AD-A095 856 ROYAL AIRCRAFT ESTABLISHMENT FARNBOROUGH (ENGLAND) F/G 22/3
LUMPED HARMONICS OF 15TH AND 30TH ORDER IN THE GEOPOTENTIAL, FR--ETC(U)
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ROYAL AIRCRAFT ESTABLISHMENT

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Technical Report 80088

July 1980

LUMPED HARMONICS OF 15TH AND 30TH ORDER IN THE GEOPOTENTIAL FROM THE RESONANT ORBIT OF 1971-54A

by

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Farnborough, Hants

REPORT DOCUMENTATION PAGE

Overall security classification of this page

UNCLASSIFIED UNLIMITED

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1. DRIC Reference (to be added by DRIC)	2. Originator's Reference	3. Agency Reference	4. Report Security Classification/Marking
	RAE TR 80088	N/A	UNCLASSIFIED UNLIMITED

5. DRIC Code for Originator	6. Originator (Corporate Author) Name and Location
7673000W	Royal Aircraft Establishment, Farnborough, Hants, UK

5a. Sponsoring Agency's Code	6a. Sponsoring Agency (Contract Authority) Name and Location
N/A	N/A

7. Title Lumped harmonics of 15th and 30th order in the geopotential, from the resonant orbit of 1971-54A

7a. (For Translations) Title in Foreign Language

7b. (For Conference Papers) Title, Place and Date of Conference

8. Author 1. Surname, Initials	9a. Author 2	9b. Authors 3, 4	10. Date	Pages	Refs.
King-Hele, D.G.			July 1980	31	28

11. Contract Number	12. Period	13. Project	14. Other Reference Nos.
N/A	N/A		Space 582

15. Distribution statement

(a) Controlled by – Head of Space Dept, RAE (RAL)

(b) Special limitations (if any) –

16. Descriptors (Keywords) (Descriptors marked * are selected from TEST)

Earth's gravity field. Resonance. Geopotential. Orbit analysis. GEM 10.

17. Abstract

The satellite 1971-54A entered a near-circular orbit with period 95.9 minutes and inclination 90.2°. Between 1972 and 1978 the orbit passed slowly through 15th-order resonance, when the track over the Earth repeats after 15 revolutions, and the 15th- and 30th-order harmonics in the geopotential may produce substantial orbital perturbations.

The values of orbital inclination and eccentricity from 269 weekly US Navy orbits between November 1972 and January 1978 have been analysed to determine 12 lumped harmonic coefficients of order 15 and 30. The analysis of inclination yields 15th-order coefficients accurate to 1.5% and 2.8%, and 30th-order coefficients accurate to 7%. The analysis of eccentricity gives two 15th-order coefficients accurate to 3% and 4%.

These lumped harmonic coefficients are used to test the accuracy of the Goddard Earth Model 10B, which is complete to order and degree 36. The agreement with GEM 10B is excellent, for both 15th and 30th order, and shows that GEM 10B is more accurate than was expected.

The 12 values of lumped harmonics obtained give 12 linear equations between individual coefficients of order 15 and 30, which will be used in a future solution for the individual coefficients.

F5910/1

RAE Form A143

UDC 521.6 : 521.4 : 517.564.4

ROYAL AIRCRAFT ESTABLISHMENT

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Received for printing 9 July 1980

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SUMMARY

The satellite 1971-54A entered a near-circular orbit with period 95.9 minutes and inclination 90.2°. Between 1972 and 1978 the orbit passed slowly through 15th-order resonance, *when the track over the Earth repeats after 15 revolutions*, and the 15th- and 30th-order harmonics in the geopotential may produce substantial orbital perturbations.

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Departmental Reference: Space 582

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1 INTRODUCTION

The Earth's gravitational potential is usually expressed as a double infinite series of tesseral harmonics of degree ℓ and order m $(\leqslant \ell)$. The order m of the harmonic specifies variations between one meridian and another, and if the Earth were sliced through the equatorial plane, a harmonic of order m would express itself as a shape with m oscillations as the longitude goes through 360°: thus there would be 'humps' at intervals of $360^\circ/m$.

The orbit of an Earth satellite exhibits mth-order resonance when its orbital period is such that its path over the Earth repeats after m revolutions. Successive tracks over the Earth are then spaced at $360^\circ/m$ in longitude, and the harmonics of order m, $2m$, $3m$, ... have the same effect on the orbit on successive transits. So their effects build up.

In practice, a satellite close to the Earth suffers appreciable air drag, as a result of which its orbital period steadily decreases, and the satellite is slowly drawn through the resonance by drag. The effects of the resonance may last for several years, however, for a low-drag satellite, and the orbit may be altered by as much as 5 km.

A thorough analysis of such changes can give accurate values for 'lumped harmonics' of order m, a lumped harmonic being the linear sum of a series of individual harmonic coefficients of order m and degree ℓ_0, $\ell_0 + 2$, $\ell_0 + 4$, ..., where $\ell_0 = m$ or $m + 1$. The analysis also usually yields values of lumped harmonics of order $2m$, of poorer but still useful accuracy; and sometimes gives lumped harmonics of order $3m$. In these lumped harmonics there are often significant contributions from individual harmonics up to degree 50 or more.

Accurate values of lumped harmonics are now particularly useful, because they provide an exacting independent test of the accuracy of the comprehensive models of the gravity field determined from other data. Until recently, the tests were often inconclusive because the models did not go beyond order and degree 36 - the limit of GEM 10B[1], which was derived from some 800000 accurate observations of Earth satellites, including about 200000 each of laser, Doppler and camera observations, together with $5^\circ \times 5^\circ$ surface gravity measurements and radio altimeter data on 700 passes of the Geos 3 satellite. By the addition of 28000 $1^\circ \times 1^\circ$ areas of sea-surface geoid derived from 2300 passes of altimetry from Geos 3, a new geopotential model, GEM 10C, has been derived[2], which goes up to order and degree 180, but is the same as GEM 10B up to order and degree 36. The existence of GEM 10C is a spur to new and more accurate determinations of lumped harmonics.

Many examples of orbital resonance have been analysed in recent years. Individual harmonics of 15th order[3,4] and 14th order[5], and also tentative values of 30th-order coefficients[6], have been determined from analysis of a number of orbits at different inclinations. In addition, lumped harmonics have been determined from examples of 29:2 resonance[7] (29 revolutions in 2 days), and 31:2 resonance[8,9,10], as well as from 11th- and 13th-order resonances[11,12]. Many studies of near-resonant orbits have also been made.

The most accurate values of lumped harmonics are likely to be obtained from satellites of low drag, and 1971-54A was chosen for analysis because it was appreciably affected by 15th-order resonance for more than 5 years, from mid-1972 to the end of 1977.

Through the courtesy of the US Naval Research Laboratory, we have received US Navy orbital elements for 1971-54A, and the Navspasur observations of the satellite, at weekly intervals since launch. The present analysis is entirely based on 269 weekly US Navy orbits, for epochs between 12 November 1972 and 14 January 1978. The orbital elements used are the inclination and eccentricity.

As it happens, high-degree harmonics have relatively little effect on 1971-54A, because the orbit is polar, so the results provide a test for GEM 10B rather than GEM 10C.

2 THE SATELLITE 1971-54A (SESP-1) AND ITS ORBIT

The satellite designated 1971-54A was launched by the USAF on 8 June 1971 into a near-circular orbit of inclination 90.2°, with perigee and apogee heights near 550 and 590 km. The satellite consisted of a Burner 2 rocket with a payload attached: called the 'Space Experiments Support Program 1' (SESP-1), its purpose is said to have been 'to test an infra-red celestial mapping sensor system'[13]. The size, shape and mass of the satellite are not exactly known, but it was probably[14] about 3 m long and 1.3 m in diameter, with a mass of about 260 kg. The satellite was launched at a time when the solar activity, and hence the upper-atmosphere density, was decreasing after the maximum of 1968-70. Consequently, the rate of orbital decay was very slow between 1972 and 1977, while the solar activity remained low. The decrease in orbital period between 1971 and 1979 is shown in Fig 1.

By great good fortune, the 15th-order resonance occurred during the time of lowest drag, thus maximizing its effects. The orbit has been analysed between 12 November 1972 and 14 January 1978, as indicated in Fig 1: throughout this time the effects of 15th-order resonance were appreciable because the anomalistic period was within 0.1 minute of its resonant value, 95.68 minutes. The analysis was stopped at 14 January 1978 because of the great increase in drag as a result of the increased solar activity. It will be seen from Fig 1 that for a short time early in 1975 there was an increase in orbital period because the effect of the resonance was, temporarily, strong enough to overcome the effect of drag. Although 1971-54A was too low in the atmosphere to become trapped in resonance, it hesitated before escaping, and nearly turned back to experience resonance for a second time.

3 PREPARATION OF THE ORBITS FOR THE RESONANCE ANALYSIS

The resonance analysis is conducted with the aid of the THROE computer program[15], which requires orbital elements in the 5-card format of the PROP orbit refinement program[16]. The US Navy orbits differ from PROP orbits in two ways:

(a) The epoch is at the last ascending node before midnight, rather than at midnight as in PROP;

(b) The values of eccentricity have a part of the odd zonal harmonic oscillation removed, and the argument of perigee also consequently suffers modification.

The 269 US Navy orbits were converted to PROP elements at the nearest midnight using the program ELTRAN2, which also restores the perturbation in eccentricity on the assumption that the values of the geopotential coefficients J_3 and J_5 used were $J_3 = -2.551 \times 10^{-6}$ and $J_5 = -0.191 \times 10^{-6}$. After conversion by ELTRAN2, the 269 values of eccentricity e and argument of perigee ω closely followed the expected basic circular pattern of variation - that is, a circle in the $(e \cos \omega, e \sin \omega)$ cartesian plane. So the conversion was considered successful.

Before the resonance can be analysed, other perturbations must be removed from the orbital elements. For the inclination, the most important perturbations are those due to lunisolar gravitational attraction and the zonal harmonics in the geopotential. These were calculated by running the PROD computer program[17] with 1-day integration steps through the 1889 days of the analysis, with restarts at intervals of about 100 days, because the running integration gradually drifts off as a result of the air-drag and resonance effects not included in PROD. The lunisolar gravitational perturbations are themselves resonant for a polar orbit[18], and therefore near-resonant for 1971-54A at an inclination of 90.2°. During the 5 years of the integration by PROD the inclination irregularly but inexorably increased by about 0.05°. Since this perturbation was so large, its zero was taken to occur at a time close to resonance, so that the maximum departure of the adjusted values from the observed values would be as small as possible. To remove the effects of lunisolar and zonal harmonic perturbations, the amount to be added to the observed value of inclination was 0.0170° initially; 0.0091° at MJD 42000; -0.0078° at MJD 42500; -0.0143° at MJD 43000; and -0.0296° at the end of the analysis.

The second important perturbation of the inclination is that due to atmospheric rotation. For a circular polar orbit with a period decreasing by ΔT minutes, the expected decrease in inclination due to an atmosphere rotating at Λ rev/day is $0.007\Lambda\ \Delta T$ degrees. Here $\Delta T = 0.2$ over the 5 years, so the inclination decreases by 0.0014Λ degrees; previous results[19] indicate an average atmospheric rotation rate of 0.8 ± 0.1 rev/day at heights near 450 km, so the expected decrease in inclination is $0.0011^\circ \pm 0.0001^\circ$ over the 5 years. This perturbation is computed within THROE, together with that due to the precession of the Earth's axis. Perturbations due to Earth and ocean tides are expected to be about 50 m at most, and need not be considered, since the accuracy of the observational values of inclination is about 200 m. Perturbations due to solar radiation pressure for 1971-54A, which has a mass/area ratio of about 200 kg/m^2, are estimated at 0.00004°, and are also negligible.

By far the most important perturbation in eccentricity is that due to the odd zonal harmonics in the Earth's gravitational potential, which produce an oscillation with the same period as ω (about 100 days) and an amplitude of 0.0014, which is equivalent to about 10 km in perigee distance. This perturbation is removed within THROE using a 20-coefficient set of zonal harmonics, the odd zonal harmonics being those of Ref 20. Since the eccentricity is small (its average value being 0.0022) the lunisolar perturbations to e should also be small. They were evaluated by running PROD with and without lunisolar perturbations: it was found that the maximum difference in e between the two runs was 0.000001, which is negligible. Earth tide and radiation pressure perturbations to e should also be negligible.

If the perturbations in inclination and eccentricity are successfully removed, the remaining variations in these two orbital parameters should be caused only by the effects of 15th-order resonance.

4 THEORY FOR GENERAL $\beta:\alpha$ RESONANCE

The longitude-dependent part of the Earth's gravitational potential at an exterior point (r,θ,λ) may be written in normalized form[21] as

$$\frac{\mu}{r}\sum_{\ell=2}^{\infty}\sum_{m=1}^{\ell}\left(\frac{R}{r}\right)^{\ell} P_{\ell}^{m}(\cos\theta)\left\{\bar{C}_{\ell m}\cos m\lambda + \bar{S}_{\ell m}\sin m\lambda\right\}N_{\ell m} , \qquad (1)$$

where r is the distance from the Earth's centre, θ is co-latitude, λ is longitude (positive to the east), μ is the gravitational constant for the Earth ($398600\ \mathrm{km}^3/\mathrm{s}^2$), R is the Earth's equatorial radius (6378.1 km), $P_{\ell}^{m}(\cos\theta)$ is the associated Legendre function of order m and degree ℓ, and $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ are the normalized tesseral harmonic coefficients. The normalizing factor $N_{\ell m}$ is given by[21]

$$N_{\ell m}^{2} = \frac{2(2\ell+1)(\ell-m)!}{(\ell+m)!} . \qquad (2)$$

When a satellite orbit experiences $\beta:\alpha$ resonance - that is, when the satellite makes β revolutions while the Earth makes α revolutions relative to the satellite's orbital plane - the rates of change of inclination and eccentricity depend on the resonance angle Φ, defined by

$$\Phi = \alpha(\omega + M) + \beta(\Omega - \nu) , \qquad (3)$$

where M is the mean anomaly, Ω is the right ascension of the ascending node, and ν is the sidereal angle. The perturbation produced in i and e by a relevant pair of harmonic coefficients, $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$, may be written[22,23]

$$\frac{di}{dt} = \frac{n(1-e^2)^{-\frac{1}{2}}}{\sin i}\left(\frac{R}{a}\right)^{\ell}\bar{F}_{\ell mp}G_{\ell pq}(k\cos i - m)\,\mathscr{R}\left[j^{\ell-m+1}(\bar{C}_{\ell m} - j\bar{S}_{\ell m})\exp\{j(\gamma\Phi - q\omega)\}\right] , \qquad (4)$$

$$\frac{de}{dt} = n(1-e^2)^{-\frac{1}{2}}\left(\frac{R}{a}\right)^{\ell}\bar{F}_{\ell mp}G_{\ell pq}\left\{\frac{q - \frac{1}{2}(k+q)e^2}{e}\right\}\mathscr{R}\left[j^{\ell-m+1}(\bar{C}_{\ell m} - j\bar{S}_{\ell m})\exp\{j(\gamma\Phi - q\omega)\}\right] . \qquad \ldots\ldots (5)$$

Here $\bar{F}_{\ell mp}$ is Allan's normalized inclination function[22], $G_{\ell pq}$ is a function of eccentricity e for which explicit forms have been derived by Gooding[23], $\mathscr{R}$ denotes 'real part of' and $j = \sqrt{-1}$. The indices γ, p, k and q are integers, with γ taking the values 1, 2, 3,..., and q the values 0, ±1, ±2,... . The equations linking ℓ, m, k and p are: $m = \gamma\beta$; $k = \gamma\alpha - q$; $2p = \ell - k$.

For a particular $\beta:\alpha$ resonance, the relevant value of m in the $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ coefficients is decided by the choice of γ, since $m = \gamma\beta$. There is a series of possible values of ℓ - any values such that $\ell \geqslant m$ and $(\ell - k)$ is even. The successive $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ coefficients that arise (for given γ and q) may usefully be combined in 'lumped' form and written as

$$\bar{C}_m^{q,k} = \sum_{\ell} Q_\ell^{q,k} \bar{C}_{\ell m} , \quad \bar{S}_m^{q,k} = \sum_{\ell} Q_\ell^{q,k} \bar{S}_{\ell m} , \qquad (6)$$

where ℓ increases in steps of 2 from its minimum permissible value ℓ_0; the $Q_\ell^{q,k}$ are functions of inclination that can be taken as constant for a particular satellite; and $Q_\ell^{q,k} = 1$ when $\ell = \ell_0$.

5 ANALYSIS OF THE RESONANT VARIATION OF INCLINATION FOR 1971-54A

5.1 The form of the equation for di/dt

In equation (4) for di/dt, the functions $G_{\ell pq}$ are of order $\frac{(\frac{1}{2}\ell e)^{|q|}}{(|q|)!}$ and, since e is so small ($\simeq 0.002$) for 1971-54A, the q = 0 terms in equation (4) should be the largest. For 15th-order resonance, with $\beta = 15$ and $\alpha = 1$, and with only the $(\gamma,q) = (1,0),(2,0)$ and $(3,0)$ terms written explicitly, equation (4) becomes:

$$\begin{aligned}\frac{di}{dt} = \frac{n}{\sin i}\left(\frac{R}{a}\right)^{15}(15 - \cos i)\Bigg[&\bar{F}_{15,15,7}\left\{\bar{C}_{15}^{0,1}\sin\Phi - \bar{S}_{15}^{0,1}\cos\Phi\right\} \\ &+ 2\left(\frac{R}{a}\right)^{15}\bar{F}_{30,30,14}\left\{\bar{C}_{30}^{0,2}\sin 2\Phi - \bar{S}_{30}^{0,2}\cos 2\Phi\right\} \\ &+ 3\left(\frac{R}{a}\right)^{30}\bar{F}_{45,45,21}\left\{\bar{C}_{45}^{0,3}\sin 3\Phi - \bar{S}_{45}^{0,3}\cos 3\Phi\right\} \\ &+ \text{terms in } \frac{(\frac{1}{2}\ell e)^{|q|}}{(|q|)!}\begin{matrix}\cos\\ \sin\end{matrix}(\gamma\Phi - q\omega)\Bigg] . \qquad (7)\end{aligned}$$

The coefficients $\bar{F}_{15,15,7}$, $\bar{F}_{30,30,14}$ etc, and the multiplying factors $Q_\ell^{q,k}$, have been calculated with the aid of the computer program PROF for 1971-54A, taking mean values of the elements at a time near the exact resonance, namely

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$a = 6930$ km, $e = 0.0022$ and $i = 90.21^\circ$. The numerical values of the $Q_\ell^{q,k}$ can be used to make a rough estimate of the order of magnitude of $\bar{C}_m^{q,k}$ in the various (γ,q) terms in equation (7), on the assumption that the individual coefficients $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ are of order $10^{-5}/\ell^2$. We assume $\bar{C}_m^{q,k}$ is of order $10^{-5}\left\{\sum_\ell \left(Q_\ell^{q,k}/\ell^2\right)^2\right\}^{\frac{1}{2}}$. By inserting numerical values for the $\bar{F}$ functions and R/a (= 0.9204), the relative magnitudes of the various (γ,q) terms in equation (7) can be calculated. The results are shown in Table 1, the leading term being scaled up to 100: the explicit forms of the $(\gamma,q) = (1,\pm 1)$ terms were taken from Ref 24.

Table 1

Relative orders of magnitude of (γ,q) terms in (7), for 1971-54A

q \ γ	1	2	3
0	100	10	2
1	0.1	-	-
-1	1	-	-

Terms marked with a dash and those with $|q| \geqslant 2$ or $\gamma \geqslant 4$ are not considered large enough to be worth evaluating.

Table 1 shows that the three terms given explicitly in equation (7) should be the largest, If the main term can be evaluated with an accuracy of 2% or better, it may be necessary to include the $(\gamma,q) = (3,0)$ term in the fitting.

5.2 The resonance angle Φ

From equation (3), the resonance angle Φ for 15th-order resonance is given by

$$\Phi = \omega + M + 15(\Omega - \nu) . \tag{8}$$

The variations of Φ and $\dot{\Phi}$ during the 5 years of the analysis are shown in Fig 2.

The variation of Φ follows the usual near-parabolic form, but the 'hesitation' after resonance is clearly visible in Fig 2.

For a satellite of fairly high drag, the increase of $\dot{\Phi}$ is usually nearly linear, with variations in slope in response to variations in drag. Here, however, $\dot{\Phi}$ pursues a sinuous course through the resonance, with its slope dependent on Φ as well as on the drag. The equation for the variation of $\dot{M}$ due to resonance[22] shows that $\dot{M}$ - and hence $\dot{\Phi}$ - tends to decrease whenever di/dt is negative. The numerical values of $\bar{C}^{0,1}_{15}$ and $\bar{S}^{0,1}_{15}$ for 1971-54A are such that di/dt is negative wherever $-18^\circ < \Phi < 162^\circ$, and has its maximum negative value when $\Phi = 72^\circ$. Inspection of the values of Φ in Fig 2 shows that the resonance effect is reducing $\dot{M}$, and hence $\dot{\Phi}$, for 530 days between MJD 42054 and 42583, when $57^\circ < \Phi < 162^\circ$. It was only at the end of this time interval, in June-July 1975, when air density fell to its minimum value in the solar cycle, that the effects of resonance were able to overcome drag. If the mininum density had occurred a little earlier, the resonance effect could have pulled the satellite back to resonance, where it might have become trapped until the density increased as the solar activity began its increase in September 1977.

The points where the maximum (negative) effect of resonance on $\dot{\Phi}$ is expected - *ie* $\Phi = 432^\circ, 792^\circ, 1152^\circ, \ldots$ - are marked on Fig 2, and it can be seen that they correspond to the minima in the slope of $\dot{\Phi}$.

If the effect of drag could be independently assessed and removed, the variation of $\dot{\Phi}$ would provide an easy and accurate method of determining lumped harmonics: but there seems little prospect of being able to estimate the drag with adequate accuracy by other methods.

The other obvious conclusion to be drawn from Fig 2 is that the evaluation of air density from a satellite's orbital decay rate should not be attempted near a resonance, unless the effect of resonance is very much smaller than that of drag.

5.3 The fitting of the variations

The analysis of the variation in inclination presented no problems, and Fig 3 shows the theoretical curve fitted by THROE to the 269 values after removal of the four perturbations specified in section 3 (lunisolar, zonal harmonic, atmospheric rotation and precession of the Earth's axis). The curve fitted is that specified by equation (7) with $(\gamma, q) = (1,0), (2,0)$ and $(3,0)$. The *a priori* error assigned to the values of inclination was 0.003°, but the residuals are nearly all less than 0.003°, so that the weighted residuals are less than 1 and $\varepsilon = 0.55$ (where ε^2 is the sum of the squares of the weighted residuals, divided by the number of degrees of freedom). Thus the *a posteriori* estimate of accuracy is 0.0016°, equivalent to 200 m in position.

The fitting of the theoretical curve in Fig 3 is as nearly perfect as is ever seen in the real world. It is particularly significant that the amplitudes of the minor oscillations before and after resonance are correctly modelled by the theoretical curve, thus banishing any doubts about the theory which may have arisen from poor modelling of the subsidiary oscillations in some previous analyses[25]. The phase of the minor oscillations after resonance lags behind that of the theoretical curve, but this can be partially explained by the fact that the epoch of the US Navy orbits is at the end of the 7 days of observations, so that the inclination values might arguably be regarded as applying to an epoch $3\frac{1}{2}$ days earlier.

The main decrease in inclination in Fig 3, from 90.25° in November 1973 to 90.15° in October 1975, is equivalent to a positional change of 12 km.

5.4 Values of the lumped coefficients

The fitting shown in Fig 3 gives numerical values of $\bar{C}_{15}^{0,1}$ and $\bar{S}_{15}^{0,1}$ having standard deviations of 2% and 3% respectively; so, on the basis of Table 1, it is questionable whether the 45th-order terms are needed. A fitting with (γ,q) = (1,0) and (2,0) only was therefore made: the value of ε was 0.552, almost the same; nearly all the weighted residuals were within ±0.1; and the standard deviations of the 15th- and 30th-order coefficients were smaller. Table 2 gives the values of the lumped coefficients obtained from the two fittings.

Table 2

Values of lumped harmonics obtained from fitting of i

(γ,q) pairs included	$10^9\bar{C}_{15}^{0,1}$	$10^9\bar{S}_{15}^{0,1}$	$10^9\bar{C}_{30}^{0,2}$	$10^9\bar{S}_{30}^{0,2}$	$10^9\bar{C}_{45}^{0,3}$	$10^9\bar{S}_{45}^{0,3}$
(1,0) (2,0) (3,0)	-16.2 ± 0.33	-5.2 ± 0.18	-6.7 ± 1.2	11.7 ± 1.0	6.0 ± 3.4	-2.5 ± 3.0
(1,0) (2,0)	-16.4 ± 0.24	-5.4 ± 0.15	-8.2 ± 0.6	11.1 ± 0.8	-	-

(The standard deviations of the 15th-order coefficients are given to two decimal places so as to allow comparisons between them.)

Since the 45th-order terms are ill-determined and the (1,0),(2,0) solution gives better standard deviations, it is probably to be preferred. However, the $\bar{C}_{45}^{0,3}$ term, when multiplied by the ratio of the $\bar{F}$ terms etc in (7) to provide a measure of its effect relative to $\bar{C}_{15}^{0,1}$, has a value 1.1 ± 0.6, which exceeds the standard deviation of $\bar{C}_{15}^{0,1}$: so it could be argued that the 45th-order terms are needed in the fitting. Thus it seemed best to give both solutions: it is very satisfactory that they agree to within the sum of their standard deviations. The values of both the 15th- and 30th-order coefficients in the second solution are nominally the most accurate yet obtained from any satellite.

Now that the actual values of the lumped harmonics are known, the prediction of their orders of magnitude in Table 1 can be checked. The actual ratios of the (1,0):(2,0):(3,0) terms are 31:11:2 rather than 100:10:2 in the order-of magnitude prediction in Table 1. So the 15th-order terms are relatively smaller than expected on the simple $10^{-5}/\ell^2$ prediction.

6 ANALYSIS OF THE RESONANT VARIATION OF ECCENTRICITY FOR 1971-54A

6.1 The form of the equation for de/dt

In analysing the eccentricity, we have first to decide which of the (γ,q) terms in equation (5) are likely to be important. Assuming that $G_{\ell pq}$ is of order $(\frac{1}{2}\ell e)^{|q|}/(|q|)!$, the term $G_{\ell pq}\left\{\frac{q - \frac{1}{2}(k + q)e^2}{e}\right\}$ in (5) is of order $\frac{1}{2}\gamma e$ when $q = 0$; of order $\frac{1}{2}\ell$ when $|q| = 1$; of order $\frac{1}{4}\ell^2 e$ when $|q| = 2$; and of order $\left\{(\frac{1}{2}\ell)^{|q|}e^{|q|-1}\right\}/(|q - 1|)!$ for general $|q| \geqslant 1$. Since the mean value of e for 1971-54A is 0.0022, the $|q| = 1$ terms are expected to be dominant. With the same assumptions as in section 5.1, the relative magnitudes of the relevant (γ,q) terms for 1971-54A are given in Table 3, the explicit algebraic forms for the terms again being taken from Ref 24.

Table 3

Relative orders of magnitude of (γ,q) terms in (5), for 1971-54A

q \ γ	1	2	3
0	0.1	-	-
1 -1	5 100	8 12	2 2
2 -2	3 1	- -	- -

Terms marked with a dash and those with $|q| \geqslant 3$ or $\gamma \geqslant 4$, are not considered large enough to be worth evaluating.

The very small value of the $(\gamma,q) = (1,1)$ term in Table 3 is unusual, and arises because its multiplying factor $\bar{F}_{16,5,8}$ has a zero at $i = 90^\circ$ and is therefore very small at $i = 90.2^\circ$. The disparity between the (1,1) and (1,-1) terms does lead to some problems in the fitting.

From Table 3 it appears that the only terms needed in the fitting will probably be those with $(\gamma,q) = (1,\pm1)$ and $(2,\pm1)$, and, if so, equation (5) for de/dt becomes

$$\begin{aligned}\frac{de}{dt} = n\left(\frac{R}{a}\right)^{16}\Bigg[& -\frac{17}{2}\bar{F}_{16,15,8}\left\{\bar{S}^{1,0}_{15}\sin(\Phi-\omega) + \bar{C}^{1,0}_{15}\cos(\Phi-\omega)\right\} \\ & + \frac{13}{2}\bar{F}_{16,15,7}\left\{\bar{S}^{-1,2}_{15}\sin(\Phi+\omega) + \bar{C}^{-1,2}_{15}\cos(\Phi+\omega)\right\} \\ & - 17\left(\frac{R}{a}\right)^{15}\bar{F}_{31,30,15}\left\{\bar{S}^{1,1}_{30}\sin(2\Phi-\omega) + \bar{C}^{1,1}_{30}\cos(2\Phi-\omega)\right\} \\ & + 13\left(\frac{R}{a}\right)^{15}\bar{F}_{31,30,14}\left\{\bar{S}^{-1,3}_{30}\sin(2\Phi+\omega) + \bar{C}^{-1,3}_{30}\cos(2\Phi+\omega)\right\}\Bigg] \quad . \quad (9)\end{aligned}$$

This will be the basic equation for fitting the observed values, though other terms will be added and dropped on a trial basis.

6.2 The fitting of the variations

The 269 values of eccentricity were fitted with equation (9) using THROE, zonal harmonic and atmospheric perturbations being removed within THROE. The fitting was unacceptable: the maximum weighted residual was 8.4; there were several long runs of residuals of the same sign; and the overall measure of fit, ε, was 3.3. Re-runs with $(\gamma,q) = (3,\pm1)$ terms added, and then with $(1,\pm2)$ terms added, somewhat reduced ε, to between 2.5 and 2.8, but the fitting was still unsatisfactory. Finally, $(\gamma,q) = (0,1)$ terms were added to the basic four pairs: this introduces terms $(A \sin\omega - B\cos\omega)$, which do not arise in the resonance equation (5), and their inclusion represents an attempt to remove any spurious oscillations of the same frequency as ω and unconnected with the resonance. The $(\gamma,q) = (0,1)$ terms added were well determined, ε was reduced from 3.3 to 2.2, and the fitting was much more satisfactory. So the run with $(\gamma,q) = (0,1)$ terms added can be regarded as "the best of a bad bunch", and on that basis it is shown in Fig 4.

The fitting of the points in Fig 4 is good in one respect: the detail in the observational values is well mirrored by the curve, which has ups and downs in nearly perfect unison with the data. But the curve is usually either above or below the general run of the values by a margin too large to be acceptable.

If we ignore its defects for the moment, Fig 4 has some interesting features. First, it shows that the variation in eccentricity near the $\dot\Phi = \dot\omega$ resonance is much smaller than the variation near the $\dot\Phi = -\dot\omega$ resonance: this is as expected, because the $(\gamma,q) = (1,1)$ term is much smaller than the $(\gamma,q) = (1,-1)$ term in Table 3. The strong variation in e during 1976-7 would have continued into 1978 if the great increase in drag had not carried the orbit away from resonance.

A second feature of interest in Fig 4 is the form of the variation near the central ($\dot\Phi = 0$) resonance. Equation (9) shows that the variation in eccentricity is governed by terms of the form $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\omega \pm \Phi)$ and $\begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(\omega + 2\Phi)$: so near $\dot\Phi = 0$ the oscillations should have the same period as ω, and this expectation is borne out, as shown by the $\omega = 270^\circ$ values marked, the first five of which coincide with maxima in the observational values.

The most important feature of Fig 4, however, is the deficiency in fitting. This was a severe setback, which provoked a thorough examination of the *modus operandi* of THROE and PROD, including tests of omitting alternate points, running time backwards, etc. No flaw was found. So a reconsideration of the procedures was required. In the past, the inclination has been successfully analysed for approximately equal times before and after the exact resonance ($\dot\Phi = 0$), and preferably for a time long enough to include a few of the subsidiary oscillations (as in Fig 3). The eccentricity has in the past then been analysed over the same time interval as i. The greatest effects of resonance on eccentricity are likely to occur near the times when $\dot\Phi = \dot\omega$ and $\dot\Phi = -\dot\omega$, since the $(\Phi - \omega)$ and $(\Phi + \omega)$ terms in equation (9) will then be nearly constant. Although the $\dot\Phi = 0$ resonance has no relevance for eccentricity, the 'eccentricity resonances', $\dot\Phi = \pm\dot\omega$, usually occur within a few months of the 'inclination resonance' and the analysis can be carried through from before $\dot\Phi = \dot\omega$ to after $\dot\Phi = -\dot\omega$ (or from $\dot\Phi = -\dot\omega$ to $\dot\Phi = \dot\omega$ if $i < 63^\circ$). For 1971-54A, however, $\dot\Phi \triangleq \dot\omega$ occurs in December 1972, nearly 2 years before $\dot\Phi = 0$; and $\dot\Phi \triangleq -\dot\omega$ occurs in July 1977, $2\frac{3}{4}$ years after $\dot\Phi = 0$ *. Thus 1971-54A really experienced two separate 'eccentricity resonances'

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* These are the months when $\dot\Phi$ is equal to the *mean* value of $\dot\omega$ (or $-\dot\omega$) over a cycle: because of the variations in $\dot\omega$ for such a nearly circular orbit, an exact date cannot be specified, but the envelope of the minor oscillations in $\dot\Phi + \dot\omega$ has its minimum at about these times.

centred more than $4\frac{1}{2}$ years apart; and the variation in e near $\dot{\Phi} = 0$, which, as we have seen, is effectively dependent only on ω, may well contain no useful information about the 15th- and 30th-order coefficients in the geopotential. So the obvious procedure is to abandon previous practices and to treat the two e resonances separately. Values of e near $\dot{\Phi} = 0$ can be excluded unless they are helpful.

In analysing the two resonances, not all the terms in equation (9) are needed: the small terms will have a negligible effect at the 'other' resonance, where they will only generate small periodic terms. Thus for the first resonance we shall expect to use the $(\gamma,q) = (1,1),(1,-1)$ and $(2,1)$ terms, the $(1,-1)$ terms being included because they are so large; and for the second resonance just the $(1,-1)$ and $(2,-1)$ terms would be used. Again, however, it may be necessary to add $(\gamma,q) = (0,1)$ terms.

In the first e-resonance 95 orbits were used initially (MJD 41633-42291). The fitting with $(\gamma,q) = (1,1),(1,-1)$ and $(2,1)$ was fairly satisfactory, with $\varepsilon = 1.23$; but there was a considerable improvement when the $(0,1)$ terms were added, ε being reduced to 0.74 and the standard deviation of the $(1,1)$ terms being reduced by 40%. The addition of other terms, such as $(3,1)$ and $(2,-1)$, was tested, but gave no advantage. The fitting with $(\gamma,q) = (1,1),(1,-1),(2,1)$ and $(0,1)$ was quite satisfactory, except that the last 16 values did not fit well. When these values were omitted, ε decreased from 0.74 to 0.58, the values of the $(1,1)$ coefficients changed by less than 0.4 sd, and the standard deviations decreased by more than 10%. This shortened fitting, of 79 values, seems slightly preferable. A further shortening, to 71 values, was also tried. There was a decrease in ε, to 0.53, and a slight decrease in most but not all of the standard deviations. The relative merits of these last two fittings are debatable, but the argument against the shorter run (namely the increased danger of overfitting) led to a preference for the 79-value fitting. This is taken as the final choice and shown in Fig 5.

In the second e-resonance, 117 orbits were included (MJD 42697-43522). The fit with $(\gamma,q) = (1,-1)$ and $(2,-1)$ only was fairly good, with $\varepsilon = 1.66$, but the inclusion of the $(0,1)$ terms again improved the fit, ε being reduced to 0.94, with large reductions in two of the four standard deviations. This fitting is shown in Fig 6 and was taken as final, since the addition of other terms, such as $(3,-1)$ and $(1,1)$ gave no advantage; reducing or increasing the number of orbits to 97 or 131 was also tried, but found to be of no advantage.

Though the fittings shown in Figs 5 and 6 are entirely satisfactory, we have to ask (a) why the $(\gamma, q) = (0,1)$ terms are needed, and (b) whether their presence disturbs the values of the lumped harmonics obtained. The (0,1) terms represent an oscillation with the same period as ω, so the need for them suggests that the ω-dependent oscillation in eccentricity, which is partially removed from the US Navy values of e, has not been correctly restored. In some previous analyses such an error was certainly present and (0,1) terms were included to correct it. But ELTRAN2 was intended to solve the problem. It has not done so, and the question will be pursued further. In answer to the second question, it seems that, since no pure ω terms due to resonance appear in the equation (5), there is no reason why the fitting of an ω term should degrade the accuracy of the lumped harmonic coefficients obtained: on the contrary, the inclusion of a term which greatly improves the fitting should greatly improve the values.

If the ω term arises from the failure to remove ω-dependent terms from the original data, the values found for the coefficients of the (0,1) terms from analysis of the first and second e-resonances and all 269 values should show similarities. In fact the values of the $\sin\omega$ and $\cos\omega$ coefficients from the first resonance (Fig 5) are

$$-8.0 \pm 0.9 \quad \text{and} \quad -10.8 \pm 1.1 \; ;$$

the corresponding values from the second resonance (Fig 6) are

$$-10.2 \pm 0.8 \quad \text{and} \quad -8.5 \pm 1.4 \; ;$$

while the values from the 269-orbit fitting (Fig 4) are

$$-13.0 \pm 1.0 \quad \text{and} \quad -11.8 \pm 2.1 \; .$$

These three pairs of values (all are $\times 10^{-9}$) agree with each other to within 2.6 sd (sin) and 1.0 sd (cos), and so can be regarded as consistent.

6.3 Values of the lumped harmonics

The values of the lumped harmonics obtained from the three fittings of e are given in Table 4. The preferred pairs of values are underlined.

Table 4

Values of lumped harmonics obtained from fitting of e

	(γ,q) pairs	$10^9\bar{C}_{15}^{1,0}$	$10^9\bar{S}_{15}^{1,0}$	$10^9\bar{C}_{15}^{-1,2}$	$10^9\bar{S}_{15}^{-1,2}$	$10^9\bar{C}_{30}^{1,1}$	$10^9\bar{S}_{30}^{-1,1}$	$10^9\bar{C}_{30}^{-1,3}$	$10^9\bar{S}_{30}^{-1,3}$
All 269 values	(1,1) (1,-1) (2,1) (2,-1) (0,1)	-205 ± 102	665 ± 110	-69.3 ± 4.3	-39.9 ± 2.8	90 ± 11	-137 ± 19	-16 ± 6	-10 ± 11
First resonance (79 values)	(1,1) (1,-1) (2,1) (0,1)	*-92 ± 48*	*-170 ± 56*	-51.7 ± 5.5	-29.3 ± 4.3	*15 ± 8*	*-39 ± 12*	-	-
Second resonance (117 values)	(1,-1) (2,-1) (0,1)	-	-	*-62.9 ± 2.6*	*-53.4 ± 1.6*	-	-	*-8 ± 3*	*-13 ± 7*

The results in the first row, from all 269 values, have already been discounted because of the poor fit; but some of the lumped harmonic values are in agreement with those from the two divided fittings. The preferred values of $(\bar{C},\bar{S})_{15}^{-1,2}$ are those from the second resonance, of course, because this is the main term at that resonance. The values of $(\bar{C},\bar{S})_{15}^{1,0}$ were expected to be - and are - of poor accuracy, because their influence on the resonance was only about 1/20 of that of the $(\bar{C},\bar{S})_{15}^{-1,2}$ terms (Table 3).

On taking the mean of each underlined pair of values of $|\bar{C}|$ and $|\bar{S}|$ in Table 4, and multiplying them by the appropriate $\bar{F}$ factors, etc, in equation (9), we find that the relative magnitudes of the four harmonics are 9:100:13:9, as compared with the *a priori* order-of-magnitude expectation (Table 3) of 5:100:8:12. Thus the order of magnitude estimates were excellent, being within a factor of 2 of the real values.

7 THE LUMPED HARMONICS EXPRESSED IN TERMS OF THE INDIVIDUAL $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$

The lumped harmonics $\bar{C}_m^{q,k}$ and $\bar{S}_m^{q,k}$ are linear functions of the individual harmonic coefficients $\bar{C}_{\ell m}$ and $\bar{S}_{\ell m}$ given by equations (6). The numerical values of the coefficients $Q_\ell^{q,k}$ in equations (6) for 1971-54A have been evaluated with the aid of the computer program PROF, to express each of the 6 pairs of lumped harmonics, given in Tables 2 and 4, in terms of the individual harmonics. The numerical values of the $Q_\ell^{q,k}$ decrease when the degree ℓ is increased far enough, and only a limited number of terms need to be evaluated, depending on the accuracy of the values of the lumped harmonics. Here we have ignored terms which are not expected to contribute (relative to the main term) an amount more than half the proportional error in the value of the more accurate of the two lumped coefficients. Thus, for the first pair of lumped coefficients, the more accurate is $\bar{C}_{15}^{0,1}$, which has a value of $(-16.4 \pm 0.24) \times 10^{-9}$, so that the proportional error is 0.015. The expected value of $Q_\ell^{0,1}\bar{C}_{\ell m}$ relative to the first term (which is the largest) is $Q_\ell^{0,1}(\ell_0/\ell)^2$, if we assume $\bar{C}_{\ell m}$ is of order $10^{-5}/\ell^2$. Thus, since $\ell_0 = 15$ for $\bar{C}_{15}^{0,1}$, the series is truncated when $Q_\ell^{0,1}(15/\ell)^2 < 0.0075$: this applies for $\ell \geqslant 33$, so the series is taken to $\ell = 31$. Under these criteria the equations for the 12 lumped harmonics are as follows.

$$\bar{C}_{15}^{0,1} : \bar{C}_{15,15} + 0.513\bar{C}_{17,15} + 0.327\bar{C}_{19,15} + 0.221\bar{C}_{21,15} + 0.154\bar{C}_{23,15} + 0.110\bar{C}_{25,15} + 0.079\bar{C}_{27,15} + 0.058\bar{C}_{29,15} + 0.042\bar{C}_{31,15} = (-16.4 \pm 0.24) \times 10^{-9}. \quad (10)$$

$\bar{S}_{15}^{0,1}$: the equation is the same as (10), with S instead of C, and the numerical value on the right-hand side changed to $(-5.4 \pm 0.15) \times 10^{-9}$. (11)

$\bar{C}_{30}^{0,2}$: $\bar{C}_{30,30} + 0.430\bar{C}_{32,30} + 0.213\bar{C}_{34,30} + 0.100\bar{C}_{36,30} = (-8.2 \pm 0.6)\times 10^{-9}$. (12)

$\bar{S}_{30}^{0,2}$: the equation is the same as (12), with S instead of C , and the numerical value on the right-hand side changed to $(11.1 \pm 0.8) \times 10^{-9}$ ...... (13)

$\bar{C}_{15}^{1,0}$: $\bar{C}_{16,15} + 1.177\bar{C}_{18,15} + 1.250\bar{C}_{20,15} + 1.270\bar{C}_{22,15} + 1.256\bar{C}_{24,15}$
$+ 1.220\bar{C}_{26,15} + 1.169\bar{C}_{28,15} + 1.108\bar{C}_{30,15} + 1.040\bar{C}_{32,15}$
$+ 0.970\bar{C}_{34,15} + 0.899\bar{C}_{36,15} = (-92 \pm 48) \times 10^{-9}$ (14)

$\bar{S}_{15}^{1,0}$: the equation is the same as (14), with S instead of C , and the numerical value on the right-hand side changed to
$(-170 \pm 56) \times 10^{-9}$ (15)

$\bar{C}_{15}^{-1,2}$: $\bar{C}_{16,15} + 0.957\bar{C}_{18,15} + 0.839\bar{C}_{20,15} + 0.714\bar{C}_{22,15} + 0.598\bar{C}_{24,15}$
$+ 0.496\bar{C}_{26,15} + 0.410\bar{C}_{28,15} + 0.337\bar{C}_{30,15} + 0.276\bar{C}_{32,15}$
$+ 0.226\bar{C}_{34,15} + 0.185\bar{C}_{36,15} + 0.151\bar{C}_{38,15} + 0.123\bar{C}_{40,15}$
$= (-62.9 \pm 2.6) \times 10^{-9}$ (16)

$\bar{S}_{15}^{-1,2}$: the equation is the same as (16), with S instead of C , and the numerical value on the right-hand side changed to
$(-53.4 \pm 1.6) \times 10^{-9}$ (17)

$\bar{C}_{30}^{1,1}$: $\bar{C}_{31,30} + 1.040\bar{C}_{33,30} + 0.988\bar{C}_{35,30} + 0.907\bar{C}_{37,30} + 0.817\bar{C}_{39,30}$
$+ 0.728\bar{C}_{41,30} + 0.644\bar{C}_{43,30} + 0.567\bar{C}_{45,30} + 0.496\bar{C}_{47,30}$
$+ 0.433\bar{C}_{49,30} = (15 \pm 8) \times 10^{-9}$ (18)

$\bar{S}_{30}^{1,1}$: the equation is the same as (18), with S instead of C , and the numerical value on the right-hand side changed to
$(-39 \pm 12) \times 10^{-9}$ (19)

$\bar{C}_{30}^{-1,3}$: $\bar{C}_{31,30} + 0.874\bar{C}_{33,30} + 0.695\bar{C}_{35,30} + 0.531\bar{C}_{37,30} + 0.394\bar{C}_{39,30}$
$= (-8 \pm 3) \times 10^{-9}$ (20)

$\bar{S}_{30}^{-1,3}$: the equation is the same as (20), with S instead of C , and the numerical value on the right-hand side changed to
$(-13 \pm 7) \times 10^{-9}$. (21)

Equations (10) to (21) provide 12 equations of condition for the individual coefficients of order 15 and 30, which will be used in future determinations of the individual coefficients.

8 TESTS OF COMPREHENSIVE GRAVITY FIELD MODELS BY COMPARISON OF THEIR LUMPED HARMONICS WITH OBSERVATIONAL VALUES FROM 1971-54A

In recent years a number of gravity field models, complete up to a certain order and degree, have been determined from a combination of satellite observations, terrestrial gravity measurements and altimeter data. Of these models the Smithsonian Standard Earth IV.3 [26] goes to order and degree 24, the European model GRIM 2 [27] goes to order and degree 30, and the Goddard Earth Model 10B [1] goes to order and degree 36; while GEM 10C [2] consists of GEM 10B up to order and degree 36, together with 31000 extra coefficients determined from the altimeter geoid and going up to order and degree 180.

Inspection of equations (10) to (21) shows that terms of degree higher than 24 are involved in all the equations, so comparisons with SSE IV.3 cannot be pursued. Comparisons with GRIM 2 are also fruitless, because the 15th-order terms in GRIM 2 rely heavily on our 1975 sets of individual harmonics [3,4] which utilize our analysis of 1971-54A for the years 1972-4. In fact the values of $10^9\bar{C}_{15}^{0,1}$ and $10^9\bar{S}_{15}^{0,1}$ from GRIM 2 are -19 and -4, as compared with -16.4 ± 0.2 and -5.4 ± 0.1 in Table 3. The agreement is within the expected standard deviation of GRIM 2.

This leaves only GEM 10B (or GEM 10C if terms beyond $\ell = 36$ are required) for comparison. In making the comparisons, the last four equations (18)-(21), will be omitted, because the lumped values are not accurate enough to provide a satisfactory test - their standard deviations range between 30% and 55%. With these omissions, only one of the pairs of equations, namely (16) and (17), includes terms of degree higher than 36, and these are small, so that the 180 × 180 field GEM 10C is not really being tested and the comparison is with GEM 10B. Substituting the values of $\bar{C}_{\ell m}$, $\bar{S}_{\ell m}$ from GEM 10B into equations (10), (11), (14), (15), (16) and (17), ignoring the small terms involving GEM 10C in (16) and (17), we obtain values for the 15th-order lumped coefficients which are compared with those from the analysis of 1971-54A in Table 5. It was tentatively suggested in Ref 28 that the individual coefficients in GEM 10B might be given standard deviations of 5×10^{-9}, and the standard deviation of the lumped coefficients has been estimated on this basis.

Table 5 shows that the second pair of coefficients $(\bar{C},\bar{S})_{15}^{1,0}$ is, nominally, better determined from GEM 10B than from the resonance analysis. This was to be expected, because the $(\gamma,q) = (1,1)$ terms have very little influence on the resonance, as indicated by Table 3. So the second pair of lumped values from 1971-54A is of no use as a test of GEM 10B.

Table 5

Values of 15th-order lumped harmonics from 1971-54A resonance, compared with corresponding values from GEM 10B

	$10^9\bar{C}_{15}^{0,1}$	$10^9\bar{S}_{15}^{0,1}$	$10^9\bar{C}_{15}^{1,0}$	$10^9\bar{S}_{15}^{1,0}$	$10^9\bar{C}_{15}^{-1,2}$	$10^9\bar{S}_{15}^{-1,2}$
1971-54A	-16.4 ± 0.2	-5.4 ± 0.1	-92 ± 48	-170 ± 56	-63 ± 3	-53 ± 2
GEM 10B	-22 ± 6	-8 ± 6	-77 ± 18	-55 ± 18	-64 ± 10	-46 ± 10

For the other four coefficients in Table 5, the resonance values are nominally much more accurate than GEM 10B, and provide the required test. GEM 10B emerges from the comparison extremely well, all the GEM 10B values being within 1 standard deviation of the values from resonance - the actual differences being 0.9, 0.4, 0.1 and 0.7 of the estimated GEM 10B standard deviations. These four values have an rms of 0.6, thus indicating that a value of 3×10^{-9}, rather than 5×10^{-9} as assumed, is appropriate as an approximate standard deviation for the 15th-order coefficients in GEM 10B.

Only one of the pairs of 30th-order coefficients from the resonance analysis, namely $(\bar{C},\bar{S})_{30}^{0,2}$, is determined accurately enough to provide a real test of the 30th order coefficients in GEM 10B, and fortunately the numerical coefficients in equation (12) decrease rapidly, so that terms of degree higher than 36 are not needed. The comparisons are made in Table 6, where the assumed standard deviation of the individual GEM 10B coefficients has been reduced to 3×10^{-9}, as suggested in the previous paragraph.

Table 6

Values of $(\bar{C},\bar{S})_{30}^{0,2}$ from 1971-54A resonance, compared with corresponding values from GEM 10B

	$10^9\bar{C}_{30}^{0,2}$	$10^9\bar{S}_{30}^{0,2}$
1971-54A	-8.2 ± 0.6	11.1 ± 0.8
GEM 10B	-8.4 ± 3.3	11.2 ± 3.3

The agreement between these quite independent results from the resonance analysis and GEM 10B is amazing, and should be treated as yet another example of very low probabilities occurring more often than they ought to*. If we "season our admiration for a while", we may conclude from Table 6 that GEM 10B has passed a stringent examination with distinction, and that the standard deviation of the individual values may be less than 3×10^{-9}.

9 FURTHER WORK

Large numbers of observations of 1971-54A have been made by a variety of sensors, including the Baker-Nunn cameras, the Hewitt camera at Malvern, the US Navy Navspasur system and visual observers. If orbits were determined from all these observations for dates near the two e-resonances, a better analysis of the variations in eccentricity should be possible, without the need for the $(\gamma,q) = (0,1)$ terms. It would then be possible to make a combined fitting of i and e together using the SIMRES program[23]; this is not possible at present because the presence of the (0,1) terms would spoil the fitting of i.

10 CONCLUSIONS

The variations in the inclination and eccentricity of 1971-54A between November 1972 and January 1978, as given by the 269 US Navy weekly orbits, have been analysed to determine 12 lumped geopotential coefficients of order 15 and 30, given in Tables 2 and 4.

The fitting of the variations in inclination is nearly perfect (Fig 3) and yields the most accurate values yet obtained for lumped coefficients, $\bar{C}_{15}^{0,1}$ having an accuracy of 1.5%, and $\bar{C}_{30}^{0,2}$ an accuracy of 7%.

Difficulties were encountered in fitting the variations in eccentricity over the full 5¼ years, and the problem was solved by separate analyses of the two eccentricity resonances, which are 4 years apart (Figs 5 and 6).

Each of the 12 lumped coefficients obtained is a linear function of the individual harmonic coefficients, and the 12 equations, given as equations (10)-(21), will shortly be used in a new solution for the individual harmonic coefficients of 15th and 30th order.

Comparison of the lumped harmonics of 15th order with those obtained from the Goddard Earth Model 10B shows excellent agreement, and suggests that the accuracy of the high-order coefficients in GEM 10B is better than had been expected, being about 3×10^{-9}.

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* The values we obtained in 1975 for the lumped coefficients from 1971-54A were (-10.3 ± 1.5) and (15.3 ± 1.3); so the cynical suggestion that our 1975 values somehow found their way into GEM 10B can be withdrawn before it is voiced.

Only one pair of 30th-order coefficients is accurate enough to provide a test of GEM 10B. The values agree astonishingly well, thus suggesting the possibility that the GEM 10B coefficients may be even more accurate than indicated in the previous paragraph.

Acknowledgments

I am most grateful to Dr R.L. Easton and Dr R.R. Zirm of the US Naval Research Laboratory for kindly arranging for the unfailing supply over so many years of the weekly US Navy orbital elements, on which this analysis entirely relies.

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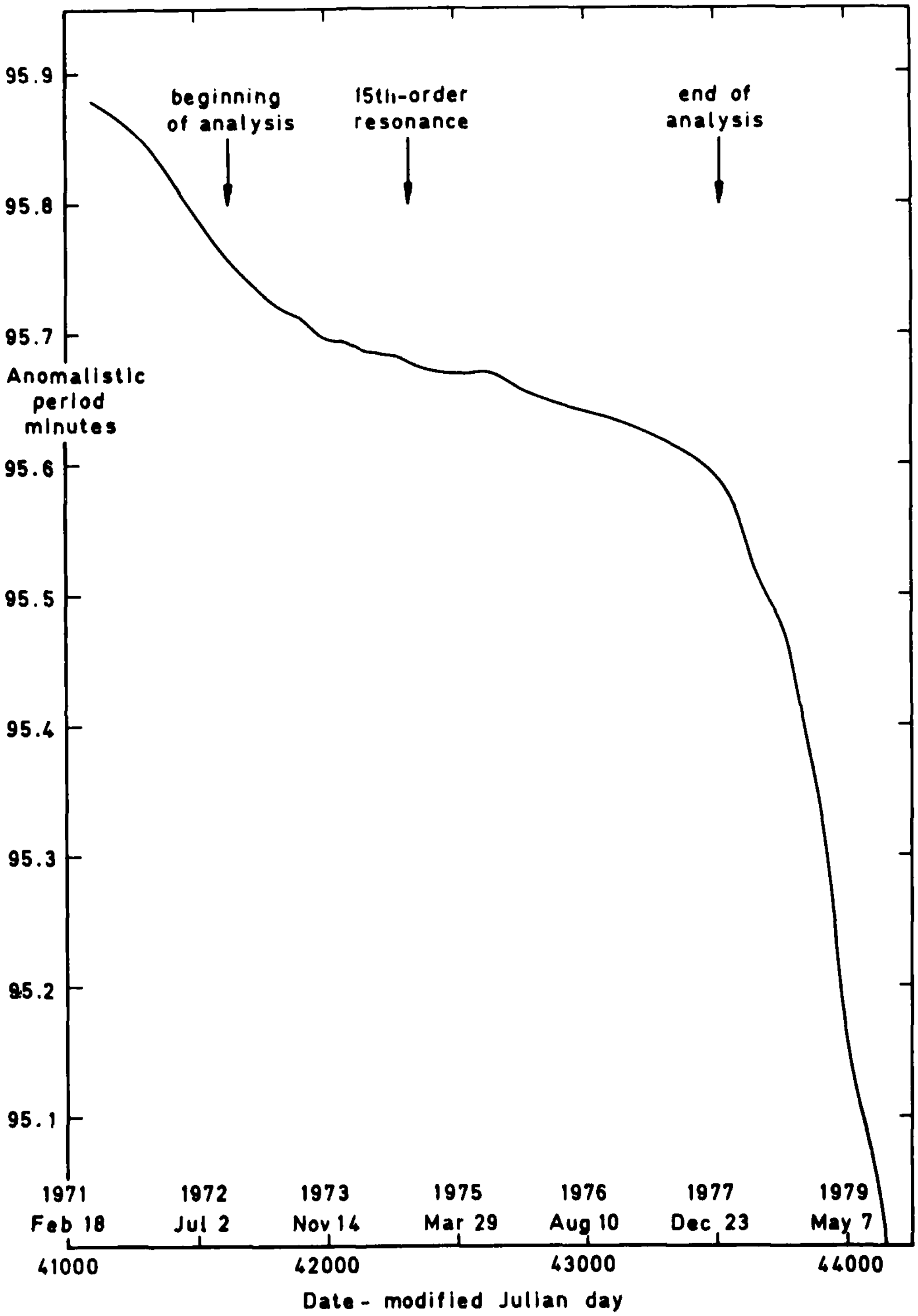

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Fig 1 Orbital period of 1971-54A from launch (1971) to 1979

Fig 2

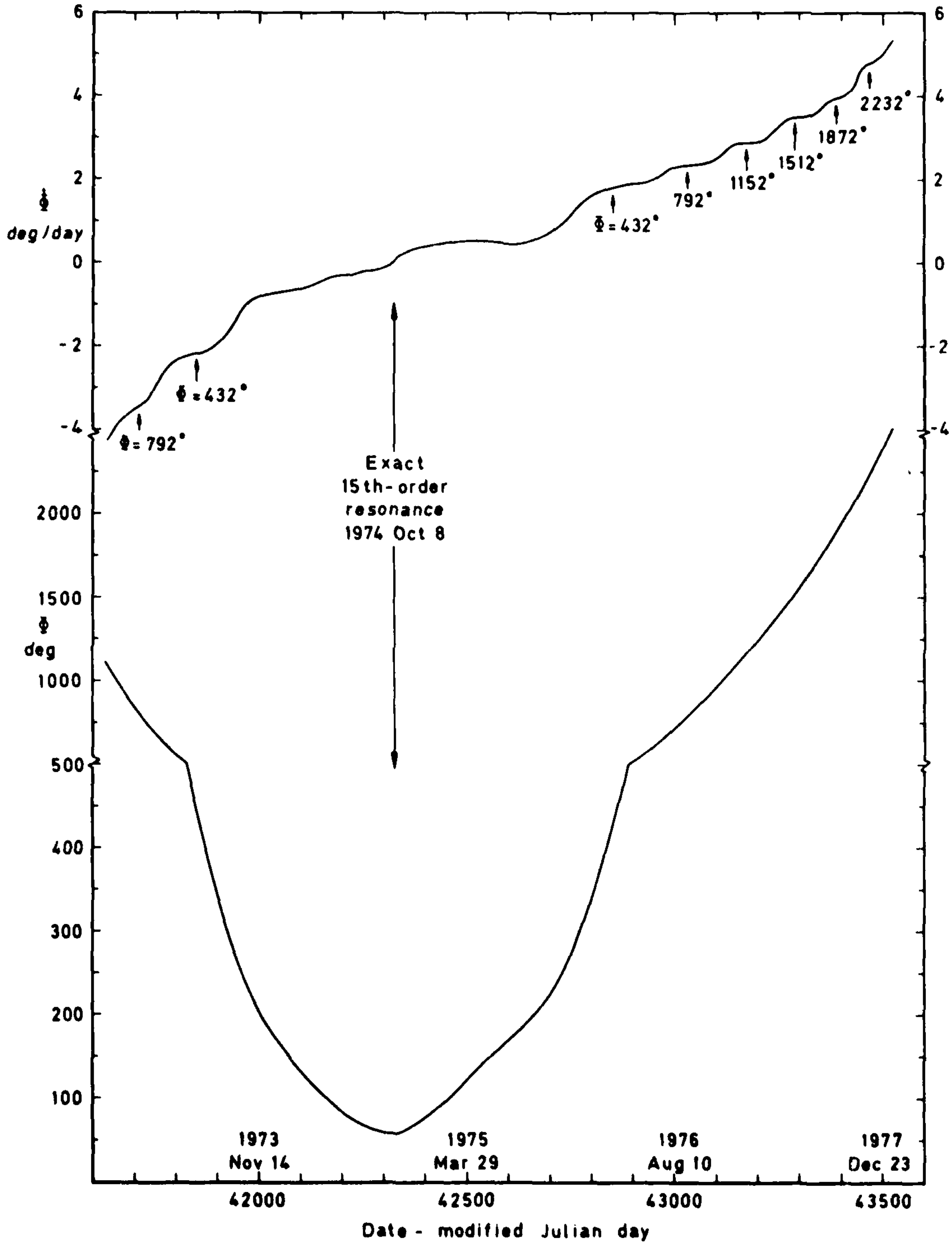

Fig 2 Variation of Φ and $\dot{\Phi}$

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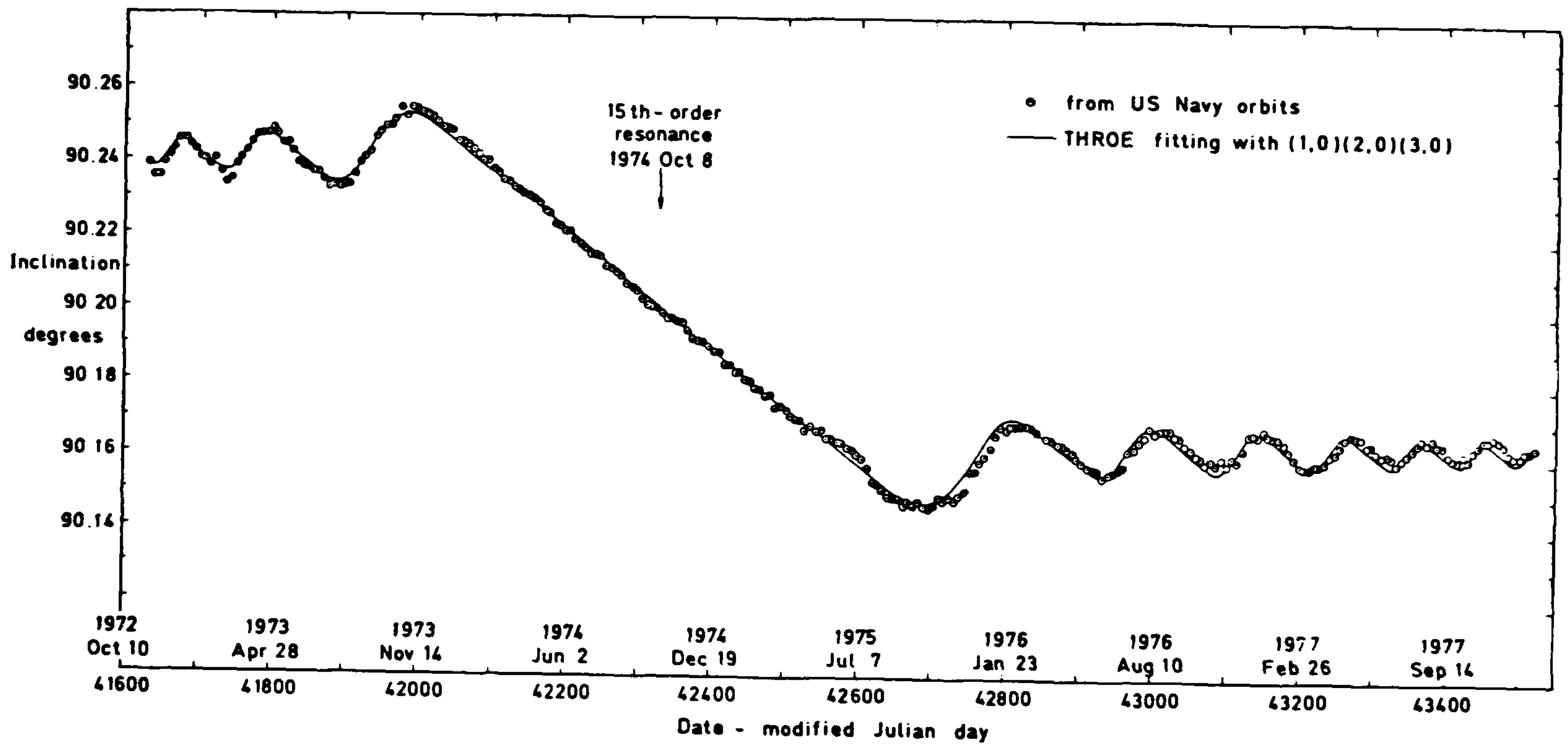

Fig 3 1971-54A: variation of inclination near 15th-order resonance

Fig 4

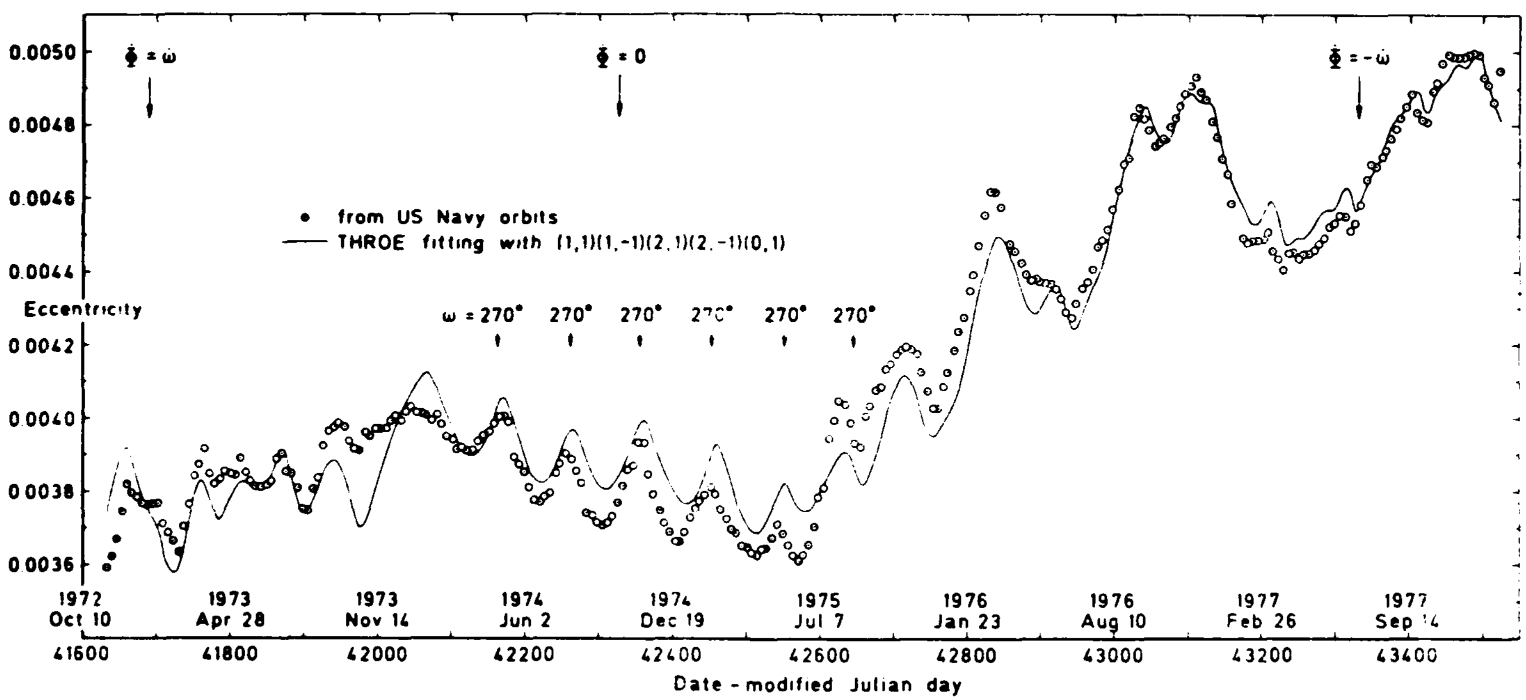

Fig 4 1971-54A: variation of eccentricity from Nov 1972 to Jan 1978

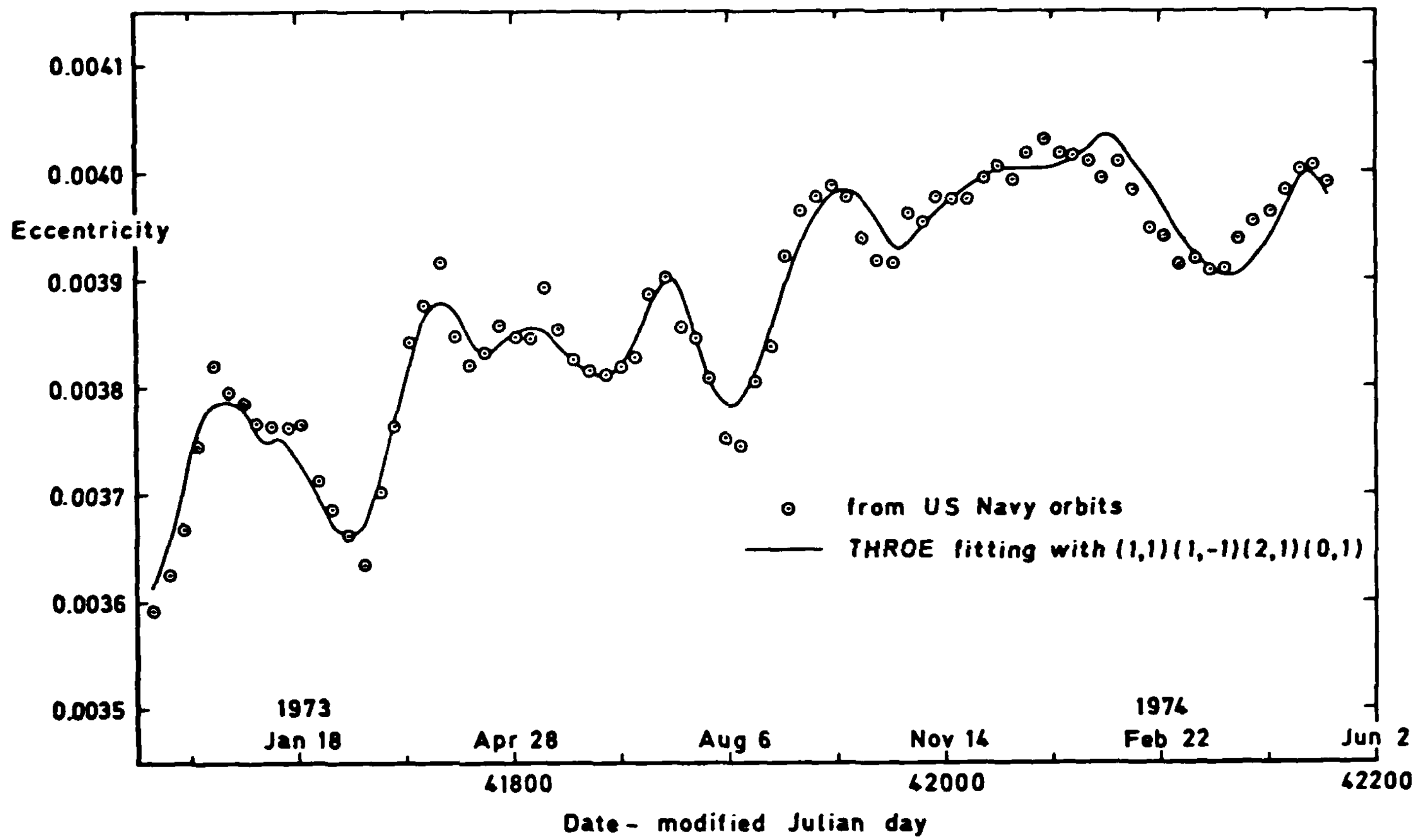

Fig 5 1971-54A: variation of eccentricity near the $(\gamma,q) = (1,1)$ resonance

Fig 6

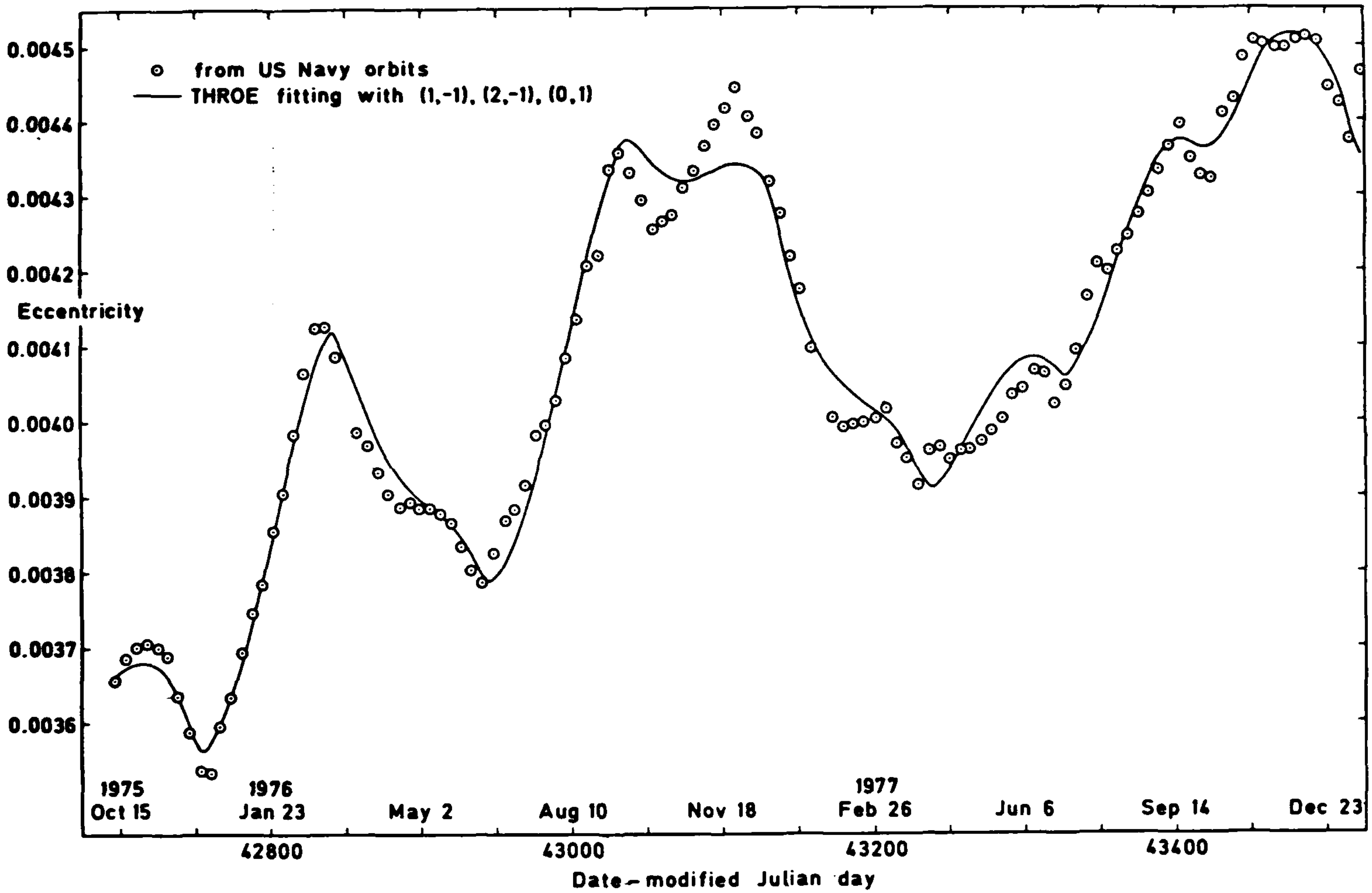

Fig 6 1971-54A: variation of eccentricity near the $(\gamma, q) = (1,-1)$ resonance

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