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411278 SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) solution of a non-linear operator equation. Necessary and sufficient conditions for to exist are also found. In the case of the GI/C/1 queueing chain above these are exactly the usual stability conditions.

OPERATOR-GEOMETRIC STATIONARY DISTRIBUTIONS FOR MARKOV CHAINS, WITH APPLICATION TO QUEUEING MODELS

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ABSTRACT

This paper considers a class of Markov chains on a bivariate state space $(\mathbb{N},E)$, whose transition probabilities have a particular "block-partitioned" structure. Examples of such chains include those studied by Neuts (1978) who took E to be finite; they also include chains studied in queueing theory, such as (N_n,S_n) where N_n is the number of customers in a GI/G/1 queue immediately before, and S_n the remaining service time immediately after, the n^{th} arrival.

We show that the stationary distribution Π for these chains has an "operator-geometric" nature, with $\Pi(k,\cdot) = \int \Pi(0,dy)\ S^k(y,\cdot)$, where the operator S is the minimal solution of a non-linear operator equation. Necessary and sufficient conditions for Π to exist are also found. In the case of the GI/G/1 queueing chain above these are exactly the usual stability conditions.

Keywords: GI/G/1 queue, phase-type, invariant measure, Foster's conditions.

1. INTRODUCTION

In [8], [9] and [10] Neuts has shown that certain classes of Markov chains admit a stationary distribution of a form which he has termed "matrix-geometric". Such chains have a bivariate state space (i,j), $i = 0,1,\ldots,$ $j = 1,\ldots,n$; the stationary distribution $\Pi(i,j)$ has the geometric property that

$$\Pi(0,j) = {}_0\Pi(j), \qquad j = 1,\ldots,n,$$
$$\Pi(i,j) = ({}_0\Pi S^{i})_j, \qquad j = 1, \ldots,n,$$

for a particular $n\times n$ matrix S.

The class of chains with this structure includes embedded Markov chains associated with GI/PH/1 queues, where the service time distribution is "phase-type" [7]. The second variable j then indexes the phase of the service time.

In this paper we investigate more general chains with bivariate space (i,x), where the first coordinate i takes on integer values $0,1,\ldots,$ but the second coordinate x is in E, some general measure space. In particular examples such as the GI/G/1 queue in §4 below, E will be taken as $[0,\infty)$. We let $\mathcal{E}$ denote σ-field on E; when $E = [0,\infty)$ we will take $\mathcal{E}$ as the Borel σ-field. We denote by $\mathbb{N}$ the set $\{0,1,\ldots\}$, and the set $i\times E$ will be called the *level* i, and denoted by $\underset{\sim}{i}$.

The basic Markov chain $\{X_n\}$ that we shall study is assumed to have a transition probability law denoted by $P(i,x;\ j,A)$ for $i,j \in \mathbb{N}$, $x \in E$, $A \in \mathcal{E}$. As usual it is taken to be a measurable function in x for each $A \in \mathcal{E}$ and a substochastic measure on $\mathcal{E}$ for each $i,j \in \mathbb{N}$ and $x \in E$; we also assume $\sum_j P(i,x;\ j,E) = 1$. The n-step iterates of P are defined by

$$P^n(i,x;\ j,A) = \sum_k \int_E P^{n-1}(i,x;\ k,dy)\, P(k,y;\ j,A)\ ,$$

and have the interpretation

$$P^n(i,x;\ j,A) = \mathbb{P}\{X_n \in j\times A \mid X_0 = (i,x)\} .$$

We now impose the following "spatial homogeneity" pattern on the transition law P. We assume that

$$P(i,x;\ j,E) = 0\ ,\quad j > i+1\ ,$$

and for each $k = 0,1,\ldots$ there exists a substochastic transition kernel A_k (that is, a collection $A_k(x,A)$, $x \in E$, $A \in \mathcal{E}$, measurable in x for each A and a measure on $\mathcal{E}$ for each $x \in E$, with $A_k(x,E) \le 1$, $x \in E$) such that for $j \neq 0$,

(1.1) $$P(i,x;\ j,A) = A_{i-j+1}(x,A)\ ;$$

and we write

$$P(i,x;\ 0,A) = B_i(x,A)\ .$$

Since P is stochastic, from (1.1) we have

$$\begin{aligned} P(i,x;\ 0,E) &= 1 - \sum_{j=1}^{\infty} A_{i-j+1}(x,E) \\ &= 1 - \sum_{j=0}^{i} A_j(x,E) \end{aligned}$$

(1.2) $$= B_i(x,E)\ .$$

Diagramatically, (1.1) and (1.2) show that we can write P as

(1.3) $$P = \begin{pmatrix} B_0 & A_0 & 0 & \cdots & & \\ B_1 & A_1 & A_0 & 0 & \cdots & \\ B_2 & A_2 & A_1 & A_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

by labelling the states in increasing order on the first variable. Typically (see [8] and §4) in queueing applications we have $\sum_{k=0}^{\infty} A_k(x,E) = 1$, so that $B_j(x,E) = \sum_{j+1}^{\infty} A_k(x,E)$.

2. THE OPERATOR-GEOMETRIC FORM OF Π

A stationary probability measure Π for the chain $\{X_n\}$ is a collection of measures $\Pi(j,\cdot)$ on E, $j \in \mathbb{N}$, with $\sum_j \Pi(j,E) = 1$, and satisfying

$$\Pi(j,A) = \sum_k \int_E \Pi(k,dy)\ P(k,y;\ j,A)\ ,\quad A \in \mathcal{E},\ j \in \mathbb{N}\ .$$

From (1.1) and (1.2), for $j > 0$

$$\Pi(j,A) = \sum_k \int_E \Pi(k,dy)\ A_{k-j+1}(y,A)\ , \tag{2.1}$$

whilst for $j = 0$,

$$\Pi(0,A) = \sum_k \int_E \Pi(k,dy)\ B_k(y,A)\ . \tag{2.2}$$

In order to use the theory of general Markov chains to find conditions under which a measure Π satisfying (2.1)-(2.2) exists, we shall assume the following.

<u>Irreducibility condition</u>: There is a measure $\Phi(\cdot)$ on $\mathbb{N} \times E$ such that, whenever $\Phi(j,A) > 0$ for some $j \in N$, $A \in \mathcal{E}$, for each $i \in \mathbb{N}$ and $x \in E$ we have

$$0 < \sum_{n=1}^{\infty} P^n(i,x;\ j,A)\ ; \tag{2.3}$$

and

$$\Phi(0,E) > 0\ . \tag{2.4}$$

The condition (2.3) is just the standard Φ-irreducibility condition (see [14], §2, or [11], [12]). We impose (2.4) specifically because of the behaviour we wish to investigate : the chain $\{X_n\}$ satisfying (2.4) and (2.3) is such that from any pair (i,x), the zero-level $\underset{\sim}{0}$ can be reached eventually with positive probability.

The key to the operator geometric structure of any solution Π to (2.1) and (2.2) is a last exit-time representation of Π, familiar in the countable case (cf. [1]). We write, for $i,j \in \mathbb{N}$, $x \in E$, $A \in \mathcal{E}$,

$$ {}_0P^1(i,x;\ j,A) = P(i,x;\ j,A) $$

$$ {}_0P^n(i,x;\ j,A) = \sum_{k>0} \int_E {}_0P^{n-1}(i,x;\ k,dy)\ P(k,y;\ j,A)\ . $$

These are the taboo probabilities of $\{X_n\}$ where the level $\underset{\sim}{0}$ is taboo; and we write

$$ L(i,x;\ j,A) = \sum_{n=1}^{\infty} {}_0P^n(i,x;\ j,A)\ . $$

If $\tau_0 = \inf(n > 0 : X_n \in \underset{\sim}{0})$, then

$$ \begin{aligned} L(0,x;\ 0,E) &= \mathbb{P}(\tau_0 < \infty \mid X_0 = (0,x)) \\ &\le 1\ , \end{aligned} $$

and $L(0,x;\ j,A)$ for $j > 0$ is just the expected number (possibly infinite) of visits to state $j \times A$ before returning to level $\underset{\sim}{0}$.

We now have

<u>Theorem 1</u>: (i) If the chain $\{X_n\}$ has a stationary probability measure Π, then $\Pi(0,E) > 0$. For Π-almost every $x \in E$, we have $L(0,x;\ 0,E) = 1$. The chain $\{{}_0X_n\}$ defined on E with transition law $L(x,A) = L(0,x;\ 0,A)$ has a stationary probability measure, ${}_0\Pi(\cdot)$, satisfying

(2.5) $$ {}_0\Pi(A) = \int_E {}_0\Pi(dy)\ L(y,A), \quad A \in \mathcal{E}\ . $$

The measure Π is then given, for $j > 0$, by

(2.6) $$ \Pi(j,A) = c \int_E {}_0\Pi(dy)\ L(0,y;\ j,A), \quad A \in \mathcal{E}, $$

and

(2.7) $$ \Pi(0,A) = c\ {}_0\Pi(A), \quad A \in \mathcal{E}\ , $$

where

(2.8) $$ C = [\Pi(0,E)]^{-1} > 0\ . $$

(ii) If ${}_0\Pi(A)$, $A \in \mathcal{E}$ is a probability measure on $\mathcal{E}$ satisfying (2.5), then Π defined by (2.6) and (2.7) is a σ-finite measure satisfying

(2.1) and (2.2).

Proof: Most of these results are well-known in a general context. If Π exists as in (i), then $\Pi >> \Phi$ and so $\Pi(0,E) > 0$ from (2.4). The existence of Π implies the chain $\{X_n\}$ is 1-positive recurrent (see [14]) and so, since $\Pi(0,E) > 0$, we have $L(i,x;\ 0,E) = 1$ for Π-almost all $(i,x) \in \mathbb{N} \times E$. It is then easy to see ([11], pp. 32-33) that ${}_0\Pi(\cdot) = \Pi(0,\cdot)/\Pi(0,E)$ is invariant for the chain with transition law $L(x,A)$; one has to check that, writing $N = \{y \in E : L(y,E) < 1\}$, the set $\bar{N} = \{y \in E : \sum_n L^n(y,N) > 0\}$ also has $\Pi(\bar{N}) = 0$, which is a standard result.

Since $\{X_n\}$ is a Φ-irreducible chain, Π is unique; it is not difficult ([11], p. 32) to check that the measure given by the right-hand sides of (2.6), (2.7) is invariant for $\{X_n\}$, and so must be Π. This also shows that (ii) is true. $\square$

This theorem basically shows that the only invariant measure is given by (2.6) and (2.7), and motivates the investigation below into the structure of the quantities $L(i,x;\ j,A)$. We shall define, analogously with the quantities $L(i,x;\ 0,A)$, the more general taboo probabilities

$$ {}_{\ell}L(i,x;\ j,A) = \sum_n {}_{\ell}P^n(i,x;\ j,A)\ , $$

where

$$ {}_{\ell}P^n(i,x;\ j,A) = \mathbb{P}(X_n \in \underset{\sim}{j} \times A,\ X_r \notin \underset{\sim}{\ell},\ r = 1,\ldots,n-1 \mid X_0 = (i,x))\ . $$

The first crucial aspect of the structure (1.3) of P that we use is that the chain moves in a "right-continuous" way between levels; from $\underset{\sim}{i}$ one can increase levels to $\underset{\sim}{i}+\underset{\sim}{1}$ but not in one step to $\underset{\sim}{i}+\underset{\sim}{k}$, $k > 1$. Hence, if $\ell \le i$, $\ell \le j$, ${}_{\ell}P^n(i,x;\ j,A) = P\{X_n \in \underset{\sim}{j} \times A,\ X_r \notin \{\underset{\sim}{0},\underset{\sim}{1},\ldots,\underset{\sim}{\ell}\}\ r = 1,\ldots, n-1 \mid X_0 = (i,x)\}$. We can thus define, as in [9],

$$ S^{(k)}(x,A) = {}_iL(i,x;\ i+k,A),\ i \in \mathbb{N},\ x \in E,\ A \in \mathcal{E}, \tag{2.9} $$

independent of i. (This can be checked by writing out $S^{(k)}$ in terms of the A_j). We write

$$S(x,A) = S^{(1)}(x,A) .$$

The following result then parallels Theorem 1 of [9].

Theorem 2: Suppose the Markov chain $\{X_n\}$ has an invariant probability measure Π.

(i) For $k \geq 0$,

$$\Pi(k,A) = c \int_E {}_0\Pi(dy)\ S^{(k)}(y,A) ,$$

where $c = [\Pi(0,E)]^{-1}$.

(ii) For $k \geq 0$,

$$S^{(k)}(x,A) = S^k(x,A), \quad x \in E,\ A \in \&,$$

where

$$S^1(x,A) = S(x,A), \quad x \in E,\ A \in \&,$$

and

$$S^k(x,A) = \int_E S^{k-1}(x,dy)\ S(y,A), \ x \in E,\ A \in \&,$$

are the usual iterates of the kernel $S(x,A)$.

(iii) The kernel S satisfies the non-linear operator equation

$$S = A[S] \tag{2.10}$$

where we define, following [8], the formal series

$$D[Q](x,B) = \sum_{k=0}^{\infty} \int_E Q^k(x,dy)\ D_k(y,B), \ x \in E,\ B \in \&.$$

for any set D_j of kernels and the iterates Q^k of any kernel Q.

(iv) The kernel S is 1-transient (see [14]) and in fact for ${}_0\Pi$-almost all $x \in E$,

$$\sum_k S^k(x,E) < \infty . \tag{2.11}$$

(v) If $\hat{S}$ is another kernel satisfying $\hat{S} = A[\hat{S}]$, then

$$\hat{S}(x,A) \geq S(x,A), \quad x \in E,\ A \in \mathcal{E}. \tag{2.12}$$

Proof: (i) This is just a restatement of the general result in (2.6) and (2.7), using the fact that by definition (2.9), $L(0,y;\ k,A) = S^{(k)}(x,A)$.

(ii) We have, decomposing over the time of last entrance to level $\underset{\sim}{k}$,

$$_0P^n(0,x;\ k+1,A) = \sum_{r=0}^{n} \int_E {_0P^r}(0,x;\ k,dy)\ {_kP^{n-r}}(k,y;\ k+1,A)$$

$$= \sum_{r=0}^{n} \int_E {_0P^r}(0,x;\ k,dy)\ {_0P^{n-r}}(0,y;\ 1,A)\ , \tag{2.13}$$

and summing (2.13) over n gives the result

$$S^{(k+1)}(x,A) = \int_E S^{(k)}(x,dy)\ S(y,A)$$

as required.

(iii) In this case we decompose $_0P^n$ over the position at time n-1. By construction, $_0P(0,x;\ 1,B) = A_0(x,B)$ and for $n > 1$,

$$_0P^n(0,x;\ 1,B) = \sum_{k=1}^{\infty} \int_E {_0P^{n-1}}(0,x;\ k,dy)\ A_k(y,B)\ ; \tag{2.14}$$

summing over n leads, using (ii), to the required result.

(iv) Since $S^k = S^{(k)}$, we have that $\sum_{k=1}^{\infty} S^k(x,E) = E(\tau_0 \mid X_0 = (0,x))$.

It is well-known that, since $\Phi(0,E) > 0$, this quantity is finite for Π-almost all $x \in E$; to see this one uses the fact that, from (i) above,

$$1 = \Pi(\mathbb{N} \times E) = c \int_E {_0\Pi}(dy) \sum_k S^k(x,E)\ .$$

(v) As in [9], we first define another sequence of kernels by setting $X_0(x,B) \equiv 0$, and $X_{N+1} = A[X_N]$ for $N \geq 0$. For every $x \in E$, $B \in \mathcal{E}$ we have $X_0(x,B) \leq S(x,B)$, and by induction

$$X_{N+1}(x,B) = A[X_N](x,B) \leq A[S](x,B) = S(x,B)\ . \tag{2.15}$$

A similar calculation also shows $X_N(x,B)$ to be monotonically increasing in N for every $x \in E$ and $B \in \mathcal{E}$. Hence the quantity $X_*(x,B) = \lim_{N\to\infty} \uparrow X_N(x,B)$ exists for every $x \in E$ and $B \in \mathcal{E}$. For each fixed x, this setwise convergence guarantees that $X_*(x,\cdot)$ is a measure on $\mathcal{E}$; for each $B \in \mathcal{E}_1$, $X_*(\cdot,B)$ is clearly a measurable function on E which is finite except perhaps on the set $N_S = \{y \in E : S(y,E) = \infty\}$. The kernel X_* is a solution to (2.10), since the monotone convergence of $X_N(x,B)$ to $X_*(x,B)$ implies firstly that

$$X_N^k(x,B) \uparrow X_*^k(x,B), \text{ and then that } X_*(x,B) = \lim_N X_N(x,B) = A[X_*](x,B) .$$

Moreover (2.15) shows that $X_*(x,B) \leq \hat{S}(x,B)$ for any solution $\hat{S}$ satisfying (2.10), so that X_* is the minimal solution of (2.10).

We now show that in fact $X_* = S$, as in [9]. Define

$$S_N(x;\ k,B) = \sum_{n=1}^{N} {}_0P^n(0,x;\ k,B) ,$$

so that clearly $S_N(x;\ 1,B) \uparrow S(x,B)$. Now from (2.14), we have

$$S_N(x;\ 1,B) = A_0(x,B) + \sum_{k=1}^{\infty} \int_E S_{N-1}(x;\ k,dy)\ A_k(y,B) . \tag{2.16}$$

Moreover, the last exit equations (2.13) give

$$S_{N-1}(x;\ k+1,B) \leq \int_E S_{N-1}(x;\ k,dy)\ S_{N-1}(y;\ 1,B),\ B \in \mathcal{E} ,$$

so that $S_{N-1}(x;\ k+1,B) \leq S_{N-1}^{k+1}(x;\ 1,B)$, $B \in \mathcal{E}$.

Substituting in (2.16) gives

$$S_N(x;\ 1,B) \leq \sum_k \int_E S_{N-1}^k(x;\ 1,dy)\ A_k(y,B) . \tag{2.17}$$

Finally, note that $S_1(x;\ 1,B) = A_0(x,B) = X_1(x,B)$, and so from (2.17), we have by induction

$$S_N(x;\ 1,B) \leq X_N(x,B) .$$

Taking limits as $N \to \infty$ gives $S(x,B) \leq X_*(x,B)$ as required. □

This result shows that Theorem 1 of [9] and Theorem 3 of [8] are essentially a product of the standard construction of Π as in Theorem 1 above, with the geometric part S^k coming from the right-continuity and translation invariance of P. The multiplier ${}_0\Pi$ is shown in Theorem 3 of [8] to be an eigenvector of the matrix B[S]; our next result gives the probabilistic explanation for this.

Proposition 1 : The Markov chain $\{{}_0\dot{X}_n\}$ has the transition law

$$L(x,A) = B[S](x,A)$$

and hence when $\{X_n\}$ has a stationary measure Π, for ${}_0\Pi$-almost all $x \in E$,

$$B[S](x,E) = 1 \ . \tag{2.18}$$

Proof : As in (2.14), ${}_0P(0,x;\ 0,A) = B_0(x,A)$ and for $n > 1$

$${}_0P^n(0,x;\ 0,A) = \sum_{k=1}^{\infty} \int_E {}_0P^{n-1}(0,x;\ k,dy)\ B_k(y,A) \ ,$$

so that the form of L(x,A) follows on summing over n. The result (2.18) is then immediate from Theorem 1(i). □

The probabilistic interpretation here explains (2.18) which is derived directly in [8]. The direct derivation as in [8] does give us the following uniqueness result for the kernel S.

Theorem 3 : Suppose $\hat{S}$ satisfies $\hat{S} = A[\hat{S}]$. For any $\alpha \in E$, either $\alpha \in \hat{N} = \{y : \sum_k \hat{S}^k(y,E) = \infty\}$, or $\hat{S}(\alpha,A) \equiv S(\alpha,A)$, $A \in \mathcal{E}$.

Proof : From (2.12), $\hat{S}(x,A) \geq S(x,A)$ for all $x \in E$, $A \in \mathcal{E}$ (note that this proof does not require Π to exist); and so for all A,

$$B[\hat{S}](x,A) \geq B[S](x,A) \ ; \tag{2.19}$$

moreover, for any x, A, the inequality in (2.19) is strict if there exists B with $\hat{S}^k(x,B) > S^k(x,B)$ and $B_k(y,A) > 0$, $y \in B$. Now note from (1.2) that

$B_k(y,E) \leq B_1(y,E)$ for all $y \in E$; and so, since $\Phi(0,E) > 0$, it follows that $B_1(y,E) > 0$ for all $y \in E$, from (2.3). Hence from (2.19) if for some $x \in E$ and $B \in \mathcal{E}$, $\hat{S}(x,B) > S(x,B)$, then

$$B[\hat{S}](x,E) > B[S](x, E) \,. \tag{2.20}$$

Now suppose $\alpha \notin \hat{N}$. Then

$$\begin{aligned}
B[\hat{S}](x,E) &= \sum_{k=0}^{\infty} \int_E \hat{S}^k(x,dy)\, B_k(y,E) \\
&= \sum_{k=0}^{\infty} \int_E \hat{S}^k(x,dy)\, [1 - \sum_{\ell=0}^{k} A_\ell(y,E)] \\
&= \sum_{k=0}^{\infty} \hat{S}^k(x,E) - \sum_{\ell=0}^{\infty} \sum_{k=\ell}^{\infty} \int_E \hat{S}^k(x,dy)\, A_\ell(y,E) \\
&= \sum_{k=0}^{\infty} \hat{S}^k(x,E) - \int_E \sum_{k=0}^{\infty} \hat{S}^k(x,dw) \sum_{\ell=0}^{\infty} \int_E \hat{S}^\ell(w,dy)\, A_\ell(y,E) \qquad (2.21) \\
&= \sum_{k=0}^{\infty} \hat{S}^k(x,E) - \int_E \sum_{k=0}^{\infty} \hat{S}^k(x,dw)\, \hat{S}(w,E) \\
&= 1
\end{aligned}$$

from (2.10) and the fact that $\hat{S}^0(x,E) \equiv 1$. Since $\alpha \notin \hat{N}$, the minimality of S implies $\alpha \notin N$ so (2.21) also shows $B[S](\alpha,E) = 1$. Hence (2.20) cannot hold with $\alpha \notin \hat{N}$, and so $\hat{S}(\alpha,B) \equiv S(\alpha,B)$ as required. □

This result is in some sense an analogue of Theorem 2 of [8], although the proof is quite different, and the uniqueness result is not as strong. We need more structure than is generally available (unless E is finite) to gain the complete analogue of Theorem 2; we now investigate this. Following [15], we call the kernel $\{S(x,A)\}$ *R-recurrent* if (i) S is μ-irreducible for some μ, (ii) for any $r < R$ there is some $A \in \mathcal{E}$ with $\mu(A) > 0$ and some $x \in E$ with $\sum_k S^k(x,A)r^k < \infty$, (iii) for all $A \in \mathcal{E}$ with $\mu(A) > 0$, and all $x \in E$, the series $\sum_k S^k(x,A)R^k$ diverges. Recall from [15] that there is then a unique

R-subinvariant measure $Q(\cdot)$ satisfying

$$Q(A) \geq R \int Q(dy)\, S(y,A), \quad A \in \mathcal{E}, \tag{2.22}$$

which is R-*invariant*, i.e. satisfies (2.22) with equality. The quantity R^{-1} is the natural analogue of the Perron-Frobenius eigenvalue for finite matrices, with Q the corresponding eigenvector.

Hence the restriction in [8] to solutions of (2.10) with spectral radius $sp(S) \leq 1$ corresponds to restricting ourselves to solutions of (2.10) with radius of convergence $R \geq 1$. If S is 1-recurrent then this is the natural analogue of $sp(S) = 1$, whilst if S is R-recurrent for $R > 1$, then this is analogous to $sp(R) < 1$. Our next result thus covers the dichotomy of Theorem 2 of [8] exactly if we recall that when E is finite and $R > 1$, then $\sum S^k(x,E) < \infty$ for all x; this does not happen when E is infinite.

<u>Theorem 4</u> : Suppose that for some measure ψ the kernel $A(x,B) \equiv \sum_k A_k(x,B)$ is ψ-irreducible, and that the minimal solution of $S = A[S]$ is μ-irreducible for some μ. Then the convergence norm R of the kernel S satisfies $R \geq 1$. Suppose further that S is R-recurrent. Either

(i) $R = 1$: and then S is the unique (up to definition on a ψ-null set) solution of $S = A[S]$ with convergence norm at least unity; or

(ii) $R > 1$: and then if $\hat{S} = A[\hat{S}]$, either $\psi(\hat{N}) = 0$ and so from Theorem 3 $S \equiv \hat{S}$ ψ-a.e.; or $\hat{N} = E$, and so $S(x,E) < \hat{S}(x,E)$ for every $x \notin N$. In particular, if $\{X_n\}$ has a stationary measure Π then either $S \equiv \hat{S}$ ψ-a.e. or $S < \hat{S}$ ${}_0\Pi$-a.e.

<u>Proof</u> : Since $\{A(x,\cdot)\}$ is ψ-irreducible, it has at least one 1-subinvariant measure ν (see[15]). Let X_N be the sequence of kernels approximating S as in the proof of Theorem 2(v). Clearly $\nu(B) \geq \int \nu(dy)\, X_0(y,B)$, and by induction

$$\int_E \nu(dy)\ X_{N+1}(y,B) = \int_E \nu(dy) \sum_k \int_E X_N^k(y,dw)\ A_k(w,B)$$

$$(2.23) \qquad \le \int_E \nu(dw) \sum_k A_k(w,B)$$

$$\le \nu(B)\ ;$$

hence $\nu(B) \geq \int \nu(dy)\ S(y,B)$, and hence [15], since S has a 1-subinvariant measure, S must have convergence norm $R \geq 1$. Assume now that S is R-recurrent.

(*i*) *Let* $\hat{S}$ *be any other kernel satisfying* $\hat{S} = A[\hat{S}]$. If S is 1-recurrent, and $\hat{S}$ has convergence norm $\hat{R} \geq 1$, then $S \leq \hat{S}$ implies that $\hat{S}$ is 1-recurrent also. Let $\hat{Q}$ be the unique 1-invariant measure of $\hat{S}$; from (2.22), and (2.10),

$$\hat{Q}(B) = \int_E \hat{Q}(dy)\ \hat{S}(y,B)$$

$$(2.24) \qquad = \int_E \hat{Q}(dy) \sum_k \int_E \hat{S}^k(y,dw)\ A_k(w,B)$$

$$= \int_E \hat{Q}(dw) \sum_k A_k(w,B)\ .$$

Hence $\hat{Q}$ is 1-invariant for the kernel $\{A(x,\cdot)\}$ and from (2.23), we have that $\hat{Q}$ is 1-subinvariant for S. Since S is 1-recurrent, it has a unique 1-subinvariant measure Q which is thus $\hat{Q}$. Thus we have for all $B \in \mathcal{E}$

$$(2.25) \qquad \int_E Q(dw)\ S(w,B) = Q(B) = \int_E Q(dw)\ \hat{S}(w,B)\ ,$$

and from (2.25) and the minimality of S it follows that for each $B \in \mathcal{E}$, $S(w,B) = \hat{S}(w,B)$, except perhaps for $w \in D_B$ where $Q(D_B) = 0$. By taking a sequence $B_j \uparrow E$ with $Q(B_j) < \infty$ for each j it follows that there exists $D = \bigcup_j D_{B_j}$ with $Q(D) = 0$ such that $\hat{S}(w,B) \equiv S(w,B)$, $w \notin D$. Since $Q \gg \psi$ we have proved (i).

(ii) As in [8], we write $A_z^*(x,B) = \sum_{k=0}^{\infty} A_k(x,B)\, z^k$, $0 \leq z \leq 1$.

Suppose that S is R-recurrent with R > 1, and let Q be the unique R-invariant measure for S. As in (2.24) we have

$$Q(B) = R \int_E Q(dw) \sum_k R^{-k} A_k(w,B) \tag{2.26}$$

so that $Q(\cdot)$ is also R-invariant for $A^*_{R^{-1}}(w,A)$. Write $C(w,A) = A^*_{R^{-1}}(w,A)$; and note that, since $\{A(x,\cdot)\}$ is ψ-irreducible so is $\{C(x,\cdot)\}$.

Now suppose $\hat{S}$ satisfies (2.10), and assume that $\psi(\hat{N}) > 0$. Note first of all that $\hat{N} \supseteq \{y: \sum_k \hat{S}^k(y,\hat{N}) > 0\}$, and hence $\hat{F} = \hat{N}^c$ is "closed" in the sense that if $y \in \hat{F}$, then $\hat{S}(y,\hat{N}) = 0$. But from Theorem 3, on $\hat{F}$ we have $\hat{S}(y,\cdot) \equiv S(y,\cdot)$; so $S(y,\hat{N}) = 0$, $y \in \hat{F}$. Since S is μ-irreducible, this implies that either $\hat{F}$ is empty (i.e. $\hat{N} = E$ as required) or else $\mu(\hat{F}) > 0$; and in the latter case from (2.22) we have $Q(\hat{F}) > 0$.

Let $Q_1(A) = Q(A \cap \hat{F})$; standard results give that Q_1 is also R-invariant for S, when S is R-recurrent (but not necessarily otherwise), and so from (2.26),

$$Q_1(A) = R \int Q_1(dw)\, C(w,A) . \tag{2.27}$$

But iterating (2.27) and summing shows

$$Q_1(A)\, [1-R^{-1}]^{-1} = \int Q_1(dw) \sum_k C^k(w,A) ;$$

taking $A = \hat{N}$ then leads to a contradiction since $\psi(\hat{N}) > 0$ implies $\sum_k C^k(w,\hat{N}) > 0$ for all w but $Q_1(\hat{N}) = 0$ by construction. □

<u>Remark</u> : If we assume that E is finite then the assumption that A is irreducible is enough to ensure that S is irreducible, using Perron-Frobenius theory: see [8], pp. 187-188.

§3 Neuts' "mean-drift" condition

In Theorems 2 and 3 of [8], Neuts shows that the existence for $\{X_n\}$ of an invariant probability measure of matrix geometric form is equivalent, when the matrix $A = \sum_k A_k$ is stochastic and irreducible, to

$$(3.1) \qquad \int_E \nu(dw)\ \beta(w) > 1 \ ,$$

where $\beta(w) = \sum_{n=0}^{\infty} n\ A_n(w,E)$ and ν is the unique invariant measure for $\{A(x,\cdot)\}$. In his context, where E is finite, ν is guaranteed to exist from Perron-Frobenius theory, and the proof that (3.1) is equivalent to the existence of a stationary distribution relies heavily on that theory.

In the general context we need other tools. We have already shown that any stationary measure must be "operator geometric"; in this section we investigate the probabilistic significance of (3.1) and show that it is very closely related to the positive recurrence of $\{X_n\}$, provided $B_k(x,\cdot)$ has suitable structure. Specifically, we let $\{\tilde{X}_n\}$ be a chain as in §1 with the zero-level probability structure

$$(3.2) \qquad \tilde{B}_k(x,A) = \sum_{j=k+1}^{\infty} A_j(x,A) \ ,$$

and let $\tilde{P}^n(i,x;\ j,B)$ denote the n-step transition probabilities of $\{\tilde{X}_n\}$. We shall need the following relationships between $\{\tilde{X}_n\}$ and $\{A(x,\cdot)\}$.

Proposition 2 : (i) For every n, i and x

$$(3.3) \qquad \tilde{P}^n(i,x;\ \mathbb{N}\times B) = A^n(x,B) \ .$$

(ii) If $\{\tilde{X}_n\}$ is $\tilde{\Phi}$-irreducible, then $\{A(x,\cdot)\}$ is ψ-irreducible, when $\psi(B) = \tilde{\Phi}(\mathbb{N}\times B)$, $B \in \mathcal{E}$.

(iii) If $\{\tilde{X}_n\}$ is $\tilde{\Phi}$-irreducible, and has a stationary probability measure $\tilde{\Pi}$, then $\{A(x,\cdot)\}$ has a unique stationary probability measure ν, with $\nu(\cdot) = \tilde{\Pi}(\mathbb{N}\times\cdot)$.

(iv) If $\{\tilde{X}_n\}$ is $\tilde{\Phi}$-irreducible, and $\{A(x,\cdot)\}$ has a stationary probability measure, then either

(a) $\{\tilde{X}_n\}$ also has a stationary probability measure; or

(b) for $\tilde{\Phi}$-almost all (i,x), and any level $\underset{\sim}{j}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \tilde{P}^m(i,x;\ j,E) = 0 . \tag{3.4}$$

<u>Proof</u> : (i) From (3.2) we have $\tilde{P}(i,x;\ \mathbb{N},B) = A(x,B)$, for all i; and thence by induction

$$\begin{aligned}\tilde{P}^n(i,x;\ \mathbb{N},B) &= \sum_k \int_E \tilde{P}^{n-1}(i,x;\ k,dw)\ \tilde{P}(k,w;\ \mathbb{N},B) \\ &= \int_E \tilde{P}^{n-1}(i,x;\ \mathbb{N},dw)\ A(w,B) \\ &= A^n(x,B) .\end{aligned}$$

(ii) If $\tilde{\Phi}(\mathbb{N}\times B) > 0$ then for each (i,x), there exists n such that $0 < \tilde{P}^n(i,x;\ \mathbb{N},B) = A^n(x,B)$ so that $\{A(x,\cdot)\}$ is ψ-irreducible.

(iii) If $\tilde{\Pi}$ exists, then from (3.3)

$$\begin{aligned}\tilde{\Pi}(\mathbb{N}\times B) &= \sum_k \int \tilde{\Pi}(k,dw)\ \tilde{P}(k,w;\ \mathbb{N},B) \\ &= \int \tilde{\Pi}(\mathbb{N}\times dw)\ A(w,B) ,\end{aligned}$$

and so $\nu(\cdot) = \tilde{\Pi}(N\times\cdot)$ is stationary for $\{A(x,\cdot)\}$: the uniqueness of ν follows since $\{A(x,\cdot)\}$ is ψ-irreducible.

(iv) Suppose that ν exists but $\tilde{\Pi}$ does not. From Proposition 4.2 of [14] there is then a sequence of sets $B(k) \uparrow \mathbb{N}\times E$ such that for all (i,x),

$$\tilde{P}^n(i,x;\ B(k)) \to 0,\ n \to \infty .$$

Fix the level $\underset{\sim}{j}$ and let $(j\times C(k)) = B(k) \cap \underset{\sim}{j}$. Since ν is a probability measure we can certainly choose k sufficiently large that $\nu(C(k)) \geq 1 - \varepsilon$, for given $\varepsilon > 0$. Now Proposition 4.2 of [14] also shows that for ψ-almost all x,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{1}^{n} A^m(x,C(k)) = \nu(C(k)) \geq 1 - \varepsilon , \tag{3.5}$$

and so from (3.5) and (3.3) we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \tilde{P}^m(i,x;\ j,E)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \tilde{P}^m(i,x;\ j,C(k)) + \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \tilde{P}^m(i,x;\ j,E\backslash C(k))$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \tilde{P}^m(i,x;\ B(k)) + \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} A^m(x,E\backslash C(k))$$

$$\leq \varepsilon ;$$

hence (3.4) holds. □

Proposition 2 (iv) shows that when ν exists the level sets $\underset{\sim}{j}$ are "status sets" for $\{\tilde{X}_n\}$; see [14], §5. We need this result in order to prove the sufficiency of (3.1) for the existence of a stationary measure for $\{\tilde{X}_n\}$.

<u>Theorem 5</u> : Suppose $\{\tilde{X}_n\}$ is $\tilde{\Phi}$-irreducible. If $A(x,\cdot)$ admits an invariant probability measure ν such that (3.1) holds, then $\{\tilde{X}_n\}$ admits a stationary probability measure $\tilde{\Pi}$.

<u>Proof</u> : Suppose (3.1) holds. It follows that

$$0 \leq \int_E \nu(dw)[1 - \sum_{\ell=0}^{\infty} \ell A_\ell(w,E)]$$

$$= \lim_{k\to\infty} + \int_E \nu(dw)[\sum_{\ell=0}^{k} [1-\ell]\ A_\ell(y,E) - k \sum_{\ell=k+1}^{\infty} A_\ell(y,E)] . \tag{3.6}$$

Now write the mean change of level of $\{\tilde{X}_n\}$ as

$$\mu(k,w) = \sum_{j=0}^{\infty} \tilde{P}(k,w;\ j,E)\ j - k ;$$

clearly $|\mu(k,w)| \leq d+1$ if $k \leq d$, and

$$\mu(k,w) \leq \mu(d,w) \ , \ w \in E, \ k > d \ . \tag{3.7}$$

From (3.6), we can choose d sufficiently large that

$$\int_E \nu(dw) \ \mu(d,w) < 0 \ . \tag{3.8}$$

If we write $D_a = \{y : \mu(d,y) \geq -a\}$, then $|\mu(d,w)| \leq a+1$ on D_a and by choosing a sufficiently large, from (3.8) we can ensure

$$\int_{D_a} \nu(dw) \ \mu(d,w) < 0 \ . \tag{3.9}$$

Now consider, for fixed i,x, the quantity

$$\phi_n(i,x) = \frac{1}{n} \sum_{m=1}^{n} \int_E \sum_j \tilde{P}^m(i,x;\ j,dw)\ \mu(j,w)$$

$$= \frac{1}{n} \sum_{m=1}^{n} \int_E \sum_{j=0}^{d-1} \tilde{P}^m(i,x;\ j,dw)\ \mu(j,w) \tag{3.10}$$

$$+ \frac{1}{n} \sum_{m=1}^{n} \int_E \sum_{j=d}^{\infty} \tilde{P}^m(i,x;\ j,dw)\ \mu(j,w) \ .$$

Suppose $\{\tilde{X}_n\}$ does not admit a stationary measure. From (3.4) and the boundedness of $\mu(j,w)$ for $j \leq d$ we have that the first term on the right of (3.10) tends to zero as $n \to \infty$, and so, using (3.7),

$$\begin{aligned} \liminf_{n\to\infty} \phi_n(i,x) &\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \int_E \sum_{j=d}^{\infty} \tilde{P}^m(i,x;\ j,dw)\ \mu(d,w) \\ &\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \int_{D_a} \sum_{j=d}^{\infty} \tilde{P}^m(i,x;\ j,dw)\ \mu(d,w) \ . \end{aligned} \tag{3.11}$$

But from (3.3), we have for any $B \in \mathcal{B}$ and any i

$$\frac{1}{n} \sum_{m=1}^{n} A^m(x,B) = \frac{1}{n} \sum_{m=1}^{n} \sum_{j=d}^{\infty} \tilde{P}^m(i,x;\ j,B) + \frac{1}{n} \sum_{m=1}^{n} \sum_{j=0}^{d} \tilde{P}^m(i,x;\ j,B) \ ; \tag{3.12}$$

we have already shown that the second term in (3.12) tends to zero, so using the boundedness of $\mu(d,w)$ on D_a we have, putting (3.12) into (3.11).

$$(3.13) \qquad \liminf_{n\to\infty} \phi_n(i,x) \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \int_{D_a} A^m(x,dy)\, \mu(d,w) \ .$$

Now we have assumed that $\{\tilde{X}_n\}$ is $\tilde{\Phi}$-irreducible, so from Proposition 2 (ii), $\{A(x,\cdot)\}$ is ψ-irreducible, and moreover has by assumption a stationary measure ν. Hence (see Proposition 4.2 of [14]), for Π-almost all x, and all $B \in \mathcal{B}$

$$(3.14) \qquad \frac{1}{n} \sum_{m=1}^{n} A^m(x,B) \to \nu(B) \ .$$

Since $\nu >> \psi$, we have from (3.14), (3.13) and (3.9) that for $\tilde{\Phi}$-almost all (i,x)

$$\liminf_{n\to\infty} \phi_n(i,x) \leq \int_{D_a} \nu(dw)\, \mu(d,w) < 0 \ .$$

However, we can emulate the proof of (9.11) of [14] with $g(j,y) = j$ to show that for *all* (i,x),

$$\liminf_{n\to\infty} \phi_n(i,x) \geq 0 \ ;$$

this shows that our assumption that $\{\tilde{X}_n\}$ does not have a stationary probability measure (and specifically, that (3.4) holds for individual j) cannot be true, and so $\tilde{\Pi}$ does exist as required. □

The proof above shows that Neuts' condition (3.1) is effectively an average "mean-drift" condition, where the averaging is done over the distribution ν. Theorem 9.2 of [14] shows that such an average mean drift condition may also be necessary, under some extra conditions, for positive recurrence.

We now turn to the necessity of condition (3.1) for existence of a stationary measure for the chain $\{\tilde{X}_n\}$.

In order to do this, we first define a further chain $\{X_n^*\}$ with transition probabilities on the state space $\{\ldots,-1,0,1,2,\ldots\} \times E$ given by, for $x \in E$, $B \in \mathcal{E}$,

$$P^*(i,x;\ j,B) = A_{j-i+1}(x,B)\ ,\quad i \geq 0,\ j = \ldots,-1,0,1,\ldots$$

$$P^*(i,x;\ 0,B) = \delta(x,B)\ ,\ i < 0\ .$$

The chain $\{X_n^*\}$ has the same motion as $\{\tilde{X}_n\}$ except that, when $\tilde{X}_n$ reaches the level $\underset{\sim}{0}$, X_n^* actually takes on the negative level that $\tilde{X}_n$ is "trying" to reach; and X_n^* is then replaced at level $\underset{\sim}{0}$ at the next step, but in the same "E-position". Trivially, if $\{\tilde{X}_n\}$ is $\tilde{\Phi}$-irreducible with $\tilde{\Phi}(0 \times E) > 0$ then so is $\{X_n^*\}$. We also have

<u>Proposition 3</u> : If $\tilde{X}_n$ has a stationary probability measure $\tilde{\Pi}$, then $\{X_n^*\}$ has a stationary probability measure Π^*, and for some constant b, we have $\Pi^*(i,A) = b\ \tilde{\Pi}(i,A)$, $i \geq 0$.

<u>Proof</u> : The stationary equations for $\{X_n^*\}$ are:

$$\Pi^*(j,B) = \sum_{k=0}^{\infty} \int_E \Pi^*(k,dy)\ P^*(k,y;\ j,B)$$

$$(3.15) \qquad = \sum_{k=j-1}^{\infty} \int_E \Pi^*(k,dy)\ A_{k-j+1}(y,B)\ ,\quad j > 0\ ;$$

$$(3.16) \qquad \Pi^*(j,B) = \sum_{k=0}^{\infty} \int_E \Pi^*(k,dy)\ A_{k-j+1}(y,B)\ ,\quad j < 0\ ;$$

and, for $j = 0$,

$$\Pi^*(0,B) = \sum_{k=-\infty}^{\infty} \int_E \Pi^*(k,dy)\ P^*(k,y;\ 0,B)$$

$$(3.17) \qquad = \sum_{k=0}^{\infty} \int_E \Pi^*(k,dy)\ A_{j+1}(y,B) + \sum_{k=-\infty}^{-1} \Pi^*(k,B)$$

$$= \sum_{k=0}^{\infty} \int_E \Pi^*(k,dy) \sum_{\ell=k+1}^{\infty} A_{\ell}(y,B)\ ,$$

using (3.16).

Now (3.15) and (3.17) are the stationary equations for $\{\tilde{X}_n\}$; these have the solution $\tilde{\Pi}(k,A)$, and this is unique up to multiplication by a constant b ([14], Proposition 4.3). From (3.16) we can construct $\Pi^*(j,B)$ for $j < 0$, once we have $\Pi^*(j,B)$ for $j \geq 0$, and the proof is completed by noting that, from (3.16),

$$\sum_{j=-\infty}^{-1} \Pi^*(j,B) = \sum_{k=0}^{\infty} \int_E \Pi^*(k,dy) \sum_{k+2}^{\infty} A_\ell(y,B)$$

$$\leq \sum_{k=0}^{\infty} \Pi^*(k,E) < \infty ,$$

since for $k \geq 0$, $\Pi^*(k,E) = b\,\tilde{\Pi}(k,E)$ and $\sum_0^\infty \tilde{\Pi}(k,E) = 1$. □

We can now prove

<u>Theorem 6</u> : Suppose $\{\tilde{X}_n\}$ has an invariant measure $\tilde{\Pi}$. Then $\{A(x,\cdot)\}$ has an invariant measure ν, and ν satisfies (3.1) provided

(3.18) $$\sum_{j=0}^{\infty} j\tilde{\Pi}(j,E) < \infty$$

<u>Proof</u> : From Proposition 2 (iii), ν exists and satisfies $\nu(B) = \sum_{k=0}^{\infty} \tilde{\Pi}(k,B)$; so from Proposition 3, $\nu(B) = b \sum_0^\infty \Pi^*(k,B)$. Hence (3.1) will hold provided

(3.19) $$\sum_{k=0}^{\infty} \int_E \Pi^*(k,dw) \, [\sum_{j=-\infty}^{\infty} P^*(k,w;\, j,E)\, j - k] < 0 .$$

Now let $h(j,x) = j$, $j \geq 0$, and 0 otherwise; and let

$$\mu_h(k,w) = \sum_{j=-\infty}^{\infty} P^*(k,w;\, j,dx)\, h(j,x) - h(k,w) .$$

Clearly (3.19) will hold provided we can prove

(3.20) $$\sum_{k=0}^{\infty} \int_E \Pi^*(k,dw)\, \mu_h(k,w) < 0 .$$

We can now emulate the necessity half of Theorem 9.2 of [14] to deduce that

(3.20), as in (9.13) of [14], always holds when $\{X_n^*\}$ has a stationary measure, and the result will be proved. The only condition that needs to be checked is that the interchange of integration in (9.14) of [14] is valid, and it is for this that we need (3.18). □

<u>Remarks</u> : (i) The need for a bound on $\int \Pi(dy)\, h(y)$ in (9.14) of [14] was overlooked in the conditions given there; it is claimed that (9.6) suffices but this not in fact enough.

(ii) It is of some historical interest that the condition (3.1) of a long-term average negative mean drift, shown above to be essentially equivalent to positive recurrence of $\{\tilde{X}_n\}$, was also first investigated by Neuts [5] in a situation where only two "drift values" were possible.

It is possible to say a little more about the finiteness of the mean (3.18) of the measure $\tilde{\Pi}$. We have

<u>Proposition 4</u> : The stationary measure $\tilde{\Pi}$ has finite mean if either

(i) the measure ν has a bounded density with respect to the measure ${}_0\tilde{\Pi}$; or

(ii) when $\tilde{X}_0$ has measure ${}_0\tilde{\Pi}$, then the variance of the return time to level $\underset{\sim}{0}$ is finite; i.e.

$$\int_E {}_0\tilde{\Pi}(dw)\, \mathbb{E}_w(\tau_0^2) < \infty . \tag{3.21}$$

<u>Proof</u> : Using the operator-geometric form of $\tilde{\Pi}$ given by Theorem 2, we have that (3.18) holds provided

$$\begin{aligned}
\infty &> \sum_j j\, c \int_E {}_0\tilde{\Pi}(dw)\, S^j(w,E) \\
&= c \int_E {}_0\tilde{\Pi}(dw) \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} S^j(y,E) \\
&= c \int_E \int_E {}_0\tilde{\Pi}(dw) \sum_{k=1}^{\infty} S^k(w,dy) \sum_{j=0}^{\infty} S^j(y,E) \\
&= \int_E \sum_{k=1}^{\infty} \tilde{\Pi}(k,dy) \sum_{j=0}^{\infty} S^j(y,E) .
\end{aligned} \tag{3.22}$$

Using the identification of ν in Proposition 2 (iii) shows that this is

$$(3.23) \qquad \int_E \nu(dy) \sum_{j=0}^{\infty} S^j(y,E) - \sum_j \tilde{\Pi}(j,E) .$$

If ν has a bounded density with respect to ${}_0\tilde{\Pi}$, then (3.23) is finite since

$$\int_E {}_0\tilde{\Pi}(dy) \sum_{j=0}^{\infty} S^j(y,E) = c \sum_j \tilde{\Pi}(j,E) \leq c ;$$

this proves (i). To prove (ii) we use the fact that

$$\sum_{j=0}^{\infty} S^j(y,E) = 1 + \mathbb{E}(\tau_0 \mid X_0 = (0,y))$$

so that (3.22) is also

$$(3.24) \qquad \int_E \sum_{k=1}^{\infty} \tilde{\Pi}(k,dy) \, [1 + \mathbb{E}(\tau_0 \mid X_0 = 0,y)]$$

$$\leq 1 + \int_E \sum_{k=0}^{\infty} \tilde{\Pi}(k,dy) \, \mathbb{E}(\tau_0 \mid X_0 = k,y) ;$$

from Cogburn [2] Corollary 3.1, we have that (3.24) is finite provided (3.21) holds. □

In this section we have proved that Neuts' mean drift condition is essentially related probabilistically to the positive recurrence of the chain $\{\tilde{X}_n\}$. It is of some interest to know what the positive recurrence of $\{\tilde{X}_n\}$ implies for an arbitrary chain $\{X_n\}$ with structure (1.1). This seems difficult to say. From (1.2) and (2.17), $L(x,E) = \tilde{L}(x,E)$, and this positive recurrence of $\{\tilde{X}_n\}$ implies at least that $\{{}_0X_n\}$ is well-defined in the sense that $L(x,E) = 1$ for ${}_0\tilde{\Pi}$-almost all x. Moreover $\{y: L(y,E) = 1\}$ is stochastically closed, so provided $\{X_n\}$ is Φ-irreducible and $B_k(y,\cdot) << \tilde{B}_k(y,\cdot)$ for all k,y, we can deduce that $L(x,E) = 1$ for Φ-almost all x.

However, this does not even guarantee the recurrence of the process $\{{}_0X_n\}$. The following example illustrates this.

Example : Let $E = \{0,1,\dots\}$, and let the transition law take the form (1.1) with only A_0 and A_2 being non-zero, and given by

$$
\begin{array}{lll}
 & A_0(x,x-1) = pq\ , & x \geq 1\ , \\
 & A_0(x,x+1) = (1-p)q\ , & x \geq 1\ , \\
\text{and} & A_0(0,0) = pq,\ A_0(0,1) = (1-p)q\ ; & \\
 & A_2(x,x-1) = p(1-q)\ , & x \geq 1 \\
 & A_2(x,x+1) = (1-p)(1-q)\ , & x \geq 1 \\
\text{and} & A_2(0,0) = p(1-q)\ ,\ A_2(0,1) = (1-p)(1-q)\ , & \\
\text{with} & 0 < p < 1\ ,\ \ 0 < q < 1\ . &
\end{array}
$$

Then the marginal chain with transition matrix $\{A(x,\cdot)\}$ is a random walk on a half-line, with a stationary probability measure ν if and only if $p > \frac{1}{2}$. In the other dimension the mean change in level is, for levels greater than zero, always of the same sign; and this is negative, i.e. $\beta(x) > 1$, if and only if $q < \frac{1}{2}$. Hence the necessary and sufficient condition for $\{\tilde{X}_n\}$ to have a stationary measure is $q < \frac{1}{2} < p$, from Theorems 5 and 6.

Let us alter the transition law to define $\{X_n\}$ by setting for $x \geq 1$

$$B_0(x,x-1) = r(1-q)$$

$$B_0(x,x+1) = (1-r)(1-q)\ ,$$

but leave $B_1(x,x-1) = p(1-q)$, $B_1(x,x+1) = (1-p)(1-q)$ as for $\{\tilde{X}_n\}$. For the embedded chain $\{{}_0X_n\}$ we have, by considering whether the chain $\{X_n\}$ leaves level zero or not, for $x \geq 1$,

$$\text{(3.25)} \qquad \mathbb{E}({}_0X_n \mid {}_0X_n = x) - x \geq (1-2r)(1-q) + (1-2p) \cdot \frac{1-q}{1-2q} \cdot q\ ;$$

the second term comes because, once the chain $\{X_n\}$ leaves level $\underset{\sim}{0}$ (with probability q) the expected number of steps to return is given by $(1-q)/(1-2q)$ and on each such step the expected drift to the left is, independent of level change, $1-2p$ (unless the chain hits $(\mathbb{N} \times 0)$, which gives the inequality in

(3.25)).

From (3.25) we see that, provided q and r are small enough, and p is not too much greater than ½, we can ensure that for all $x \geq 1$

$$\mathbb{E}({}_0X_n \mid {}_0X_n = x) - x \geq \delta \quad .$$

This is enough to ensure that $\{{}_0X_n\}$ is transient, as can be seen by comparing $\{{}_0X_n\}$ with a random walk with positive drift. Hence $\{{}_0X_n\}$ does not have a stationary measure, and thus neither does $\{X_n\}$. □

In order to deduce positive recurrence of $\{X_n\}$ we need to know, essentially, that ${}_0\Pi$ exists and has a bounded density with respect to ${}_0\tilde{\Pi}$. If this happens then we have, for some κ,

$$\int {}_0\Pi(dy)\, \mathbb{E}(\tau_0 \mid X_0 = (0,y)) \leq \kappa \int \tilde{\Pi}(dy)\, \mathbb{E}(\tau_0 \mid X_0 = (0,y))$$

$$= \kappa \int {}_0\tilde{\Pi}(dy)\, \mathbb{E}(\tilde{\tau}_0 \mid \tilde{X}_0 = (0,y)) < \infty ,$$

and so $\{X_n\}$ is positive recurrent also. The case when E is finite is the only situation where this is automatically true, and we can at least prove

Proposition 5 : If E is finite and $\{A(x,\cdot)\}$ is an irreducible stochastic matrix, then (3.1) is necessary and sufficient for $\{X_n\}$ to have a stationary distribution. □

Proof : From the argument above and Proposition 4 (i), our result will follow from Theorems 5 and 6 provided only that ${}_0\tilde{\Pi}(x) > 0$ for each $x \in E$. It is easy to show that irreducibility of $\{A(x,\cdot)\}$ implies irreducibility of $\{L(x,\cdot)\}$, which gives the required positivity. □

§4 Some queueing models

(i) The GI/PH/1 queue

This model is described in detail in Neuts [8], and we do not repeat details here. The bivariate chain $\{X_n\}$ consists of (N_n, P_n), where N_n is the number of customers immediately before the n^{th} arrival and P_n is the phase of service immediately after the n^{th} arrival.

The main result that is new in this situation comes from Proposition 1. Recall that the chain $\{_0X_n\}$ gives the phase on successive returns to level (0,E); this is precisely the phase of service of the arriving customer who finds the queue empty, and so $\{_0X_n\}$ consists of a sequence of independent and identically distributed random variables; from [8] we have that the distribution of these phases is given by a vector $\underline{\alpha}$. The chain $\{_0X_n\}$, and hence $\{X_n\}$, then trivially satisfies the irreducibility Condition I, with $\Phi(j) = \alpha_j$. Neuts ([8], Lemma 7) shows that (3.1) is equivalent to the usual stability condition for the GI/PH/1 queue.

From the discussions of §3 and §2 it follows that, since E is finite, under these stability conditions the invariant measure for $\{X_n\}$ has the form

$$\Pi(k,j) = c \sum_m \alpha_m S^k(m,j) \;;$$

clearly $_0\Pi(j) = \alpha_j$ since $\{_0X_n\}$ is a sequence of i.i.d. variables. Since E is finite, S is R-recurrent when it is irreducible, and so from Theorem 4 we have that S is the unique solution of $S = A[S]$ with convergence norm $R > 1$ in this case. This shows that Theorem 4 of [8] follows from our results without extra conditions being necessary.

The identification of $\underline{x}_0 = c\underline{\alpha}$ is deduced in [8] algebraically, but the fact that, from Proposition 1, $B[S] = \underline{\alpha}$ also, is not noticed there.

(ii) The GI/G/1 queue

There are various embedded Markov chains on which the results above might be demonstrated. In order to compare results with the GI/PH/I queue above, we will take $\{X_n\} = \{(N_n,S_n)\}$ as the chain with N_n as the number of customers immediately before the n^{th} interarrival time and S_n as the residual service time immediately after the n^{th} interarrival time.

We let G denote the distribution function of service times, and write $\mu = \int_0^\infty t\, dG(t)$ for the mean service time; we let F denote the distribution function of interarrival times, and write $\lambda = \int_0^\infty t\, dF(t)$ for its mean. We assume both λ and μ are finite.

Let $\sigma_1,\sigma_2,\sigma_3,\ldots$ denote a renewal process with $\sigma_n - \sigma_{n-1}$ having the service distribution function G; and let R_t denote the residual life-time at time t in this process, i.e. $R_t = t - \sigma_{N(t)}$, where N(t) is the number of renewals in [0,t]. If $R_0 = x$ then $\sigma_1 = x$. Now write

$$P_n^t(x,y) = \mathbb{P}(\sigma_n \le t < \sigma_{n+1},\ R_t \le y \mid R_0 = x) \tag{4.1}$$

for the probability that n renewals occur in $[0,\tau]$ and that the residual lifetime at t is in [0,y] given $R_0 = x$. It is easy to verify that the chain $\{X_n\}$ has the form (1.1) with

$$A_n(x,[0,y]) = \int_0^\infty P_n^t(x,y)\, dF(t) \tag{4.2}$$

and

$$B_n(x,[0,y]) = [\sum_{n+1}^\infty A_j(x,[0,\infty))]\, G(y) .$$

Hence $\{_0X_n\}$ again consists of i.i.d. variables with distribution function G, and so $_0\Pi[0,y] = G(y)$ *provided it exists*; for this we need to ensure that $\{_0X_n\}$ does not terminate. To check that $\{_0X_n\}$ is a proper chain we can use (3.1), since $\{_0X_n\}$ terminates if and only if $\{_0\tilde{X}_n\}$ does.

Our main result is

Theorem 7 : (i) A necessary and sufficient condition for $\{A(x,\cdot)\}$ to have a stationary probability measure ν satisfying (3.1) is that $\lambda > \mu$.

(ii) When $\lambda > \mu$, $\{X_n\}$ has an invariant probability measure $\Pi(j,\cdot)$ given by

$$\Pi(k,\cdot) = c \int_0^\infty dG(x)\ S^k(x,\cdot) \tag{4.3}$$

where $S(x,\cdot)$ is the minimal solution of

$$S(x,[0,y]) = \sum_j \int_0^\infty S^j(x,dw)\ A_j(w,[0,y]) \ , \quad y \in [0,\infty) \ ;$$

and the constant c is given by

$$c = 1 + \{\int_0^\infty [\sum_{n=0}^\infty F^{n*}(x)]\ dG(x)\}\ \{\exp \sum_{n=1}^\infty (1-a_n) / n\} \tag{4.4}$$

where

$$a_n = \int_0^\infty [1 - F^{n*}(x)]\ dG^{*n}(x) \ . \tag{4.5}$$

Proof : (i) From (4.1) and (4.2),

$$A(x,[0,y]) = \int F(dt)\ P^t(x,y) \tag{4.6}$$

where $P^t(x,y) = \mathbb{P}(R_t \leq y \mid R_0 = x)$. The kernel A is thus the transition probability kernel corresponding to the residual life-time process sampled at points of an (independent) renewal process generated by F. Provided $\mu < \infty$, the residual life-time process has invariant measure ν with

$$\nu[0,x] = \frac{1}{\mu} \int_0^x [1 - G(x)]dx \ ,$$

and this transfers immediately to the chain with kernel (4.6) (see [13] for other results concerning the relationship of the process $\{R_t\}$ to its "sampled" version.)

On the other hand, from (4.1)

$$\sum_{n=0}^{\infty} n \, P_n^t(x,\infty) = \mathbb{E}(N(t) \mid R_0 = x) .$$

The stationarity of ν then gives, by Fubini's Theorem

$$\int \nu(dx)\ \beta(x) = \int_0^{\infty} [\int_0^{\infty} \nu(dx)\ \mathbb{E}(N(t) \mid R_0 = x)]\ dF(t)$$

$$= \int_0^{\infty} [t/\mu]\ dF(t) = \lambda/\mu ,$$

which gives our first result.

(ii) From Theorem 5, $\{\tilde{X}_n\}$ will have a stationary probability measure when $\lambda > \mu$, and hence $\{{}_0X_n\}$ will at least return to $\underline{0}$ with probability one. Hence ${}_0\Pi$ exists and has distribution function $G(x)$, and Π given by (4.3) is at least a σ-finite stationary measure for $\{X_n\}$ from Theorem 1 (ii).

We have to prove that Π is a probability measure, and this does not follow from the general theory used so far: see the end of §3 for a discussion of the difficulties. We need to prove

$$\int {}_0\Pi(dy)\ \mathbb{E}(\tau_0 \mid X_0 = (0,y)) < \infty , \tag{4.7}$$

and to do this we need to evaluate $\mathbb{E}(\tau_0 \mid X_0 = (0,y))$ for this particular chain. Suppose $X_0 = (0,y)$ and that during the service time y of this first arriving customer, n_y further customers arrive. By rearranging the order of service in the usual way, it is clear that in this situation the expected number of customers served before $\underline{0}$ is again reached is $n_y\gamma$, where γ is the expected number of customers in a busy period in the GI/G/1 queue; from [4] it is known that, when $\lambda > \mu$,

$$\gamma = \exp\{\sum_{n=1}^{\infty} (1-a_n)/n\} < \infty \tag{4.8}$$

where a_n is given by (4.5).

Thus

$$\mathbb{E}(\tau_0 \mid X_0 = (0,y)) = 1 + \mathbb{E}(n_y\gamma \mid X_0 = (0,y))$$

(4.9) $$= 1 + [\sum_{n=0}^{\infty} F^{n*}(y)]\gamma \quad ,$$

and standard renewal theory [3] shows that for any ε and y large,

(4.10) $$0 \le [\sum_{n=0}^{\infty} F^{n*}(y) - y/\lambda] \le y\varepsilon$$

Since ${}_0\Pi[0,y] = G(y)$ and G has a finite mean, (4.7) follows from (4.9) and (4.10). The (finite) value (4.4) of (4.7), which is the constant c in (4.3), follows from (4.8) and (4.9). □

As a corollary to (4.10) and (4.11) we see that, if $\{X_n'\}$ is any chain with structure (1.1) where A_n is given by (4.2) and B_n has the "replacement" value

$$B_n'(x,[0,y]) = [\sum_{n+1}^{\infty} A_j(x,[0,\infty))]\, H(y)$$

for some distribution function H, then $\{X_n'\}$ will have a σ-finite invariant measure

$$\Pi'(k,[0,y]) = \int_0^{\infty} dH(x)\; S^k(x,[0,y]) \;;$$

and Π' will be a probability measure if and only if H has a finite mean.

We can also obtain, for the queueing chain $\{X_n\}$,

<u>Proposition 6</u> : The stationary distribution Π has finite mean $\mu^* = \sum_k k\Pi(k,[0,\infty))$ if and only if F has finite variance. In this case

(4.11) $$\mu^* = \frac{1}{\mu}\int_0^{\infty} [1 - G(x)][\sum_{n=0}^{\infty} F^{n*}(x)]\gamma\, dx$$

where γ is given by (4.8).

Proof : From (3.23) we have that μ^* is given by (4.11), and from (4.10) μ^* will be finite if and only if ν has a finite mean. It is well-known that this holds if and only if F has a finite variance. □

(iii) The M/G/1 queue

The expression [4.8] for γ simplifies when the inter-arrival time is exponential: from Neuts [7], we have

$$\gamma = \lambda/(\lambda - \mu) .$$

In this case also we have the expected number of arrivals in [0,y] is just y/λ, and so from (4.4), $c = \lambda/(\lambda - \mu)$ also.

This is a computationally useful fact. The standard method of deriving $\{\Pi(k,A)\}$ from the theory above is to solve the non-linear operator equation $S = A[S]$ to the desired accuracy then substitute in (4.3). In general c can only be estimated as the normalising constant

$$c_N = [\int_0^\infty dG(x) \sum_0^N S^k(x,[0,\infty))]^{-1} ,$$

where N is the number of terms for which the iterates are calculated. The difference $[1 - c/c_N]$ represents the total probability mass in the levels above N, i.e.

$$[1 - c/c_N] = \sum_{N+1}^\infty \Pi(j,[0,\infty)) ,$$

which is the intrinsic error in computation of Π when only a finite number of the iterates S^k are used.

Finally, we note that in this case, from (4.11) we can recover the fact that the mean queue length at inter-arrival times in equilibrium is

$$\mu^* = \tfrac{1}{2}[\int_0^\infty y^2\, dG(y)] \;/\; \mu(\lambda-\mu) .$$

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REFERENCES

[1] Chung, K.L. (1967) *Markov Chains with Stationary Transition Probabilities*. (2nd Ed), Springer-Verlag, New York.

[2] Cogburn, R. (1975) A uniform theory for sums of Markov chains transition probabilities. *Ann Probability* 3, 191-214.

[3] Feller, W. (1971) *An Introduction to Probability Theory and its Applications, Volume II*. (2nd Ed), Wiley, New York.

[4] Finch, P.D. (1961) On the busy period in the queueing system GI/G/1. *J. Austral. Math. Soc.* 2, 217-228.

[5] Neuts, M.F. (1967) Two Markov chains arising from examples of queues with state dependent service times. *Sankhyā Ser. A* 29, 255-264.

[6] Neuts, M.F. (1969) The queue with Poisson input and general service times, treated as a branching process. *Duke Math. J.* 36, 215-231.

[7] Neuts, M.F. (1975) Probability distributions of phase type. In *Liber Amicorum Professor Emeritus H. Florin*. Department of Mathematics, University of Louvain, Belgium, 173-206.

[8] Neuts, M.F. (1978) Markov chains with applications in queueing theory, which have a matrix-geometric invariant probability vector. *Adv. Appl. Prob.* 10, 185-212.

[9] Neuts, M.F. (1980) The probabilistic significance of the rate matrix in matrix-geometric invariant vectors. *J. Appl. Probl.* 17, 291-266.

[10] Neuts, M.F. (1981) *Matrix-Geometric Solutions in Stochastic Models - An Algorithmic Approach.* The Johns Hopkins University Press (to appear).

[11] Orey, S. (1971) *Lecture Notes on Limit Theorems for Markov Transition Probabilities.* Van Nostrand-Reinhold, London.

[12] Revuz, D. (1975) *Markov Chains.* North-Holland, Amsterdam.

[13] Tuominen, P. and Tweedie, R.L. (1979) The recurrence structure of general Markov processes. *Proc. Lond. Math. Soc.* (3) 39, 554-576.

[14] Tweedie, R.L. (1976) Criteria for classifying general Markov chains. *Adv. Appl. Prob.* 8, 737-771.

[15] Tweedie, R.L. (1974) R-theory for Markov chains on a general state space I. *Ann. Probability,* 2, 840-864.

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