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BAYESIAN OPTIMAL DESIGNS FOR

SURVEILLANCE PROCEDURES IN

ATTRIBUTE LIFE TESTING

By

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BAYESIAN OPTIMAL DESIGNS FOR
SURVEILLANCE PROCEDURES IN
ATTRIBUTE LIFE TESTING(*)

By

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ABSTRACT

Design of experiments for estimation of parameters in non-linear models is studied in a Bayesian framework, with the objective of maximization of the anticipated Fisher information. Two stage optimal designs are proposed in attribute life testing situations.

1.   Introduction

Consider a system of N components working independently and having identical cumulative distribution functions (c.d.f.) of the time till failure $F(t;\theta)$. F is a known function and θ is an unknown parameter belonging to a parameter space Θ. It is assumed that the number of components, N, is a fixed positive integer. The components fail randomly at unobservable times. We inspect the system after x units of time and count the number of failed components. The replacement of the components could follow one of the following two policies:

(A)   Only failed components are replaced at each inspection.

(B)   All items in the system are replaced at each inspection (frequent replacement policy, or block replacement policy).

The policy to ascribe depends upon the type of system under consideration. For example, policy (B) is preferred when it costs more to inspect and change only failed components as compared to changing the whole system. Such examples are encountered in the replacement of street light bulbs, etc. Sometimes it is practically impossible to change only the failed components without effecting the whole system. Systems composed of transistors built in modules. For more applications see Barlow and Proschan (1967). In quantal response bioassay studies policy (B) is followed, where; after experimentation, the whole batch of experimental units (mice, fish, etc.) is replaced by a new one. Finney (1978) gives an exhaustive reference list of bioassay studies of this kind.

Let $J(x_1), J(x_2), \ldots, J(x_n), \ldots$ denote the number of components failing during the intervals $(0, x_1)$, $(0, x_1)$, $(x_1, x_1+x_2), \ldots, (\sum_{i=1}^{n-1} x_1, \sum_{i=1}^{n} x_i), \ldots$. Intuitively, we would like to use the information $(J(x_1), \ldots, J(x_n), x_1, \ldots, x_n)$ to define x_{n+1} so that $J(x_{n+1})$ will provide as much information on θ as possible. To define the best or optimal interinspection time at the (n+1)st stage, we shall use the criterion of maximizing the conditional Fisher information about θ, given $(J(x_1), \ldots, J(x_n), x_1, \ldots, x_n)$. More specifically let F_n denote the sigma algebra generated by $(J(x_1), \ldots, J(x_n), x_1, \ldots, x_n)$ and let $I(\theta; x_{n+1} | F_n)$ denote the conditional Fisher information of θ at the (n+1)st stage given F_n. Generally, $I(\theta; x_{n+1} | F_n)$ depends on θ. Hence, the optimal value of x_{n+1} is a function of the unknown parameter θ. This problem can be overcome by changing the criterion of optimality in a suitable manner.

In a Bayesian framework we assume that θ is a random variable having a specified distribution function, called the prior distribution. The Bayesian interinspection time x_{n+1}^* is a number maximizing the predictive Fisher information i.e.,

$$E\{I(\theta; x_{n+1}^* | F_n)\} \geq E\{I(\theta; x | F_n)\} \qquad (1.1)$$

for n=1,2,..., and $\forall$ x ϵ X, where the expectation is with respect to the posterior distribution of θ given F_n and X is an appropriate design space. x_1^* is defined as the value belonging to X such that

$$E\{I(\theta;x_1^*)\} \geq E\{I(\theta;x)\} \quad \forall \ x \ \epsilon \ X \ , \tag{1.2}$$

where the expectation is taken with respect to the prior distribution of θ.

Zacks (1973,1977) discussed this problem when the time-till failure follows an exponential distribution

$$F(x,\theta) = 1 - \exp\{-\theta x\},$$

and θ follows a gamma prior distribution. This case will be studied along with some other distribution functions $F(x,\theta)$ and different prior distributions of θ.

It is readily seen that $J(x_1)$ is a binomial random variable with parameters N and $F(x_1;\theta)$. Accordingly, the Fisher information function of $F(x_1;\theta)$ given x_1 is

$$I(F(x_1;\theta)) = N/\{F(x_1;\theta)(1-F(x_1;\theta))\}. \tag{1.3}$$

Therefore the Fisher information of θ given x_1 is (Khan (1980))

$$I(\theta;x_1) = I(F(x_1;\theta))(\frac{\partial}{\partial\theta} F(x_1;\theta))^2 \ . \tag{1.4}$$

Unfortunately, the conditional distribution of $J(x_2)$ given F, is not necessarily binomial under replacement policy A, unless $F(x;\theta)$ is a negative exponential. This complication arises due to the fact that at the 2nd stage we have two kinds of failure distributions - for those components which failed in the previous interval and were replaced by identical components we still have the failure distribution $F(x_2;\theta)$, however, for those components which did not fail during $(0,x_1^*)$ the failure distribution is

$$G(x_2; x_1^*, \theta) = [F(x_1^* + x_2; \theta) - F(x_1^*; \theta)]/[1 - F(x_1^*; \theta)] \quad .$$

Clearly, if F is negative exponential $G(x_2; x_1^*; \theta) = F(x_2; \theta)$. So

$$J(x_2) | F_1 = Y + Z$$

where

$$Y \sim B(J(x_1^*), \ F(x_2; \theta))$$

$$Z \sim B(N - J(x_1^*), \ G(x_2; x_1^*, \theta))$$

and Y, Z are conditionallly independent.

Now $I(\theta; x_2 | F_1)$ does not have the same form as (1.4) for general failure distribution $F(x; \theta)$. This problem, however, can be overcome by the method used by Zacks and Fenske (1973). If $F(x, \theta)$ is the negative exponential distribution, then

$$I(\theta, x_2 | F_1) = I(\theta, x_2)$$

and the second stage optimal interinspection time x_2^* is the value maximizing

$$E\{I(\theta; x_2)\}$$

where the expectation is with respect to the posterior distribution of θ given F_1. Under replacement policy B, on the other hand,

$$J(x_n) | F_{n-1} \sim B(N, F(x_n; \theta)) \ ; \quad n=1,2,\dots \ ,$$

for all distributions $F(x; \theta)$. Hence,

$$I(\theta; x_n | F_{n-1}) = I(F(x_n; \theta))(\frac{\partial}{\partial \theta} F(x_n, \theta))^2 \quad . \tag{1.5}$$

Thus, under replacement policy B the problem has the same structure in all the stages. In the case of negative exponetial distribution the problem remains the same in both kind of replacement policies.

2. One-Stage Designs

In this section we shall discuss the determination of interinspection times for the following cumulative distribution functions:

(i) $F(x,\theta) = 1 - \exp\{-x/\theta\}$; the negative exponential distribution.

(ii) $F(x,\theta) = 2(1 + \exp\{-\theta x\})^{-1} - 1$; $x > 0$, the truncated logistic distribution.

(iii) $F(x;\theta) = (e-1)^{-1}\{e \cdot \exp\{-e^{-\theta x}\} - 1\}$; $x > 0$; the truncated extreme value distribution.

(iv) $F(x,\theta)$ having a symmetric density function and θ is the shift parameter.

These examples will suffice to show the complexity of the algebraic manipulations involved for the solution of this problem. Nevertheless the method is straightforward and could be applied to any $F(x;\theta)$.

(i) The negative exponential distribution:

If $F(x,\theta) = 1 - \exp\{-x/\theta\}$; $x > 0$, $\theta > 0$, then by equation (1.4)

$$I(\theta;x) = Nx^2/\{\theta^4(\exp(x/\theta)-1)\} . \qquad (2.1)$$

The design level x maximizing (2.1) is the solution of the equation

$$\exp(x/\theta)\{2 - x/\theta\} - 2 = 0 ,$$

which is, $x^0 = 1.5936\ \theta$.

The prior information with respect to a prior distribution $G(\theta)$ is defined as the prior expectation of $I(\theta;x)$ and is given by

$$E\{I(\theta;x)\} = Nx^2E\{e^{-x/\theta}/\{\theta^4(1-e^{-x/\theta})\}\}$$

$$= Nx^2E\{\sum_{i=1}^{\infty} \exp\{-kx/\theta\}/\theta^4\}$$

$$= Nx^2 \sum_{k=1}^{\infty} E\{\exp\{-kx/\theta\}/\theta^4\} . \qquad (2.2)$$

The exchange of summation and expectation in (2.2) is permissible, since the function is non-negative.

The Bayes optimal design level, $x^0(G)$, is a value of x satisfying

$$\sum_{k=1}^{\infty} \int_0^{\infty} \frac{e^{-kx/\theta}}{\theta^4} (1 - kx/2\theta) \, dG(\theta) = 0 \qquad (2.3)$$

provided the left hand side of (2.3) converges uniformly in x.

Consider the case when $G(\theta)$ is the inverted gamma distribution with parameters (λ, m). The corresponding density function $g(\theta)$ is given by

$$g(\theta) = \frac{\lambda^m}{\Gamma(m)} \theta^{-m-5} \exp(-\lambda/\theta) \; ; \quad \theta > 0 \quad .$$

For this prior distribution (2.2) reduced to

$$E\{I(\theta;x)\} = cx^2 \sum_{k=1}^{\infty} (\lambda + xk)^{-m-4}$$

$$= c'x^2 \sum_{k=1}^{\infty} (1 + xk/\lambda)^{-m-4}$$

when c and c' are appropriate constants.

Without loss of generality, we discuss the problem of choosing an x to maximize

$$f(x) = \sum_{k=1}^{\infty} x^2/(1 + xk)^{m+4} \quad .$$

The solution x_0 will provide the optimal design level x_1^* for all λ by the relationship

$$x_1^* = \lambda x_0 \quad .$$

Let,

$$g(k,x) = x^2/(1 + xk)^{m+4} \quad .$$

<u>Lemma 2.1.</u>

(i) $\sum\limits_{k=1}^{\infty} g(k,x)$ is uniformly convergent in $X \varepsilon (0,\infty)$.

(ii) $\dfrac{d}{dx} f(x) = \sum\limits_{k=1}^{\infty} \dfrac{d}{dx} g(k,x)$.

(iii) $x_0 \sim 1.5936/(m+2)$ for large values of m.

<u>Proof.</u>

Each function $g(k,x)$ attains a unique maximum at

$$x_k = 2/(k(m+2)) , \quad k=1,2,\dots .$$

Let,

$$M_k = g(k,x_k) = \sup_x g(k,x) , \quad k=1,2,\dots .$$

It follows that

$$\sum\limits_{k=1}^{\infty} M_k \approx \sum\limits_{k=1}^{\infty} \dfrac{1}{k^2} < \infty .$$

Thus (i) is proved by the Weirstrass M-test (Widder 1961). By similar arguments (ii) follows.

To prove (iii) we notice that

$$g(k,x) > g(k+1,x) \quad V \ x \ \varepsilon \ (0,\infty), \quad k=1,2,\dots .$$

Since $g(k,x)$ attains its maximum value at $x_k = 2/(m+2)k$, it readily follows that for each $k=1,2,\dots$

$$g(k,x) = \begin{cases} \downarrow & \text{for} \quad x \ \varepsilon \ [2/(m+2)k \ , \ \infty) \\ \uparrow & \text{for} \quad x \ \varepsilon \ (0,2/(m+2)k) \end{cases} \qquad (2.4)$$

So if $f(x)$ attains its maximum value at $x_{m,0}$ then

$$x_{m,0} \in (0, 2/(m+2)) \quad .$$

Indeed,

$$g(k,x) \leq g(1,x) \; ; \quad k=1,2,\ldots$$

and

$$x_k \leq x_1 \qquad , \quad k=1,2,\ldots \; .$$

According to (ii) one can differentiate under the summation and

$$\frac{d}{dx} f(x) = 0 \quad \text{is equivalent to}$$

$$\sum_{k=1}^{\infty} \{2 - (m+2)kx\} (1 + kx)^{-m-5} = 0 \quad . \tag{2.5}$$

Let $x_m = \alpha_m (m+2)^{-1}$ where for each m, $0 < \alpha_m < 2$. The values of x_m (or α_m) can be determined numerically, for each m from equation (2.5). We show now that for large values of m the solution has a simple approximation. For this purpose we establish first that α_m is a convergent sequence. Indeed, there exists a sub-sequence and a limit point α such that

$$\alpha_{m_\nu} \to \alpha \in (0,2] \quad \text{as} \quad \nu \to +\infty \quad .$$

Hence,

$$\sum_{k=1}^{\infty} \{2 - \alpha_{m_\nu} \cdot k\}(m_\nu + 2 + \alpha_{m_\nu} k)^{-m_\nu-5} = 0 \quad \forall \; \nu > 0 \quad .$$

Equivalently,

$$2 - \alpha_{m_\nu} = \sum_{k=2}^{\infty} (k\alpha_{m_\nu} - 2) \left(\frac{m_\nu + 2 + \alpha_{m_\nu}}{m_\nu + 2 + \alpha_{m_\nu} k} \right)^{m_\nu+5} \quad .$$

Taking the limit as $\nu \to +\infty$, one obtains the equation

$$2 - \alpha = \sum_{k=2}^{\infty} (\alpha k - 2) e^{-\alpha(k-1)} \ ,$$

or

$$2 - \alpha = e^{-\alpha} \left[\frac{\alpha(2-e^{-\alpha})}{(1-e^{-\alpha})^2} - \frac{2}{1-e^{\alpha}} \right] \ ,$$

this equation is further reduced to

$$2 - \alpha - 2e^{-\alpha} = 0 \ .$$

The solution of this equation is approximately $\alpha = 1.5936$. i.e., α_m has an unique limit point. Hence, $x_{m,0} \sim 1.5936/(m+2)$, as $m \to \infty$.

(Q.E.D.)

Some values of α_m, $x_{m,0}$ and x_m^* for various values of m are given in Table 2.1.

The results obtained so far are summarized in the following theorem.

Table 2.1

Values of α_m, x_m^* and $x_{m,0}$ for $\lambda = 1$

m	α_m	x_m^*	$x_{m,0}$
0	1.530	.765	.797
1	1.549	.516	.531
2	1.556	.389	.398
3	1.560	.318	.318
4	1.564	.261	.266
5	1.567	.224	.228
6	1.568	.196	.199
7	1.573	.175	.177
8	1.577	.158	.159
9	1.580	.144	.145
15	1.582	.093	.094
20	1.584	.072	.072
$+\infty$	1.594	0	0

Theorem 2.1.

If N *components are working independently with identical failure distribution*

$$F(x,\theta) = 1 - \exp(-x/\theta)$$

where θ has the inverted gamma (λ, m), $\lambda, m > 0$, *prior distribution, then the first stage optimal interinspection time* $x^*_{m,1}$ *is given by*

$$x^*_{m,1} = \lambda \; \alpha_m (m+2)^{-1} \; ,$$

where the values of α_m *are given in* Table 2.1.

(ii) The truncated logistic distribution.

We consider the logistic distribution truncated to the left, i.e.,

$$F(x,\theta) = 2(1 + \exp(-\theta x))^{-1} - 1 \; ; \quad x > 0 \; , \; \theta > 0 \; .$$

In this case the failure (hazard) rate function,

$$h(x) = \theta(1 + \exp(-\theta x))^{-1} \; ,$$

is an increasing function of x.

Also according to (1.4) the Fisher information function of θ given x is

$$I(\theta,x) = \frac{2Nx^2 \exp(-\theta x)}{\{1-\exp(-\theta x)\}\{1+\exp(-\theta x)\}^2} \; . \tag{2.6}$$

The design level x maximizing (2.6) is the unique solution of the equation

$$2 - \theta x = \theta x(3 - e^{-\theta x})(e^{2\theta x}-1)^{-1}$$

which is $x^0(\theta) = 2.1651/\theta$.

For a given prior distribution of θ, the expected Fisher information function is given by

$$E\{I(\theta,x)\} = 2Nx^2 E\ \{\frac{e^{-\theta x}}{(1-e^{-\theta x})(1+e^{-\theta x})^2}\}$$

$$= 2Nx^2 \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j(j+1)E\{e^{-(k+j)x\theta}\} \tag{2.7}$$

$$= 2Nx^2 \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j(j+1)M_\theta(-(k+j)x)$$

where M_θ denotes the Laplace Steiltjes transform of the c.d.f. of θ. It is difficult to characterize the design level x^* which maximizes (2.7) in the general case. Therefore, consider the special case where θ has a prior gamma distribution, $G(\lambda,m)$, with $m > 2$.

In this case

$$M_\theta(-(k+j)x) = (1 + x(k+j)/\lambda)^{-m} .$$

Hence, without loss of generality we can assume that $\lambda = 1$, and consider, for $m > 2$, the expression

$$E\{I(\theta,x) = cx^2 \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j(j+1)(1 + x(k+j))^{-m} \tag{2.8}$$

Differentiating with respect to x under the summation signs we obtain

$$\frac{\partial}{\partial x} E\{I(\theta;x)\} = c \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j(j+1) \frac{2-x(k+j)(m-2)}{(1+x(k+j))^{m+1}} .$$

Let $\alpha_m = x_m(m-2)$ where x_m is the solution of the equation

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (-1)^j(j+1) \frac{2 - \alpha_m(k+j)}{(1 + (k+j)\alpha_m(m-2)^{-1})^{m+1}} = 0\ \forall\ m , \tag{2.9}$$

obtained by equating the partial derivative to zero. To see why α_m is a convergent sequence as $m \to +\infty$, we note that for large m

$$(1 + (k+j)\alpha_m(m-2)^{-1})^{m+1} \approx e^{\alpha_m(k+j)}$$

i.e., $\sum_{k=1}^{\infty} \sum_{j=0}^{\infty} (-1)^j (j+1) \dfrac{2 - \alpha_m(k+j)}{e^{\alpha_m(k+j)}} \approx 0$ for large m

$$\Rightarrow 2(1 - e^{-2\alpha_m})(1 + e^{-\alpha_m}) - \alpha_m(1 + 2e^{-3\alpha_m} + e^{-2\alpha_m}) \approx 0 \quad \text{for large m}$$

$$\Rightarrow \alpha_m \approx 2.1651 \quad \text{for large m}$$

Hence, $\alpha_m \to 2.1651$ as $m \to +\infty$. As before, when $\lambda \neq 1$, we have

$$x_m = \lambda \alpha_m (m-2)^{-1}.$$

In the following table we give the optimal solution in terms of α_m and x_m for small values of m, as determined by numerical solution of (2.9).

Table 2.2

$\lambda = 1$

m	α_m	x_m^*	$x_m = 2.1651(m-2)^{-1}$
3	2.04687	2.04687	2.1651
4	2.06844	1.03422	1.0825
5	2.08836	0.69612	0.7217
6	2.10172	0.52543	0.5413
9	2.12331	0.30333	0.3093
10	2.12768	0.26596	0.2706
15	2.14032	0.16464	0.1665
$+\infty$	2.16509	0.0	0.0

(iii) <u>The truncated extreme value distribution.</u>

The truncated extreme value distribution is given by

$$F(x,\theta) = (e-1)^{-1}(e \exp(e^{-\theta x}) - 1) \; ; \; x > 0 \, , \; \theta > 0 \, .$$

The Fisher information function of θ at a design level x is

$$I(\theta,x) = \frac{Nex^2 \exp(-2e^{-\theta x})\exp(-2\theta x)}{[e\,\exp(e^{-\theta x})-1][1-\exp(-e^{-\theta x})]} \ .$$ (2.10)

The design level x maximizing (2.10) satisfies the equation

$$2(1 + 1/\ell n\,z)/z = [(1+e)\exp(-z)-2]/[(1-e^{-z})(e\,e^{-z}-1)] \ ,$$

where $z = \exp(-\theta x)$. The solution of this equation yields

$$x = -\ell n(.144188)/\theta = 1.9366/\theta \ .$$ (2.11)

In analogy to the previous two cases we may conjecture that if the prior distribution of θ is gamma $G(\lambda,m)$, the optimal design level, at which $E\{I(\theta;x)\}$ is maximized, is

$$x_m \approx \lambda(1.9366)/(m-2) \quad \text{for large m.}$$

Theorem 2.2

Suppose that the Fisher information function is of the form $I(\theta;x) = \sum_k a_k x^p \exp\{-\theta kx\}; p \geq 0$, *where* a_k *do not depend on* θ *and x; and that* $I(\theta;x)$ *is maximized at* $x^0(\theta) = \alpha/\theta$. *Furthermore, assume that*

(i) $\sum_k a_k x^{p-1}(p-xk\theta)e^{-\theta xk}$ is uniformly convergent in x,

(ii) $\sum_k a_k \theta^{m-1} e^{-(\lambda+xk)\theta}$ is uniformly convergent in θ,

(iii) $\sum_k a_k x^{p-1} \frac{(p-(m-p)xk)}{(1+xk)^{m+1}}$ is uniformly convergent in x,

then if the prior distribution of θ is the gamma $G(\lambda,m)$, $E\{I(\theta;x)\}$ is maximized at $\lambda\alpha/(m-p)$ for large values of m.

Proof. Since $I(\theta;x)$ is maximized at $x = \alpha/\theta$, therefore

$$\frac{\partial}{\partial x} I(\theta;x) = 0 \quad \text{for} \quad x = \alpha/\theta . \tag{2.12}$$

It follows from (i) that $\sum\limits_{k} a_k (p - xk\theta) e^{-\theta x k} = 0$ for $x = \alpha/\theta$. From assumption (ii) we obtain that

$$E\{I(\theta;x)\} = \sum_{k} a_k \; x^p \; E\{e^{-\theta x k}\}$$

$$= \sum_{k} a_k \; x^p (1 + xk/\lambda)^{-m} . \tag{2.13}$$

without loss of generality, assume that $\lambda = 1$.

Furthermore, from (iii) we get

$$\frac{\partial}{\partial x} E\{I(\theta;x)\} = x^{p-1} \sum_{k} a_k \; \frac{p - (m-p)xk}{(1+xk)^{m+1}} . \tag{2.14}$$

Let x_m be the value of x for which (2.12) is equal to zero, and let $\alpha_m = (m-p)x_m$. We thus obtain the equation

$$\sum_{k} a_k (p - \alpha_m k)(1 + \frac{\alpha_m}{m-p} k)^{-m-1} = 0 \; \forall \; m. \tag{2.15}$$

Moreover,

$$0 = \sum_{k} a_k (p - \alpha_m k)(1 + \frac{\alpha_m k}{m-p})^{-m-1}$$

$$\approx \sum_{k} a_k (p\alpha_m k) e^{-\alpha_m k} \quad \text{for large } m$$

$$\Rightarrow \alpha_m \approx \alpha \quad \text{for large } m ,$$

since $(1 + \frac{\alpha_m k}{m-p})^{m+1} \approx e^{\alpha_m k}$. Hence, $x_m \sim \lambda\alpha/(m-p)$, for large m.

(Q.E.D.)

Remarks. 1. For the truncated extreme value distribution

$$I(\theta;x) = \frac{Nex^2 \exp(-2e^{-\theta x})e^{-2\theta x}}{[e \exp(-e^{-\theta x}) - 1][1 - \exp(-e^{-\theta x})]} \quad ,$$

could be written as

$$I(\theta;x) = Nx^2 \sum_{j,k,\ell,q,\nu=0}^{\infty} a_{j,k,\ell,q,\nu} e^{-(2+j+\ell+\nu)\theta x} \quad ,$$

where

$$a_{j,k,\ell,q,\nu} = \frac{(-2)^j(-k)^\ell(q+1)^\nu e^{-q}}{j!\ \ell!\ q!\ \nu!}$$

2. The conditions (i), (ii), (iii), of Theorem 2.2 could be relaxed after assuming uniform convergence of the series

$$\sum_k a_k \exp(-\theta xk) \quad .$$

The following theorem generalizes the last theorem for the prior distributions belonging to the class of infinitely divisible distributions.

Theorem 2.3.

If the Fisher information function of θ *is of the following form*

$$I(\theta;x) = \sum_k a_k x^p \exp(-\theta xk) \ ;\ p \geq 0 \ ,$$

where a_k *do not depend on* θ *or x; and if* $I(\theta;x)$ *is maximized at the design level* α/θ *and* θ *has a prior distribution belonging to the class of infinitely divisible distributions with parameter* m, *then* $E\{I(\theta;x)\}$ *is maximized at the design level* $x_m \approx \alpha/E\{\theta\}$ *for large* m *provided*

(i) $\sum_k a_k x^{p-1}(p-xk\theta)\exp(-\theta xk)$ *is uniformly convergent in* x.

(ii) $\sum_k a_k g(\theta) \exp(-\theta xk)$ *is uniformly convergent in* θ.

(iii) $\sum_k a_k x^{p-1} E\{(p-x\theta k)\exp(-\theta xk)\}$ *is uniformly convergent in* x.

(iv) $m\, x_m$ *is a bounded sequence.*

Proof.

Since $I(\theta;x)$ is maximized at $x = \alpha/\theta$,

$$\frac{\partial}{\partial x}\, I(\theta;x) = 0 \quad \text{for} \quad x = \alpha/\theta \tag{2.16}$$

Now, according to (ii),

$$E\{I(\theta;x)\} = \sum_k a_k x^p E\{e^{-\theta xk}\}$$

$$= \sum_k a_k x^p \phi(xk)\ ,$$

where $\phi(t) = \exp(-m\psi(t))$, such that $\psi(0) = 0$.

Hence, by (iii)

$$\frac{\partial}{\partial x}\, E\{I(\theta;x_m)\} = \sum_k a_k\{p + mxk\ \psi'(xk)\}\ \exp\{-m\psi(xk)\} = 0.$$

Let $\alpha_m = m\, x_m$. We thus have the equation

$$\sum_k a_k\{p + k\alpha_m \psi'(\frac{k\alpha_m}{m})\}\exp(-m\ \psi(\frac{k\alpha_m}{m})) = 0 \quad \forall\, m\ , \tag{2.17}$$

since $\alpha_m = mx_m$ is bounded, there exists a subsequence m_ν such that $m_\nu \cdot x_{m_\nu}$ converges to a finite limit, say α', as $\nu \to +\infty$. Hence, $x_{m_\nu} \to 0$ as $\nu \to +\infty$.

Expanding,

$$\psi(x) = \psi(0) + x\psi'(0) + \frac{x^2}{2}\ \psi''(0) + \dots$$

we have

$$\psi(\frac{\alpha_m}{m}\, k) = \frac{\alpha_m}{m}\, k\ \psi'(0) + (\frac{\alpha_m}{m})^2\, \frac{k^2}{2}\ \psi''(0) + \dots$$

$$-m\psi(\frac{\alpha_m}{m}\, k) = -\alpha_m\, k\ \psi'(0) - \frac{(\alpha_m k)^2}{2m}\ \psi''(0) + \dots\ .$$

Therefore,

$$\exp(-m_\nu \psi(\frac{k\alpha_{m_\nu}}{m_\nu})) \to \exp(-\alpha'k\psi'(0)) \quad \text{as} \quad \nu \to +\infty .$$

Or,

$$\sum_k a_k \{p - \alpha'k(-\psi'(0))\}\exp(-\alpha'k\psi'(0)) = 0 .$$

Hence, $\alpha' = \alpha/\{-\psi'(0)\}$ and $\alpha_m \to \alpha/\{-\psi'(0)\}$, which implies that

$$x_m \approx \alpha/E\{\theta\} \quad \text{for large values of } m .$$

$$(Q.E.D.)$$

Remarks:

(i) As the previous examples show, the approximation of x_m is better if $F(x) = -m\psi'(0)$ is replaced by $-(m-p)\psi'(0)$.

(ii) Zacks (1973,1977) considered the case when

$$F(x;\theta) = 1 - \exp(-\theta x)$$

$$with \quad \theta \sim G(\lambda,m) ; \ m > 2 ,$$

by numerical results it was conjectured that

$$x_m = 1.5936 \ \lambda/(m-2) .$$

By the above theorem we see that this is an aysmptotic result for large values of m. However for small values of m, $x_m = \alpha_m \lambda/(m-2)$ where α_m are given in Table 2.1.

(iv)   F(x,θ) having symmetric density function.

So far we have discussed the cases when the parameter of interest is a scale parameter. However, some times we are interested in the shift parameter.

Consider the case where $F(x;\theta)$ is the logistic distribution with a shift parameter

$$F(x;\theta) = [1 + \exp(-x+\theta)]^{-1} , \quad -\infty < x < \infty, \ -\infty < \theta < \infty .$$

It is readily verified that

$$I(\theta;x) = \frac{Ne^{-(x-\theta)}}{[1 + \exp(-x+\theta)]^2}$$

$$= NF(x;\theta)[1 - F(x;\theta)] .$$

Therefore $I(\theta;x)$ is maximized at the median of $F(x;\theta)$, furthermore, since $F(x;\theta)$ is symmetric, $x^0(\theta) = \theta$. The Bayesian estimation of the median of the logistic distribution was discussed by Freeman (1970). However, dynamic programming methods were used to obtain sequentially optimal designs, up to three stages, when the prior distribution of θ was taken to be the conjugate prior. We shall consider this problem from the point of view of maximizing the Fisher information.

For any symmetric c.d.f. $F(x;\theta)$, the Fisher information function of θ given x is

$$I(\theta;x) = N\{\frac{\partial}{\partial\theta} F(x;\theta)\}^2/[F(x;\theta)(1 - F(x;\theta))]$$

$$= N f^2(x;\theta)/[F(x;\theta)(1 - F(x;\theta))] , \qquad (2.18)$$

where $f(x;\theta)$ is the density of $F(x;\theta)$.

Consider the case when $f(x;\theta)$ is symmetric about $x = \theta$. Note that $f(x;\theta)$ is defined on the real line R. Otherwise, the Fisher information function of θ does not exist. Furthermore,

$$\frac{d}{dx} f(x;\theta) \to 0 \quad \text{as} \quad x \to +\infty.$$

We also note that in order that $I(\theta;x)$ be maximized at $x = \theta$ we need the necessary conditions that

    (i) $f'(\theta;\theta) = 0$

    (ii) $f''(\theta;\theta) < 0$

where prime denotes differentiation with respect to x.

This implies that the density $f(x;\theta)$ is maximized at $x = \theta$. Moreover, if $f(x;\theta)$ attains a minimum value at $x = \theta$, then $I(\theta;x)$ is minimized in a neighborhood around θ.

Accordingly, we consider only those symmetric densities which are bell shaped (e.g., logistic, normal, etc., commonly known as Logit and Probit models in bioassay).

Theorem 2.4.

Let $I(\theta;x)$ *be the Fisher information function of the shift parameter* θ *at the design level* x *for distributions* $F(x;\theta)$ *symmetric about* θ. *If*

(i) $f'(x;\theta) \leq 0 \ V \ x \geq \theta$,

(ii) $f''(\theta,\theta) + 2 \ f^{(3)}(\theta,\theta) < 0$.

Then $x = \theta$ *is a point of maxima of* $I(\theta;x)$.

Further more, if

(iii) $\int_0^x f(t,\theta)dt < 1/2[1 - (f(x,\theta)/f(\theta,\theta))^2]^{1/2} \ V \ x > \theta$.

Then

$$I(\theta;x) \ attains \ its \ maximum \ value \ at \ x = \theta.$$

Proof.

$$I(\theta;x) = N \ f^2(x;\theta)/[F(x;\theta)(1 - F(x;\theta))] \ .$$

Since

$$I(\theta;x) \sim N \ f^2(x;\theta)/[1 - F(x;\theta)] \ \ as \ \ x \to +\infty \ ,$$

we obtain

$$\lim_{x \to +\infty} I(\theta;x) = -2 \ N \lim_{x \to +\infty} f'(x;\theta) = 0 \ .$$

Similarly $\lim_{x \to -\infty} I(\theta;x) = 0$.   Thus, there exists a design level x such that $I(\theta;x)$ is maximized. Now differentiating $\log I(\theta;x)$ and equating to zero we get

$$\frac{2f'(x,\theta)}{f(x;\theta)} - \frac{f(x;\theta)}{F(x;\theta)} + \frac{f(x;\theta)}{1 - F(x;\theta)} = 0 \ .$$

If $x = \theta$, then $f'(x;\theta) = 0$ and $F(x;\theta) = 1 - F(x;\theta)$. Therefore $\frac{\partial}{\partial x} \log I(\theta;x)\big|_{x=\theta}$

$= 0$ i.e., $x = \theta$ is a point of extremum of $I(\theta;x)$.

By twice differentiating the $\log I(\theta;x)$ we get

$$\frac{\partial^2}{\partial x^2} \log I(\theta;x) = 2 \frac{f(x;\theta)f''(x;\theta) - (f'(x;\theta))^2}{f^2(x;\theta)}$$

$$+ \frac{(1-F(x;\theta))f'(x;\theta) + f^2(x;\theta)}{[1 - F(x;\theta)]^2} - \frac{F(x;\theta)f'(x;\theta) + f^2(x;\theta)}{(F(x;\theta))^2}$$

Now,

$$f'(\theta;\theta) = 0 \ , \ 1 - F(\theta,\theta) = F(\theta;\theta) = 1/2$$

we get

$$\frac{\partial^2}{\partial x^2} \log I(\theta;x)\bigg|_{x=\theta} = 2 \frac{f''(\theta,\theta)}{f(\theta,\theta)} + 4 \ f^2(\theta,\theta) < 0 \quad \text{by (ii)} \ ,$$

i.e., $I(\theta;x)$ attains a local maximum value at $x = \theta$.

Now by (iii), we get

$$F(x;\theta)(1 - F(x;\theta)) = (\tfrac{1}{4} - (\int_0^x f(t;\theta)dt)^2)$$

$$> \tfrac{1}{4} \frac{f^2(x,\theta)}{f^2(\theta,\theta)} \quad \forall \ x > \theta$$

therefore, $$4 \ f^2(\theta,\theta) > \frac{f^2(x,\theta)}{F(x,\theta)(1 - F(x;\theta))} \quad \forall \ x > \theta$$

or,

$$I(\theta,\theta) > I(\theta,x) \quad \forall \ x > \theta.$$

Since $f^2(x,\theta)$ is symmetric about $x = \theta$ one can imply that $I(\theta,x)$ is also symmetric about $x = \theta$. Hence,

$$I(\theta,\theta) > I(\theta,x) \quad \forall \ x \neq \theta \ .$$

Therefore $I(\theta,x)$ attains its maximum at $x = \theta$.

(Q.E.D.)

Corollary.

If $F(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{- \tfrac{1}{2} (t-\theta)^2\}dt$. Then $I(\theta;x)$ is maximized at $x = \theta$.

To see that the normal distribution satisfies the conditions of the above theorem, we see that,

(i) is trivial.

(ii) $f''(\theta,\theta) = \frac{1}{\sqrt{2\pi}}$ and $f^{(3)}(\theta,\theta) = \frac{1}{(2\pi)^{3/2}}$

$f''(\theta,\theta) + 2f^{(3)}(\theta,\theta) = \frac{2}{(2\pi)^{3/2}} - \frac{1}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} (\frac{1}{\pi} - 1) < 0$.

(iii) It is well known that (see, for example, D'Ortenzio (1965) or, Johnson and Kotz (1970) Chapter 13).

$$\phi(x) \leq \tfrac{1}{2}[1 + \sqrt{1 - e^{-x^2}}\ ] \quad \forall\ x\ .$$

Therefore

$$F(x,\theta) = \phi(x-\theta) \leq \tfrac{1}{2}\left[1 + \sqrt{1 - e^{-(x-\theta)^2}}\right]$$

$$= \tfrac{1}{2}\left[1 + \sqrt{1 - (\frac{f(x,\theta)}{f(\theta,\theta)})^2}\right] \quad .$$

Hence,

$$\int_{0}^{x} f(t,\theta)dt < \tfrac{1}{2}[1 - (f(x,\theta)/f(\theta,\theta))^2]^{1/2} \quad \forall\ x > \theta.$$

Hence, by the above theorem, the corollary is proved.

In order to construct first stage Bayes optimal interinspection design level, we first consider the simplest case when the prior distribution of θ is rectangular (a,b). In this case,

$$E\{I(\theta,x)\} = E\ \frac{N\ f^2(x-\theta)}{F(x-\theta)[1 - F(x-0)]}$$

$$= N \int_{a}^{b} \frac{f^2(x-\theta)}{F(x-\theta)(1 - F(x-\theta))}\ d\theta$$

$$= N \int_{x-b}^{x-a} \frac{f^2(y)}{F(y)[1 - F(y)]}\ dy$$

$$\leq N \int_{\frac{b-a}{2}}^{\frac{b-a}{2}} \frac{f^2(y)}{F(y)[1 - F(y)]}\ dy$$

i.e. $x^* = \dfrac{a + b}{2}$.

Hence the first stage optimal Bayes design level is the median of the uniform distribution. We can generalize the above result to any symmetric prior distribution as follows.

In general let $g(t)$ be a symmetric bell shaped density symmetric about θ. Then for the prior distribution of θ having density $g(\theta-\lambda)$ defined over the real line, the expected Fisher information function of θ is

$$E\{I(\theta,x)\} = N \int_{-\infty}^{\infty} \frac{f^2(x-\theta)}{F(x-\theta)(1 - F(x-\theta))} \, g(\theta-\lambda) \, d\theta$$

$$= N \int_{-\infty}^{\infty} \frac{f^2(y)}{F(y)\{1 - F(y)\}} \, g(x-y-\lambda) \, dy$$

$$= \int_{-\infty}^{\infty} I(y) \, g(x-y-\lambda) \, dy$$

where $I(y) = f^2(y)/F(y)\{1 - F(y)\}$.

For the sake of maximization of $E\{I(\theta,x)\}$ we see that without loss of generality we can assume that $\lambda = \theta$. For the following theorem we assume that $F(x,\theta) = \int_{-\infty}^{x} f(t-\theta) \, dt$ when f satisfies the conditions of the previous theorem such that $I(\theta,x)$ is also bell shaped.

Theorem 2.5.

Let $I(\theta,x)$ be the Fisher information function of θ. Suppose that θ has a prior distribution $G(\theta-\lambda)$ defined on R, such that its p.d.f. g is symmetric about the median λ. Then $x^* = \lambda$ is the point of maxima of $E\{I(\theta,x)\}$ provided

$$\int_{-\infty}^{\infty} g''(\theta) \, d\theta \le 0 \quad . \tag{2.19}$$

Proof. Assume that $\lambda = 0$. Then

$$E\{I(\theta;x)\} = \int_{-\infty}^{\infty} I(y)\ g(x-y)\ dy \ .$$

Therefore,

$$\frac{\partial}{\partial x} E\{I(\theta;x)\} = \int_{-\infty}^{\infty} I(y)\ g'(x-y)\ dy$$

$$= 0 \quad \text{if} \quad x = 0 = \lambda .$$

This follows by considering the fact that $g'(\cdot)$ is an odd function and $I(y)$ is symmetric about $y = 0$. Hence, $x = \lambda = 0$ is an extremal point of the expected Fisher information function. Furthermore,

$$\frac{\partial^2}{\partial x^2} E\{I(\theta;x)\} = \int_{-\infty}^{\infty} I(y)\ g''(x-y)\ dy \ .$$

Let $\pm K$ denote the inflexion points of the density $g(\cdot)$. Hence,

$$\frac{\partial^2}{\partial x^2} E\{I(\theta,x)\}\bigg|_{x=0} = \int_{-\infty}^{\infty} I(y)\ g''(-y)\ dy$$

$$= \int_{-\infty}^{\infty} I(y)\ g''(y)\ dy \quad \text{because } g'' \text{ is an even function.}$$

$$= (\int_{-\infty}^{-K} + \int_{-K}^{K} + \int_{K}^{\infty})\ I(y)\ g''(y)\ dy$$

$$= 2 \int_{-\infty}^{-K} I(y)\ g''(y)\ dy + \int_{-K}^{K} I(y)\ g''(y)\ dy$$

$$< 2\ I(K) \int_{-\infty}^{-K} g''(y)\ dy + I(K) \int_{-K}^{K} g''(y)\ dy \ .$$

This result follows by noting that

$$g''(\theta) \begin{cases} < 0 \ \forall\ \theta \in (-K,K) \\ \geq 0 \ \forall\ \theta \notin (-K,K) \end{cases} .$$

Hence, $\dfrac{\partial^2}{\partial x^2} E\{I(\theta,x)\}\Big|_{x=0} < I(K) \displaystyle\int_{-\infty}^{\infty} g''(y)\ dy \le 0 = x^* = \lambda$ is the point of maxima of the expected Fisher information function.

(Q.E.D.)

Example

If $\theta \sim N(0,\sigma^2)$, then

$$\int_{-\infty}^{\infty} g''(y)\ dy = E(\theta^2/\sigma^4 - \frac{1}{\sigma^2})$$

$$= 0 \quad .$$

Therefore the normal density satisfies the condition of the above theorem.

3.   Optimal Design of Two Consecutive Design Levels

We shall consider the exponential fail .... section. Therefore the results obtained are applicable under both policies (A) and (B).

If $J_1(x_1^*)$, $J_2(x_2)$ denote the number of components failing during $(0,x_1^*)$ and $(x_1^*, x_2^* + x_2)$ respectively, then

$$J_2(x_2)\,|\,J_1(x_1^*), x_2 \sim B(N, F(x_2,\theta))$$

and the conditional Fisher information of θ given $J_1(x_1^*)$ and x_2 is

$$I(\theta,x_2) = \frac{N\,x_2^2}{\theta^4 (e^{x_2/\theta} - 1)} \quad .$$

THus, if θ has an inverted gamma (λ,m) prior distribution, then the posterior distribution of θ given $J_1(x_1^*) = j$ is

$$f(\theta\,|\,j) = C(x_1^*,j) b(j,N,F(x_1^*,\theta))\, \frac{1}{\theta^{m+1}}\, e^{-\lambda/\theta} \ , \quad 0 < \theta \quad .$$

It follows that

$$E\{I(\theta;x_2)|J_1(x_1^*) = j,x_2\} = c'(x_1^*,j)x_2^2 \int_0^\infty \frac{1}{\theta^{m+5}} \frac{(1-e^{-x_1^*/\theta})^j \exp(-\frac{x_1^*(N-j)+\lambda+x_2)}{\theta})}{(1-e^{-x_2/\theta})} \, d\theta$$

$$= c'(x_1^*,j)x_2^2 \sum_{k=1}^\infty \sum_{\ell=0}^j \binom{j}{\ell}(-1)^\ell \frac{1}{(\lambda + x_1^*(N=j+\ell) + kx_2)^{m+4}} \tag{3.1}$$

In the special case of $j = 0$, the optimal second stage interinspection time, say, $x_{2,0}^*$ is the point x_2 at which

$$\sum_{k=1}^{\cdots} \frac{x_2^2}{(\lambda + Nx_1^* + x_2 k)^{m+4}} \quad \text{is maximized} \quad .$$

Applying Lemma 2.1 we obtain,

$$x_{2,0}^* = \frac{(\lambda + N x_1^*)\alpha_m}{m + 2} \tag{3.2}$$

where α_m is given in Table 2.1 and $\alpha_m \to 1.5936$ as $m \to +\infty$.

Let $x_{2,j}^*$ denote the second stage optimal interinspection time given that j number of components failed during $(0,x^*)$. We shall call an inspection "redundant" if the number of components failed is zero.

$$\text{Intuitively} \quad x_1^* \div x_{2,0}^* \quad .$$

This is because no component failed during $(0,x_1^*)$. So we would give the system more than x_1^* units of time so that some of the components may fail and avoid any redundant inspection.

Also if $j(>0)$ number of components failed during $(0,x_1^*)$, then the inspection was not redundant. So the next interinspection time $x_{2,j}^*$ should not be as large as had the first inspection produced a redundant inspection i.e.,

$$x_{2,j}^* \leq x_{2,0}^* \quad \forall \quad j = 0,1,2,\ldots,N \quad .$$

On the other hand, if j = N, i.e., if all the components had failed at the time of inspection, we let the system run too long without any inspection. Therefore,

$$x^*_{2,N} \leq x^*_1 \ .$$

And if j(<N) number of components failed during $(0, x^*_1)$, then the 2nd stage interinspection time $x^*_{2,j}$ should not be as small as $x^*_{2,N}$, i.e.,

$$x^*_{2,N} \leq x^*_{2,j} \quad \forall \quad j = 0,1,2,\ldots,N.$$

We can combine the above intuitive results as

$$x^*_{2,N} \leq x^*_{2,N-1} \leq \cdots \leq x^*_{2,j_0} \leq x^*_1 \leq x^*_{2,j_0+1} \leq \cdots \leq x^*_{2,1} \leq x^*_{2,0} \ , \qquad (3.3)$$

where j_0 is some integer belonging to the set $\{1,2,\ldots,N-1\}$. Again, intuitively, we feel that $j_0 = .80(N)$, because the maximal Fisher information of θ is obtained at the 80th percentile of the exponential distribution.

The following lemma verifies the above intuitive results.

Lemma 3.1.

$E\{I(\theta,x_2|F_1)\}$ *is maximized at* $x^*_{2,j}$ *where*

$$x^*_{2,j} \approx \frac{\alpha_m}{m+2} \{\lambda + (N-j)x^*_1 - jx^2_1/(e^{\alpha_m/\lambda} - 1)\} \qquad (3.4)$$

and $\alpha_m \to 1.5936$ *as* $m \to +\infty$. (α_m are given in Table 2.1).

Proof.

$$E\{I(\theta;x_2|F_1)\} = C(x^*_1,j)x^2_2 \sum_{k=1}^{\infty} \sum_{\ell=0}^{j} \binom{j}{\ell}(-1)^\ell \frac{1}{(\lambda+x^*_1(N-j+\ell)+kx_2)^{m+4}} \ .$$

Let $f(x) = x^2 \sum\limits_{k=1}^{\infty} \sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell} \dfrac{1}{(a_{\ell,m} + xk)^{m+4}}$, where $a_{\ell,m} = \lambda + x_1^*(N - j + \ell)$.

Notice that x_1^* may depend on m. Approximate the function $f(x)$ by using integrals instead of the summation $\sum\limits_{k=1}^{\infty}$.

Note that there exists a constant $c_{m,j,\ell}$ such that

$$f(x) = x^* \sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1) \int\limits_{c_{m,j,\ell}} \dfrac{dk}{(a_{\ell,m} + xk)^{m+4}} \ .$$

Furthermore,

$$f(x) \approx x^2 \sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell} \int\limits_{c_m} \dfrac{dk}{(a_{\ell,m} + xk)^{m+4}} \ ,$$

where $c_m \in [0,1)$. (We are approximating $f(x)$ by choosing c_m independent of ℓ and j.) Now,

$$\int\limits_{c_m}^{\infty} \dfrac{dk}{(a_{\ell,m}+xk)^{m+4}} = \dfrac{1}{x(m+3)(a_{\ell,m} + c_m x)^{m+3}} \ .$$

Therefore,

$$f(x) = \sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell} \dfrac{x}{(m+3)(a_{\ell,m} + c_m x)^{m+3}} \ .$$

Moreover, $\dfrac{d}{dx} f(x) = 0$ implies the equation

$$\sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1) \dfrac{a_{\ell,m} + c_m x - c_m x(m+3)}{(a_{\ell,m} + c_m x)^{m+4}} = 0 \ .$$

Let $y_m = c_m x(m+2)$. Therefore,

$$\sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell} \dfrac{a_{\ell,m} - y_m}{\left(\lambda + \dfrac{\alpha_m(N-j+\ell)}{m+2} + \dfrac{y_m}{m+2}\right)^{m+4}} = 0 \ ,$$

and for large m,

$$\sum\limits_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell} \dfrac{a_{\ell,m} - y_m}{e^{\alpha_m(N-j)/\lambda} \cdot e^{\alpha_m \cdot \ell/\lambda} \, e^{y_m/\lambda}} = 0 \ ,$$

or

$$\sum_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell}\left[1 + \frac{x_1^*}{\lambda}(N-j) - \frac{y_m}{\lambda} + \frac{x_1^*\ell}{\lambda}\right]e^{\alpha_m\ell/\lambda} = 0 \quad .$$

From this equation we obtain

$$(1 + \frac{x_1^*}{\lambda}(N-j) - \frac{y_m}{\lambda})(1 - e^{-\alpha_m/\lambda})^j - \frac{x_1^*}{\lambda} \cdot j\, e^{-\alpha_m/\lambda}(1-e^{-\alpha_m/\lambda})^{j-1} = 0 \quad ,$$

from which

$$\frac{y_m}{\lambda} = [1 + \frac{x_1^*}{\lambda}(N-j) - \frac{x_1^*}{\lambda}\, j/(e^{\alpha_m/\lambda} - 1)] \quad ;$$

or

$$y_m = \lambda + x_1^*(N-j) - x_1^*\, j/(e^{\alpha_m/\lambda} - 1) \quad ,$$

$$x_{m,j} = \frac{1}{c_m(m+2)}\, [\lambda + x_1^*(N-j) - x_1^*\, j/(e^{\alpha_m/\lambda} - 1)] \quad .$$

Now we note that if $j = 0$, then $c_m = \frac{1}{\alpha_m}$. Therefore,

$$x_{2,j}^* \approx \frac{\alpha_m}{m+2}\, [\lambda + x_1^*(N-j) - x_1^*\, j/(e^{\alpha_m/\lambda} - 1)]$$

for large values of m.

(Q.E.D.)

The following table shows that the approximation is close even for small values of m.

Incidently, it is readily seen that the optimal Bayes interinspection time $x_{2,j}^b = E(\theta|j, x_1^*)$ is

$$x_{2,j}^b = \frac{\sum_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell}\, [\lambda + (N-j+\ell)x_1^*]^{-m+1}}{(m-1)\sum_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell}\, [\lambda + (N-j+\ell)x_1^*]^{-m}} \quad . \tag{3.5}$$

Table 3.1

$$\lambda = 1, \; N = 1, \; j = 0$$

m	1	2	3	4	5
$x_{2,0}^{*}$	.79	.54	.41	.33	.27
$\hat{x}_{2,0}$	.79	.54	.41	.33	.27
x_{1}^{*}	.516	.389	.312	.261	.224

Note that $x_{2,0}^{*}$ is always greater than x_{1}^{*} as expected.

$$\lambda = 1, \; N = 5$$

	j	0	1	2	3	4	5	α_m
m=1	$x_{2,j}^{*}$	1.88	1.31	0.95	0.70	0.51	0.36	1.549
	$\hat{x}_{2,j}$	1.88	1.53	1.10	0.84	0.49	0.48	
m=5	$x_{2,j}^{*}$	0.48	0.39	0.33	0.27	0.22	0.18	1.567
	$\hat{x}_{2,j}$	0.48	0.41	0.35	0.28	0.22	0.16	
m=9	$x_{2,j}^{*}$	0.25	0.22	0.19	0.16	0.14	0.12	1.574
	$\hat{x}_{2,j}$	0.25	0.22	0.19	0.17	0.14	0.12	
m=20	$x_{2,j}^{*}$	0.10	0.09	0.08	0.08	0.07	0.07	.583
	$\hat{x}_{2,j}$	0.10	0.09	0.08	0.08	0.07	0.07	

$$\hat{x}_{2,j} = \frac{\alpha_m}{m+2} \left(1 + (N-j)x_1^* - jx_1^*(e^{\alpha_m} - 1)^{-1} \right)$$

$$x_1^* = 1.5936/(m+2) \; .$$

4. Two Stage Optimal Designs.

By combining the results of Sections 2 and 3 we can construct two stage optimal designs by using a procedure discussed by Zacks (1973) and (1977).

Now

$$I(\theta, x) = N\{\frac{\partial}{\partial\theta} F(x,\theta)\}^2 \{F(x,\theta)(1-F(x,\theta))\}^{-1}.$$

In order to construct two stage optimal Bayes design we select n components,

$$0 \le n \le N ,$$

and perform our experiment at the design level x, using only n components. Then at the second stage the experiment is performed at the design level x_2 for the remaining N-n components. The construction of the two stage design involves finding the vector (n_0, x_1^0, x_2^0) such that

$$(N-n)E\{I(\theta; x_2)|F_1\} + nE\{I(\theta; x_1)\} \le (N-n_0)E\ I\{I(\theta; x_2^0)|F\} + n_0 E\{I(\theta; x_1^0)\} \quad (4.1)$$

for all $n \in \{0, 1, \ldots, N\}$, $x_1, x_2 \in X$.

That is the use of (n_0, x_1^0, x_2^0) gives global maximal Fisher information during two stages.

We obtain (n_0, x_1^0, x_2^0) by the following steps:

(i) Find the optimal x_2^0 given F_1 and n. Since x_2^0 is independent on n, x_1 and $J(x_1)$, define

$$g(n, x_1) = E[E\{I(\theta; x_2^0)|F_1\} (N-n) + n\ I(\theta; x_1)] .$$

(ii) Determine n_0, x_1^0 such that $g(n, x_1)$ is maximized.

(iii) Redefine x_0^0 by using n_0 and x_1^0 and $J(x_1^0)$.

By the lemma (3.1)

$$x_2^0(n, x_1, J(x_1)) = \alpha_m (m+2)^{-1}\{\lambda + (n-J(x_1))x_1 - (e^{\alpha_m/\lambda}-1)^{-1}x_1 J(x_1)\}.$$

In order to determine n_0, x_1^0, one can use the computer for specified values of N, λ, m. If n = 0 or N, the solution is simple to characterize.

Relative efficiency of the design is defined as follows:

For the first stage, the relative efficiency $RE(\theta|\lambda,m)$ is

$$RE(\theta|\lambda,m) = I(\theta;x_1^b)/I(\theta;x_m^*)$$

where x_1^b denotes the Bayes design level for the first stage, i.e.,

$$x_1^b = 1.5936 \ \lambda/m \ .$$

For the two stage design the relative efficiency function of θ is defined in a similar manner, by using x_2^b, where x_2^b is given by (3.5).

For numerical values of the relative efficiency function for different values of m, see Zacks (1973).

5. Kth Stage Optimal Interinspection Times.

In the present section we consider the exponential failure distribution.

If $j_1, j_2, \ldots, j_k$ denote the number of components failing during $(0, x_1^*)$, $(x_1^*, x_1^* + x_2^*), \ldots, (\sum_{i=1}^{K-1} x_1^*, \sum_{i=1}^{K} x_1^*)$, respectively then, we would like to find the (K+1)st stage optimal interinspection time x_{K+1}^* given $j_1, j_2, \ldots, j_K, \ x_1^*, \ldots, x_K^*$.

Now $J_{K+1}(x)|\sigma(j_1(x_1^*), \ldots, J_K(x_K^*), x_1^*, \ldots, x_K^*)$ $B(N, F(x; \ ))$ where $J_{K+1}(x) = $ number of components failing during $(\sum_{i=1}^{K} x_i^*, \sum_{i=1}^{K} x_i^* + x)$ and the conditional Fisher information of θ is

$$I(\theta;x) = \frac{N \ x^2 \ e^{-x/\theta}}{\theta^4 (1 - e^{-x/\theta})} \ .$$

Now if θ has inverted gamma (λ, m) as the prior distribution, then the posterior distribution of θ given $(j_1, j_2, \ldots, j_K, \ x_1^*, x_2^*, \ldots, x_K^*)$ is

$$f(\theta|\underset{\sim}{j}) = C \sum_{i=1}^{K} b(j_i;N,F(x_i^*;\theta)) \frac{1}{\theta^{m+1}} e^{-\lambda/\theta} \quad ; \quad 0 < \theta < +\infty$$

where C is a constant. Therefore,

$$E\{I(\theta,x)|\underset{\sim}{j}\} = C \int_0^\infty \sum_{i=1}^{K} b(j_1,N,F(x_i^*;\theta)) \frac{1}{\theta^{m+1}} e^{-\lambda/\theta} \frac{N x^2 e^{-x/\theta}}{\theta^4(1-e^{-x/\theta})} d\theta$$

$$= CN \prod_{i=1}^{K} \binom{N}{j_i} \int_0^\infty \sum_{i=1}^{K} [1-e^{-x_i^*/\theta})^{j_i} \exp\{-(N-j_1)x_i^*/\theta\}] \frac{x^2}{\theta^{m+5}} \frac{e^{(\lambda+x)/\theta}}{(1-e^{x/\theta})} d\theta$$

$$= CN \sum_{i=1}^{K} \binom{N}{j_i} x^2 \int_0^\infty \prod_{i=1}^{K} (1-e^{-x_i^*/\theta})^{j_i} \frac{1}{\theta^{m+5}(1-e^{-x/\theta})}$$

$$\exp\{-1/\theta(\sum_{i=1}^{K} (N-j_i)x_i^* + \lambda + x)\} \, d\theta.$$

Now $(1 - e^{-x_i^*/\theta})^{j_i} = \sum_{\nu_i=1}^{j_i} \binom{j_i}{\nu_i}(-1)^{\nu_i} \exp\{-x_i^* \nu_i/\theta\}$.

Therefore,

$$\prod_{i=1}^{K} (1-e^{-x_i^*/\theta})^{j_i} = \prod_{i=1}^{K} \left\{ \sum_{\nu_i=1}^{j_i} \binom{j_i}{\nu_i}(-1)^{\nu_i} e^{-\nu_i x_i^*/\theta} \right\}$$

$$= \sum_{\nu_1=1}^{j_i} \sum_{\nu_2=1}^{j_2} \cdots \sum_{\nu_K=1}^{j_K} \binom{j_1}{\nu_1}\binom{j_2}{\nu_2} \cdots \binom{j_K}{\nu_K}(-1)^{\sum_{i=1}^{K} \nu_i}$$

$$\exp(- \sum_{i=1}^{K} \nu_i x_i^*/\theta) \ .$$

Therefore,

$$E\{I(\theta;x)|\underset{\sim}{j}\} = C' \; x^2 \sum_{\nu_1} \sum_{\nu_2} \dots \sum_{\nu_K} \binom{j_1}{\nu_1}\binom{j_2}{\nu_2}\binom{j_K}{\nu_K} (-1)^{\sum_{i=1}^{K} \nu_i}$$

$$\int_0^\infty \frac{e^{-1/\theta[\Sigma_1^K \nu_i x_i^* + \Sigma_1^K (N-j_i)x_i^* + \lambda + x]}}{\theta^{m+5}(1 - e^{-x/\theta})} \; d\theta$$

$$= C' \; x^2 (\sum_{i=1}^{K} \sum_{\nu_i}^{j_i})(\prod_{i=1}^{K} \binom{j_i}{\nu_i})(-1)^{\sum_{i=1}^{K} \nu_i}$$

$$\sum_{k=1}^{x} \int_0^\infty \frac{e^{-1/\theta[\Sigma_1^K x_i^*[\nu_i+N-j_i] + \lambda + kx]}}{\theta^{m+5}} \; d\theta$$

$$= C' \; x^2 \sum_{k=1}^{\infty} \sum_{\nu_1}^{j_1} \sum_{\nu_2}^{j_2} \dots \sum_{\nu_K}^{j_K} \{ \prod_{i=1}^{K} \binom{j_i}{\nu_i}\}(-1)^{\Sigma_1^K \nu_i}$$

$$\frac{1}{(\Sigma_1^K x_i^*(\nu_i+N-j_i) + \lambda + kx)^{m+4}} \quad .$$

Let

$$a_{\nu_i} = x_i^*(\nu_i+N-j_1) = C'' \; x^2 \sum_{k=1}^{\infty} \sum_{\nu_1}^{j_1} \dots \sum_{\nu_K}^{j_K} (\sum_{i=1}^{K} \binom{j_i}{\nu_i}(-1)^{\nu_i}) \frac{1}{(\Sigma_{i=1} a_{\nu_i} + \lambda + kx)^{m+4}} \quad .$$

Differentiating with respect to x, we get after equating to zero;

$$\frac{\partial}{\partial x} E\{I(\theta;x)|\underset{\sim}{j}\} = \sum_{k=1}^{\infty} \sum_{\nu_1}^{j_1} \dots \sum_{\nu_K}^{j_K} (\prod_1^K \binom{j_i}{\nu_i}(-1)^{\nu_i}) \left[\frac{2(\Sigma_{i=1}^{K} a_{\nu_i} +\lambda)-x(m+2)k}{(\lambda + \Sigma_{i=1}^{K} a_{\nu_i} +kx)^{m+5}} \right] = 0 \quad .$$

Let $x_{m,\underset{\sim}{j}}$ be the point for which the above equation is satisfied, define $\alpha_{m,\underset{\sim}{j}} = (x_{m,\underset{\sim}{j}})(m+2)$.

Therefore, $\frac{\partial}{\partial x} E\{I(\theta;x)|\underset{\sim}{j}\} = 0$ implies

$$\sum_{k=1}^{\infty} \sum_{\nu_1=0}^{j_1} \cdots \sum_{\nu_K=0}^{j_K} [\sum_{i=1}^{K} \binom{j_i}{\nu_i}(-1)^{\nu_i}] \frac{2[\lambda + \Sigma_{i=1}^{K} a_{\nu_i}) - \alpha_{m,j} \cdot k}{[(m+2)(\Sigma_{i=1}^{K} a_{\nu_i} + \lambda) = k\alpha_{m,j}]^{m+5}} = 0$$

$$a_{\nu_i} = x_i^*(\nu_i + N - j_i) .$$

If all $j_i = 0$, $i=1,2,\ldots,K$, then trivially,

$$E\{I(\theta;x)|\underset{\sim}{j}\} \text{ is maximized at } x_{m,\underset{\sim}{i}=0}^*$$

where

$$x_{m,\underset{\sim}{j}=0}^* = \frac{\alpha_m(\lambda + N \Sigma_{i=1}^{K} x_i^*)}{m + 2} .$$

We thus <u>conjecture</u> that if $\underset{\sim}{j} \neq 0$, then

$$x_{m,\underset{\sim}{j}}^* \approx \frac{\alpha_m(\lambda + \Sigma_{i=1}^{K}(N-j_i)x_i^* - \Sigma_{i=1}^{K} j_i x_i^*/(e^{\alpha_m/\lambda} - 1))}{m + 2} .$$

But this result needs to be varified.

Also, we note that if $\underset{\sim}{j} = (0,0,0,\ldots,0,j_i,0,0,\ldots,0)$ then

$$x_{m,\underset{\sim}{j}}^* \approx \frac{\alpha_m}{m+2} \{\lambda + (N-j_i)x_i^* - j_i x_j^* (e^{\alpha_m/\lambda} - 1)\}$$

as varified in Lemma 3.1.

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Design of experiments for estimation of parameters in non-linear models is studied in a Bayesian framework, with the objective of maximization of the anticipated Fisher information. Two stage optimal designs are proposed in attribute life situations.

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