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A DECOMPOSITION FOR MULTISTATE MONOTONE SYSTEMS.(U)

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A Decomposition for
Multistate Monotone Systems

by

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ABSTRACT

A decomposition theorem for multistate structure functions is proven. This result is applied to obtain bounds for the system performance function. Another application is made to interpret the multistate structures of Barlow and Wu. Various concepts of multistate importance and coherence are also discussed.

AMS Subject Classification: Primary 62N05, Secondary 60K10.

Key Words: Multistate structure function, bounds, system performance function, coherence, relevance, importance, associated components.

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§ 1. Introduction

Multistate structure functions have been studied by Barlow and Wu (1978), El-Newehi, Proschan and Sethuraman (1978) (henceforth EPS) and Griffith (1980). These structure functions have been developed in order to deal with the situations where components and systems have more than two states. Prior to these papers most research had concentrated on the situation where the components and system could only take binary values corresponding to "failed" and "operating" states.

In this paper we obtain a decomposition for multistate structure functions. This is Theorem 2.8 (see also Theorem 2.11) of Section 2. This result is applied to find system bounds in Section 3 and to interpret the multistate structures of Barlow and Wu in Section 6. Concepts of multistate importance and coherence are discussed in Sections 4 and 5.

Extensions to the continuous case for the decomposition have been completed and will appear elsewhere.

§ 2. Multistate structure functions and decomposition results.

Let $S = \{0, 1, \dots, M\}$ and $\Phi: S^n \rightarrow S$ be a nondecreasing function. The values taken by Φ will represent the system performance and, for each i , x_i will denote the performance of the i^{th} component. We distinguish $M+1$ performance levels ranging from perfect functioning (level M) to complete failure (level 0). The assumption that Φ is non-decreasing corresponds to the notion that improvement of a component cannot lead to a worsening of the system.

Before we state any of our results we should first note that the above set up is really the most general in the finite state case. For suppose that we can distinguish among M_i+1 performance levels for com-

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ponent i , which we designate by $S_i = \{0, 1, \dots, M_i\}$, and that we can distinguish among $N+1$ performance levels for the system ϕ , which we designate by $E = \{0, 1, \dots, N\}$. Thus $\phi: S_1 \times \dots \times S_n \rightarrow E$. Now let $M = \max\{M_1, \dots, M_n, N\}$ and set $S = \{0, 1, \dots, M\}$. We define a new system function

$\tilde{\phi}: S^n \rightarrow S$ by $\tilde{\phi}(x_1, \dots, x_n) = \phi(x_1 \wedge M_1, \dots, x_n \wedge M_n)$. Clearly $\tilde{\phi}$ is nondecreasing

if ϕ is. Later on we also impose the condition that $\tilde{\phi}(M, \dots, M) = M$.

Since this may not be the case above, we simply redefine $\tilde{\phi}$ to be M on the set where it had the value N . Of course, if one prefers to work directly with the given ϕ instead of $\tilde{\phi}$, all of our results can be appropriately modified to handle this case.

We now give some elementary conditions and implications concerning the monotonicity of ϕ . The first two parts of the following result are consequences of results in Griffith (1980).

(2.1) Proposition. Let $\phi: S^n \rightarrow S$ where $S = \{0, 1, \dots, M\}$.

1. ϕ is nondecreasing if and only if either of the following conditions hold:

$$(i) \quad \phi(\underline{x} \vee \underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y}) \quad \text{for all } \underline{x}, \underline{y} \in S^n,$$

$$(ii) \quad \phi(\underline{x} \wedge \underline{y}) \leq \phi(\underline{x}) \wedge \phi(\underline{y}) \quad \text{for all } \underline{x}, \underline{y} \in S^n$$

2. If ϕ is nondecreasing, then for all $\underline{x} = (x_1, \dots, x_n) \in S^n$

$$(i) \quad \min_i x_i \leq \phi(\underline{x}) \quad \text{if and only if } \phi(\underline{k}) \geq k \quad \text{for all } k \in S,$$

$$(ii) \quad \phi(\underline{x}) \leq \max_i x_i \quad \text{if and only if } \phi(\underline{k}) \leq k \quad \text{for all } k \in S.$$

Consequently, $\min_i x_i \leq \phi(\underline{x}) \leq \max_i x_i$ if and only if $\phi(\underline{k}) = k$ for

all $k \in S$.

3. If ϕ is nondecreasing, then

$$(i) \quad \max_i \phi(x_i, 0) \leq \phi(\underline{x}) \leq \max_i \phi(x_i, M_i),$$

$$(ii) \quad \phi(\min_i x_i) \leq \phi(\underline{x}) \leq \phi(\max_i x_i).$$

Furthermore, these bounds are not compatible in the sense that there exist systems ϕ for which (i) is a better bound than (ii) and vice-versa.

4. (i) $\phi(\underline{x} \vee \underline{y}) = \phi(\underline{x}) \vee \phi(\underline{y})$ for all $\underline{x}, \underline{y} \in S^n$ if and only if

$$\phi(\underline{x}) = \max_i h_i(x_i) \text{ where } h_i(j) = \phi(j_i, \underline{0}).$$

(ii) $\phi(\underline{x} \wedge \underline{y}) = \phi(\underline{x}) \wedge \phi(\underline{y})$ for all $\underline{x}, \underline{y} \in S^n$ if and only if

$$\phi(\underline{x}) = \min_i \Pi_i(x_i) \text{ where } \Pi_i(j) = \phi(j_i, \underline{M}).$$

Here we use the notation $\phi(x_i, \underline{y}) = \phi(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n)$ and

$$\underline{k} = (k, \dots, k).$$

For the next results besides assuming ϕ is nondecreasing, we impose the condition that $\phi(\underline{0}) = 0$ and $\phi(\underline{M}) = M$. This merely states that if all components fail, the system fails and if all components are functioning perfectly, the system functions perfectly. We do not make the assumption imposed by EPS and Griffith that $\phi(\underline{k}) = k$ for $k = 1, \dots, M-1$. We will call such a function ϕ a multistate monotone structure function (MMS).

(2.2) Definition. A vector $\underline{x}$ called an upper (lower) vector for level k of an MMS if $\phi(\underline{x}) \geq k$ ($\phi(\underline{x}) \leq k$). It is called a critical upper (lower) vector for level k if in addition $\underline{y} < \underline{x}$ and $\underline{y} \neq \underline{x}$ implies $\phi(\underline{y}) < k$ (if $\underline{y} \geq \underline{x}$ and $\underline{y} \neq \underline{x}$ implies $\phi(\underline{y}) > k$).

The set of all critical upper (lower) vectors for level k is denoted by U_k or $U_k(\phi)$ if necessary (L_k or $L_k(\phi)$). If $\underline{x} \in U_k$, $k = 1, 2, \dots, M$, let

$$U_k(\underline{x}) = U_k(\phi; \underline{x}) = \{ (i, x_i) : x_i \neq 0 \};$$

if $\underline{x} \in L_k$, $k = 0, 1, \dots, M-1$, let

$$L_k(\underline{x}) = L_k(\phi; \underline{x}) = \{ (i, x_i) : x_i \neq M \}.$$

As we will see, these sets play the role of min path sets and min cut sets respectively.

As usual the concept of duality changes upper vector concepts to lower vector concepts. More precisely, if ϕ is an MMS, then $\phi^D(\underline{x}) = M - \phi(M - \underline{x})$ is also an MMS called the dual of ϕ . The proofs of the following two results are obvious.

(2.3) Theorem. The vector $\underline{x}$ is an upper vector for level k of ϕ if and only if $M - \underline{x}$ is a lower vector for level $M - k$ of ϕ^D . Furthermore, $\underline{x} \in U_k(\phi)$ if and only if $M - \underline{x} \in L_{M-k}(\phi^D)$.

(2.4) Theorem. For $k > 0$, $\phi(\underline{x}) \geq k$ if and only if $\underline{x} \geq \underline{x}^0$ for some $\underline{x}^0 \in U_k$.

(2.5) Remark. The assumption $\phi(M) = M$ implies $U_k \neq \emptyset$ for $k = 1, \dots, M$ and $U_M \neq \emptyset$ implies $\phi(M) = M$.

Now we define the binary function ϕ_k of $M \cdot n$ binary variables $\underline{y} = (y_{ij}: 1 \leq i \leq n, 1 \leq j \leq M)$ by

$$(2.6) \quad \phi_k(\underline{y}) = \max_{\underline{x} \in U_k} \min_{(i,j) \in U_k(\underline{x})} y_{ij}, \quad k = 1, \dots, M.$$

Although this function is defined for all $M \cdot n$ values of $\underline{y}$, we are only interested in this function on the domain given by the image of the following function. We define $\alpha: S^n \rightarrow \{0,1\}^{M \cdot n}$ by $\alpha(\underline{x}) = (\alpha_{ij}(\underline{x}): 1 \leq i \leq n, 1 \leq j \leq M)$, where $\underline{x} \in S^n$ and $\alpha_{ij}(\underline{x}) = 1$ if $x_i \geq j$ and 0 otherwise.

(2.7) Lemma. For $k > 0$, $\phi(\underline{x}) \geq k$ if and only if $\phi_k(\alpha(\underline{x})) = 1$.

$$(2.8) \text{ Theorem. } \phi(\underline{x}) = \sum_{k=1}^M \phi_k(\alpha(\underline{x})).$$

Since the proofs are straightforward, we omit them. Theorem (2.8) is a type of decomposition result analogous to those using min path sets in the binary case.

(2.9) Remarks. (1) Note that $\phi_1(\alpha(\underline{x})) \geq \phi_2(\alpha(\underline{x})) \geq \dots \geq \phi_M(\alpha(\underline{x}))$; equivalently, $\phi_1 \geq \phi_2 \geq \dots \geq \phi_M$ on $\Delta = \alpha(S^n) = \{\underline{y} = (y_{ij}): \text{if } y_{ij} = 1, \text{ then } y_{i\ell} = 1 \text{ for all } \ell = 1, \dots, j\}$.

(ii) The min path sets of the binary ϕ_k in the sense of Barlow and Proschan (1975) are precisely the sets $\{U_k(\underline{x}): \underline{x} \in \Pi_k\}$.

(iii) Although the ϕ_k are binary structure functions, other concepts discussed in Barlow and Proschan such as min path vectors, min cut vectors, etc. don't correspond exactly to the concepts discussed here; however, if these binary concepts are modified for ϕ_k restricted to Δ , then there is a correspondence between the two notions.

(iv) If $\gamma_1, \dots, \gamma_M$ are binary monotone structure functions of the binary variables $(y_{ij}: 1 \leq i \leq n, 1 \leq j \leq M)$ satisfying (i), then $\phi(\underline{x}) =$

$$\bigwedge_{k=1}^M \gamma_k(\alpha(\underline{x})) \text{ is a multistate structure function; furthermore, if } \phi_1, \dots, \phi_M$$

are the binary monotone structure functions in the decomposition Theorem 2.8 of ϕ then $\phi_k = \gamma_k$ on Δ for all $k = 1, \dots, M$.

(2.10) Example. An example will serve to illustrate the procedure. Let $\phi: \{0,1,2\}^2 \rightarrow \{0,1,2\}$ with $0 = \phi(0,0) \neq \phi(1,0)$, $1 = \phi(0,1) = \phi(0,2) = \phi(1,2) = \phi(1,1)$, $2 = \phi(2,0) = \phi(2,1) = \phi(2,2)$. Then

$$U_1 = \{(2,0), (0,1)\}, U_2 = \{(2,0)\}$$

and

$$U_1(2,0) = \{(1,2)\} = U_2(2,0),$$

$$U_1(0,1) = \{(2,1)\}.$$

Then

$$\phi_2(\underline{y}) = \max_{\underline{x} \in U_2} \min_{(i,j) \in U_2(\underline{x})} y_{ij} = y_{12}$$

and

$$\phi_1(\underline{y}) = \max_{\underline{x} \in U_1} \min_{(i,j) \in U_1(\underline{x})} y_{ij} = \max(y_{21}, y_{12})$$

where we have

$\underline{x}$	$\alpha(\underline{x})$	$\phi_1(\alpha(\underline{x}))$	$\phi_2(\alpha(\underline{x}))$	$\phi(\underline{x}) = \sum_{k=1}^2 \phi_k(\alpha(\underline{x}))$
(0,0)	(0,0,0,0)	0	0	0
(0,1)	(0,0,1,0)	1	0	1
(0,2)	(0,0,1,1)	1	0	1
(1,0)	(1,0,0,0)	0	0	0
(1,1)	(1,0,1,0)	1	0	1
(1,2)	(1,0,1,1)	1	0	1
(2,0)	(1,1,0,0)	1	1	2
(2,1)	(1,1,1,0)	1	1	2
(2,2)	(1,1,1,1)	1	1	2

A similar decomposition can be obtained using critical lower vectors. More precisely, define the binary structure function ψ_k of the $M \cdot n$ binary variables $\underline{z} = (z_{ij} : 1 \leq i \leq n, 0 \leq j \leq M-1)$ by

$$\psi_k(\underline{z}) = \min_{\underline{x} \in L_k} \max_{(i,j) \in L_k(\underline{x})} z_{ij}$$

for $k = 0, 1, \dots, M-1$. As in the previous case we restrict the domain of $\psi_k(\underline{z})$ to the image of $\beta : S^n \rightarrow \{0,1\}^{Mn}$ where $\beta(\underline{x}) = (\beta_{ij}(\underline{x}) : 1 \leq i \leq n, 0 \leq j \leq M-1)$ and $\beta_{ij}(\underline{x}) = 0$ if $x_i \leq j$ and 1 otherwise.

(2.11) Theorem. $\phi(\underline{x}) = \sum_{k=0}^{M-1} \psi_k(\beta(\underline{x}))$.

Proof. The proof is most easily obtained by duality arguments.

(2.12) Remark. It should be noted that $\phi_k(\alpha(\underline{x})) = \psi_{k-1}(\beta(\underline{x}))$ and

$\alpha_{ij}(\underline{x}) = \beta_{i,j-1}(\underline{x})$ for all $k=1, \dots, M, i=1, \dots, n, j=1, \dots, M$ and $\underline{x} \in S^n$.

We will make use of this observation in Section 3.

We now consider the stochastic behavior. Let $X_i(t)$ be a right-continuous nonincreasing stochastic process with values in S ; i.e., $X_i(t)$ represents the statistical behavior of component i . Set $\underline{X}(t) = (X_1(t), \dots, X_n(t))$. We define

$$T_{ij} = \inf \{t \geq 0 : X_i(t) \leq j\}$$

$$T_k = \inf \{t \geq 0 : \phi(\underline{X}(t)) \leq k\}$$

for $i = 1, \dots, n$ and $j, k = 0, 1, \dots, M-1$.

(2.13) Theorem. For $k = 0, 1, \dots, M-1$,

$$\begin{aligned} T_k &= \max_{\underline{x} \in U_{k+1}} \min_{(i,j) \in U_{k+1}(\underline{x})} T_{i,j-1} \\ &= \min_{\underline{x} \in L_k} \max_{(i,j) \in L_k(\underline{x})} T_{ij} \end{aligned}$$

Proof. First we observe that $\phi(\underline{X}(t)) \leq k$ if and only if $\phi_{k+1}(\alpha(\underline{X}(t))) = 0$.

Consequently, $T_k = \tau^{k+1}$ where $\tau^{k+1} = \inf \{t \geq 0 : \phi_{k+1}(\alpha(\underline{X}(t))) = 0\}$. But from the results in the binary case,

$$\tau^{k+1} = \max_{\underline{x} \in U_{k+1}} \min_{(i,j) \in U_{k+1}(\underline{x})} \tau_{ij},$$

where $\tau_{ij} = \inf \{t \geq 0 : \alpha_{ij}(\underline{X}(t)) = 0\}$. Since

$$\begin{aligned} \tau_{ij} &= \inf \{t \geq 0 : \alpha_{ij}(\underline{X}(t)) = 0\} = \inf \{t \geq 0 : X_i(t) < j\} \\ &= \inf \{t \geq 0 : X_i(t) \leq j-1\} = T_{i,j-1}, \end{aligned}$$

we are done. The second half follows similarly.

§ 3. Bounds

EPS (1978) have obtained bounds on the system performance function $E[\phi(\underline{X})]$ where $\underline{X} = (X_1, \dots, X_n)$ is the state vector which is assumed to have independent components and ϕ is an MMS which satisfies $\phi(k) = k$ for all $k \in S$. The bounds given are

$$\sum_{j=1}^M \prod_{i=1}^n \bar{P}_i(j-1) \leq E[\phi(\underline{X})] \leq \sum_{j=1}^M \{1 - \prod_{i=1}^n P_i(j-1)\}$$

where $P_i(j) = P(X_i \leq j)$ and $\bar{P}_i(j) = 1 - P_i(j)$. It should be observed that this result still holds if the components are assumed to be associated.

In this section we use the decomposition (2.8) and (2.11) along with the remark (2.12) to obtain bounds based on the upper and lower critical vectors. Let $P(j) = P(\phi(\underline{X}) \leq j)$ and $\bar{P}(j) = 1 - P(j)$.

(3.1) Lemma. Let ϕ be an MMS and $k=0, 1, \dots, M-1$.

(a) The following bounds always hold:

$$\max_{\underline{y} \in U_{k+1}} P(\bigcap_{(i,j) \in U_{k+1}(\underline{y})} \{X_i > j-1\}) \leq \bar{P}(k) \leq \min_{\underline{y} \in I_k} P(\bigcup_{(i,j) \in I_k(\underline{y})} \{X_i > j\})$$

(b) If the X_i are associated, then

$$\max_{\underline{y} \in U_{k+1}} \prod_{(i,j) \in U_{k+1}(\underline{y})} \bar{P}_i(j-1) \leq \bar{P}(k) \leq \min_{\underline{y} \in I_k} \prod_{(i,j) \in I_k(\underline{y})} P_i(j)$$

and

$$\prod_{\underline{y} \in I_k} P(\bigcup_{(i,j) \in I_k(\underline{y})} \{X_i > j\}) \leq \bar{P}(k) \leq \prod_{\underline{y} \in U_{k+1}} P(\bigcap_{(i,j) \in U_{k+1}(\underline{y})} \{X_i > j-1\})$$

(c) If the X_i are independent, then

$$\prod_{\underline{y} \in I_k} \prod_{(i,j) \in I_k(\underline{y})} \bar{P}_i(j) \leq \bar{P}(k) \leq \prod_{\underline{y} \in U_{k+1}} \prod_{(i,j) \in U_{k+1}(\underline{y})} P_i(j-1)$$

Proof. (a) This is easy.

(b) The first set of bounds is a consequence of (a) and association.

The second set follows since the collections

$$\left\{ \max_{(i,j) \in L_k(\underline{y})} \beta_{ij}(\underline{X}), \underline{y} \in L_k \right\} \text{ and } \left\{ \min_{(i,j) \in U_{k+1}(\underline{y})} \alpha_{ij}(\underline{X}), \underline{y} \in U_{k+1} \right\}$$

are each collections of associated random variables.

(c) This follows from the second set of inequalities in (b).

(3.2) Note. The bounds in b) are also valid under the weaker assumptions that the components satisfy an appropriate type of orthant dependence as discussed by Ahmed et al (1978).

(3.3) Theorem. Under the assumption that the components of the MMS ϕ are independent,

$$\begin{aligned} & \sum_{k=1}^M \max \left\{ \max_{\substack{\underline{y} \in U_k \\ (i,j) \in U_k(\underline{y})}} \prod_{(i,j) \in L_{k-1}(\underline{y})} \bar{P}_i(j-1), \prod_{\substack{\underline{y} \in L_{k+1} \\ (i,j) \in L_{k-1}(\underline{y})}} \prod_{(i,j) \in U_{k+1}(\underline{y})} \bar{P}_i(j) \right\} \leq E(\phi(\underline{X})) \\ & \leq \sum_{k=1}^M \min \left\{ \min_{\substack{\underline{y} \in L_{k-1} \\ (i,j) \in L_{k-1}(\underline{y})}} \prod_{(i,j) \in U_{k-1}(\underline{y})} \bar{P}_i(j), \prod_{\substack{\underline{y} \in U_k \\ (i,j) \in U_k(\underline{y})}} \prod_{(i,j) \in L_k(\underline{y})} \bar{P}_i(j-1) \right\} \end{aligned}$$

Proof. We need only use the fact that

$$E(\phi(\underline{X})) = \sum_{j=1}^M \bar{P}(j-1) \quad \text{and apply the Lemma.}$$

(3.4) Note. Using the notation of Theorem 2.13 and recognizing that $P\{X_{ij}(t) > t\} = P\{T_{ij} > t\}$, if the T_{ij} are IER or IERA the results of Chapter 4, Section 6 of Barlow and Proschan (1975) can be applied to obtain bounds on $P\{T_{ij} > t\}$ and so on $E(\phi(Y(t)))$ by the above theorem.

§ 4. Some remarks about coherence assumptions.

As was remarked earlier, EPS (1978) and Griffith (1980) also studied the deterministic properties of multistate monotone structure functions in the finite state case. Besides the basic monotonicity assumption, however, they assumed that $\phi(\underline{k}) = k$ for all $k \in S$ plus a type of coherence assumption. In Griffith (1980) three distinct coherence conditions are delineated which we list below.

- (SC): ϕ is said to be strongly coherent if for any component i and any level j , there exists $\underline{x}$ such that $\phi(j_i, \underline{x}) = j$ while $\phi(l_i, \underline{x}) \neq j$ for $l \neq j$.
- (C): ϕ is said to be coherent if for any component i and any level $j \geq 1$, there exists $\underline{x}$ such that $\phi((j-1)_i, \underline{x}) < \phi(j_i, \underline{x})$.
- (WC): ϕ is said to be weakly coherent if for any component i , there exists $\underline{x}$ such that $\phi(0_i, \underline{x}) < \phi(M_i, \underline{x})$.

EPS (1978) assumed condition (SC) for their class whereas Griffith (1980) showed that all of the results of (1978) hold under the assumption of (C), but some are false under (WC). Loosely speaking, condition (SC) says that every level of every component is relevant to the same level of the system ϕ ; condition (C) says that every level of every component is relevant to the system ϕ ; condition (WC) says that every component is relevant to the system ϕ .

In terms of the decomposition (2.8), we can paraphrase the above as follows. The system ϕ is coherent if and only if for every i and j , y_{ij} is relevant to some ϕ_k ; ϕ is weakly coherent if and only if there exists j such that y_{ij} is relevant to some ϕ_k . The condition of strong coherence and the conditions $\phi(\underline{k}) \geq k$, $\phi(\underline{k}) \leq k$ for $k \in S$ can be similarly rephrased,

but since they are somewhat more complicated to state, and lose their intuitive content, we will not state them here.

Part 4) of Proposition 2.1 has a stronger form. In EPS (1978), under the assumption of (SC), it was concluded that $h_i(j) = j$ and $H_i(j) = j$ for all $i = 1, \dots, n$, $j = 0, \dots, M$. Griffith (1980) concluded the same result assuming the weaker condition (C). Griffith also showed that the result was false under (WC).

§ 5. Multistate importance and coherence.

Besides introducing the three coherence assumptions mentioned in the preceding section, Griffith (1980) has introduced a concept of multistate importance. Previously Barlow and Wu (1978) had discussed a measure of multistate importance. In this section we discuss a simple connection between relevance, coherence and importance and consequently relate the various concepts of these authors. This also leads to new concepts of importance. The simple principle follows.

(5.1) Importance Principle. A component i is relevant at a state j if there exists a state vector $\underline{x}$ such that a certain condition involving i, j and the structure function holds. A system is coherent if a relevance condition holds for all components and certain states. The importance of a component i at a state j is a measure of the number of state vectors $\underline{x}$ for which the relevance condition holds.

(5.2) Example. Consider a binary nondecreasing structure function $\phi(\underline{x})$. Then i is relevant if there is an $\underline{x}$ such that $\phi(0_i, \underline{x}) < \phi(1_i, \underline{x})$. The system is coherent if the condition holds for every component i . Two measures of importance are

$$I_{\phi}(i) = \frac{1}{2^{n-1}} \text{Card} \{ (1_i, \underline{x}) \mid \phi(0_i, \underline{x}) < \phi(1_i, \underline{x}) \}$$

which is called the structural importance and

$$I(1) = P\{\phi(0_1, \underline{X}) < \phi(1_1, \underline{X})\}$$

which is called the Birnbaum importance.

Now the connection between the various types of multistate conditions should be clear. The importance measure of Barlow and Wu is

$$I_j^{SC}(1) = P\{\phi(j_1, \underline{X}) = j, \phi(k_1, \underline{X}) \neq j, k \neq j\}$$

where ϕ is the particular multistate structure function of those authors. However the condition inside the probability statement is the basic form of the coherence condition of EPS, i.e. these authors say the system is coherent if the condition holds for every component i and every state j . This is also the condition which Griffith called strongly coherent (SC).

In a similar way we may extend the other two coherence concepts of Griffith to importance concepts. Concepts (C) and (WC) of Section 4 become

$$I_j^C(1) = P\{\phi((j-1)_1, \underline{X}) < \phi(1_1, \underline{X})\}$$

and

$$I^{WC}(1) = P\{\phi(0_1, \underline{X}) < \phi(M_1, \underline{X})\}.$$

Taking $P\{X_1=j\} = \frac{1}{M+1}$ we obtain two structural concepts of importance.

These can now be compared with the concept of multistate importance defined by Griffith. For utilities $a_\ell = \ell$ for $\ell=1, \dots, M$, the concept of

Griffith becomes

$$\begin{aligned} I_j(1) &= \sum_{\ell=1}^M P\{\phi((j-1)_1, \underline{X}) < \ell \leq \phi(1_1, \underline{X})\} \\ &= E[\phi(j_1, \underline{X})] - E[\phi((j-1)_1, \underline{X})]. \end{aligned}$$

Another importance measure that this suggests is

$$I_j^{(1)}(i) = P\{\phi((j-1)_i, \underline{X}) < \phi(j_i, \underline{X}) = j\}$$

All but one of the above can be interpreted as the importance of component i at state j ($I_j^{WC}(i)$ can be interpreted as the importance of component i). Griffith considers also the importance of a component i as the vector

$$\underline{I}(i) = (I_1(i), I_2(i), \dots, I_M(i)).$$

However it is also possible to consider the importance of the component i as some numerical measure of the $I_j(i)$, e.g. $\sum_{j=1}^M I_j(i)$. The value of

Griffith's concept however is that it preserves a property basic to the binary case, i.e., from Proposition 4.1 of Griffith (1980)

$$E(\phi(\underline{X})) = E(\phi(0_1, \underline{X})) + \underline{I}(i) \cdot (P(X_{i-1} \geq 1), P(X_{i-1} \geq 2), \dots, P(X_{i-1} \geq M)) .$$

This gives that if component i is stochastically improved, i.e., $P(Y_{i-1} \geq j) \leq P(X_{i-1} \geq j)$ for $j=1, \dots, M$, then the improvement in the system is

$$\underline{I}(i) \cdot (P^*(X_{i-1} \geq 1) - P(X_{i-1} \geq 1), \dots, P^*(X_{i-1} \geq M) - P(X_{i-1} \geq M)) .$$

Some relations which hold between the various importance relations introduced are

$$I_j^{SC}(i) \leq I_j^{(1)}(i) \leq I_j^C(i) \leq I_j^{WC}(i)$$

and

$$I_j^C(i) \leq I_j(i).$$

§ 6. Application of the decomposition result.

We now apply the results of Section 2 to the multistate system of

Barlow and Wu (1978). According to their definition, ζ is a multistate structure function if we can write

$$(6.1) \quad \zeta(\underline{z}) = \max_{1 \leq r \leq p} \min_{i \in P_r} z_i = \min_{1 \leq s \leq k} \max_{i \in K_s} z_i,$$

where $P_1, \dots, P_p$ are the min path sets and $K_1, \dots, K_k$ are the min cut sets of some binary coherent system ϕ and $\underline{z} = (z_1, \dots, z_n)$ is the component state vector with $0 \leq z_i \leq M$ for $i=1, \dots, n$.

It is clear that

$$U_k = \{k\underline{x} \mid \underline{x} \text{ is a min path vector of } \phi\}$$

and

$$L_k = \{(M-k)\underline{y} + k \mid \underline{y} \text{ is a min cut vector of } \phi\}.$$

For $\underline{z} \in U_k$, where $\underline{z} = k\underline{x}$, it follows that

$$U_k(\underline{z}) = \{(i, k) \mid i \in C_1(\underline{x})\}$$

where $C_1(\underline{x})$ is the min path set of ϕ (in the sense of Barlow and Proschan (1975)) corresponding to $\underline{x}$; i.e., one of the $P_1, \dots, P_p$. Similarly, for

$\underline{z} \in L_k$, where $\underline{z} = (M-k)\underline{y} + k$,

$$L_k(\underline{z}) = \{(i, k) : i \in C_0(\underline{y})\}$$

where $C_0(\underline{y})$ is the min cut set of ϕ corresponding to $\underline{y}$; i.e., one of the $K_1, \dots, K_k$.

(6.2) Remark. Notice that the condition in Theorem 2.4 becomes: for $k > 0$, $\zeta(\underline{z}) \geq k$ if and only if there is a min path vector $\underline{x}^0$ of the associated binary coherent system ϕ such that $\underline{z} \geq k\underline{x}^0$.

The following equivalent conditions can now be stated. Here ζ_k is the binary function related to ζ by (2.6).

(6.3) Lemma. The following conditions are equivalent. Let $k > 0$.

$$(i) \quad \zeta(\underline{z}) \geq k$$

$$(ii) \quad \zeta_k(\alpha(\underline{z})) = 1$$

$$(iii) \quad \phi(\alpha_k(\underline{z})) = 1$$

where $\alpha_k(\underline{z}) = (\alpha_{1k}(\underline{z}), \dots, \alpha_{nk}(\underline{z}))$ and $\alpha_{ik}(\underline{z})$ is defined in Section 2.

Proof. We've already proven the equivalence of (i) and (ii). Now suppose that $\zeta_k(\alpha(\underline{z})) = 1$. Then there is a min path vector $\underline{x}$ of ϕ such that $\underline{z} = k\underline{x}$ and $\alpha_{ij}(\underline{z}) = 1$ for all $(i,j) \in U_k(k\underline{x})$; i.e., $\alpha_{ik}(\underline{z}) = 1$ for all $i \in C_1(\underline{x})$. Hence $\alpha_k(\underline{z})$ is a path vector for ϕ and so $\phi(\alpha_k(\underline{z})) = 1$. On the other hand, if $\phi(\alpha_k(\underline{z})) = 1$, then $\alpha_k(\underline{z})$ is a path vector for ϕ . Consequently there is a min path vector $\underline{x}^0$ of ϕ such that $\alpha_k(\underline{z}) \geq \underline{x}^0$. Thus $z_i \geq k$ for all $i \in C_1(\underline{x}^0)$, which implies that $\underline{z} \geq k\underline{x}^0 \in U_k$. Hence $\zeta(\underline{z}) \geq k$ by Remark 6.2.

We now have the alternate way of expressing the decomposition Theorem 2.8 for this class of multistate structure functions. It follows immediately from the Lemma.

$$(6.4) \quad \text{Theorem.} \quad \zeta(\underline{z}) = \bigvee_{k=1}^M \phi(\alpha_k(\underline{z}))$$

Among the class of all monotone structure functions, we can now describe those of type (6.1). A similar result exists if min cut sets instead of min path sets are used.

(6.5) Theorem. Let ϕ be an MMS. Then the following are equivalent.

$$(i) \quad \phi \text{ is of the form (6.1).}$$

$$(ii) \quad \text{For } k = 1, 2, \dots, M, U_k = kU_1.$$

Proof. It follows from the remarks below (6.1) that (i) implies (ii), so we need only show that (ii) implies (i). Thus let ϕ be an MMS

satisfying (ii). Suppose that $\underline{z} \in U_1$. Then $U_1(\underline{z}) = \{ (i, z_i) :$

$z_i \neq 0 \}$. Hence if $z_i \neq 0$, it must be that $z_i = 1$ for otherwise $M\underline{z} \notin U_M$.

Let $P(\underline{z}) = \{ i : z_i = 1 \}$ be its "path set". From this it is easy to conclude that

$$\phi_1(\underline{y}) = \max_{1 \leq r \leq p} \min_{i \in P_r} y_{i1}$$

where $p = \text{card } U_1$ and $P_1, \dots, P_p$ are the path sets corresponding to each of the path vectors in U_1 . Since $U_k = kU_1$, we get then that

$$\phi_k(\underline{y}) = \max_{1 \leq r \leq p} \min_{i \in P_r} y_{ik}$$

for $k=1, \dots, M$. Thus if we define the binary structure function ϕ of the binary vector $\underline{x} = (x_1, \dots, x_n)$ by $\phi(\underline{x}) = \max_{1 \leq r \leq p} \min_{i \in P_r} x_i$, we have

$$\phi_k(\alpha(\underline{z})) = \phi(\alpha_k(\underline{z})).$$

Since $\phi(\underline{z}) = \sum_{k=1}^M \phi_k(\alpha(\underline{z})) = \sum_{k=1}^M \phi(\alpha_k(\underline{z}))$, it readily follows that

$$\phi(\underline{z}) = \max_{1 \leq r \leq p} \min_{i \in P_r} z_i ;$$

i.e., ϕ is the form (6.1).

(6.6) Remark. Borges and Rodrigues (1980) have also obtained a characterization of the Barlow and Wu multistate system which is different than the one here.

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