .$$

Close to the shore, this description is no longer valid and it is seen that nonlinear effects play a dominate role. The resultant solution and formula for the break point are given by equations (7) and (8). The relation of the near-shore variable $\hat{x}$ to the variables $\tilde{x}$ and x are

$$\tilde{x} = x\delta = 1 + \Delta\hat{x} = 1 + \varepsilon^{\frac{4}{7\alpha-4}} \hat{x} ,$$

where $\alpha > \frac{4}{7}$. Thus, breaking occurs when $\tilde{x}-1 = O(\varepsilon^{\frac{4}{7\alpha-4}})$ and the depth h_b at which breaking occurs is

$$h_b = O(\varepsilon^{\frac{4\alpha}{7\alpha-4}}),$$

where (5) has been used. At breaking, the actual wave height,

$$\bar{\eta} = \epsilon h_o h^{-\frac{1}{4}} u,$$

is therefore of order $h_o \epsilon^{\frac{6\alpha-4}{7\alpha-4}} \delta^{\frac{\alpha}{7\alpha-4}}$.

It is interesting to note that dispersive effects play no role in the wave evolution, at least in the case considered here. In the off-shore or outer region they are negligible for the same reason nonlinear effects are; that is, they are only important over distances which are large compared to the distance to shore. Near the shore, h is small and the dispersive term in (2) is negligible compared to the nonlinear term. In fact, we see that this case may be modeled by the nondispersive version of (2), viz.,

$$u_{\tilde{x}} + \frac{3}{2}\varepsilon h^{-\frac{7}{4}} uu_{\tau} = 0$$

$$u(0,\tau) = F(\tau).$$

To lowest order, the solutions to this problem are identical to those obtained here.

It should also be noted that the results obtained here are valid for solitary waves, i.e., $\epsilon = O(\mu^2)$. Because $\epsilon << \delta << 1$ we might refer to these as strongly shoaling solitary waves. In fact, in terms of the ordering of the parameters ϵ, μ^2 and δ, this is exactly the case studied by Tappert and Zabusky (1971). These authors have considered the case of a shelving solitary wave and related the number of solitons generated to the depth ratio of the shelf. The problem discussed in this section may be applied to shelving waves and it is clear that for sufficiently small shelf depths the wave will break without fissioning. In the example discussed earlier in this section the shelf depth must be of order $(\frac{\epsilon}{\delta})^{\frac{4}{5}}$ in order for breaking to occur. Thus, this theory provides limits on that of Tappert and Zabusky and, in particular, limits the number of solitons which can be generated by the shelving process.

In Section 6, the results obtained here are compared to numerical solutions of (2). It is also desirable to compare these results with calculations which include reflected waves. One such study is due to Peregrine (1967). The appropriate Boussinesq equations were derived and solved numerically. The resultant evolution was interpreted by Peregrine to include breaking near the shore. The present study neglects reflected waves; the fact that these remain small up to breaking is certainly suggested by Peregrine's calculations. Furthermore, the wave growth appears to be in reasonable agreement as well.

To see how this fits in with previous theories, consider a solitary wave of fixed amplitude and length, i.e., fixed ϵ and μ, $\epsilon = O(\mu^2)$. If we then let this approach a very gradual beach, e.g., $\epsilon \gg \delta$ or $\varepsilon \gg 1$, we obtain the gradually varying solitary wave solution of Johnson (1973b). If we let the same wave approach a second beach which is steeper, say $\delta = O(\epsilon)$ or $\varepsilon = O(1)$, we find that the dispersion, modulation and nonlinearity all act with equal strength and the wave fissions as described by Madsen and Mei (1969). For an even steeper beach, i.e., $\delta \gg \epsilon$ or $\varepsilon = o(1)$, the wave will come relatively close to shore without significant distortion. Once the wave is very near shore the dispersion is very weak and the wave must break as described by the present theory.

5. <u>The General Dispersive Case</u>

We now consider shorter waves which exhibit a dispersive behavior before breaking occurs. The solution to the initial value problem (2) will now be outlined for the case $\nu = O(1)$ and $\varepsilon = o(1)$, that is, $\mu^2 = O(\delta)$ and $\epsilon = o(\delta)$. We now assume an off-shore expansion

$$u = U_o + o(1),$$

where there should be no confusion between the notation in this section and the previous. The initial value problem (2) therefore becomes

$$U_{o\tilde{x}} + \frac{\nu}{6} h^{\frac{1}{2}}(\tilde{x}) U_{o\tau\tau\tau} = 0$$

$$U_o(0,\tau) = F(\tau),$$

to lowest order. We can transform this into a problem in water of constant depth by the following transformation

$$Z \equiv \int_o^{\tilde{x}} h^{\frac{1}{2}}(p)\,dp,$$

which yields

$$\left.\begin{aligned} U_{oZ} + \frac{\nu}{6} U_{o\tau\tau\tau} &= 0 \\ U_o(0,\tau) &= F(\tau). \end{aligned}\right\} \tag{9}$$

General solutions to (9) may be obtained by transform techniques. Because $\nu \equiv O(1)$ these are strongly dispersive solutions. If we assume that

$$F(\tau) \equiv \cos\tau,$$

an exact solution to (9) is

$$U_o(\tilde{x},\tau) \equiv \cos T,$$

where $T \equiv \tau + \frac{\nu}{6} Z = \tau + \frac{\nu}{6} \int_o^{\tilde{x}} h^{\frac{1}{2}}(p)\,dp.$

As in Section 4, we will need a second approximation to (2) near the shore. By the same procedure as used in Section 4 we find that

$$u_o(\hat{x},\tau) = G(\tau_\infty)$$

on

$$\tau = \tau_\infty + \frac{3}{2} G(\tau_\infty) I(\hat{x}),$$

where

$$I(\hat{x}) \equiv \int_{-\infty}^{\hat{x}} \hat{h}^{-\frac{7}{4}}(p)dp$$

and G is an unknown function to be determined by the matching. Here the near-shore expansions

$$u = u_o + o(1)$$

$$\hat{x} \equiv (\tilde{x}-1)\, \varepsilon^{\frac{4}{4-7\alpha}} = O(1)$$

have been used. Again, one should not confuse the notation of this section and the last.

Because of the dispersion present in the off-shore solution, see, e.g., (9), it is clear that the result of the matching will not yield G = F. Essentially G will be the solution to (9) evaluated at $\tilde{x}=1$. For example, when $F(\tau) \equiv \cos \tau$ the near-shore solution is found to be

$$u_o(\tilde{x},\tau) = \cos T_\infty$$

on

$$\tau = \tau_\infty + \frac{3}{2} I(\hat{x}) \cos T_\infty,$$

where $T_\infty \equiv \tau_\infty + \frac{\nu}{6} \int_o^1 h^{\frac{1}{2}}(p)dp$, or, equivalently,

$$u_o(\hat{x},\tau) = \cos(\tau + \frac{\nu}{6} \int_o^1 h^{\frac{1}{2}}(p)dp - \frac{3}{2} I(\hat{x}) \cos T_\infty)$$

on $\tau = \tau_\infty + \frac{3}{2} I(\hat{x}) \cos T_\infty$.

A breaking condition may also be derived; in this case, $\hat{x}_b$ and $\hat{h}_b$ will depend on ν through T_∞. As in Section 4, higher order matchings have been carried out; these show that the solutions presented above are, in fact, the lowest order approximations to (2) in the appropriate regions.

Thus, we see that, although the dispersion plays a major role in the off-shore, $\tilde{x} = O(1)$, evolution of the wave, it is negligible near the shore. The breaking point depends on ν or μ^2 only insofar as the "incoming wave" does. It is clear that the results sketched here will contain those of Section 4 when ν is small.

6. Numerical Results

In this section we compare numerical solutions of (2) to the asymptotic solutions derived in this report. We shall concentrate on the behavior of solitary waves and therefore the results of Section 4 will be of interest. The finite difference scheme developed by Zabusky and Kruskal (1965), Vliegenthart (1971) and Johnson (1972) was applied to (2) for various values of ν and ε. As in the previous investigations, the differential equation was replaced by the difference formula

$$u_{i,j+1} = u_{i,j-1} - \frac{\nu}{6} h^{\frac{1}{2}} \frac{\Delta\tilde{x}}{(\Delta\tau)^3} (u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} - u_{i-2,j})$$
$$- \frac{\varepsilon}{2} h_j^{-\frac{7}{4}} \frac{\Delta\tilde{x}}{\Delta\tau} (u_{i+1,j} + u_{i,j} + u_{i-1,j})(u_{i+1,j} - u_{i-1,j})$$

where i and j correspond to the τ and $\tilde{x}$ directions, respectively, $\Delta\tau$ and $\Delta\tilde{x}$ are the step sizes in the τ and $\tilde{x}$ directions and $h_j \equiv h(j\Delta\tilde{x})$. The first step was calculated by a simple uncentered scheme. For this problem, the stability criterion reads

$$\frac{\Delta\tilde{x}}{\Delta\tau} (h_j^{-\frac{7}{4}} |u| \frac{3}{2} \varepsilon + \frac{2}{3} \frac{h_j^{\frac{1}{2}}}{(\Delta\tau)^2} \nu) \leqslant 1 .$$

It was found that step sizes of $\Delta\tau = 0.1$ and $\Delta\tilde{x} = 0.0001$ satisfy the above criterion and provided satisfactory results in all the calculations described below.

We can expect differences between the asymptotic results of Section 4 and those obtained by the finite difference scheme for two reasons. The first is that the results of Section 4 are only valid in the limit of small ε. As we would expect, the agreement between the two solutions becomes better and better as ε approaches zero. The second source of discrepancy is due to the numerical dispersion inherent in the finite difference scheme. For the sake of illustration, consider the following initial value problem involving the constant coefficient Burger's equation.

$$v_x + vv_\tau = 0$$

$$v(0,\tau) = F(\tau)$$

Exact solutions to this problem are easily obtained by the method of characteristics. It is well-known that the waveform steepens until the slope at some point on the front face of the wave becomes infinite. After this time, the solution is seen to become triple-valued. However, when this problem is solved using the finite difference scheme, the predicted behavior is quite different. Although during the initial stages the behavior is similar to the exact solution and although the wave steepens noticeably, the wave slope never becomes infinite. In fact, the wave is observed to fission in a manner similar to that of the constant coefficient Korteweg-deVries equation and is seen to emit a large number of very narrow spike-like solitons. If, in a Korteweg-deVries equation, the physical dispersion is weak enough, e.g., of the order of the numerical dispersion, erroneous results are also expected. Thus, near breaking, some care should be taken when comparing the results of our asymptotic theory to the numerical results. A crude measure of this error was obtained by comparing exact and finite difference solutions to the appropriate Burger's equation; this is discussed later.

The waves considered were of the solitary wave type. The initial condition used was

$$F(\tau) \equiv \text{sech}^2 \tau \ .$$

In order to illustrate the strength of the shoaling we have chosen

$$\nu = \frac{4}{3} \varepsilon \ .$$

As a result, this wave will fission in water of uniform depth. However, we will see that the strength of the shoaling is such that, even in this case, the fissioning cannot occur and the wave breaks as described in Section 4. For the sake of simplicity, the beach slope was taken to be constant. That is,

$$h = \begin{cases} 1 & \tilde{x} \leqslant 0 \\ 1 - \tilde{x} & 0 \leqslant \tilde{x} \leqslant 1; \end{cases}$$

thus, the shore is located at $\tilde{x} = 1$ or $x = \delta^{-1}$.

In each case, the asymptotic solution has been calculated from equations (6) and (7). For the beach and wave profile described above, the solution may be calculated from

$$u(\hat{x},\tau) = \text{sech}^2\, \tau_\infty$$
$$\tau = \tau_\infty + 2(-\hat{x})^{-\frac{3}{4}}\, \text{sech}^2\, \tau_\infty, \tag{10}$$

where τ_∞ may be regarded as a parameter and $\hat{x} = (\tilde{x}-1)\varepsilon^{-\frac{4}{3}}$. The breaking point is calculated from (8) to be

$$h_b \equiv 1-\tilde{x}_b = (2\varepsilon|F'|_{max})^{\frac{4}{3}}$$

or

$$h_b = \left(\frac{8}{3\sqrt{3}}\,\varepsilon\right)^{\frac{4}{3}},$$

since $|F'|_{max} = \frac{4}{3\sqrt{3}}$ for this case.

In Figure 2, the evolution of a wave having $\varepsilon = 0.1$ has been depicted. The solid lines are the finite-difference solutions and the dotted lines the asymptotic theory. Figure 2(a) is the waveform at $\tilde{x} = 0$, Figure 2(b) is the waveform at $\tilde{x} = 0.9$ and Figure 2(c) is the waveform at breaking $\tilde{x}_b \approx 0.918$. The discrepancy is expected to be due to the small ε approximation. Figure 3 depicts the evolution for a smaller value of ε; $\varepsilon = 0.05$. As we would expect, the agreement is much better; again, the error is within the bounds of the small ε approximation. As predicted by the theory of Section 4, there is very little steepening of the wave until very near the shore, see, e.g. Figure 3(b). In Figure 4, the waveform corresponding to $\varepsilon = 0.025$ is shown. Here the agreement is quite good, even at the breaking point; see Figure 4(c). The main source of discrepancy is expected to be due to numerical dispersion. As a check on some of the assertions made in this report, we have recalculated the waveform using the Burger's equation:

$$u_{\bar{x}} + \frac{3}{2}\varepsilon h^{-\frac{7}{4}} uu_{\tau} = 0$$
$$u(0,\tau) = \text{sech}^2\tau, \tag{11}$$

where $\varepsilon = 0.025$. We first obtained the exact solution to this problem by the method of characteristics and then an approximate solution was calculated using the above finite difference scheme. In Figure 5(a), we have compared the asymptotic solution (10) to the exact solution of (11); as asserted at the end of Section 4, we see that these agree very well. In Figure 5(b), we have compared the finite difference solutions of (2) and (11). These are also in agreement suggesting that the physical dispersion in (2) is very small; at least in the problem considered here. Because the only difference between the finite difference and exact solutions to (11) is due to numerical dispersion, a comparison of all four plots in Figure 5 strongly suggests that the discrepancy in Figure 4(c) is primarily due to the numerical dispersion inherent in the finite difference scheme.

7. Conclusion

This report describes a particular class of solutions to the variable coefficient Korteweg-deVries equation. For the most part, nonlinear effects were negligible and the wave evolution was described by a linear and, in some cases, dispersive, equation. Because of the amplification and effective lengthening of the wave, we find that near the shore dispersive effects are negligible and the nonlinear steepening effects predominate; this leads to a breaking of the wave. A careful consideration of higher order terms has shown that the perturbation and matching procedure is, in fact, correct; furthermore, the numerical solutions presented here give further support for this. Thus, we expect that when exact solutions to (2) are obtained, these will approach ours in the limit of vanishing $\varepsilon \equiv \frac{\epsilon}{\delta}$.

We have also provided estimates for the break point and the wave height at breaking. In the case of solitary waves, or, more generally, $\epsilon = O(\mu^2)$, we find that these depend on the parameters ϵ, μ and δ only through $\varepsilon \equiv \frac{\epsilon}{\delta}$. For the general dispersive case, i.e., $\nu = O(1)$, there is an additional dependence on $\nu = \frac{\mu^2}{\delta}$ through the initial condition for the near-shore problem.

It is expected that the results obtained here will also provide a good approximation to actual waves on a beach. While we can find no numerical or experimental studies which permit direct comparison, we find that our results are in qualitative agreement with the numerical study of Peregrine (1967) and the laboratory studies of Street and Camfield (1966) and Galvin (1968); in particular, the results of Section 4 correspond to the plunging breaker regime of Street and Camfield.

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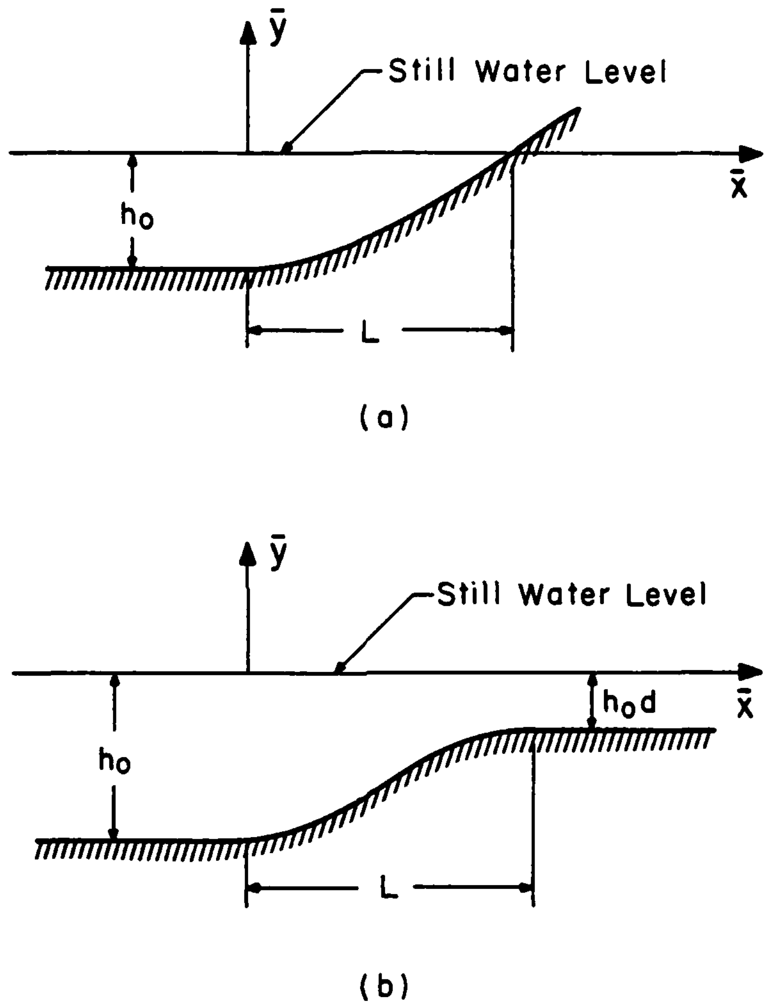

Figure 1: Typical Bathymetries. The undisturbed water level is taken to be $\overline{y} = 0$ and L may be considered to be the length scale characteristic of the depth variation. (a) Closed beach. (b) Shelf; d gives the ratio of the shelf depth to the deep water depth.

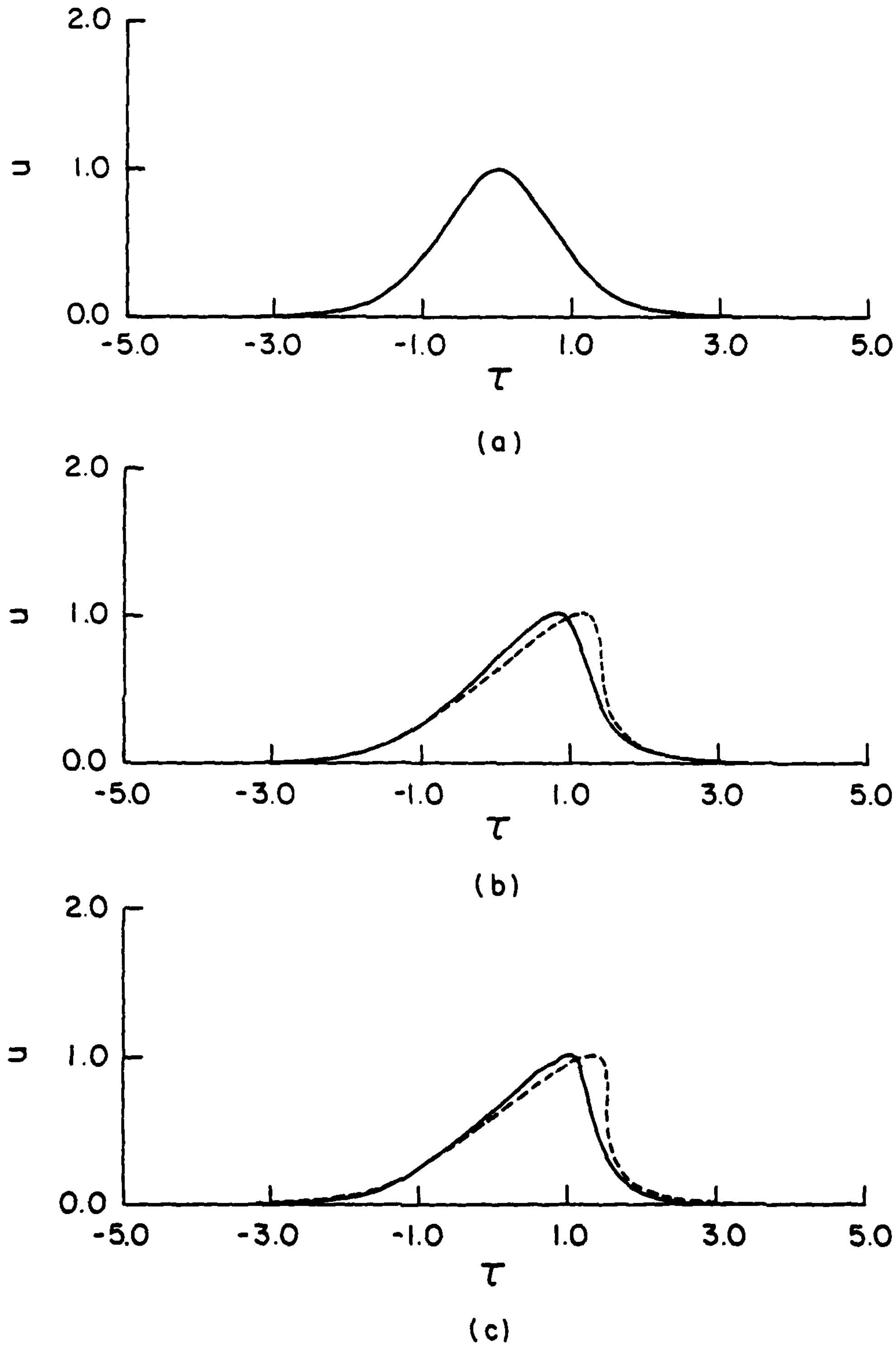

Figure 2: Wave Evolution for Solitary Wave Profile $u(0,\tau) = \text{sech}^2\,\tau$ and $\varepsilon = 0.1$, $\nu = 0.133$. Asymptotic Theory - - - -, Finite Difference Solution ——— . (a) $\hat{x} = 0$, (b) $\hat{x} = 0.9$, (c) $\hat{x} = 0.918$.

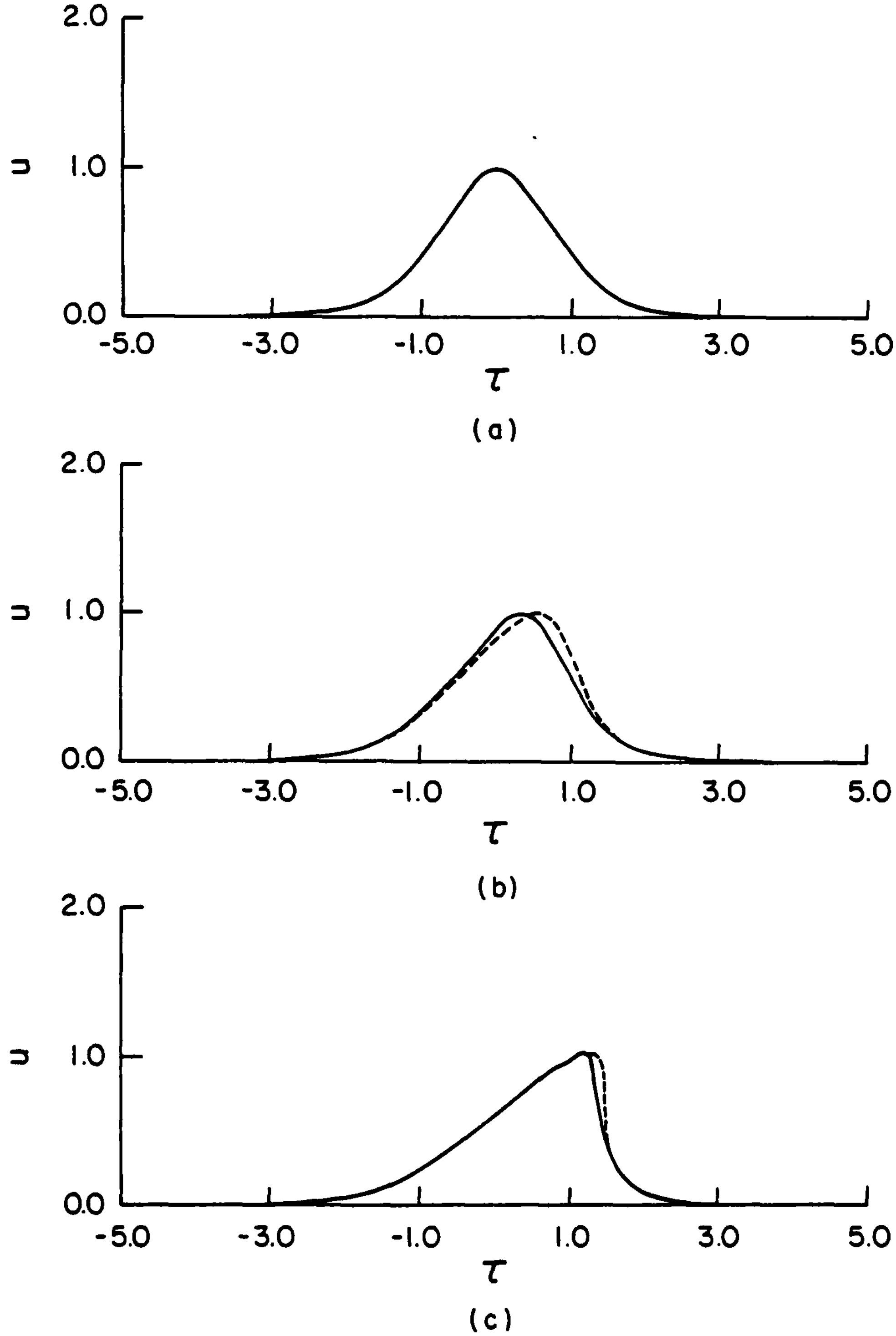

Figure 3: Wave Evolution for $\varepsilon = 0.05$, $\nu = .0667$. Asymptotic Theory - - - - , Finite Difference Solution ——— . (a) $\dot{x} = 0$, (b) $\dot{x} = 0.9$, (c) $\dot{x} = .967$.

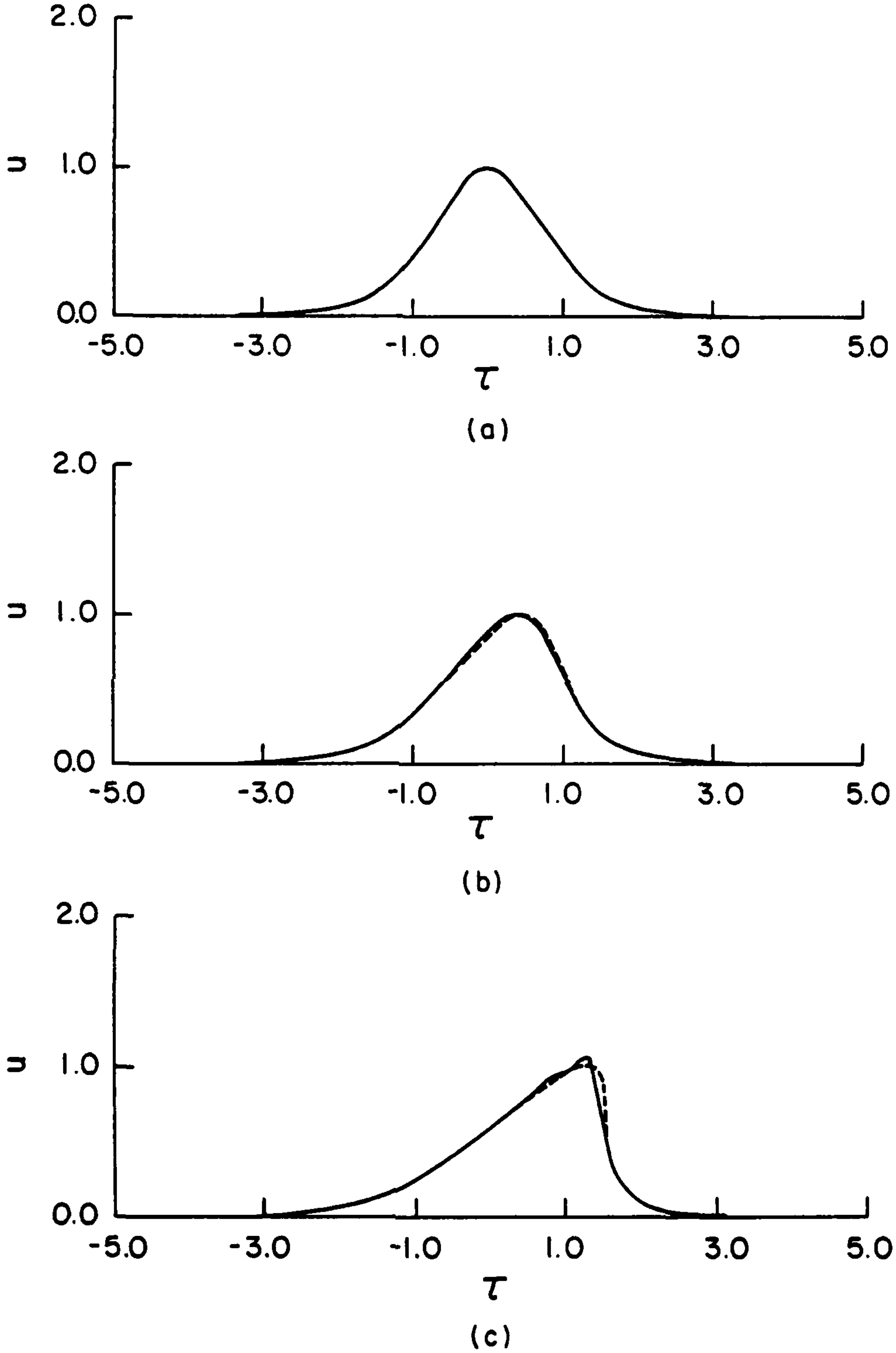

Figure 4: Wave Evolution for $\varepsilon = 0.025$, $\nu = .0333$. Asymptotic Theory - - - - , Finite Difference Solution ——— . (a) $\dot{x} = 0$, (b) $\dot{x} = 0.95$, (c) $\dot{x} = .987$.

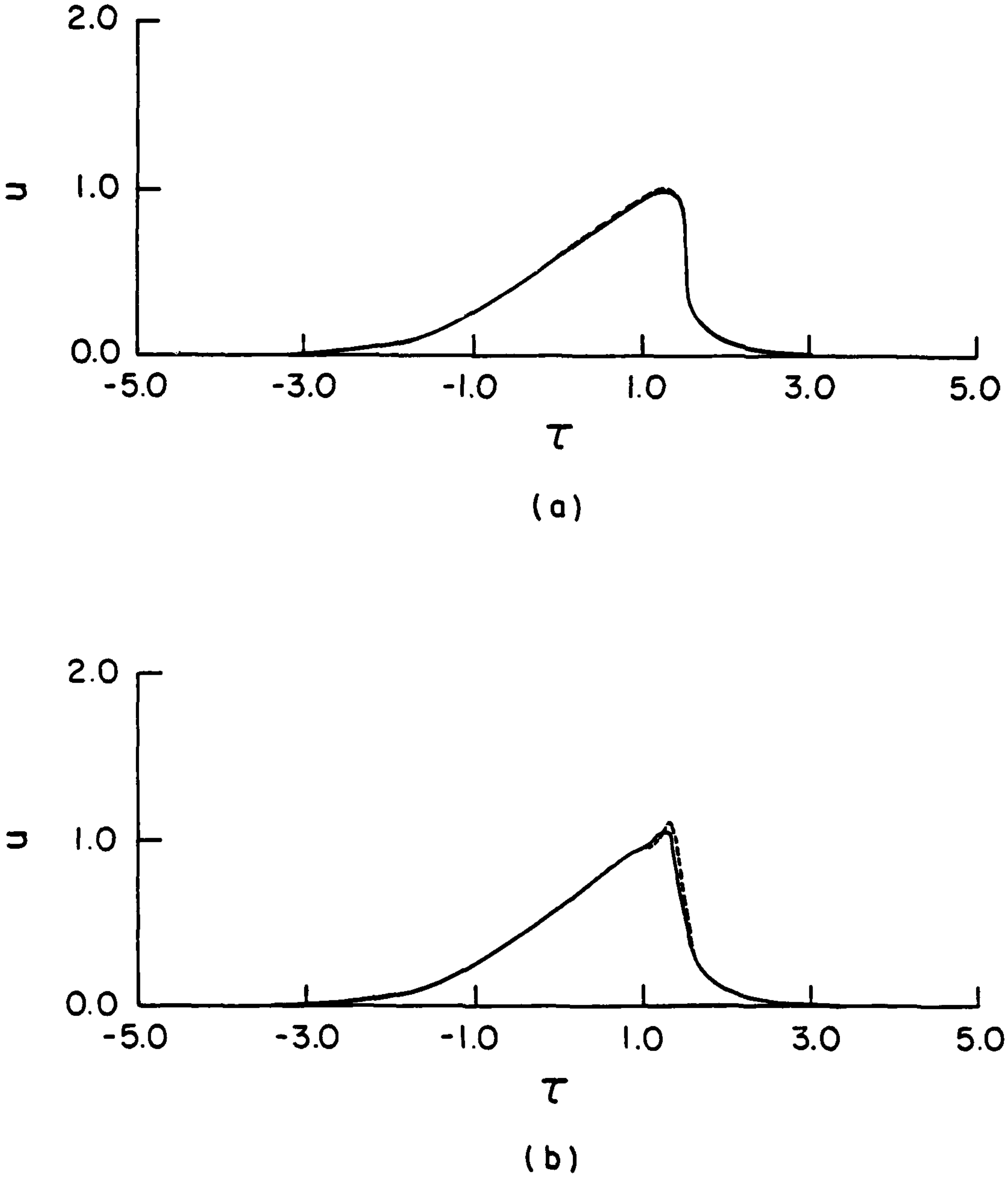

Figure 5: Comparison of Various Theories for $u(0,\tau) = \text{sech}^2\, \tau$, $\varepsilon = 0.025$, $\nu = .0333$, $\hat{x} = .987$. (a) Asymptotic Solution ——— , Exact Solution of Burger's Equation - - - - , (b) Finite Difference Solutions: Full Korteweg-deVries Equation ——— , Burger's Equation - - - - .

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