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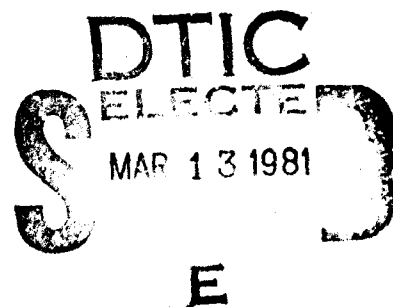
LEVEL II



AN H-FIELD SOLUTION FOR A CONDUCTING BODY OF REVOLUTION

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Air Force Systems Command
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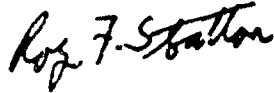
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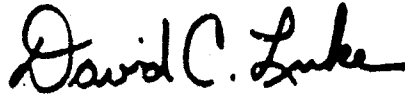
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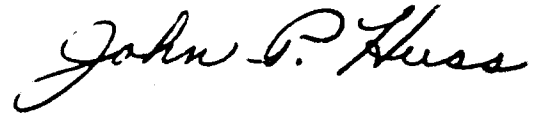
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on S , t being the arc length along the generating curve of S and ϕ the azimuthal angle. θ at a

A numerical solution is obtained by means of a computer program which is described and listed. This computer program is designed to handle oblique plane wave incidence efficiently.

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PART ONE

SOLUTION PROCEDURE AND NUMERICAL RESULTS

I. INTRODUCTION

The purpose of the present report is to develop an H-field solution which can be used alone or used in conjunction with the E-field solution in [1] to obtain a combined field solution which is hopefully more accurate than the one in [2]. Here, H-field solution means a solution to the magnetic field integral equation for a perfectly conducting body immersed in an incident electric field. Similarly, E-field solution means a solution to the electric field integral equation. In [1] and in the present report, solution is by the method of moments with a Fourier series in ϕ . For compatibility, the expansion functions used in the present report must be the same as those in [1]. In [1] and in the present report, the t dependence of the expansion functions is subsectional. Pulses are used for the ϕ component of the unknown electric current induced on the surface S of the body of revolution. Triangles divided by the cylindrical coordinate radius are used for the t component. Here, t and ϕ are orthogonal coordinates on S , t being the arc length along the generating curve of S and ϕ the azimuthal angle.

Computations indicate that the E-field solution of [1] is more accurate than the E-field solution of [2] and that the present H-field solution is nearly as accurate as the H-field solution of [2]. Hence, the combined field solution obtained by putting together the present H-field solution and the E-field solution of [1] should compare favorably with the combined field solution of [2].

Early work on H-field solutions is summarized in [3]. Two recent H-field solutions are presented in [4] and [2]. The solution in [4] uses pulse expansion functions for both t and ϕ components of the unknown electric current induced on S . H-field formulations are basic ingredients in solutions to problems of scattering by a dielectric body of revolution [5], [6], [7] and a dielectrically clad conducting body of revolution [8]. The expansion functions in [5] and [8] are triangles divided by the cylindrical

coordinate radius for both components of the current. The H-field formulation of [2] is used in [6] and [9]. The expansion functions in [7] are pulses for the ϕ component of current and shifted pulses divided by the cylindrical coordinate radius for the t component.

The magnetic field integral equation for a perfectly conducting body of revolution illuminated by an incident magnetic field $\underline{H}^i$ is

$$-\underline{n} \times \underline{H}^S(\underline{J}) = \underline{n} \times \underline{H}^i \quad \text{just inside } S \quad (1)$$

where S is the surface of the body of revolution and $\underline{n}$ is the unit vector which is normal to S and which points outward from S . In (1), $\underline{J}$ is the electric current induced on S and $\underline{H}^S(\underline{J})$ is the magnetic field due to $\underline{J}$ radiating in the absence of the body of revolution. It is assumed that the medium outside S is linear, isotropic, and homogeneous. "Outside S " means outside the body of revolution and "inside S " means inside it. "Absence of the body of revolution" means that the body of revolution has been removed and that the ensuing void has been filled with the outside medium. The incident magnetic field $\underline{H}^i$ is the field which would exist in the absence of the body of revolution.

Equation (1) states that the tangential components of the total magnetic field ($\underline{H}^S + \underline{H}^i$) vanish just inside S . Consequently, (1) is valid only if S is closed. By contrast, the E-field integral equation (40) of [2] is valid for both open and closed surfaces of revolution.

II. METHOD OF MOMENTS SOLUTION

Equation (1) is solved by the method of moments. The moments solution to (1) is approached by writing

$$\underline{J} = \sum_n \left(\sum_j I_{nj}^t \underline{J}_{nj}^t + \sum_j I_{nj}^\phi \underline{J}_{nj}^\phi \right) \quad (2)$$

where $\underline{J}_{nj}^t$ and $\underline{J}_{nj}^\phi$ are expansion functions defined by

$$\frac{J^t}{n_j} = \frac{u}{u_t} \frac{T_j(t)}{\rho} e^{jn\phi} \quad \begin{cases} n = 0, \pm 1, \pm 2, \dots \\ j = 1, 2, \dots, P-2 \end{cases} \quad (3a)$$

$$\frac{J^\phi}{n_j} = \frac{u}{u_\phi} \frac{P_j(t)}{\rho_j} e^{jn\phi} \quad \begin{cases} n = 0, \pm 1, \pm 2, \dots \\ j = 1, 2, \dots, P-1 \end{cases} \quad (3b)$$

and where I_{nj}^t and I_{nj}^ϕ are unknown constants to be determined. The subscript j which runs from 1 to either $(P-2)$ or $(P-1)$ in (3) is not to be confused with the j which appears in the argument of the exponential in (3). The latter j is $\sqrt{-1}$. The variables t and ϕ are orthogonal curvilinear coordinates on S . Specifically, t is the arc length along the generating curve of S and ϕ is the azimuthal angle. The generating curve of S is the plane curve which, when rotated about the z axis, generates S . In (3), $\underline{u}_t$ and $\underline{u}_\phi$ are unit vectors in the t and ϕ directions, respectively. These directions are chosen so that

$$\underline{n} = \underline{u}_\phi \times \underline{u}_t \quad (4)$$

The function $T_j(t)$ is the triangle function defined by

$$T_j(t) = \begin{cases} (t - t_j^-)/\Delta_j & , \quad t_j^- \leq t \leq t_{j+1}^- \\ (t_{j+2}^- - t)/\Delta_{j+1} & , \quad t_{j+1}^- \leq t \leq t_{j+2}^- \\ 0 & , \quad \text{otherwise} \end{cases} \quad (5)$$

and $P_j(t)$ is the pulse function defined by

$$P_j(t) = \begin{cases} 1 & , \quad t_j^- \leq t < t_{j+1}^- \\ 0 & , \quad \text{otherwise} \end{cases} \quad (6)$$

Here, t_1^- , t_2^- , ..., t_p^- are consecutive but not necessarily equally spaced points on the generating curve of S . t_1^- is the value of t at the beginning of the generating curve and t_p^- is the value of t at the end. Henceforth, the generating curve is assumed to be a series of straight line segments joining the points t_1^- , t_2^- , ..., t_p^- . In (5),

$$\Delta_j = t_{j+1}^- - t_j^- \quad (7)$$

In (3), ρ is the distance of the point t on the generating curve from the z axis and ρ_j is the value of ρ at $t = t_j$ where

$$t_j = \frac{1}{2} (t_j^- + t_{j+1}^-) \quad (8)$$

It is assumed that all the ρ_j 's are positive.

The moments solution to (1) is obtained by substituting (2) into (1), integrating over S the dot product of (1) with each member of a set of testing functions, and solving the resulting matrix equation for I_{nj}^t and I_{nj}^ϕ . There are two kinds of testing functions, W_{ni}^t and W_{ni}^ϕ defined by

$$W_{ni}^t = u_t \frac{T_i(t)}{\rho} e^{-jn\phi} \quad \begin{cases} n = 0, \pm 1, \pm 2, \dots \\ i = 1, 2, \dots, P-2 \end{cases} \quad (9a)$$

$$W_{ni}^\phi = u_\phi \frac{P_i(t)}{\rho_i} e^{-jn\phi} \quad \begin{cases} n = 0, \pm 1, \pm 2, \dots \\ i = 1, 2, \dots, P-1 \end{cases} \quad (9b)$$

The matrix equation for I_{nj}^t and I_{nj}^ϕ decomposes into

$$\begin{bmatrix} Y_n^{tt} & Y_n^{t\phi} \\ Y_n^{\phi t} & Y_n^{\phi\phi} \end{bmatrix} \begin{bmatrix} \vec{I}_n^t \\ \vec{I}_n^\phi \end{bmatrix} = \begin{bmatrix} \vec{I}_n^{it} \\ \vec{I}_n^{i\phi} \end{bmatrix}, \quad n = 0, \pm 1, \pm 2, \dots \quad (10)$$

where the superscripted $\vec{I}_n$'s are column vectors and the superscripted Y_n 's are submatrices. The j th element of $\vec{I}_n^t$ is I_{nj}^t and the j th element of $\vec{I}_n^\phi$ is I_{nj}^ϕ . The i th elements of $\vec{I}_n^{it}$ and $\vec{I}_n^{i\phi}$ are given by

$$I_{ni}^{it} = \int dt T_i(t) \int_0^{2\pi} d\phi (\underline{u}_\phi \cdot \underline{H}^i) e^{-jn\phi} \quad (11a)$$

$$I_{ni}^{i\phi} = -\frac{1}{\rho_i} \int dt \rho P_i(t) \int_0^{2\pi} d\phi (\underline{u}_t \cdot \underline{H}^i) e^{-jn\phi} \quad (11b)$$

The second subscript on I in (11) denotes the i th element. The superscript i on I in (11) is the same as that in (10). This superscript indicates dependence on the incident magnetic field $\underline{H}^i$.

For a particular value of n in (10), the matrix of the superscripted Y_n 's on the left-hand side of (10) is called the moment matrix for that value of n . The ij th elements of the superscripted Y_n 's are given by

$$(Y_n^{tt})_{ij} = \pi \int_{t_i}^{t_{i+2}} \frac{T_i(t) T_j(t)}{\rho} dt + k^3 \int_{t_i}^{t_{i+2}} dt T_i(t) \int_{t_j}^{t_{j+2}} dt' T_j(t')$$

$$[((\rho' - \rho) \cos v' - (z' - z) \sin v') G_2 - G_1 \rho \cos v'] \quad (12a)$$

$$(Y_n^{\phi t})_{ij} = \frac{jk^3}{\rho_i} \int_{t_i}^{t_{i+1}} dt \rho P_i(t) \int_{t_j}^{t_{j+2}} dt' T_j(t') (\rho' \sin v \cos v' - \rho \sin v' \cos v - (z' - z) \sin v \sin v') G_3 \quad (12b)$$

$$(Y_n^{t\phi})_{ij} = \frac{jk^3}{\rho_j} \int_{t_i}^{t_{i+2}} dt T_i(t) \int_{t_j}^{t_{j+1}} dt' \rho' P_j(t') (z' - z) G_3 \quad (12c)$$

$$(Y_n^{\phi\phi})_{ij} = \frac{\pi}{\rho_i \rho_j} \int_{t_i}^{t_{i+1}} \rho P_i(t) P_j(t) dt + \frac{k^3}{\rho_i \rho_j} \int_{t_i}^{t_{i+1}} dt \rho P_i(t) \int_{t_j}^{t_{j+1}} dt' \rho' P_j(t')$$

$$[((\rho' - \rho) \cos v - (z' - z) \sin v) G_2 + G_1 \rho' \cos v] \quad (12d)$$

The derivation of (12) is not included here because it is similar to that of (20)-(23) of [2]. Actually, (12) was obtained by replacing $f_i(t)$ by $\frac{1}{\rho} T_i(t)$ in (20) and (22) and by $\frac{1}{\rho_i} P_i(t)$ in (21) and (23) and by replacing $f_j(t')$ by $\frac{1}{\rho'} T_j(t')$ in (20) and (21) and by $\frac{1}{\rho_j} P_j(t')$ in (22) and (23). The equations (20)-(23) referred to in the previous sentence are in [2].

In (12), k is the propagation constant in the medium outside S . Also, v is the angle between the tangent to the generating curve and the z axis. v is positive if the generating curve is moving away from the z axis. Otherwise, $v \leq 0$. In (12), ρ , z , and v are evaluated at the point t on the generating curve and ρ' , z' , and v' are the values of ρ , z , and v at the point t' . The subscripted G 's in (12) are given by

$$G_1 = 2 \int_0^\pi G \sin^2(\phi/2) \cos(n\phi) d\phi \quad (13a)$$

$$G_2 = \int_0^\pi G \cos \phi \cos(n\phi) d\phi \quad (13b)$$

$$G_3 = \int_0^\pi G \sin \phi \sin(n\phi) d\phi \quad (13c)$$

where

$$G = \frac{1 + jkR}{k^3 R^3} e^{-jkR} \quad (14)$$

where

$$R = \sqrt{(\rho' - \rho)^2 + (z' - z)^2 + 4\rho\rho' \sin^2(\phi/2)} \quad (15)$$

The regions of integration in (12a)-(12c) overlap. To avoid integrating more than once over the same region, we account for the contributions in (12a)-(12c) by regions of integration rather than by matrix elements. If A_q is the region which $t_q^- \leq t \leq t_{q+1}^-$, then A_q contributes

$$(\dot{Y}_n^{tt})_{ij} = \pi \int_{t_q^-}^{t_{q+1}^-} \frac{T_i(t) T_j(t)}{\rho} dt \quad (16)$$

to the single integral with respect to t in (12a). Because the domains of the triangle functions are limited, (16) is nonzero only for

$$\left. \begin{aligned} i &= q-1, q \\ i &\neq 0 \\ i &\neq P-1 \end{aligned} \right\} \quad (17a)$$

$$\left. \begin{aligned} j &= q-1, q \\ j &\neq 0 \\ j &\neq P-1 \end{aligned} \right\} \quad (17b)$$

The dot on the left-hand side of (16) indicates that (16) is not all of (12a) but only the contribution due to A_q .

If A_{pq} is the region for which

$$t_p^- \leq t \leq t_{p+1}^-$$

$$t_q^- \leq t' \leq t_{q+1}^- ,$$

then the contributions to (12a)-(12c) due to A_{pq} are given by

$$(\dot{Y}_n^{*tt})_{ij} = k^3 \int_{t_p^-}^{t_{p+1}^-} dt T_i(t) \int_{t_q^-}^{t_{q+1}^-} dt' T_j(t') [((\rho' - \rho) \cos v' - (z' - z) \sin v') G_2 - G_1 \rho \cos v'] \quad (18a)$$

$$(\dot{Y}_n^{*\phi t})_{pj} = \frac{jk^3}{\rho_p} \int_{t_p^-}^{t_{p+1}^-} dt \rho P_p(t) \int_{t_q^-}^{t_{q+1}^-} dt' T_j(t') (\rho' \sin v \cos v' - \rho \sin v' \cos v - (z' - z) \sin v \sin v') G_3 \quad (18b)$$

$$(\overset{*}{Y}_n^{\phi})_{iq} = \frac{jk}{\rho_q} \int_{t_p}^{t_{p+1}} dt T_i(t) \int_{t_q}^{t_{q+1}} dt' \rho' P_q(t') (z' - z) G_3 \quad (18c)$$

where

$$\left. \begin{array}{l} i = p-1, p \\ i \neq 0 \\ i \neq P-1 \end{array} \right\} \quad (19a)$$

and

$$\left. \begin{array}{l} j = q-1, q \\ j \neq 0 \\ j \neq P-1 \end{array} \right\} \quad (19b)$$

The asterisks on the left-hand sides of (18a)-(18c) indicate contributions due to A_{pq} . For instance, (18a) is not all of (12a) but only the contribution due to A_{pq} . Equation (12d) is suitable for calculation as is because it has no overlapping intervals. However, to obtain a notation consistent with that in (18a)-(18c), we replace ij by pq in (12d). The result is

$$(\overset{*}{Y}_n^{\phi})_{pq} = \frac{\pi}{2} \delta_{pq} \int_{t_p}^{t_{p+1}} \rho P_p^2(t) dt + \frac{k^3}{\rho_p \rho_q} \int_{t_p}^{t_{p+1}} dt \rho P_p(t) \int_{t_q}^{t_{q+1}} dt' \rho' P_q(t') \\ [((\rho' - \rho) \cos v - (z' - z) \sin v) G_2 + G_1 \rho' \cos v] \quad (18d)$$

where

$$\delta_{pq} = \begin{cases} 1, & p = q \\ 0, & p \neq q \end{cases} \quad (20)$$

The fact that the pulses (6) do not overlap was used to obtain the δ_{pq} term in (18d).

Each integral with respect to t in (18) is approximated by sampling the pertinent integrand at $t = t_p$ and multiplying by Δ_p . Next, the definitions (5) and (6) of the triangle and pulse functions are substituted into (18). Because the portion of the generating curve for $t_q^- \leq t' \leq t_{q+1}^-$ is assumed to be a straight line, v' is constant there so that

$$\left. \begin{aligned} \rho' &= \rho_q + (t' - t_q) \sin v_q \\ z' &= z_q + (t' - t_q) \cos v_q \end{aligned} \right\} t_q^- \leq t' \leq t_{q+1}^- \quad (21)$$

where z_q is the value of z' at $t' = t_q$. Equations (21) are also substituted into (18). As a result, (18) becomes

$$(\ddot{Y}_n^{tt})_{ij} = \frac{k^3 \Delta_p}{4} \int_{t_q^-}^{t_{q+1}^-} (1 + (-1)^{q-j} \frac{2(t' - t_q)}{\Delta_q}) [((\rho_q - \rho_p) \cos v_q - (z_q - z_p) \sin v_q) G_2 - G_1 \rho_p \cos v_q] dt' \quad (22a)$$

$$(\ddot{Y}_n^{\phi t})_{pj} = \frac{jk^3 \Delta_p}{2} \int_{t_q^-}^{t_{q+1}^-} (1 + (-1)^{q-j} \frac{2(t' - t_q)}{\Delta_q}) (\rho_q \sin v_p \cos v_q - \rho_p \sin v_q \cos v_p - (z_q - z_p) \sin v_p \sin v_q) G_3 dt' \quad (22b)$$

$$(\ddot{Y}_n^{t\phi})_{iq} = \frac{jk^3 \Delta_p}{2} \int_{t_q^-}^{t_{q+1}^-} (1 + \frac{(t' - t_q) \sin v_q}{\rho_q}) (z_q - z_p + (t' - t_q) \cos v_q) G_3 dt' \quad (22c)$$

$$\begin{aligned} (\ddot{Y}_n^{\phi\phi})_{pq} &= \frac{\pi \Delta_q}{\rho_q} \delta_{pq} + k^3 \Delta_p \int_{t_q^-}^{t_{q+1}^-} (1 + \frac{(t' - t_q) \sin v_q}{\rho_q}) [((\rho_q - \rho_p) \cos v_p - (z_q - z_p) \sin v_p \\ &+ (t' - t_q) (\sin v_q \cos v_p - \cos v_q \sin v_p)) G_2 + G_1 \rho_q (1 + \frac{(t' - t_q) \sin v_q}{\rho_q}) \cos v_p] dt' \end{aligned}$$

d)

The ranges of values of i and j in (22) are given by (19). Because ρ is linear in t , the process of sampling the integrand at $t = t_p$ and multiplying by Δ_p gave the exact value of the first integral in (18d).

The integral in (16) can be approximated either by sampling the integrand at $t = t_q$ and multiplying by Δ_q or by applying Gaussian quadrature. Gaussian quadrature was chosen because it gave more accurate values of electric currents induced on the sphere and cone-sphere examples in Section V. Thanks to (5) and (21), the Gaussian quadrature formula for the integral in (16) becomes

$$(\dot{Y}_n^{tt})_{q-1,q-1} = \frac{\pi \Delta_q}{8} \sum_{\ell=1}^{n_t} \frac{A_\ell^{(n_t)} (1 - x_\ell^{(n_t)})^2}{\hat{\rho}_\ell} \quad (23a)$$

$$(\dot{Y}_n^{tt})_{q,q-1} = (\dot{Y}_n^{tt})_{q-1,q} = \frac{\pi \Delta_q}{8} \sum_{\ell=1}^{n_t} \frac{A_\ell^{(n_t)} (1 - x_\ell^{(n_t)}) (1 + x_\ell^{(n_t)})}{\hat{\rho}_\ell} \quad (23b)$$

$$(\dot{Y}_n^{tt})_{q,q} = \frac{\pi \Delta_q}{8} \sum_{\ell=1}^{n_t} \frac{A_\ell^{(n_t)} (1 + x_\ell^{(n_t)})^2}{\hat{\rho}_\ell} \quad (23c)$$

$$\text{where} \quad \hat{\rho}_\ell = \rho_q + \frac{\Delta_q x_\ell^{(n_t)}}{2} \sin v_q \quad (23d)$$

The abscissas $x_\ell^{(n_t)}$ and weights $A_\ell^{(n_t)}$ in (23) are given in Appendix A of [10] for several values of n_t .

Equations (22) can be rewritten as

$$\begin{aligned} (\dot{Y}_n^{tt})_{ij} &= \frac{k^3 \Delta_p \Delta_q}{8} [((\rho_q - \rho_p) \cos v_q - (z_q - z_p) \sin v_q) G_{21} - G_{11} \rho_p \cos v_q] \\ &+ (-1)^{q-j} \frac{k^3 \Delta_p \Delta_q}{8} [((\rho_q - \rho_p) \cos v_q - (z_q - z_p) \sin v_q) G_{22} - G_{12} \rho_p \cos v_q] \end{aligned} \quad (24a)$$

$$\begin{aligned} (\dot{Y}_n^{\phi t})_{pj} &= \frac{jk^3 \Delta_p \Delta_q}{4} (\rho_q \sin v_p \cos v_q - \rho_p \sin v_q \cos v_p - \\ &- (z_q - z_p) \sin v_p \sin v_q) (G_{31} + (-1)^{q-j} G_{32}) \end{aligned} \quad (24b)$$

$$(\hat{Y}_n^{*t\phi})_{iq} = \frac{jk^3 \Delta_p \Delta_q}{4} [(z_q - z_p)(G_{31} + \frac{\Delta_q \sin v_q}{2\rho_q} G_{32}) + \frac{\Delta_q \cos v_q}{2} (G_{32} + \frac{\Delta_q \sin v_q}{2\rho_q} G_{33})] \quad (24c)$$

$$\begin{aligned} (\hat{Y}_n^{\phi\phi})_{pq} = & \frac{\pi \Delta_p}{\rho_p} \delta_{pq} + \frac{k^3 \Delta_p \Delta_q}{2} [((\rho_q - \rho_p) \cos v_p - (z_q - z_p) \sin v_p)(G_{21} + \frac{\Delta_q \sin v_q}{2\rho_q} G_{22}) \\ & + \frac{\Delta_q}{2} (\sin v_q \cos v_p - \cos v_q \sin v_p)(G_{22} + \frac{\Delta_q \sin v_q}{2\rho_q} G_{23}) + \\ & + \rho_q \cos v_p (G_{11} + \frac{\Delta_q \sin v_q}{\rho_q} G_{12} + (\frac{\Delta_q \sin v_q}{2\rho_q})^2 G_{13})] \end{aligned} \quad (24d)$$

where

$$G_{m\alpha} = \left(\frac{2}{\Delta_q}\right)^\alpha \int_{t_q}^{t_q+1} (t' - t_q)^{\alpha-1} G_m(t' - t_q) dt', \quad \begin{cases} m = 1, 2, 3 \\ \alpha = 1, 2, 3 \end{cases} \quad (25)$$

Here,

$$G_1(u) = 2 \int_0^\pi G(u, \phi) \sin^2(\phi/2) \cos(n\phi) d\phi \quad (26a)$$

$$G_2(u) = \int_0^\pi G(u, \phi) \cos \phi \cos(n\phi) d\phi \quad (26b)$$

$$G_3(u) = \int_0^\pi G(u, \phi) \sin \phi \sin(n\phi) d\phi \quad (26c)$$

where

$$G(u, \phi) = \frac{1 + jkR(u, \phi)}{k^3 R^3(u, \phi)} e^{-jkR(u, \phi)} \quad (27)$$

where

$$R(u, \phi) = \sqrt{(\rho' - \rho_p)^2 + (z' - z_p)^2 + 4\rho_p \rho' \sin^2(\phi/2)} \quad (28)$$

where

$$\rho' = \rho_q + u \sin v_q \quad (29a)$$

$$z' = z_q + u \cos v_q \quad (29b)$$

III. NUMERICAL EVALUATION OF THE INTEGRALS $G_{m\alpha}$

The integrals $G_{m\alpha}$ of (25) are now evaluated by means of either pure Gaussian quadrature or Gaussian quadrature fortified by the method of elimination of the singularity of the integrand. The proximity of the singularity of $G(u, \phi)$ of (26) to the region of integration falls into one of four different cases defined in this section.

If

$$\left. \begin{array}{l} p \neq q \\ \frac{1}{2} c_t \Delta_q \leq d_o \\ c_\phi \rho_q \leq d_o \end{array} \right\} \begin{array}{l} \text{Case 1} \\ \text{Pure quadrature} \end{array} \quad (30)$$

then pure Gaussian quadrature is used to evaluate the integrals in (25) and (26). In (30), c_t and c_ϕ are constants for which the values

$$\left. \begin{array}{l} c_t = 2. \\ c_\phi = 0.1 \end{array} \right\} \quad (31)$$

are suggested. Also in (30),

$$d_o = \begin{cases} d^* & , \quad |t_o^*| \leq \frac{1}{2} \Delta_q \\ \sqrt{|t_o^*| - \frac{1}{2} \Delta_q)^2 + (d^*)^2} & , \quad |t_o^*| > \frac{1}{2} \Delta_q \end{cases} \quad (32)$$

where

$$t_o^* = (\rho_q - \rho_p) \sin v_q + (z_q - z_p) \cos v_q \quad (33)$$

$$d^* = |(\rho_q - \rho_p) \cos v_q - (z_q - z_p) \sin v_q| \quad (34)$$

The quantity d_o defined by (32) is the minimum value of $R(u, \phi)$ for the ranges $|u| \leq \frac{1}{2} \Delta_q$ and $0 \leq \phi \leq \pi$ of values of u and ϕ in (26). Equations (30)-(34) were taken from pages 24 and 25 of [1].

The Gaussian quadrature formulas for the integrals in (25) and (26) are

$$G_{m\alpha} = \sum_{\ell=1}^{n_t} A_{\ell}^{(n_t)} \frac{(x_{\ell}^{(n_t)})^{\alpha-1}}{(x_{\ell}^{(n_t)})^{\alpha}} G_m\left(\frac{1}{2} \Delta_q x_{\ell}^{(n_t)}\right), \quad \begin{cases} m = 1, 2, 3 \\ \alpha = 1, 2, 3 \end{cases} \quad (35)$$

$$G_1(u) = \pi \sum_{\ell=1}^{n_\phi} A_{\ell}^{(n_\phi)} G(u, \phi_{\ell}) \sin^2\left(\frac{1}{2} \phi_{\ell}\right) \cos(n\phi_{\ell}) \quad (36a)$$

$$G_2(u) = \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_{\ell}^{(n_\phi)} G(u, \phi_{\ell}) \cos \phi_{\ell} \cos(n\phi_{\ell}) \quad (36b)$$

$$G_3(u) = \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_{\ell}^{(n_\phi)} G(u, \phi_{\ell}) \sin \phi_{\ell} \sin(n\phi_{\ell}) \quad (36c)$$

where $G(u, \phi_{\ell})$ is given by (27)-(29) with ϕ replaced by ϕ_{ℓ} . Here,

$$\phi_{\ell} = \frac{\pi}{2} (x_{\ell}^{(n_\phi)} + 1) \quad (37)$$

The abscissas $x_{\ell}^{(n_t)}$ and $x_{\ell}^{(n_\phi)}$ and weights $A_{\ell}^{(n_t)}$ and $A_{\ell}^{(n_\phi)}$ in (35) and (36) are given in Appendix A of [10] for several values of n_t and n_ϕ .

If (30) is not true, then the singularity due to the kernel (27) is eliminated [11] before applying Gaussian quadrature to the integrals in (25) and (26). The singularity analysis presented here is more general than that in [7] because the one in [7] was carried out only for $p = q$. Substitution of

$$e^{-jkR} = 1 - jkR - \frac{k^2 R^2}{2}$$

into (27) reveals that, near $R = 0$, (27) behaves like a function G^s given by

$$G^s = \frac{1}{3R^3} + \frac{1}{2kR} \quad (38)$$

Three different methods of eliminating the singularity are used.

If

$$\left. \begin{array}{l} p \neq q \\ \frac{1}{2} c_t \Delta_q > d_o \\ c_\phi \rho_q \leq d_o \end{array} \right\} \begin{array}{l} \text{Case 2} \\ \text{Method 1} \end{array} \quad (39)$$

then method 1 is used. In method 1, the orders of integration in (25) are interchanged to obtain

$$G_{1\alpha} = 2 \int_0^\pi H_\alpha(\phi) \sin^2(\phi/2) \cos(n\phi) d\phi \quad (40a)$$

$$G_{2\alpha} = \int_0^\pi H_\alpha(\phi) \cos \phi \cos(n\phi) d\phi \quad (40b)$$

$$G_{3\alpha} = \int_0^\pi H_\alpha(\phi) \sin \phi \sin(n\phi) d\phi \quad (40c)$$

where

$$H_\alpha(\phi) = \left(\frac{2}{\Delta_q}\right)^\alpha \int_{t_q}^{t_q+1} (t'-t_q)^{\alpha-1} G(t'-t_q, \phi) dt' \quad (41)$$

In method 1, the singular part of the integrand in (41) is integrated analytically, the remaining part is integrated by means of Gaussian quadrature, and then Gaussian quadrature is applied to the integrations with respect to ϕ in (40). The singular part of a singular integrand is a function which behaves like the integrand near its singularity. Evidently, the singular part is not unique.

Guided by (38) in our choice of the singular part of the integrand in (41), we write

$$H_\alpha(\phi) = \left(\frac{2}{\Delta_q}\right)^\alpha \int_{t_q}^{t_q+1} (t'-t_q)^{\alpha-1} \left(G(t'-t_q, \phi) - \frac{1}{k^3 R^3(t'-t_q, \phi)} - \frac{1}{2kR(t'-t_q, \phi)} \right) dt' + I_\alpha \quad (42)$$

where

$$I_{\alpha} = \left(\frac{2}{\Delta_q}\right)^{\alpha} \int_{t_q}^{t_q+1} (t'-t_q)^{\alpha-1} \left(\frac{1}{k^3 R^3(t'-t_q, \phi)} + \frac{1}{2kR(t'-t_q, \phi)} \right) dt' \quad (43)$$

Application of Gaussian quadrature to the first integral in (42) gives

$$H_{\alpha}(\phi) = \sum_{\ell'=1}^{n_t} A_{\ell'}^{(n_t)} (x_{\ell'}^{(n_t)})^{\alpha-1} \left(G\left(\frac{1}{2} \Delta_q x_{\ell'}^{(n_t)}, \phi\right) - \frac{1}{k^3 R^3\left(\frac{1}{2} \Delta_q x_{\ell'}^{(n_t)}, \phi\right)} - \frac{1}{2kR\left(\frac{1}{2} \Delta_q x_{\ell'}^{(n_t)}, \phi\right)} \right) + I_{\alpha} \quad (44)$$

Equation (43) can be rewritten as

$$I_{\alpha} = \left(\frac{2}{\Delta_q}\right)^{\alpha} \int_{t_o - \frac{1}{2}\Delta_q}^{t_o + \frac{1}{2}\Delta_q} (w-t_o)^{\alpha-1} \left[\frac{1}{k^3 (w^2+d^2)^{3/2}} + \frac{1}{2k(w^2+d^2)^{1/2}} \right] dw \quad (45)$$

where

$$t_o = (\rho_q - \rho_p) \sin v_q + (z_q - z_p) \cos v_q + 2\rho_p \sin v_q \sin^2(\phi/2) \quad (46a)$$

$$d^2 = r_{pq}^2 - t_o^2 \quad (46b)$$

$$r_{pq}^2 = (\rho_q - \rho_p)^2 + (z_q - z_p)^2 + 4\rho_p \rho_q \sin^2(\phi/2) \quad (46c)$$

Application of formulas 200.01., 200.03., and sequel in [12] to (45) gives

$$I_1 = \frac{2}{k\Delta_q} \left[\frac{w}{k^2 d^2 r} + \frac{1}{2} \log(w+r) \right]_{t_o - \frac{1}{2}\Delta_q}^{t_o + \frac{1}{2}\Delta_q} \quad (47a)$$

$$I_2 = \left(\frac{2}{k\Delta_q}\right)^2 \left[\frac{kr}{2} - \frac{1}{kr} \right]_{t_o - \frac{1}{2}\Delta_q}^{t_o + \frac{1}{2}\Delta_q} - \frac{2t_o}{\Delta_q} I_1 \quad (47b)$$

$$I_3 = \left(\frac{2}{k\Delta_q}\right)^3 \left[kw\left(\frac{kr}{4} - \frac{1}{kr}\right) + \left(1 - \frac{k^2 d^2}{4}\right) \log(w+r) \right]_{t_o - \frac{1}{2}\Delta_q}^{t_o + \frac{1}{2}\Delta_q} - \frac{4t_o I_2}{\Delta_q} - \left(\frac{2t_o}{\Delta_q}\right)^2 I_1 \quad (47c)$$

where

$$r = \sqrt{w^2 + d^2} \quad (47d)$$

If $\phi = 0$ in (42), then (30) guarantees that the variable t' of integration in (42) passes close to the singularity of $G(t' - t_q, \phi)$. In this case, (44) and (47) are necessary. However, as ϕ moves toward π , r_{pq} may become so large that the variable t' in (42) never comes close to the singularity of $G(t' - t_q, \phi)$. In this case, (44) and (47) are no longer necessary. Their calculation is always time consuming regardless of the value of r_{pq} . Even worse, the calculation of (44) and (47) is subject to excessive roundoff error whenever r_{pq} is appreciably larger than Δ_q . It was decided to restrict use of (44) and (47) to

$$r_{pq} < \frac{1}{2} c_t \Delta_q + \frac{1}{2} \Delta_q \quad (48)$$

For larger values of r_{pq} , recourse is to the Gaussian quadrature formula

$$H_\alpha(\phi) = \sum_{\ell'=1}^{n_t} A_{\ell'}^{(n_t)} (x_{\ell'}^{(n_t)})^{\alpha-1} G\left(\frac{1}{2} \Delta_q x_{\ell'}^{(n_t)}, \phi\right) \quad (49)$$

The term $\frac{1}{2}\Delta_q$ on the right-hand side of (48) is necessary to assure that the distance $R(t' - t_q, \phi)$ used in (41) is at least as large as $\frac{1}{2} c_t \Delta_q$ for all values of t' in (41) before (49) is invoked. This distance enters (41) through (27).

In method 1, $G_{m\alpha}$ of (40) is approximated by

$$G_{1\alpha} = \pi \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} H_\alpha(\phi_\ell) \sin^2\left(\frac{1}{2}\phi_\ell\right) \cos(n\phi_\ell) \quad (50a)$$

$$G_{2\alpha} = \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} H_\alpha(\phi_\ell) \cos \phi_\ell \cos(n\phi_\ell) \quad (50b)$$

$$G_{3\alpha} = \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} H_\alpha(\phi_\ell) \sin \phi_\ell \sin(n\phi_\ell) \quad (50c)$$

$\alpha = 1, 2, 3$

where $H_\alpha(\phi_\ell)$ is given by either (44) of (49).

If

$$\left. \begin{array}{l} p \neq q \\ c_{\phi} \rho_q > d_o \end{array} \right\} \begin{array}{l} \text{case 3} \\ \text{Method 2} \end{array} \quad (51)$$

then method 2 is used to eliminate the singularity due to the kernel (27).

In method 2, the singular parts of the integrands in (26) are integrated analytically, the remaining parts are integrated by means of Gaussian quadrature, and then (25) is approximated by the Gaussian quadrature formula (35).

Integration of (38) with respect to ϕ is difficult. With a view toward replacement of (38) by a function which behaves like (38) near $\phi = 0$ but is easier to integrate, we rewrite (28) as

$$R(u, \phi) = a\sqrt{1-b} \quad (52)$$

where

$$a = \sqrt{(\rho' - \rho_p)^2 + (z' - z_p)^2 + \rho_p \rho' \phi^2} \quad (53a)$$

$$b = \frac{\rho_p \rho'}{a^2} (\phi^2 - 4 \sin^2(\phi/2)) \quad (53b)$$

where ρ' and z' are given by (29). Near $\phi = 0$,

$$0 \leq b \leq \phi^2/12 \quad (54)$$

The left-hand side of (54) is a consequence of the fact that $|x| \geq |\sin x|$ for any value of x . The right-hand side of (54) was obtained by setting $a^2 = \rho_p \rho' \phi^2$ in (53b). Substitution of (52) into (38) and subsequent expansion of (38) in powers of b reveals that, near $\phi = 0$, G^s behaves like the function G^{ss} given by

$$G^{ss} = \frac{1}{k^3 a^3} + \frac{1}{2ka} + \frac{\rho_p \rho' \phi^4}{8k^3 a^5} \quad (55)$$

Guided by (55) in our choice of the singular parts of the integrands in (26), we approach method 2 by rewriting (26) as

$$G_1(u) = 2 \int_0^\pi (G(u, \phi) \sin^2(\phi/2) \cos(n\phi) - \frac{\phi^2}{4k^3 a^3}) d\phi + \int_0^\pi \frac{\phi^2}{2k^3 a^3} d\phi \quad (56a)$$

$$G_2(u) = \int_0^\pi (G(u, \phi) \cos \phi \cos(n\phi) \frac{1}{k^3 a^3} - \frac{1}{2ka} + \frac{(n^2+1)\phi^2}{2k^3 a^3} - \frac{\rho_p \rho' \phi^4}{8k^3 a^5}) d\phi \\ + \int_0^\pi (\frac{1}{k^3 a^3} + \frac{1}{2ka} - \frac{(n^2+1)\phi^2}{2k^3 a^3} + \frac{\rho_p \rho' \phi^4}{8k^3 a^5}) d\phi \quad (56b)$$

$$G_3(u) = \int_0^\pi (G(u, \phi) \sin \phi \sin(n\phi) - \frac{n\phi^2}{k^3 a^3}) d\phi + \int_0^\pi \frac{n\phi^2}{k^3 a^3} d\phi \quad (56c)$$

There are two integrals on the right-hand side of each of equations (56a), (56b), and (56c). If Gaussian quadrature is applied to the first integral and if the second integral is evaluated analytically by using formulas 200.01., 200.03., and sequel in [12], then (56) becomes

$$G_1(u) = \pi \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} (G(u, \phi_\ell) \sin^2(\frac{1}{2} \phi_\ell) \cos(n\phi_\ell) - \frac{\phi_\ell^2}{4k^3 a_\ell^3}) \\ - \frac{1}{2k^3 (\rho_p \rho')^{3/2}} [\frac{w}{r} - \log(w+r)] \quad (57a)$$

$$\begin{aligned}
G_2(u) = & \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} (G(u, \phi_\ell) \cos \phi_\ell \cos(n\phi_\ell) - \frac{1}{k^3 a_\ell^3} - \frac{1}{2ka_\ell} + \frac{(n^2+1)\phi_\ell^2}{2k^3 a_\ell^3} - \frac{\rho_p \rho' \phi_\ell^4}{8k^3 a_\ell^5}) \\
& + \frac{w^3}{\pi^2 k^3 (\rho_p \rho')^{3/2} r} + \frac{\log(w+r)}{2k(\rho_p \rho')^{1/2}} + \frac{(n^2+1)}{2k^3 (\rho_p \rho')^{3/2}} \left[\frac{w}{r} - \log(w+r) \right] - \\
& - \frac{1}{8k^3 (\rho_p \rho')^{3/2}} \left[\frac{w(3+4w^2)}{3r^3} - \log(w+r) \right]
\end{aligned} \quad (57b)$$

$$\begin{aligned}
G_3(u) = & \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} (G(u, \phi_\ell) \sin \phi_\ell \sin(n\phi_\ell) - \frac{n\phi_\ell^2}{k^3 a_\ell^3}) - \frac{n}{k^3 (\rho_p \rho')^{3/2}} \left[\frac{w}{r} - \log(w+r) \right]
\end{aligned} \quad (57c)$$

where a_ℓ is the right-hand side of (53a) evaluated at $\phi = \phi_\ell$. In (57),

$$w = \frac{\pi \sqrt{\rho_p \rho'}}{\sqrt{(\rho' - \rho_p)^2 + (z' - z_p)^2}} \quad (58a)$$

$$r = \sqrt{1 + w^2} \quad (58b)$$

Equation (57) is used only if the minimum value $\sqrt{(\rho' - \rho_p)^2 + (z' - z_p)^2}$ of $R(u, \phi)$ with respect to ϕ in (26) satisfies

$$\sqrt{(\rho' - \rho_p)^2 + (z' - z_p)^2} < c_\phi \rho_q \quad (59)$$

The distance $R(u, \phi)$ enters (26) through (27). If (59) is not true, then

$G_m(u)$ is approximated by the pure Gaussian quadrature formulas (36).

In method 2, $G_{m\alpha}$ is obtained by substituting either (57) or (36) into (35).

If

$$p = q \left\{ \begin{array}{l} \text{Case 4} \\ \text{Method 3} \end{array} \right. \quad (60)$$

then method 3 is used to obtain $G_{m\alpha}$. Since $p=q$, only G_{11} , G_{12} , G_{13} , G_{32} , and G_{33} are needed in order to calculate (24). From (40) and (41), these subscripted G 's are given by

$$G_{1\alpha} = 2\left(\frac{2}{\Delta_q}\right)^\alpha \int_0^\pi d\phi \int_{-\frac{1}{2}\Delta_q}^{\frac{1}{2}\Delta_q} du u^{\alpha-1} G(u, \phi) \sin^2(\phi/2) \cos(n\phi), \quad \alpha = 1, 2, 3 \quad (61a)$$

$$G_{3\alpha} = \left(\frac{2}{\Delta_q}\right)^\alpha \int_0^\pi d\phi \int_{-\frac{1}{2}\Delta_q}^{\frac{1}{2}\Delta_q} du u^{\alpha-1} G(u, \phi) \sin \phi \sin(n\phi), \quad \alpha = 2, 3 \quad (61b)$$

where $G(u, v)$ is given by (27) with

$$R(u, \phi) = \sqrt{u^2 + 4\rho_q(\rho_q + u \sin v_q) \sin^2(\phi/2)} \quad (62)$$

Substitution of (38) for $G(u, \phi)$ in (61) reveals that, in (61), the only integrand which is not bounded in the vicinity of $u = \phi = 0$ is the integrand for $\alpha = 1$ in (61a). In method 3, the singular part of this integrand is integrated analytically with respect to both u and ϕ , the remaining part is integrated by means of method 1, and then method 1 is applied directly to the integrals (40) and (41) for G_{12} , G_{13} , G_{32} , and G_{33} .

Equation (61a) for G_{11} is rewritten as

$$G_{11} = I_a + I_b \quad (63)$$

where

$$I_a = \frac{4}{\Delta_q} \int_0^\pi d\phi \int_{-\frac{1}{2}\Delta_q}^{\frac{1}{2}\Delta_q} du (G(u, \phi) \sin^2(\phi/2) \cos(n\phi) - \frac{\phi^2}{4k^3(u^2 + \rho_q^2 \phi^2)^{3/2}}) \quad (64)$$

$$I_b = \frac{1}{\Delta_q} \int_0^\pi d\phi \int_{-\frac{1}{2}\Delta_q}^{\frac{1}{2}\Delta_q} du \frac{\phi^2}{k^3(u^2 + \rho_q^2 \phi^2)^{3/2}} \quad (65)$$

It is evident from (38) and (62) that the integrand in (64) is bounded. Application of formulas 200.03. and 200.01. of [12] to (65) gives

$$I_b = \frac{1}{k^3 \rho_q^3} \log \left[\frac{2\rho_q \pi}{\Delta_q} + \sqrt{1 + \left(\frac{2\rho_q \pi}{\Delta_q} \right)^2} \right] \quad (66)$$

Integration with respect to u of the second term in (64) by means of formula 200.G3. of [12] gives

$$I_a = 2 \int_0^\pi H_1(\phi) \sin^2(\phi/2) \cos(n\phi) d\phi - \frac{1}{k^3 \rho_q^3} \int_0^\pi \frac{d\phi}{\sqrt{\phi^2 + \left(\frac{\Delta_q}{2\rho_q} \right)^2}} \quad (67)$$

where $H_1(\phi)$ is given by (41). The first integral in (67) is the right-hand side of (40a) for $\alpha = 1$. Application of method 1 to this integral and to those in (40) and (41) for G_{12} , G_{13} , G_{32} , and G_{33} , evaluation of the second integral with respect to ϕ in (67) by means of Gaussian quadrature, and use of (63) and (66) give

$$G_{11} = \pi \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} H_1(\phi_\ell) \sin^2\left(\frac{1}{2} \phi_\ell\right) \cos(n\phi_\ell) - \frac{\pi}{2k^3 \rho_q^3} \sum_{\ell=1}^{n_\phi} \frac{A_\ell^{(n_\phi)}}{\sqrt{\phi_\ell^2 + \left(\frac{\Delta_q}{2\rho_q} \right)^2}} + \frac{1}{k^3 \rho_q^3} \log \left[\frac{2\rho_q \pi}{\Delta_q} + \sqrt{1 + \left(\frac{2\rho_q \pi}{\Delta_q} \right)^2} \right] \quad (68a)$$

$$G_{1\alpha} = \pi \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} H_\alpha(\phi_\ell) \sin^2\left(\frac{1}{2} \phi_\ell\right) \cos(n\phi_\ell) \quad (68b)$$

$$G_{3\alpha} = \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} H_\alpha(\phi_\ell) \sin \phi_\ell \sin(n\phi_\ell) \quad (68c)$$

$\alpha = 2, 3$

where $H_\alpha(\phi_\ell)$ is given by either (44) or (49) depending on whether (48) is true.

Equations (68b) and (68c) and the first sum in (68a) were taken directly from (50). The rest of the right-hand side of (68a) is due to

the manipulations (63)-(67). The numerical integration with respect to ϕ required to obtain (68a) is of the bounded integrand in (64) rather than of the unbounded integrand in (40a). Hence, the portion of (68a) in excess of (50a) fortifies the numerical integration with respect to ϕ . Having been obtained through analytical integration with respect to u , this portion of (68a) does not affect the accuracy of the numerical integration with respect to u .

IV. EVALUATION OF THE PLANE WAVE EXCITATION VECTOR

The column vector which appears on the right-hand side of (10) and whose elements are given by (11) is called the excitation vector. In this section, the elements (11) are evaluated for the case in which $\underline{H}^i$ is the magnetic field of either a θ polarized or a ϕ polarized incident plane wave.

Consider a θ polarized incident plane wave defined by

$$\underline{E}^{i\theta} = \underline{u}_\theta^t k \eta e^{-j\underline{k}_t \cdot \underline{r}} \quad (69a)$$

$$\underline{H}^{i\theta} = -\underline{u}_\phi^t k e^{-j\underline{k}_t \cdot \underline{r}} \quad (69b)$$

and also a ϕ polarized incident plane wave defined by

$$\underline{E}^{i\phi} = \underline{u}_\phi^t k \eta e^{-j\underline{k}_t \cdot \underline{r}} \quad (70a)$$

$$\underline{H}^{i\phi} = \underline{u}_\theta^t k e^{-j\underline{k}_t \cdot \underline{r}} \quad (70b)$$

where k is the propagation constant, η is the intrinsic impedance of space, $\underline{r}$ is the radius vector from the origin, and

$$\begin{aligned} \underline{k}_t &= -\underline{u}_x \sin \theta_t - \underline{u}_z \cos \theta_t \\ \underline{u}_\theta^t &= \underline{u}_x \cos \theta_t - \underline{u}_z \sin \theta_t \\ \underline{u}_\phi^t &= \underline{u}_y \end{aligned} \quad (71)$$

where $\underline{u}_x$, $\underline{u}_y$, and $\underline{u}_z$ are unit vectors in the x, y, and z directions, respectively. In (71), the letter t stands for transmitter. The origin is on the axis of the body of revolution and is in the vicinity of the body of revolution although not necessarily centered inside it. In (69) and (70), the $\underline{E}$'s are electric fields and the $\underline{H}$'s are magnetic fields.

Each of the incident plane waves (69) and (70) comes from the direction for which $\theta = \theta_t$ and $\phi = 0$. Here, θ and ϕ are standard spherical coordinates. θ is measured from the positive z axis. ϕ is the angle that the projection onto the xy plane makes with the positive x axis. At the aspect ($\theta = \theta_t$, $\phi = 0$), the unit vectors $\underline{u}_\theta^t$ and $\underline{u}_\phi^t$ reduce to the unit vectors in the θ and ϕ directions, respectively. Because of the circular symmetry of the body of revolution, no generality is lost by confining $\underline{k}_t$ to the xz plane. If $\underline{k}_t$ were rotated through an angle ϕ_t from $\phi = 0$ to $\phi = \phi_t$, the response would also rotate through the same angle ϕ_t .

In view of (70a), substitution of (69b) for $\underline{H}^1$ in (11) gives

$$I_{ni}^{it\theta} = \frac{-1}{\eta} \int dt T_i(t) \int_0^{2\pi} d\phi (\underline{u}_\phi \cdot \underline{E}^{i\phi}) e^{-jn\phi} \quad (72a)$$

$$I_{ni}^{i\phi\theta} = \frac{1}{\eta\rho_i} \int dt \rho P_i(t) \int_0^{2\pi} d\phi (\underline{u}_t \cdot \underline{E}^{i\phi}) e^{-jn\phi} \quad (72b)$$

The additional superscript θ on the left-hand sides of (72) indicates the θ polarized incident plane wave. In view of (69a), substitution of (70b) for $\underline{H}^1$ in (11) gives

$$I_{ni}^{it\phi} = \frac{1}{\eta} \int dt T_i(t) \int_0^{2\pi} d\phi (\underline{u}_\phi \cdot \underline{E}^{i\theta}) e^{-jn\phi} \quad (73a)$$

$$I_{ni}^{i\phi\phi} = \frac{-1}{\eta\rho_i} \int dt \rho P_i(t) \int_0^{2\pi} d\phi (\underline{u}_t \cdot \underline{E}^{i\theta}) e^{-jn\phi} \quad (73b)$$

The third superscript on the left-hand sides of (73) indicates the ϕ polarized incident plane wave.

Substitution of (9) for the $\underline{W}$'s and (69a) for $\underline{E}^1$ in (7) and (8) of [1] reveals that the elements $V_{ni}^{t\theta}$ and $V_{ni}^{\phi\theta}$ considered in Section IV of [1] are given by

$$V_{ni}^{t\theta} = \frac{1}{\eta} \int dt T_i(t) \int_0^{2\pi} d\phi (\underline{u}_t \cdot \underline{E}^{1\theta}) e^{-jn\phi} \quad (74a)$$

$$V_{ni}^{\phi\theta} = \frac{1}{\eta\rho_i} \int dt \rho P_i(t) \int_0^{2\pi} d\phi (\underline{u}_\phi \cdot \underline{E}^{1\theta}) e^{-jn\phi} \quad (74b)$$

Substitution of (9) for the $\underline{W}$'s and (70a) for $\underline{E}^1$ in (7) and (8) of [1] gives

$$V_{ni}^{t\phi} = \frac{1}{\eta} \int dt T_i(t) \int_0^{2\pi} d\phi (\underline{u}_t \cdot \underline{E}^{1\phi}) e^{-jn\phi} \quad (75a)$$

$$V_{ni}^{\phi\phi} = \frac{1}{\eta\rho_i} \int dt \rho P_i(t) \int_0^{2\pi} d\phi (\underline{u}_\phi \cdot \underline{E}^{1\phi}) e^{-jn\phi} \quad (75b)$$

Comparison of (72) with (75) shows that if $T_i(t)$ and $(\rho P_i(t))/\rho_i$ were interchanged in (75), then

$$I_{ni}^{it\theta} = - V_{ni}^{\phi\theta} \quad (76a)$$

$$I_{ni}^{i\phi\theta} = V_{ni}^{t\phi} \quad (76b)$$

Comparison of (73) with (74) shows that if $T_i(t)$ and $(\rho P_i(t))/\rho_i$ were interchanged in (74), then

$$I_{ni}^{it\phi} = V_{ni}^{\phi\theta} \quad (76c)$$

$$I_{ni}^{i\phi\phi} = - V_{ni}^{t\theta} \quad (76d)$$

Now, $T_i(t)$ is responsible for both the asterisk on the left-hand sides of (120) and (122) of [1] and the factor

$$\frac{1}{2} \left(1 + \frac{(-1)^{p-1} 2(t-t_p)}{\Delta_p} \right)$$

on the right-hand sides of (120) and (122) of [1]. Also, $(\rho P_1(t))/\rho_1$ appears as the factor

$$1 + \frac{(t-t_p) \sin v_p}{\rho_p}$$

in (121) and (123) of [1]. Hence,

$$I_{ni}^{*it\theta} = -\frac{j^{n+1}\pi k}{2} \int_{t_p}^{t_{p+1}} \left(1 + \frac{(-1)^{p-1} 2(t-t_p)}{\Delta_p} \right) (J_{n+1} - J_{n-1}) e^{jkz \cos \theta_t} dt \quad (77a)$$

$$I_{np}^{i\phi\theta} = -j^n \pi k \int_{t_p}^{t_{p+1}} \left(1 + \frac{(t-t_p) \sin v_p}{\rho_p} \right) (J_{n+1} + J_{n-1}) \sin v_p e^{jkz \cos \theta_t} dt \quad (77b)$$

$$I_{ni}^{*it\phi} = \frac{j^n \pi k}{2} \int_{t_p}^{t_{p+1}} \left(1 + \frac{(-1)^{p-1} 2(t-t_p)}{\Delta_p} \right) (J_{n+1} + J_{n-1}) \cos \theta_t e^{jkz \cos \theta_t} dt \quad (77c)$$

$$I_{np}^{i\phi\phi} = -j^n \pi k \int_{t_p}^{t_{p+1}} \left(1 + \frac{(t-t_p) \sin v_p}{\rho_p} \right) [j \sin v_p \cos \theta_t (J_{n+1} - J_{n-1}) - 2 \cos v_p \sin \theta_t J_n] e^{jkz \cos \theta_t} dt \quad (77d)$$

where J_n , ρ , and z are given by (115), (130) and (131) of [1]. The asterisk on the left-hand sides of (77a) and (77c) indicates that the right-hand sides of (77a) and (77c) are not all of $I_{ni}^{*it\theta}$ and $I_{ni}^{*it\phi}$ but only the contributions due to the region of integration for which $t_p \leq t \leq t_{p+1}$. Both (77a) and (77c) are valid for $i = p-1$ and $i = p$.

Equation (77) can be rewritten as

$$I_{ni}^{*it\theta} = - \frac{j^{n+1} \pi k \Delta_p}{4} (F_{n+1,a} - F_{n-1,a}) - \frac{(-1)^{p-i} j^{n+1} \pi k \Delta_p}{4} (F_{n+1,b} - F_{n-1,b}) \quad (78a)$$

$$I_{np}^{i\phi\theta} = - \frac{j^n \pi k \Delta_p \sin v}{2} [F_{n+1,a} + F_{n-1,a} + \frac{\Delta_p \sin v}{2\rho_p} (F_{n+1,b} + F_{n-1,b})] \quad (78b)$$

$$I_{ni}^{*it\phi} = \frac{j^n \pi k \Delta_p \cos \theta_t}{4} (F_{n+1,a} + F_{n-1,a}) + \frac{(-1)^{p-i} j^n \pi k \Delta_p \cos \theta_t}{4} (F_{n+1,b} + F_{n-1,b}) \quad (78c)$$

$$I_{np}^{i\phi\phi} = - \frac{j^{n+1} \pi k \Delta_p \sin v \cos \theta_t}{2} [(F_{n+1,a} - F_{n-1,a}) + \frac{\Delta_p \sin v}{2\rho_p} (F_{n+1,b} - F_{n-1,b})] \\ + j^n \pi k \Delta_p \cos v \sin \theta_t [F_{na} + \frac{\Delta_p \sin v}{2\rho_p} F_{nb}] \quad (78d)$$

where the F's are given by (128) and (129) of [1]. The n_T point Gaussian quadrature formulas for them are

$$F_{ma} = \left. \sum_{\ell=1}^{n_T} A_{\ell}^{(n_T)} J_m(k\hat{\rho}_{\ell} \sin \theta_t) e^{jk\hat{z}_{\ell} \cos \theta_t} \right\} \quad (79a)$$

$$F_{mb} = \left. \sum_{\ell=1}^{n_T} A_{\ell}^{(n_T)} x_{\ell}^{(n_T)} J_m(k\hat{\rho}_{\ell} \sin \theta_t) e^{jk\hat{z}_{\ell} \cos \theta_t} \right\} \quad (79b)$$

where J_m is the Bessel function of the first kind and where $\hat{\rho}_{\ell}$ and $\hat{z}_{\ell}$ are given by (134) and (135) of [1]. The abscissas $x_{\ell}^{(n_T)}$ and weights $A_{\ell}^{(n_T)}$ are tabulated in Appendix A of [10] for several values of n_T .

V. NUMERICAL RESULTS

Computed results for the electric currents induced by an axially incident wave on a sphere, a cone-sphere, and a finite cylinder are presented in Figs. 1-14. For axial incidence, θ_t is either 0° or 180° and the only non-zero excitation vectors for the θ polarized plane wave (69) are

$$\begin{bmatrix} \hat{I}_{-1}^t \\ \hat{I}_{-1}^\phi \end{bmatrix} = \begin{bmatrix} \hat{I}_1^t \\ -\hat{I}_1^\phi \end{bmatrix} \quad (80)$$

It is evident from (12) and (13) that

$$\begin{bmatrix} Y_{-1}^{tt} & Y_{-1}^{t\phi} \\ Y_{-1}^{\phi t} & Y_{-1}^{\phi\phi} \end{bmatrix} = \begin{bmatrix} Y_1^{tt} & -Y_1^{t\phi} \\ -Y_1^{\phi t} & Y_1^{\phi\phi} \end{bmatrix} \quad (81)$$

In consequence of (80), (81), and (10), the only non-zero column vectors $\hat{I}_n^t$ and $\hat{I}_n^\phi$ are given by

$$\begin{bmatrix} \hat{I}_{-1}^t \\ -\hat{I}_{-1}^\phi \end{bmatrix} = \begin{bmatrix} \hat{I}_1^t \\ \hat{I}_1^\phi \end{bmatrix} \quad (82)$$

where the column vector on the right-hand side of (82) satisfies (10) for $n = 1$.

In view of (3), substitution of (82) into (2) and subsequent division by k give

$$\frac{\underline{J}}{|\underline{H}^1|} = 2 \underline{u}_t \cos \phi \left(\sum_j \hat{I}_{1j}^t \frac{T_j(t)}{k\rho_j} \right) + 2j \underline{u}_\phi \sin \phi \left(\sum_j \hat{I}_{1j}^\phi \frac{P_j(t)}{k\rho_j} \right) \quad (83)$$

The $|\underline{H}^1|$ written instead of k on the left-hand side of (83) is the

magnitude of the incident magnetic field. This magnitude is indeed equal to k . At $t = t_{p+1}^-$, the t component of (83) reduces to

$$\frac{J_t}{|H^i|} = \frac{2I_{1p}^t}{k\rho(t_{p+1}^-)} \cos \phi, \quad p=1, 2, \dots, P-2 \quad (84)$$

where $\rho(t_{p+1}^-)$ is the value of ρ at $t = t_{p+1}^-$. At $t = t_p$, the ϕ component of (83) reduces to

$$\frac{J_\phi}{|H^i|} = \frac{2jI_{1p}^\phi \sin \phi}{k\rho_p}, \quad p = 1, 2, \dots, P-1 \quad (85)$$

Here, J_t and J_ϕ are, respectively, the t and ϕ components of the

electric current $\underline{J}$. In Figs. 1-14, $\frac{|J_t|}{|H^i|}$ in the $\phi = 0^\circ$ plane is plotted with squares and $\frac{|J_\phi|}{|H^i|}$ in the $\phi = 90^\circ$ plane is plotted with octagons.

These currents are plotted versus t/λ where t is the arc length along the generating curve and λ is the wavelength. The horizontal axes in Figs. 1-14 were labeled T/λ because the lower case letter t could not be drawn by the plotter.

Figures 1-4 display computed values of electric current induced on a conducting sphere of radius $.2\lambda$ illuminated by a plane wave. Figure 1 shows the H-field solution of [2], Fig. 2 shows the present H-field solution, Fig. 3 shows the E-field solution of [2], and Fig. 4 shows the E-field solution of [1]. The squares and octagons represent computed values of $\frac{|J_t|}{|H^i|}$ and $\frac{|J_\phi|}{|H^i|}$, respectively. The solid curves represent the Mie series solution [13]. In Figs. 1 and 3, the squares and octagons are horizontally located at the peaks of the triangles used [2]. In Figs. 2 and 4, the squares are placed at the peaks of the triangles (5) and the octagons are placed at the centers of the pulses (6). That is why the squares and the

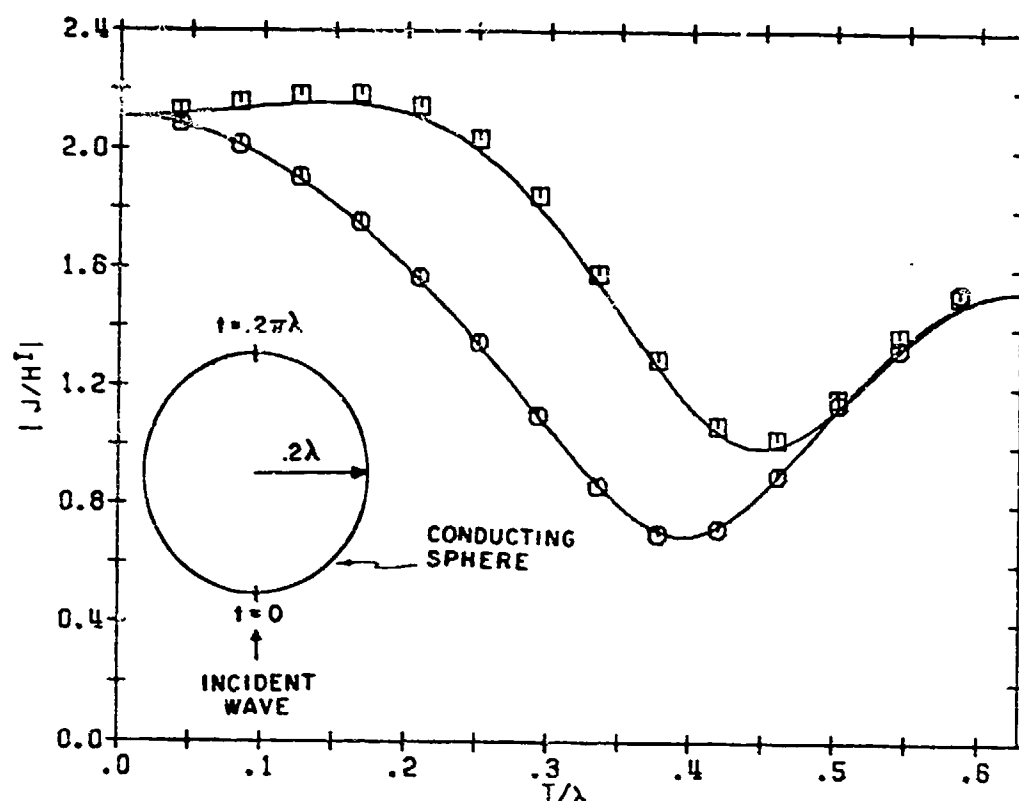


Fig. 1. Electric current on a conducting sphere of radius 0.2λ . The squares and octagons represent the H-field solution of [2]. The solid curves represent the Mie series solution.

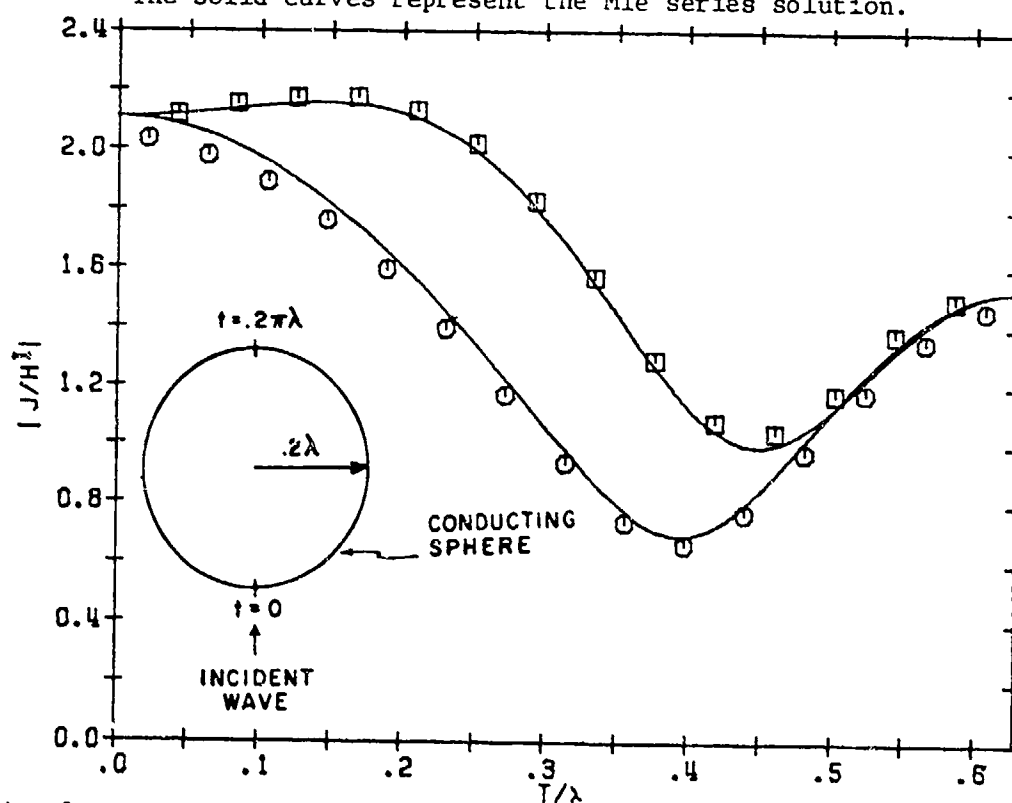


Fig. 2. Electric current on a conducting sphere of radius 0.2λ . The squares and octagons represent the present H-field solution. The solid curves represent the Mie series solution.

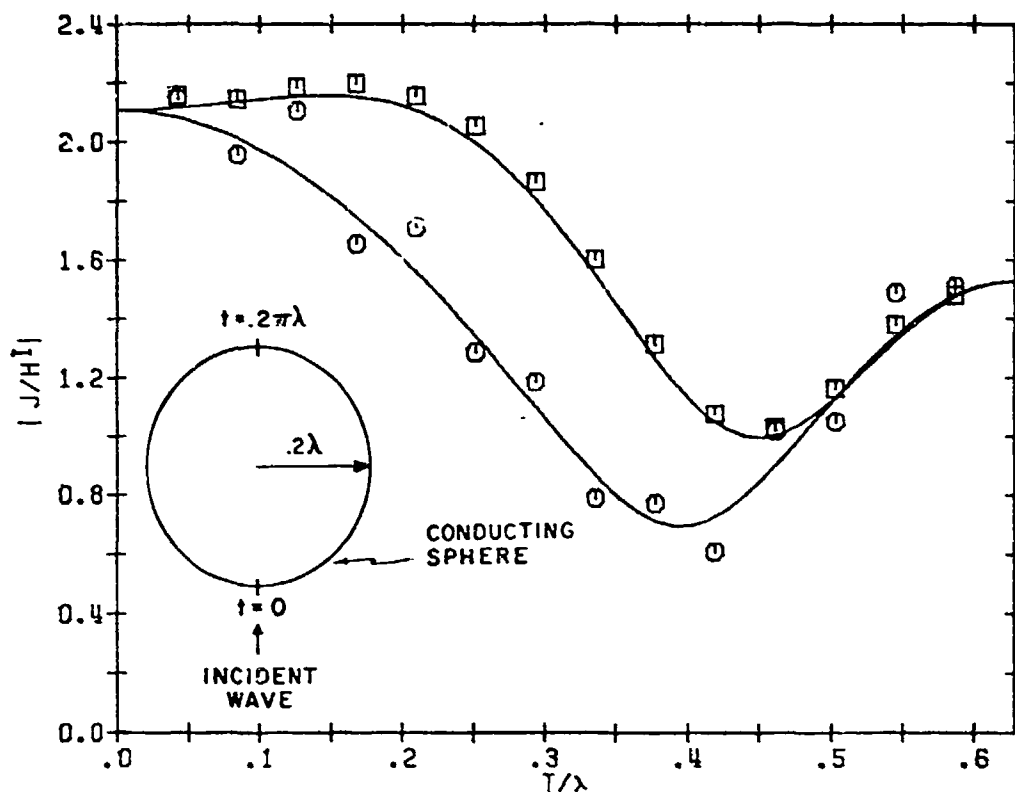


Fig. 3. Electric current on a conducting sphere of radius 0.2λ . The squares and octagons represent the E-field solution of [2]. The solid curves represent the Mie series solution.

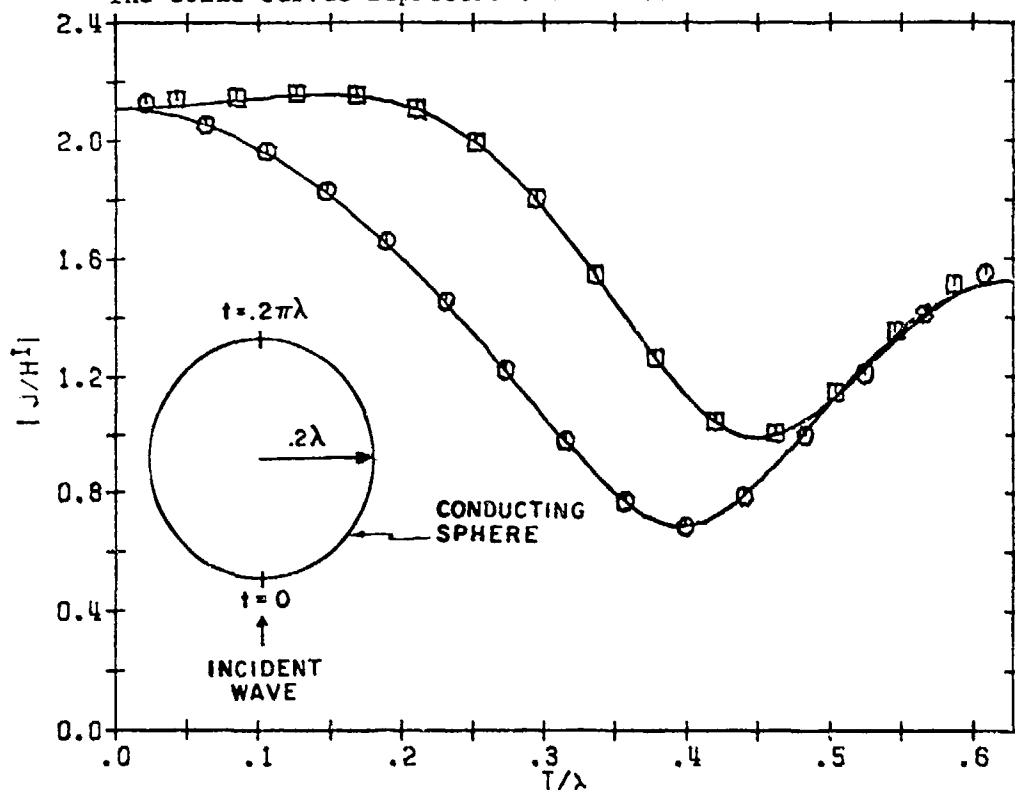


Fig. 4. Electric current on a conducting sphere of radius 0.2λ . The squares and octagons represent the E-field solution of [1]. The solid curves represent the Mie series solution.

octagons line up with each other in Figs. 1 and 3 but are staggered in Figs. 2 and 4. The oscillations of the octagons about the solid curve in Fig. 3 are not surprising because the particular size of the sphere places the electric current error in Fig. 5 on page 32 of [2] near one of its peaks.

Figures 5-10 display computed values of electric current induced on a cone-sphere by plane waves incident from both axial directions. The squares represent $\frac{|J_t|}{|H^i|}$ and the octagons represent $\frac{|J_\phi|}{|H^i|}$. These symbols are connected by straight lines in order to improve readability. The currents in Figs. 5 and 6 agree reasonably well with those in Fig. 7 of [1]. Likewise, the currents in Figs. 8 and 9 agree reasonably well with those in Fig. 8 of [1]. The jagged nature of the ϕ components of current in Figs. 7 and 10 is probably spurious.

Finally, Figs. 11-14 display computed values of the electric currents induced on a closed cylinder of length $.5\lambda$ and radius $.25\lambda$ by an axially incident plane wave.

The currents in Figs. 2,4,6,9,12, and 14 were calculated with

$$\left. \begin{array}{l} n_t = n_T = 2 \\ n_\phi = 20 \end{array} \right\}$$

The currents in the rest of the figures were calculated with $n_\phi = 20$ in [2]. The IBM System 370/155 under WATFIV took 29 seconds of execution time to calculate the H-field solution in Fig. 6. The subroutine YMAT was used for this calculation. The joint calculation of the H-field solution in Fig. 6 and the E-field solution of [1] in Fig. 7 of [1] took 45 seconds of execution time. The subroutine YZ was used for this calculation. The joint calculation of the H-field solution of [2] in Fig. 5 and the E-field solution of [2] in Fig. 7 took 39 seconds of execution time using the subroutine YZ of [14]. Hence, the present H-field solution and the E-field solution of [1] are slightly slower than the H-field and E-field solutions in [2].

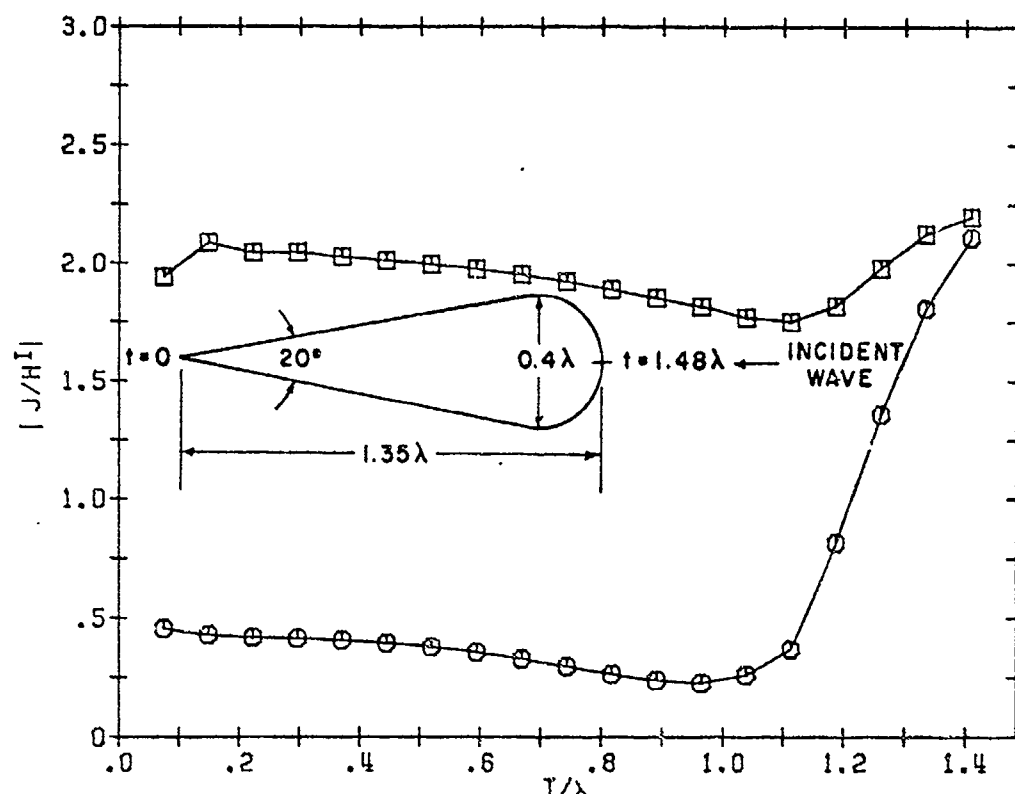


Fig. 5. Electric current induced on a cone-sphere by an axially incident plane wave. Incidence on sphere, H-field solution of [2].

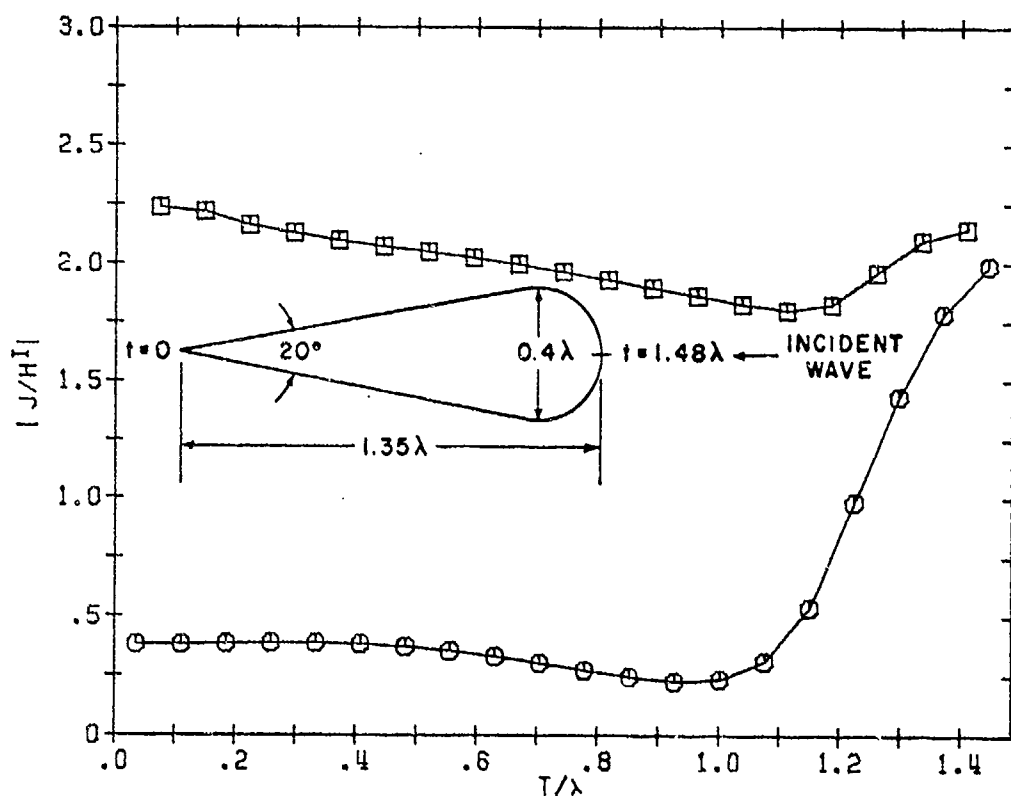


Fig. 6. Electric current induced on a cone-sphere by an axially incident plane wave. Incidence on sphere, present H-field solution.

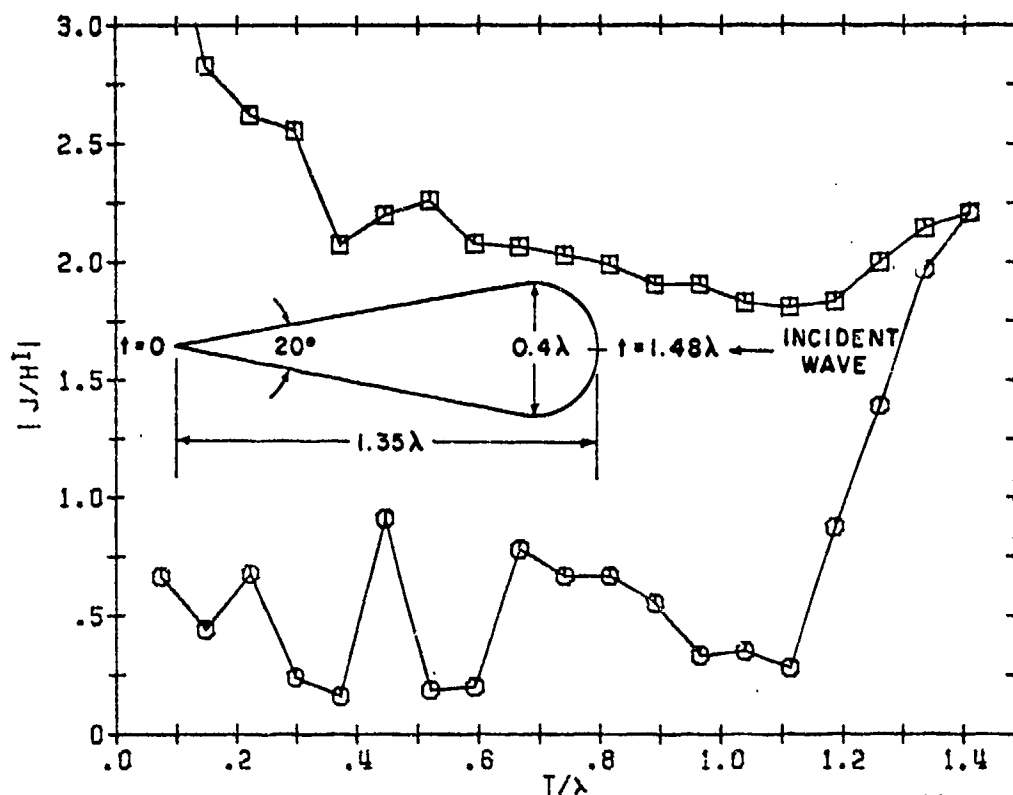


Fig. 7. Electric current induced on a cone-sphere by an axially incident plane wave. Incidence on sphere, E-field solution of [2].

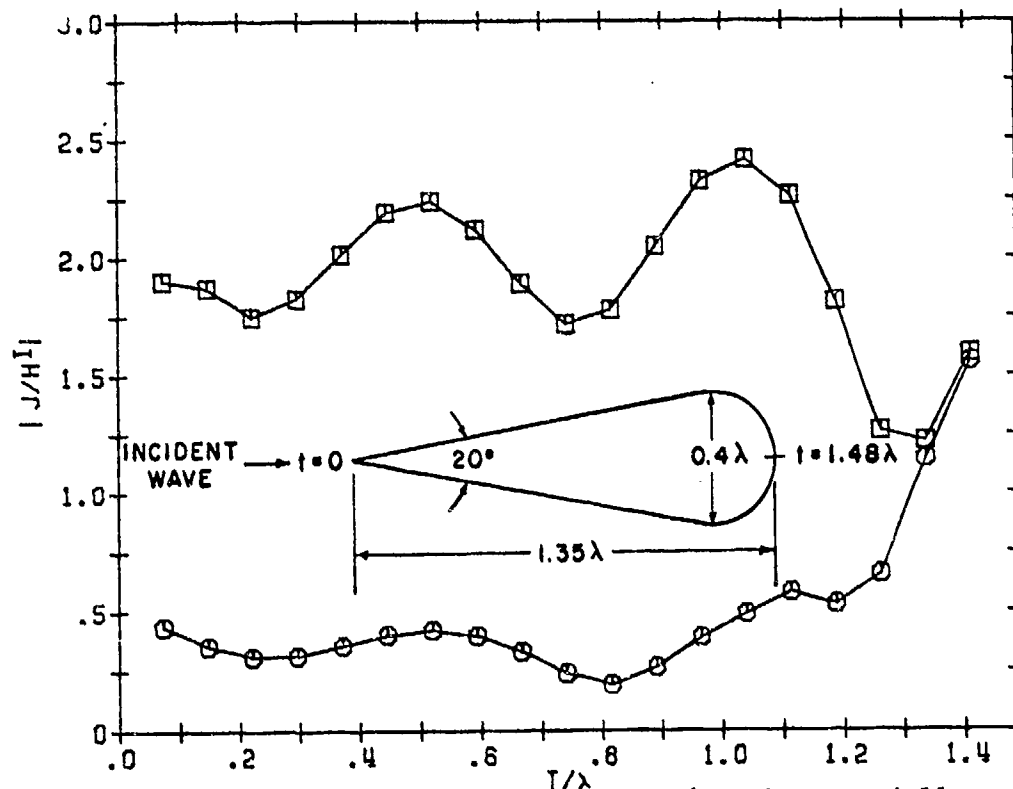


Fig. 8. Electric current induced on a cone-sphere by an axially incident plane wave. Incidence on tip, H-field solution of [2].

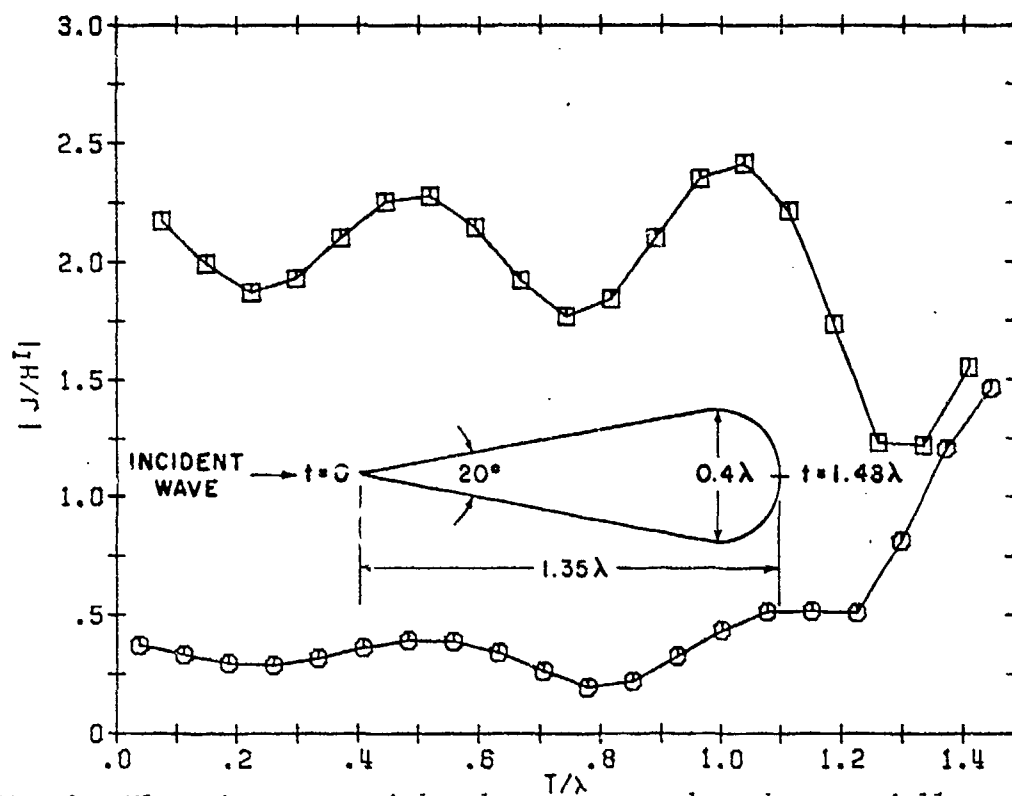


Fig. 9. Electric current induced on a cone-sphere by an axially incident plane wave. Incidence on tip, present H-field solution.

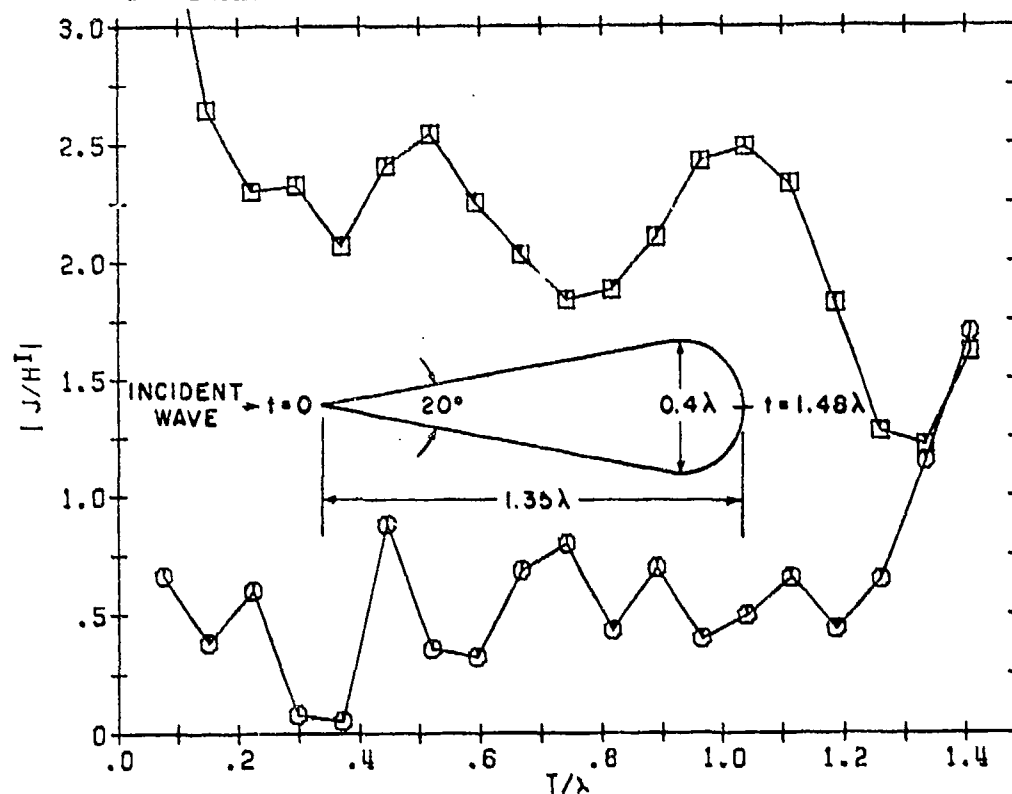


Fig. 10. Electric current induced on a cone-sphere by an axially incident plane wave. Incidence on tip, E-field solution of [2].

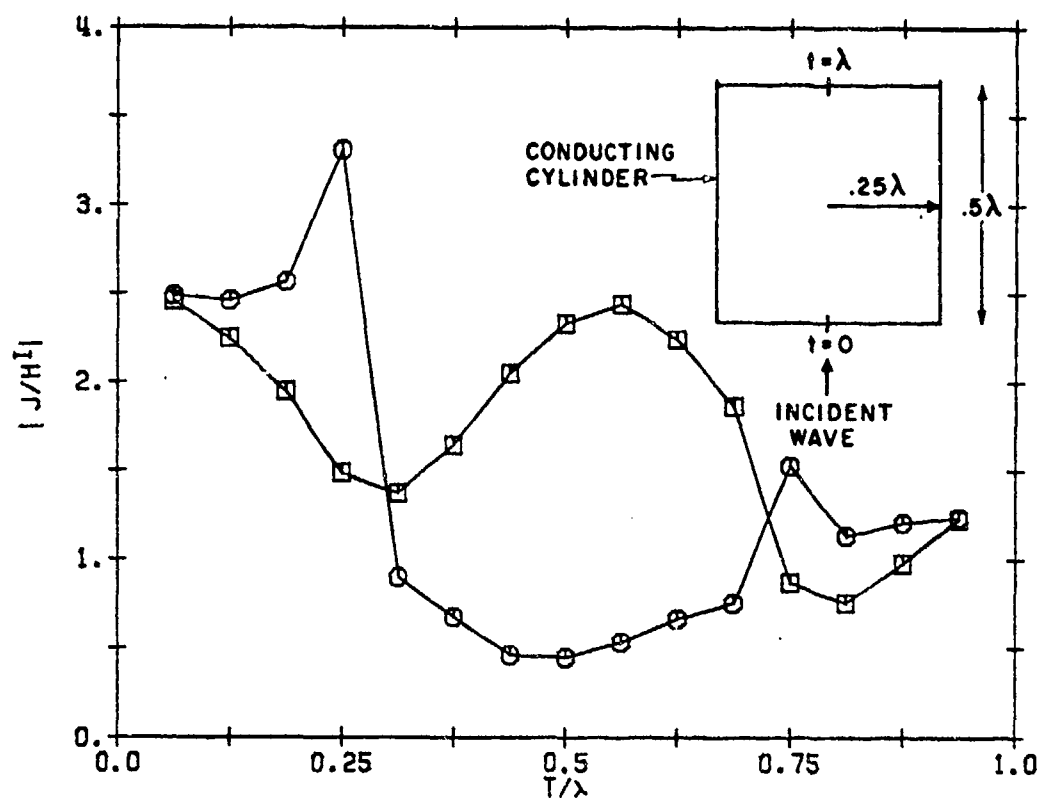


Fig. 11. Electric current induced on a closed cylinder of length 0.5λ and radius 0.25λ by an axially incident plane wave. H-field solution of [2].

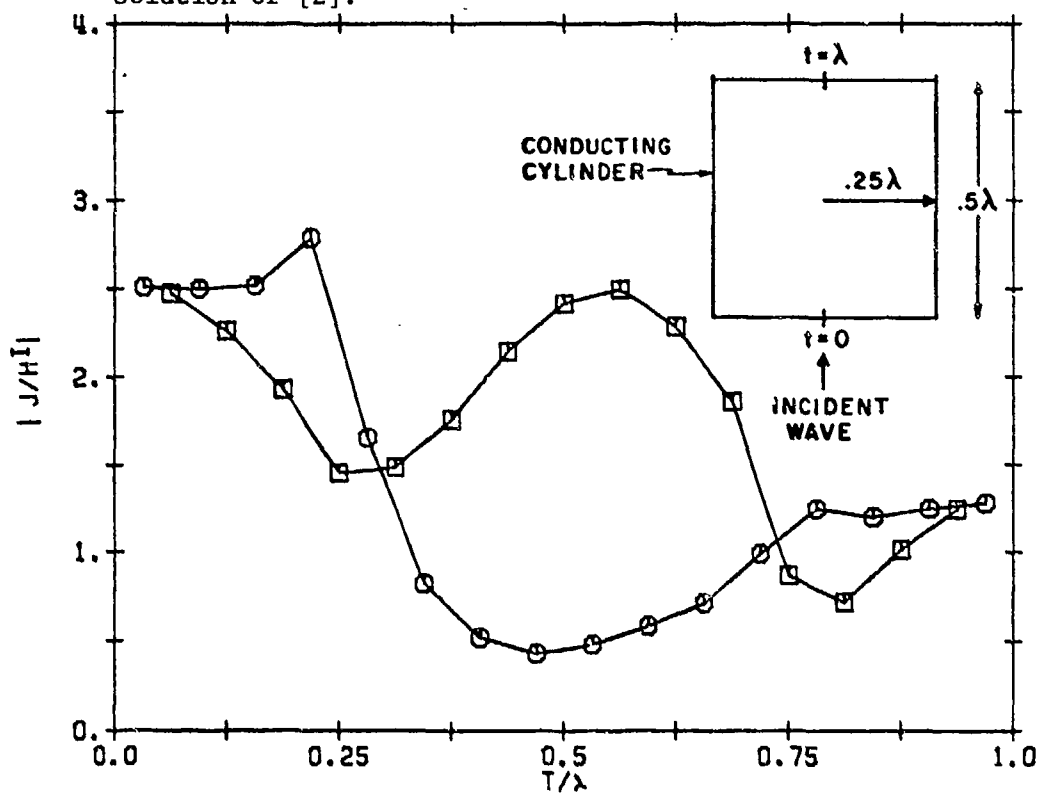


Fig. 12. Electric current induced on a closed cylinder of length 0.5λ and radius 0.25λ by an axially incident plane wave. Present H-field solution.

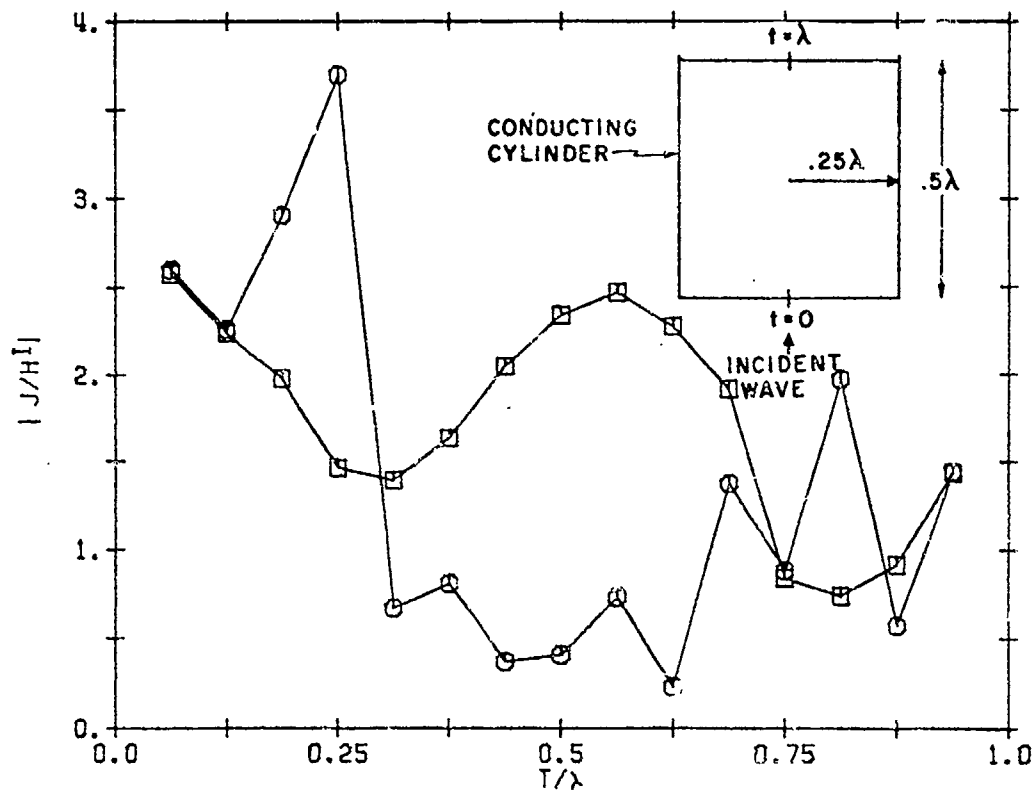


Fig. 13. Electric current induced on a closed cylinder of length 0.5λ and radius 0.25λ by an axially incident plane wave. E-field solution of [2].

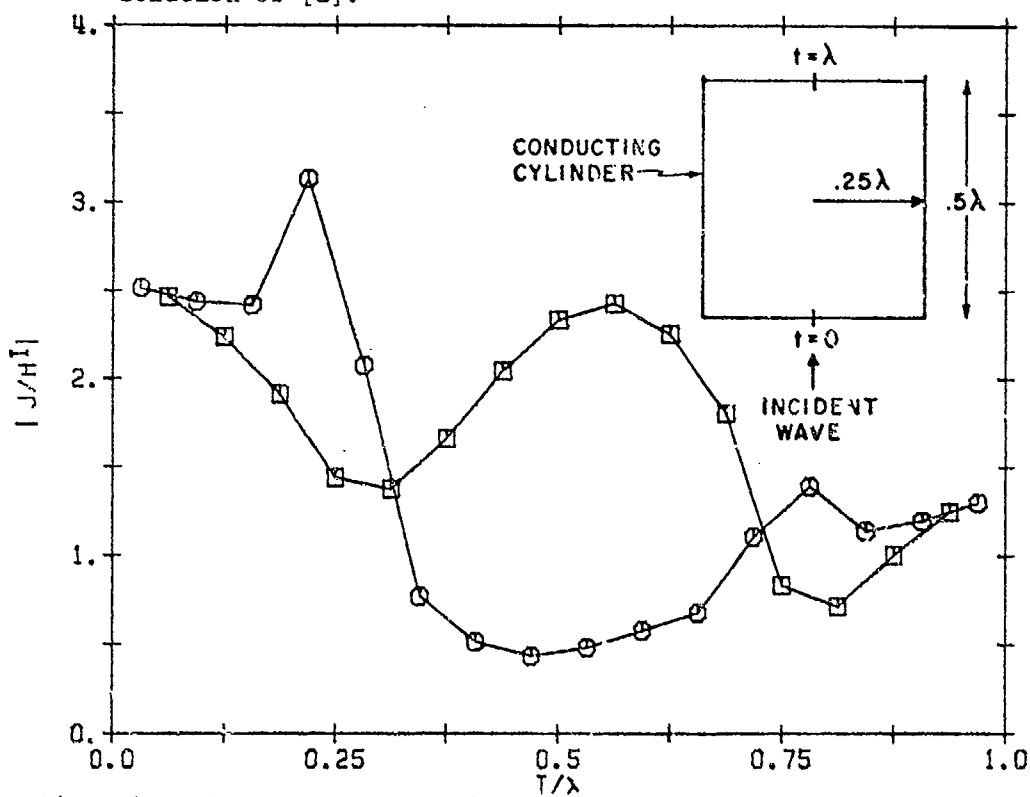


Fig. 14. Electric current induced on a closed cylinder of length 0.5λ and radius 0.25λ by an axially incident plane wave. E-field solution of [1].

The results of calculation of the present H-field solution and E-field solution of [1] for the sphere, cone-sphere and finite cylinder examples with

$$\left. \begin{array}{l} n_t = n_r = 4 \\ n_\phi = 48 \end{array} \right\}$$

were so close to the results in Figs. 2,4,6,9,12, and 14 that distinction was impossible.

PART TWO

COMPUTER PROGRAMS

I. INTRODUCTION

A computer program which implements the present H-field solution is described and listed in Part Two. This program consists of the subroutine YMAT, the function BLOG, the subroutines PLANE, DECOMP, and SOLVE, and the main program for the H-field solution. The subroutine YMAT calculates the elements (12) of the moment matrix in (10). The function BLOG is called by YMAT. The subroutine PLANE uses (78) to calculate the excitation vector on the right-hand side of (10) for a θ polarized incident plane wave. The subroutine PLANE also calculates the excitation vector for a ϕ polarized plane wave, but this vector is not used in the main program. The subroutines DECOMP and SOLVE solve the matrix equation (10) for $\vec{I}_n^t$ and $\vec{I}_n^\phi$. The main program for the H-field solution obtains the electric current induced on a conducting body of revolution by an axially incident plane wave. The subroutines YMAT and PLANE are designed not only for axial incidence but also for oblique incidence. For axial incidence, $n = \pm 1$ and θ_t of (71) is either 0° or 180° . The subroutines YMAT and PLANE admit $n = M1, M1+1, \dots, M2$ where $M2 \geq M1 \geq 0$. The subroutine PLANE also admits arbitrary values of θ_t . Formulas on pages (28) and (29) of [2] obviate calculation of moment matrices and plane wave excitation vectors for negative values of n .

A computer program which implements both the present H-field solution and the E-field solution of [1] is also described and listed in Part Two. This program consists of the previously mentioned subprograms BLOG, PLANE, DECOMP, SOLVE, a new subroutine YZ, and the main program for both the H-field and E-field solutions. The subroutine YZ calculates the moment matrices in (10) of the present report and in (6) of [1]. The subroutine YZ calls the function BLOG. The subroutine PLANE calculates the excitation vector on the right-hand side of (10) for the θ polarized plane wave. PLANE also calculates the elements

$V_{n1}^{t\theta}$ and $-V_{n1}^{\phi\theta}$ of (74) for use on the right-hand side of (6) of [1]. In addition, PLANE calculates excitation vectors for the ϕ polarized plane wave but these vectors are not used in the main program. The subroutines DECOMP and SOLVE solve the matrix equations (10) of the present report and (6) of [1]. The main program for both the H-field and E-field solutions obtains both the present H-field solution and the E-field solution of [1] for the electric current induced on a conducting body of revolution immersed in an axially incident plane wave.

II. THE SUBROUTINE YMAT

With regard to (10), the subroutine YMAT(M1, M2, NP, NPHI, NT, IN, RH, ZH, X, A, XT, AT, Y) puts in Y the moment matrices Y_n defined by

$$Y_n = \begin{bmatrix} Y_n^{tt} & Y_n^{t\phi} \\ Y_n^{\phi t} & Y_n^{\phi\phi} \end{bmatrix}, \quad n = M1, M1+1, \dots, M2 \quad (86)$$

Storage in Y is such that the i th column of Y_n goes from $Y((i-1)*N + (n-M1)*N*N+1)$ to $Y(i*N+(n-M1)*N*N)$ where

$$N = 2*NP - 3 \quad (87)$$

The only output argument of YMAT is Y. The rest of the arguments of YMAT are input arguments. The input argument IN generalizes YMAT for use with problems other than conducting body problems. The term $(\pi\Delta_q)/\rho_q$ in (22d) and the quantities $\hat{Y}_n^{tt}$ in (23) are multiplied by IN in the subroutine YMAT. Thus, the argument IN should be 1 for the present H-field solution. In this solution, the magnetic field is evaluated just inside S. Magnetic field evaluation just outside S can be obtained by setting IN = -1.

Each of the input arguments except M1, M2, and IN represents a variable in Part One of the text. The correspondence is tabulated as

Input Argument	Text Variable	Equation Number
NP	P	3
NPHI	n_ϕ	36
NT	n_t	35
RH	$k\rho(t_j^-)$, $j = 1, 2, \dots, P$	21
ZH	$kz(t_j^-)$, $j = 1, 2, \dots, P$	21
X	$x_\ell^{(n_\phi)}$, $\ell = 1, 2, \dots, n_\phi$	37
A	$A_\ell^{(n_\phi)}$, $\ell = 1, 2, \dots, n_\phi$	36
XT	$x_{\ell'}^{(n_t)}$, $\ell' = 1, 2, \dots, n_t$	35
AT	$A_{\ell'}^{(n_t)}$, $\ell' = 1, 2, \dots, n_t$	35

Here, $\rho(t_j^-)$ and $z(t_j^-)$ are the values of ρ and z at $t=t_j^-$ on the generating curve and k is the propagation constant. The generating curve starts at $t = t_1^-$ and ends at $t = t_P^-$. The input argument P must be a positive integer not less than 3. The input arguments NPHI, NT, X, A, XT, and AT are Gaussian quadrature data. Lines 11 and 12 put c_t and c_ϕ of (31) in CT and CP, respectively. The subroutine YMAT calls the function BLOG which is described and listed in Section III of Part Two.

Minimum allocations are given by

```
COMPLEX Y(M3*N*N), GA(NPHI), GB(NPHI), GC(NPHI),
      G1A(M3), G2A(M3), G3A(M3), G1B(M3), G2B(M3),
      G3B(M3), G1C(M3), G2C(M3), G3C(M3)
```

```
DIMENSION RH(NP), ZH(NP), X(NPHI), A(NPHI),
      XT(NT), AT(NT), RS(NP-1), ZS(NP-1), D(NP-1),
      DR(NP-1), DZ(NP-1), C2(NPHI), C3(NPHI),
      C4(M3*NPHI), C5(M3*NPHI), C6(M3*NPHI),
      R2(NT), Z2(NT), R7(NT), Z7(NT)
```

where N is given by (87) and

$$M3 = M2 - M1 + 1 \quad (88)$$

The elements of Y_n are calculated from (23) and (24) with $G_{m\alpha}$ evaluated as in Section III of Part One. DO loop 11 puts ϕ_ℓ^2 in C2 and $4\sin^2(\frac{1}{2}\phi_\ell)$ in C3. ϕ_ℓ is given by (37) and $4\sin^2(\frac{1}{2}\phi_\ell)$ is needed in order to evaluate (15) at $\phi = \phi_\ell$. With regard to (36), inner DO loop 29 puts

$$\pi A_\ell^{(n_\phi)} \sin^2(\frac{1}{2}\phi_\ell) \cos(n\phi_\ell) \quad \text{in C4,}$$

$$\frac{\pi}{2} A_\ell^{(n_\phi)} \cos \phi_\ell \cos(n\phi_\ell) \quad \text{in C5,}$$

and
$$\frac{\pi}{2} A_\ell^{(n_\phi)} \sin \phi_\ell \sin(n\phi_\ell) \quad \text{in C6.}$$

The index JQ of DO loop 15 obtains q in (23) and (24). With regard to (22d), line 69 puts $(IN) * (\pi \Delta_q) / \rho_q$ in P2. DO loop 12 puts $k\rho'$ and kz' of (29) in R2 and Z2, respectively. DO loop 12 also accumulates the products of IN with (23a), (23b), and (23c) in P1A, P1B, and P1C, respectively. The index IP of DO loop 16 obtains p in (24). Lines 106-112 put kd_0 of (32) in D6. If either case 1 of (30) or case 3 of (51) is obtained, branch statement 26 sends execution to statement 41. If either case 2 of (39) or case 4 of (60) is obtained, execution proceeds to line 114. Lines 114-235 calculate $G_{m\alpha}$ for case 2 and case 4. Lines 237-308 calculate $G_{m\alpha}$ for case 1 and case 3. Here, $G_{m\alpha}$ is defined by (25). In cases 2 and 4, the integration with respect to t' is done first. In cases 1 and 3, the integration with respect to ϕ is done first. In all cases,

$$\left. \begin{array}{l} G_{m1} \text{ is stored in } GmA(n-M1+1) \\ G_{m2} \text{ is stored in } GmB(n-M1+1) \\ G_{m3} \text{ is stored in } GmC(n-M1+1) \end{array} \right\} \begin{array}{l} m = 1, 2, 3 \\ n = M1, M1+1, \dots, M2 \end{array}$$

With regard to (41), DO loop 33 puts

$$\left. \begin{array}{l} H_1(\phi_K) \text{ in } GA(K) \\ H_2(\phi_K) \text{ in } GB(K) \\ H_3(\phi_K) \text{ in } GC(K) \end{array} \right\} \quad K = 1, 2, \dots, n_\phi$$

where ϕ_K is given by (37). Line 123 is based on (48). DO loop 35 accumulates the H's of (49) in H1A, H2A, and H3A. DO loop 37 accumulates in H1A, H2A, and H3A the sums on ℓ' in (44). If $kR \leq .5$, then the approximation

$$G - \frac{1}{(kR)^3} - \frac{1}{2kR} \approx -\frac{kR}{8} \left(1 - \frac{(kR)^2}{18} + \frac{(kR)^4}{720} \right) - \frac{j}{3} \left(1 - \frac{(kR)^2}{10} + \frac{(kR)^4}{280} \right) \quad (89)$$

is invoked in order to avoid excessive roundoff error. Lines 152 to 174 put the I's of (47) in W1, W2, and W3. I_1 and I_3 are even in t_o and I_2 is odd in t_o . These three I's are first calculated with t_o replaced by $|t_o|$ and then the sign of I_2 is changed if t_o is negative.

In line 161, d^2 is compared to $10^{-5} r_{pq}^2$. If $d^2 \leq 10^{-5} r_{pq}^2$, then little confidence can be placed in the calculated value of d^2 because r_{pq}^2 and t_o^2 are very close to each other in (46b). If

$$\left. \begin{array}{l} d^2 \leq 10^{-5} r_{pq}^2 \\ |t_o| < \frac{1}{2} \Delta_q \end{array} \right\}$$

then statement 52 stops execution because an accurate value of I_1 can not be obtained. However, if

$$\left. \begin{array}{l} d^2 \leq 10^{-5} r_{pq}^2 \\ |t_o| \geq \frac{1}{2} \Delta_q \end{array} \right\}$$

then accurate calculation of I_1 may be possible. If $|t_o| - \frac{1}{2} \Delta_q$ is

appreciably greater than d^2 , then the approximation

$$\left[\frac{w}{k^2 d^2 r} \right]_{|t_o| + \frac{1}{2} \Delta_q}^{|t_o| - \frac{1}{2} \Delta_q} \approx \frac{1}{2k^2} \left[\frac{1}{(|t_o| - \frac{1}{2} \Delta_q)^2} - \frac{1}{(|t_o| + \frac{1}{2} \Delta_q)^2} \right] \quad (90)$$

can be invoked. Line 163 puts the right-hand side of (90) in W4. Lines 166-169 calculate the logarithm term in (47a) according to

$$[\log(w+r)]_{w_1}^{w_2} = \begin{cases} \log \left[\frac{(w_2 + r_2)}{(w_1 + r_1)} \right] & , w_1 \geq 0 \\ \log \left[\frac{(w_2 + r_2)(-w_1 + r_1)}{d^2} \right] & , w_1 < 0 \end{cases} \quad (91)$$

where

$$\left. \begin{aligned} w_1 &= |t_o| - \frac{1}{2} \Delta_q \\ w_2 &= |t_o| + \frac{1}{2} \Delta_q \\ r_1 &= \sqrt{w_1^2 + d^2} \\ r_2 &= \sqrt{w_2^2 + d^2} \end{aligned} \right\} \quad (92)$$

This logarithm is stored in W.

Nested DO loops 45 and 46 calculate (50) for $n = M1 + M-1$ where M is the index of DO loop 45. The index K of DO loop 46 obtains ℓ in (50). With regard to (68a), lines 222-228 put

$$\frac{-\pi}{2k^3 \rho_q^3} \sum_{\ell=1}^{n_\phi} \frac{A_\ell^{(n_\phi)}}{\sqrt{\phi_\ell^2 + \left(\frac{\Delta_q}{2\rho_q}\right)^2}} + \frac{1}{k^3 \rho_q^3} \log \left[\frac{2\rho_q \pi}{\Delta_q} + \sqrt{1 + \left(\frac{2\rho_q \pi}{\Delta_q}\right)^2} \right]$$

in D8. DO loop 67 adds D8 to G_{11} . DO loop 67 also sets G_{21} , G_{22} , G_{23} , and G_{31} equal to zero. These four G's are set equal to zero when $p = q$

because the exact values of the coefficients by which they are multiplied in (24) are zero then.

The index L of DO loop 13 obtains ℓ' in (35). With regard to (36), DO loop 17 puts $G(u, \phi_K)$ in $GA(K)$. If (59) is not true, line 258 sends execution to statement 51. Lines 259-277 calculate the terms in (57) which do not involve $G(u, \phi_\ell)$. DO loop 62 accumulates

$$\sum_{\ell=1}^{n_\phi} \frac{A_\ell^{(n_\phi)} \phi_\ell^2}{k^3 a_\ell^3} \quad \text{in D6}$$

and

$$\sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} \left(\frac{1}{k^3 a_\ell^3} + \frac{1}{2ka_\ell} + \frac{\rho \rho' \phi_\ell^4}{8k^3 a_\ell^5} \right) \quad \text{in D7.}$$

Line 275 puts the coefficient of n in (57c) in D8. Line 276 puts the portion of (57a) which does not involve $G(u, \phi_\ell)$ in D6. Line 277 puts in D7 the portion of (57b) which involves neither $G(u, \phi_\ell)$ nor $(n^2 + 1)$. DO loop 32 accumulates G_1 , G_2 , and G_3 of (36) in H1A, H2A, and H3A, respectively. The index M of outer DO loop 30 obtains $n = M + M1 - 1$. If (59) is true, lines 295-297 add to H1A, H2A, and H3A the terms in (57) which are not present in (36). Lines 298-306 do the sums with respect to ℓ' in (35).

Lines 309-385 use the previously calculated $G_{m\alpha}$ to obtain the contributions (23) and (24) to the elements of the moment matrices Y_n . The index M of DO loop 31 obtains $n = M + M1 - 1$. With regard to lines 348-355, the subscripts K1 to K8 for Y are the same as the subscripts K1 to K8 for Z in Table 2 on page 50 of [1]. The variables UA, UG, UB, UH, and UF incremented for $p = q$ in lines 338-342 are intended for $Y(K1)$, $Y(K2)$, $Y(K3)$, $Y(K4)$, and $Y(K8 + MT)$, respectively. $Y(K8 + MT)$ is reserved for $(Y_n^{\phi\phi})_{pq}$ of (24d).

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001C LISTING OF THE SUBROUTINE YMAT
002C THE SUBROUTINE YMAT CALLS THE FUNCTION BLOG
003 SUBROUTINE YMAT(M1,M2,NP,NPHI,NT,IN,RH,ZH,X,A,XT,AT,Y)
004 COMPLEX Y(1600),U,H1A,H2A,H3A,GA(48),GB(48),GC(48),H1B,H2B,H3B
005 COMPLEX H1C,H2C,H3C,UA,UB,UC,UD,UE,UF,G1A(10),G2A(10),G3A(10)
006 COMPLEX G1B(10),G2B(10),G3B(10),G1C(10),G2C(10),G3C(10),CMPLX
007 COMPLEX UG,UH
008 DIMENSION RH(43),ZH(43),X(48),A(48),XT(10),AT(10),RS(42),ZS(42)
009 DIMENSION D(42),DR(42),DZ(42),C2(48),C3(48),C4(200),C5(200)
010 DIMENSION C6(200),R2(10),Z2(10),R7(10),Z7(10)
011 CT=2.
012 CP=.1
013 DO 10 I=2,NP
014 I2=I-1
015 RS(I2)=.5*(RH(I)+RH(I2))
016 ZS(I2)=.5*(ZH(I)+ZH(I2))
017 D1=.5*(RH(I)-RH(I2))
018 D2=.5*(ZH(I)-ZH(I2))
019 D(I2)=SQRT(D1*D1+D2*D2)
020 DR(I2)=D1
021 DZ(I2)=D2
022 10 CONTINUE
023 M3=M2-M1+1
024 M4=M1-1
025 P12=1.570796
026 PP=9.869604
027 DO 11 K=1,NPHI
028 PH=P12*(X(K)+1.)
029 C2(K)=PH*PH
030 SN=SN(.5*PH)
031 C3(K)=4.*SN*SN
032 A1=P12*A(K)
033 D4=.5*A1*C3(K)
034 D5=A1*COS(PH)
035 D6=A1*SN(PH)
036 M5=K
037 DO 29 M=1,M3
038 PHM=(M4+M)*PH
039 A2=COS(PHM)
040 C4(M5)=D4*A2
041 C5(M5)=D5*A2
042 C6(M5)=D6*SN(PHM)
043 M5=M5+NPHI
044 29 CONTINUE
045 11 CONTINUE
046 PNI=.7853982*(N
047 PN2=8.*PNI
048 U=(0.,1.)
049 MP=NP-1
050 MT=MP-1
051 N=MT+MP
052 N2N=MT*N
053 N2=N*N
054 JN=-1-N
055 DO 15 JQ=1,MP
056 KQ=2
057 IF(JQ.EQ.1) KQ=1
058 IF(JQ.EQ.MP) KQ=3
059 R1=RS(JQ)
060 Z1=ZS(JQ)

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061      D1=D(JQ)
062      D2=DR(JQ)
063      D3=DZ(JQ)
064      D4=D2/R1
065      D5=D1/R1
066      SV=D2/D1
067      CV=D3/D1
068      P1=PN1*D1
069      P2=PN2*D5
070      P3=2.*D1
071      P4=2.*D4
072      P5=D4*D4
073      P6=D1*D1
074      P7=P6*D1
075      T6=CT*D1
076      T62=T6*D1
077      T62=T62*T62
078      R6=CP*R1
079      R62=R6*R6
080      P1A=0.
081      P1B=0.
082      P1C=0.
083      DO 12 L=1,NT
084      D6=XT(L)
085      R2(L)=R1+D2*D6
086      Z2(L)=Z1+D3*D6
087      D7=P1*AT(L)/R2(L)
088      D8=1.-D6
089      D9=D8*D7
090      P1A=P1A+D8*D9
091      D6=1.+D6
092      P1B=P1B+D6*D9
093      P1C=P1C+D6*D6*D7
094      12 CONTINUE
095      DO 16 IP=1,MP
096      R3=RS(IP)
097      Z3=ZS(IP)
098      R4=R1-R3
099      Z4=Z1-Z3
100      DO 40 L=1,NT
101      D7=R2(L)-R3
102      D8=Z2(L)-Z3
103      R7(L)=R3*R2(L)
104      Z7(L)=D7*D7+D8*D8
105      40 CONTINUE
106      PH=R4*SV+Z4*CV
107      A1=ABS(PH)
108      A2=ABS(R4*CV-Z4*SV)
109      D6=A2
110      IF(A1.LE.D1) GO TO 26
111      D6=A1-D1
112      D6=SQRT(D6*D6+A2*A2)
113      26 IF(IP.NE.JQ.AND.(R6.GT.D6.OR.T6.LE.D6)) GO TO 41
114      Z5=R4*R4+Z4*Z4
115      R5=R3*R1
116      PHM=.5*R3*SV
117      DO 33 K=1,NPHI
118      A1=C3(K)
119      RR=Z5+R5*A1
120      H1A=0.

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121      H2A=0.
122      H3A=0.
123      IF(RR.LT.T62) GO TO 34
124      DO 35 L=1,NT
125      W=Z7(L)+R7(L)*A1
126      R=SQRT(W)
127      SN=-SIN(R)
128      CS=COS(R)
129      H1B=AT(L)/(R*W)*CMPLX(CS-R*SN,SN+R*CS)
130      H1A=H1B+H1A
131      H2B=XT(L)*H1B
132      H2A=H2B+H2A
133      H3A=XT(L)*H2B+H3A
134      35 CONTINUE
135      GO TO 36
136      34 DO 37 L=1,NT
137      W=Z7(L)+R7(L)*A1
138      R=SQRT(W)
139      IF(R.GT..5) GO TO 14
140      CS=R*(W*(.6944444E-2-W*.1736111E-3)-.125)
141      SN=W*(.3333333E-1-W*.1190476E-2)-.3333333
142      H1B=AT(L)*CMPLX(CS,SN)
143      GO TO 43
144      14 SN=-SIN(R)
145      CS=COS(R)
146      H1B=AT(L)/R*((CMPLX(CS-R*SN,SN+R*CS)-1.)/W-.5)
147      43 H1A=H1B+H1A
148      H2B=XT(L)*H1B
149      H2A=H2B+H2A
150      H3A=XT(L)*H2B+H3A
151      37 CONTINUE
152      A1=PH+PHM*A1
153      A2=ABS(A1)
154      R=RR-A2*A2
155      D6=A2-D1
156      D7=A2+D1
157      D62=D6*D6
158      D72=D7*D7
159      D8=SQRT(D62+R)
160      D9=SQRT(D72+R)
161      IF(R-(RR*1.E-5)) 52,52,53
162      52 IF(D6.LT.0.) STOP
163      W4=.5/D62-.5/D72
164      GO TO 54
165      53 W4=(D7/D9-D6/D8)/R
166      54 IF(D6.GE.0.) GO TO 38
167      W=ALOG((D7+D9)*(-D6+D8)/R)
168      GO TO 39
169      38 W=ALOG((D7+D9)/(D6+D8))
170      39 W1=(W4+.5*W)/D1
171      W5=A2/D1
172      W2=(.5*(D9-D8)-1./D9+1./D8)/P6-W5*W1
173      W3=(.25*(D7*D9-D6*D8)+W-R*(W4+.25*W))/P7-W5*(2.*W2+W5*W1)
174      IF(A1.LT.0.) W2=-W2
175      H1A=W1+H1A
176      H2A=W2+H2A
177      H3A=W3+H3A
178      36 GA(K)=H1A
179      GB(K)=H2A
180      GC(K)=H3A

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181 33 CONTINUE
182 K1=0
183 DO 45 M=1,M3
184 H1A=0.
185 H2A=0.
186 H3A=0.
187 H1B=0.
188 H2B=0.
189 H3B=0.
190 H1C=0.
191 H2C=0.
192 H3C=0.
193 DO 46 K=1,NPHI
194 K1=K1+1
195 D6=C4(K1)
196 D7=C5(K1)
197 D8=C6(K1)
198 UA=GA(K)
199 UB=GB(K)
200 UC=GC(K)
201 H1A=D6*UA+H1A
202 H2A=D7*UA+H2A
203 H3A=D8*UA+H3A
204 H1B=D6*UB+H1B
205 H2B=D7*UB+H2B
206 H3B=D8*UB+H3B
207 H1C=D6*UC+H1C
208 H2C=D7*UC+H2C
209 H3C=D8*UC+H3C
210 46 CONTINUE
211 G1A(M)=H1A
212 G2A(M)=H2A
213 G3A(M)=H3A
214 G1B(M)=H1B
215 G2B(M)=H2B
216 G3B(M)=H3B
217 G1C(M)=H1C
218 G2C(M)=H2C
219 G3C(M)=H3C
220 45 CONTINUE
221 IF (IP.NE.JQ) GO TO 47
222 A1=D5*D5
223 D8=0.
224 DO 63 K=1,NPHI
225 D8=D8+A(K)/SQRT(C2(K)+A1)
226 63 CONTINUE
227 A2=3.141593/D5
228 D8=(ELOG(A2)-P(2*D8))/(R5*R1)
229 DO 67 M=1,M3
230 G1A(M)=D8+G1A(M)
231 G2A(M)=0.
232 G2B(M)=0.
233 G2C(M)=0.
234 G3A(M)=0.
235 67 CONTINUE
236 GO TO 47
237 41 DO 25 M=1,M3
238 G1A(M)=0.
239 G2A(M)=0.
240 G3A(M)=0.

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241      G1B(M)=0.
242      G2B(M)=0.
243      G3B(M)=0.
244      G1C(M)=0.
245      G2C(M)=0.
246      G3C(M)=0.
247  25 CONTINUE
248      DO 13 L=1,NT
249      R5=R7(L)
250      Z5=Z7(L)
251      DO 17 K=1,NPHI
252      W=Z5+R5*C3(K)
253      R=SQRT(W)
254      SN=-SIN(R)
255      CS=COS(R)
256      GA(K)=CMPLX(CS-R*SN,SN+R*CS)/(W*R)
257  17 CONTINUE
258      IF(R62.LE.Z5) GO TO 51
259      D6=0.
260      D7=0.
261      DO 62 K=1,NPHI
262      W2=C2(K)
263      W=1./(Z5+R5*W2)
264      W1=A(K)*SQRT(W)
265      D6=D6+W1*W2*W
266      D7=D7+W1*(.5+W*(1.+ .125*W*R5*W2*W2))
267  62 CONTINUE
268      W1=R5/Z5
269      W2=PP*W1
270      W=SQRT(W2)
271      W3=1.+W2
272      R=SQRT(W3)
273      W4=SQRT(R5)
274      W5=ALOG(W+R)
275      D8=-P[2*C6-(W/R-W5)/(R5*W4)
276      D6=.5*D8
277      D7=((W/R*(W1-(.125+.1666667*W2)/W3)+.125*W5)/R5+.5*W5)/W4-P[2*D7
278  51 A1=AT(L)
279      A2=XT(L)*A1
280      A3=XT(L)*A2
281      K1=0
282      DO 30 M=1,M3
283      W=M+M4
284      H1A=0.
285      H2A=0.
286      H3A=0.
287      DO 32 K=1,NPHI
288      K1=K1+1
289      H1B=GA(K)
290      H1A=C4(K1)*H1B+H1A
291      H2A=C5(K1)*H1B+H2A
292      H3A=C6(K1)*H1B+H3A
293  32 CONTINUE
294      IF(R62.LE.Z5) GO TO 44
295      H1A=D6+H1A
296      H2A=D7-(W*W+1.)*D6+H2A
297      H3A=W*D8+H3A
298  44 G1A(M)=A1*H1A+G1A(M)
299      G2A(M)=A1*H2A+G2A(M)
300      G3A(M)=A1*H3A+G3A(M)

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301      G1B(M)=A2*H1A+G1B(M)
302      G2B(M)=A2*H2A+G2B(M)
303      G3B(M)=A2*H3A+G3B(M)
304      G1C(M)=A3*H1A+G1C(M)
305      G2C(M)=A3*H2A+G2C(M)
306      G3C(M)=A3*H3A+G3C(M)
307      30 CONTINUE
308      13 CONTINUE
309      47 A2=D(I P)
310      A3=.5*A2
311      W1=A3*(R4*D3-Z4*D2)
312      W2=-A3*R3*D3
313      A3=DZ(I P)
314      D6=DR(I P)
315      D7=Z4*D6
316      D9=D3*D6
317      H1C=(R1*D9-D2*(R3*A3+D7))*U
318      H3C=A2*D1*U
319      H2C=Z4*H3C
320      H3C=D3*H3C
321      W3=P3*(R4*A3-D7)
322      W4=P3*(D2*A3-D9)
323      W5=P3*R1*A3
324      JM=JN
325      DO 31 M=1,M3
326      H2A=G2A(M)
327      H1A=G1A(M)
328      H2B=G2B(M)
329      H1B=G1B(M)
330      UC=W1*H2A+W2*H1A
331      UB=W1*H2B+W2*H1B
332      UF=W3*(H2A+D4*H2B)+W4*(H2B+D4*G2C(M))+W5*(H1A+P4*H1B+P5*G1C(M))
333      UA=UC-UB
334      UB=UC+UB
335      UG=UA
336      UH=UB
337      IF(I P,NE,JQ) GO TO 48
338      UA=P1A+UA
339      UG=P1B+UG
340      UB=P1B+UB
341      UH=P1C+UH
342      UF=P2+UF
343      48 H3A=G3A(M)
344      H3B=G3B(M)
345      UC=H1C*H3A
346      UD=H1C*H3B
347      UF=H2C*(H3A+D4*H3B)+H3C*(H3B+D4*G3C(M))
348      K1=(P+JM)
349      K2=K1+I
350      K3=K1+N
351      K4=K2+N
352      K5=K2+MT
353      K6=K4+MT
354      K7=K3+N2N
355      K8=K4+N2N
356      GO TO (18,20,19),KQ
357      18 Y(K6)=UC+UD
358      IF(I P,EQ,1) GO TO 2.
359      Y(K3)=Y(K3)+UB
360      Y(K7)=Y(K7)+UE
361      IF(I P,EQ,MP) GO TO 22
362      21 Y(K4)=UH
363      Y(K8)=UE
364      GO TO 22
365      19 Y(K5)=Y(K5)+UC-UD
366      IF(I P,EQ,1) GO TO 23
367      Y(K1)=Y(K1)+UA
368      Y(K7)=Y(K7)+UE
369      IF(I P,EQ,MP) GO TO 22
370      23 Y(K2)=Y(K2)+UG
371      Y(K8)=UE
372      GO TO 22
373      20 Y(K5)=Y(K5)+UC-UD
374      Y(K6)=UC+UD
375      IF(I P,EQ,1) GO TO 24
376      Y(K1)=Y(K1)+UA
377      Y(K3)=Y(K3)+UB
378      Y(K7)=Y(K7)+UE
379      IF(I P,EQ,MP) GO TO 22
380      24 Y(K2)=Y(K2)+UG
381      Y(K4)=UH
382      Y(K8)=UE
383      22 Y(K8+MT)=UF
384      JM=JM+N2
385      31 CONTINUE
386      16 CONTINUE
387      JN=JN+N
388      15 CONTINUE
389      RETURN
390      END

```

III. THE FUNCTION BLOG

The function BLOG(x) calculates $\log(x + \sqrt{1 + x^2})$ for $x \geq 0$.
The method of calculation is described on page 56 of [1].

```

001C    LISTING OF THE FUNCTION BLOG
002    FUNCTION BLOG(X)
003    IF(X.GT..1) GO TO 1
004    X2=X*X
005    BLOG=((0.075*X2-.1666667)*X2+1.)*X
006    RETURN
007    1 BLOG=ALOG(X+SQRT(1.+X*X))
008    RETURN
009    END

```

IV. THE SUBROUTINE PLANE

The subroutine PLANE(M1, M2, NF, NP, IN, NT, RH, ZH, XT, AT, THR, RE, R) obtains the plane wave excitation vectors (72)-(75). If $IN \neq 2$, PLANE stores the plane wave excitation vectors (72) and (73) for the present H-field formulation in RE. If $IN \neq 1$, PLANE stores $V_{ni}^{t\theta}$, $-V_{ni}^{\phi\theta}$, $-V_{ni}^{t\phi}$, and $V_{ni}^{\phi\phi}$ of (74) and (75) in R. The minus signs attached to $V_{ni}^{\phi\theta}$ and $V_{ni}^{t\phi}$ transform the V's into plane wave measurement vectors. Plane wave scattering calculations are not done in this report, but if they were done, plane wave measurement vectors would be required. $I_{ni}^{it\theta}$ is put in $RE(i+(n-M1)*2*NP+(K-1)*(M2-M1+1)*2*NP)$ for

$$N = 2*NP-3$$

$$i = 1, 2, \dots, NP-2$$

$$n = M1, M1+1, \dots, M2$$

$$K = 1, 2, \dots, NF$$

$I_{ni}^{i\phi\theta}$ is displaced forward NP-2 locations from $I_{ni}^{it\theta}$ in RE. For $I_{ni}^{i\phi\theta}$, $i = 1, 2, \dots, NP-1$. $I_{ni}^{it\phi}$ is displaced forward N locations from $I_{ni}^{it\theta}$. $I_{ni}^{i\phi\phi}$ is displaced forward N locations from $I_{ni}^{i\phi\theta}$. The angle of incidence θ_t of (71) is THR(K) radians and $K = 1, 2, \dots, NF$. The storage arrangement of $V_{ni}^{t\theta}$, $-V_{ni}^{\phi\theta}$, $-V_{ni}^{t\phi}$, and $V_{ni}^{\phi\phi}$ in R is the same as that of the I's in RE. The arguments RE and R of PLANE are output arguments. The rest of the arguments of PLANE are input arguments. Some of the input arguments obtain variables in Part One of the text. The correspondence is tabulated as

Input Argument	Text Variable	Equation Number
NP	P	9
NT	n_T	79
RH	$k\rho(t_j^-)$, $j = 1, 2, \dots, P$	21
ZH	$kz(t_j^-)$, $j = 1, 2, \dots, P$	21
XT	$x_{\ell}^{(n_T)}$, $\ell = 1, 2, \dots, n_T$	79
AT	$A_{\ell}^{(n_T)}$, $\ell = 1, 2, \dots, n_T$	79

Here, k is the propagation constant. Also, $\rho(t_j^-)$ and $z(t_j^-)$ are the values of ρ and z at the point $t = t_j^-$ on the generating curve. This curve starts at $t = t_1^-$ and ends at $t = t_P^-$. It is assumed that P is not less than 3. The input arguments NT, XT, and AT are Gaussian quadrature data.

Minimum allocations are given by

```
COMPLEX RE(2*N*(M2-M1+1)*NF), R(2*N*(M2-M1+1)*NF),
      FA(M2+3), FB(M2+3)
DIMENSION RH(NP), ZH(NP), XT(NT), AT(NT), THR(NF),
      CS(NF), SN(NF), R2(NT), Z2(NT)
```

The I 's and V 's are calculated from (78) of the present report and (124)-(127) of [1]. The index IP of DO loop 12 obtains p in these equations. With regard to (79), DO loop 13 puts $\frac{1}{2} k\hat{\rho}_{\ell}$ and $k\hat{z}_{\ell}$ in R2 and Z2, respectively. The index K of DO loop 14 obtains the Kth angle of incidence. Nested DO loops 15 and 25 accumulate $S^*F_{M-2,a}$ and $S^*F_{M-2,b}$ in FA(M) and FB(M), respectively. Here, $F_{M-2,a}$ and $F_{M-2,b}$ are given by (79) and S is the normalizing constant for the Bessel functions. According to (9.1.46) on page 361 of [15],

$$S = J_0^u(x) + 2J_2^u(x) + 2J_4^u(x) + \dots \quad (93)$$

where the superscript u means unnormalized, $J_m(x)$ are the cylindrical Bessel functions of the first kind and

$$x = k\hat{\rho}_\ell \sin \theta_t \quad (94)$$

The logic inside DO loop 15 is the same as that inside the DO loop 15 which appears in the listing of the subroutine PLANE on pages 61 and 62 of [1]. If $M1 = 0$, lines 88 and 89 use the formulas

$$\left. \begin{aligned} F_{-1,a} &= -F_{1a} \\ F_{-1,b} &= -F_{1b} \end{aligned} \right\} \quad (95)$$

to store $F_{-1,a}$ and $F_{-1,b}$ in $FA(1)$ and $FB(1)$, respectively.

The index M of DO loop 27 obtains $n = M-2$. In DO loop 27,

$I_{ni}^{*it\theta}$ is added to $RE(K1)$ for $i = p-1$

$I_{ni}^{*it\theta}$ is put in $RE(K2)$ for $i = p$

$I_{np}^{i\phi\theta}$ is put in $RE(K3)$

$I_{ni}^{*it\phi}$ is added to $RE(K4)$ for $i = p-1$

$I_{ni}^{*it\phi}$ is put in $RE(K5)$ for $i = p$

$I_{np}^{i\phi\phi}$ is put in $RE(K6)$

The above I 's are given by (78). The V 's are given by (124)-(127) of [1]. Location of these V 's in R is similar to location of the I 's in RE .

```

001      LISTING OF THE SUBROUTINE PLANE
002      SUBROUTINE PLANE(M1,M2,NF,NP,IN,NT,RH,ZH,XT,AT,THR,RE,R)
003      COMPLEX RE(240),R(240),U,U1,FA(10),FB(10),UA,UB,FA1,FB1,FA2,FB2
004      COMPLEX U2,U3,U4,U5,CPLX
005      DIMENSION RH(43),ZH(43),XT(10),AT(10),THR(3),CS(3),SN(3),R2(10)
006      DIMENSION Z2(10),BJ(50)
007      MP=NP-1
008      MT=MP-1
009      N=MT+MP
010      N2=2*N
011      DO 11 K=1,NF
012      X=THR(K)
013      CS(K)=COS(X)
014      SN(K)=SIN(X)
015      11 CONTINUE
016      U=(0.,1.)
017      U1=3.141593*U**M1
018      M3=M1+1
019      M4=M2+3
020      IF(M1.EQ.0) M3=2
021      M5=M1+2
022      M6=M2+2
023      DO 12 IP=1,MP
024      K2=IP
025      I=IP+1
026      DR=.5*(RH(I)-RH(IP))
027      DZ=.5*(ZH(I)-ZH(IP))
028      D1=SQRT(DR*DR+DZ*DZ)
029      R1=.25*(RH(I)+RH(IP))
030      Z1=.5*(ZH(I)+ZH(IP))
031      D2=.5*DR
032      DR=D2/R1
033      DO 13 L=1,NT
034      R2(L)=R1+D2*XT(L)
035      Z2(L)=Z1+DZ*XT(L)
036      13 CONTINUE
037      W1=-.5*D1
038      W2=-2.*D2
039      DO 14 K=1,NF
040      CC=CS(K)
041      SS=SN(K)
042      D3=D2*CC
043      D4=-DZ*SS
044      D5=-D1*CC
045      W3=-2.*D3
046      W4=-2.*D4
047      W5=-.5*D5
048      DO 23 M=M3,M4
049      FA(M)=0.
050      FB(M)=0.
051      23 CONTINUE
052      DO 15 L=1,NT
053      X=SS*R2(L)
054      IF(X.GT..5E-7) GO TO 19
055      DO 20 M=M3,M4
056      FJ(M)=0.
057      20 CONTINUE
058      BJ(2)=1.
059      S=1.
060      GO TO 18

```

```

061      19 M=2.8*X+14.-2./X      121      U2=D3*F1A+U5*FA(M)
062      IF(X.LT..5) M=11.8+ALOG10(X) 122      U3=D3*F1B+U5*FB(M)
063      IF(M.LT.M4) M=M4      123      U4=D2*F2A
064      BJ(M)=0.      124      U5=D2*F2B
065      JM=M-1      125      IF(IP.EQ.1) GO TO 30
066      BJ(JM)=1.      126      R(K1)=R(K1)+U2-U3
067      DO 16 J=4,M      127      R(K4)=R(K4)+U4-U5
068      J2=JM      128      IF(IP.EQ.MP) GO TO 29
069      JM=JM-1      129      30 R(K2)=U2+U3
070      J1=JM-1      130      R(K5)=U4+U5
071      BJ(JM)=J1/X*BJ(J2)-BJ(JM+2) 131      29 K2=K2+N2
072      16 CONTINUE      132      UA=UB
073      S=0.      133      27 CONTINUE
074      IF(M.LE.4) GO TO 24      134      14 CONTINUE
075      DO 17 J=4,M,2      135      12 CONTINUE
076      S=S+BJ(J)      136      RETURN
077      17 CONTINUE      137      END
078      24 S=BJ(2)+2.*S
079      18 ARG=Z2(L)*CC
080      UA=AT(L)/S*CMPLX(COS(ARG),SIN(ARG))
081      UB=XT(L)*UA
082      DO 25 M=M3,M4
083      FA(M)=BJ(M)*UA+FA(M)
084      FB(M)=BJ(M)*UB+FB(M)
085      25 CONTINUE
086      15 CONTINUE
087      IF(M1.NE.0) GO TO 26
088      FA(1)=-FA(3)
089      FB(1)=-FB(3)
090      26 UA=U1
091      DO 27 M=M5,M6
092      M7=M-1
093      M8=M+1
094      F2A=UA*(FA(M8)+FA(M7))
095      F2B=UA*(FB(M8)+FB(M7))
096      UB=U*UA
097      F1A=UB*(FA(M8)-FA(M7))
098      F1B=UB*(FB(M8)-FB(M7))
099      K1=K2-1
100      K3=K2+MT
101      K4=K1+N
102      K5=K2+N
103      K6=K3+N
104      IF(IN.EQ.2) GO TO 28
105      RE(K3)=W2*(F2A+DR*F2B)
106      RE(K6)=W3*(F1A+DR*F1B)+W4*UA*(FA(M)+DR*FB(M))
107      U2=W1*F1A
108      U3=W1*F1B
109      U4=W5*F2A
110      U5=W5*F2B
111      IF(IP.EQ.1) GO TO 21
112      RE(K1)=RE(K1)+U2-U3
113      RE(K4)=RE(K4)+U4-U5
114      IF(IP.EQ.MP) GO TO 22
115      21 RE(K2)=U2+U3
116      RE(K5)=U4+U5
117      22 IF(IN.EQ.1) GO TO 29
118      28 R(K3)=D5*(F2A+DR*F2B)
119      R(K6)=D1*(F1A+DR*F1B)
120      U5=D4*UA

```

V. THE SUBROUTINES DECOMP AND SOLVE

The subroutines DECOMP(N, IPS, UL) and SOLVE(N, IPS, UL, B, X) solve a system of N linear equations in N unknowns. The input to DECOMP consists of N and the N by N matrix of coefficients on the left-hand side of the matrix equation stored by columns in UL. The output from DECOMP is IPS and UL. This output is fed into SOLVE. The rest of the input to SOLVE consists of N and the column of coefficients on the right-hand side of the matrix equation stored in B. SOLVE puts the solution to the matrix equation in X.

Minimum allocations are given by

COMPLEX UL(N*N)

DIMENSION SCL(N), IPS(N)

in DECOMP and by

COMPLEX UL(N*N), B(N), X(N)

DIMENSION IPS(N)

in SOLVE.

More detail concerning DECOMP and SOLVE is on pages 46-49 of [16].

```

001C LISTING OF THE SUBROUTINES DECOMP AND SOLVE
002 SUBROUTINE DECOMP(N, IPS, UL)
003 COMPLEX UL(1600), P(IVOT, EM
004 DIMENSION SCL(40), IPS(40)
005 DO 5 I=1, N
006 IPS(I)=I
007 RN=0.
008 J1=I
009 DO 2 J=1, N
010 ULM=ABS(REAL(UL(J1)))+ABS(AIMAG(UL(J1)))
011 J1=J1+N
012 IF(RN-ULM) 1, 2, 2
013 1 RN=ULM
014 2 CONTINUE
015 SCL(I)=1./RN
016 5 CONTINUE
017 NM1=N-1
018 K2=0
019 DO 17 K=1, NM1
020 BIG=0.
021 DO 11 I=K, N
022 IP=IPS(I)
023 IPK=IP+K2
024 SIZE=(ABS(REAL(UL(IPK)))+ABS(AIMAG(UL(IPK))))*SCL(IP)
025 IF(SIZE-BIG) 11, 11, 10
026 10 BIG=SIZE
027 IPV=I
028 11 CONTINUE
029 IF(IPV-K) 14, 15, 14
030 14 J=IPS(K)
031 IPS(K)=IPS(IPV)
032 IPS(IPV)=J
033 15 KPP=IPS(K)+K2
034 PIVOT=UL(KPP)
035 KP1=K+1
036 DO 16 I=KP1, N
037 KP=KPP
038 IP=IPS(I)+K2
039 EM=-UL(IP)/PIVOT
040 18 UL(IP)=-EM
041 DO 16 J=KP1, N
042 IP=IP+N
043 KP=KP+N
044 UL(IP)=UL(IP)+EM*UL(KP)
045 16 CONTINUE
046 K2=K2+N
047 17 CONTINUE
048 RETURN
049 END
050 SUBROUTINE SOLVE(N, IPS, UL, B, X)
051 COMPLEX UL(1600), B(40), X(40), SUM
052 DIMENSION ON(IPS(40))
053 NP1=N+1
054 IP=IPS(1)
055 X(1)=B(IP)
056 DO 2 I=2, N
057 IP=IPS(I)
058 IPB=IP
059 MI=-1
060 SUM=0.
061 DO 1 J=1, NP1
062 SUM=SUM+UL(IP)*X(J)
063 1 IP=IP+N
064 2 X(I)=B(IPB)-SUM
065 K2=N*(N-1)
066 IP=IPS(N)+K2
067 X(N)=X(N)/UL(IP)
068 DO 4 BACK=2, N
069 IP=NP1-BACK
070 K2=K2-N
071 IP=IPS(I)+K2
072 IP1=IP+1
073 SUM=0.
074 IP=IP
075 DO 3 J=IP1, N
076 IP=IP+N
077 3 SUM=SUM+UL(IP)*X(J)
078 4 X(I)=(X(I)-SUM)/UL(IP1)
079 RETURN
080 END

```

VI. THE MAIN PROGRAM FOR THE H-FIELD SOLUTION

The main program for the H-field solution calculates the H-field solution for the electric current induced on a conducting body of revolution immersed in an incident plane wave. Input data are read according to

```

      READ(1,15) NT, NPHI
15   FORMAT(2I3)
      READ(1,10)(XT(K), K = 1, NT)
      READ(1,10)(AT(K), K = 1, NT)
10   FORMAT(5E14.7)
      READ(1,10)(X(K), K = 1, NPHI)
      READ(1,10)(A(K), K = 1, NPHI)
      READ(1,16) NP, BK, THR(1)
16   FORMAT(I3, 2E14.7)
      READ(1,18)(RH(I), I = 1, NP)
      READ(1,18)(ZH(I), I = 1, NP)
18   FORMAT(10F8.4)

```

The input data obtain variables in Part One of the text. The input data are tabulated versus text variables in the following chart.

Input Data	Text Variable	Equation Number
NT	n_t	35
NPHI	n_ϕ	36
XT	$x_{\ell'}^{(n_t)}, \ell' = 1, 2, \dots, n_t$	35
AT	$A_{\ell'}^{(n_t)}, \ell' = 1, 2, \dots, n_t$	35
X	$x_\ell^{(n_\phi)}, \ell = 1, 2, \dots, n_\phi$	37
A	$A_\ell^{(n_\phi)}, \ell = 1, 2, \dots, n_\phi$	36
NP	P	3
BK	k	69
THR(1)	θ_t	71
RH	$\rho(t_j^-), j = 1, 2, \dots, P$	21
ZH	$z(t_j^-), j = 1, 2, \dots, P$	21

The Gaussian quadrature data $x_{\ell'}^{(n_t)}$, $A_{\ell'}^{(n_t)}$, $x_\ell^{(n_\phi)}$, and $A_\ell^{(n_\phi)}$ are given in Appendix A of [10]. In the main program for the H-field solution, n_T of (79) is set equal to n_t . NP controls the order of the moment matrix Y_1 of (86). This order is $(2*NP-3)$. THR is in radians. THR should be either 0 or π depending on the direction from which the incident wave approaches. The subscript on THR is for compatibility with the subroutine PLANE. $\rho(t_j^-)$ and $z(t_j^-)$ are the values of ρ and z at the point $t = t_j^-$ on the generating curve of the body of revolution. This curve starts at $t = t_1^-$ and ends at $t = t_P^-$. ρ is the distance from the axis of the body of revolution and z is the coordinate measured along this axis. RH and ZH are in meters.

The main program for the H field solution calls the subroutines YMAT, PLANE, DECOMP, and SOLVE. The function subprogram BLOG is also

needed because it is called by the subroutine YMAT.

Minimum allocations are given by

```
COMPLEX Y(N*N), R(2*N), C(N)
DIMENSION XT(NT), AT(NT), X(NPHI), A(NPHI),
          RH(NP), ZH(NP), IPS(N)
```

where

$$N = 2*NP-3$$

The t and ϕ components of the electric current are calculated from (84) and (85). The coefficients I_{1p}^t and I_{1p}^ϕ in these equations are the p th elements of the vectors $\vec{I}_1^t$ and $\vec{I}_1^\phi$ which satisfy the $n = 1$ equation in (10). DO loop 28 prepares RH and ZH for use in the subroutines YMAT and PLANE by multiplying them by k . With regard to (10) line 41 puts Y_1 of (86) in Y. Line 46 calculates IPS and changes Y. Line 47 puts the excitation vectors $\vec{I}_1^t$ and $\vec{I}_1^\phi$ for the θ polarized incident plane wave (69) in R. Line 47 also stores the excitation vectors for the ϕ polarized incident plane wave (70) further on in R but these vectors are not used in the main program. In line 50, the output IPS and Y from the subroutine DECOMP is fed along with N and R into the subroutine SOLVE. SOLVE puts $\vec{I}_1^t$ and $\vec{I}_1^\phi$ in C.

The p th line printed out under the heading JT contains the real part, the imaginary part, and the magnitude of the normalized t component $(J_t)/|H^i|$ of electric current at $\phi = 0^\circ$ and $t = t_{p+1}^-$. The p th line printed out under the heading JP contains the real part, the imaginary part, and the magnitude of the normalized ϕ component $(J_\phi)/|H^i|$ of electric current at $\phi = 90^\circ$ and $t = t_p$. According to (69b),

$$|H^i| = -\frac{u^t}{\phi} \cdot [H^i]_{z=0}$$

where $[H^i]_{z=0}$ is the incident magnetic field at $z = 0$. The sample input and output data are for the sphere example of Fig. 2. The input array ZH was constructed so that $z = 0$ at the center of the sphere.

```

001C      LISTING OF THE MAIN PROGRAM FOR THE H-FIELD SOLUTION
002C      THE SUBPROGRAMS YMAT,BLOG,PLANE,DECOMP, AND SOLVE ARE NEEDED
003//PGM JOB (XXXX,XXXX,1,1),'MAUTZ,JOE',REGION=200K
004// EXEC WATFIV
005//GO.SYS(IN DD *
006$JOB      MAUTZ,TIME=5,PAGES=60
007      COMPLEX Y(1600),R(240),C(40),U,C1
008      DIMENSION XT(10),AT(10),X(48),A(18),RH(43),ZH(43),THR(3),IPS(40)
009      READ(1,15) NT,NPHI
010      15 FORMAT(2I3)
011      WRITE(3,30) NT,NPHI
012      30 FORMAT(' NT NPHI'/(1X,I3,I5))
013      READ(1,10)(XT(K),K=1,NT)
014      READ(1,10)(AT(K),K=1,NT)
015      10 FORMAT(5E14.7)
016      WRITE(3,11)(XT(K),K=1,NT)
017      WRITE(3,12)(AT(K),K=1,NT)
018      11 FORMAT(' XT'/(1X,5E14.7))
019      12 FORMAT(' AT'/(1X,5E14.7))
020      READ(1,10)(X(K),K=1,NPHI)
021      READ(1,10)(A(K),K=1,NPHI)
022      WRITE(3,13)(X(K),K=1,NPHI)
023      WRITE(3,14)(A(K),K=1,NPHI)
024      13 FORMAT(' X'/(1X,5E14.7))
025      14 FORMAT(' A'/(1X,5E14.7))
026      READ(1,16) NP,BK,THR(1)
027      16 FORMAT(13,2E14.7)
028      WRITE(3,17) NP,BK,THR(1)
029      17 FORMAT(' NP',6X,'BK',12X,'THR'/(1X,13,2E14.7))
030      READ(1,18)(RH(I),I=1,NP)
031      READ(1,18)(ZH(I),I=1,NP)
032      18 FORMAT(10F8.4)
033      WRITE(3,19)(RH(I),I=1,NP)
034      WRITE(3,20)(ZH(I),I=1,NP)
035      19 FORMAT(' RH'/(1X,10F8.4))
036      20 FORMAT(' ZH'/(1X,10F8.4))
037      DO 28 J=1,NP
038      RH(J)=BK*RH(J)
039      ZH(J)=BK*ZH(J)
040      28 CONTINUE
041      CALL YMAT(1,1,NP,NPHI,NT,1,RH,ZH,X,A,XT,AT,Y)
042      MT=NP-2
043      N=2*MT+1
044      WRITE(3,29)(Y(J),J=1,N)
045      29 FORMAT(' Y'/(1X,6E11.4))
046      CALL DECOMP(N,IPS,Y)
047      CALL PLANE(1,1,1,NP,1,NT,RH,ZH,XT,AT,THR,R,R)
048      WRITE(3,23)(R(J),J=1,N)
049      23 FORMAT(' R'/(1X,6E11.4))
050      CALL SOLVE(N,IPS,Y,R,C)
051      U=(0.,1.)
052      WRITE(3,21)
053      21 FORMAT('      REAL JT      IMAG JT      MAG JT')
054      DO 24 J=1,MT
055      C1=2./RH(J+1)*C(J)
056      C2=CABS(C1)
057      WRITE(3,25) C1,C2
058      25 FORMAT(1X,3E11.4)
059      24 CONTINUE
060      WRITE(3,26)

```

```

061 26 FORMAT(' REAL JP IMAG JP MAG JP')
062 MP=NP-1
063 DO 27 J=1,MP
064 C1=4./(RH(J)+RH(J+1))*U*(J+MT)
065 C2=CABS(C1)
066 WRITE(3,25) C1,C2
067 27 CONTINUE
068 STOP
069 END

$DATA
2 20
-0.5773503E+00 0.5773503E+00
0.1000000E+01 0.1000000E+01
-0.9931286E+00-0.9639719E+00-0.9122344E+00-0.8391170E+00-0.7463319E+00
-0.6360537E+00-0.5108670E+00-0.3737061E+00-0.2277859E+00-0.7652652E-01
0.7652652E-01 0.2277859E+00 0.3737061E+00 0.5108670E+00 0.6360537E+00
0.7463319E+00 0.8391170E+00 0.9122344E+00 0.9639719E+00 0.9931286E+00
0.1761401E-01 0.4060143E-01 0.6267205E-01 0.8327674E-01 0.1019301E+00
0.1181945E+00 0.1316886E+00 0.1420961E+00 0.1491730E+00 0.1527534E+00
0.1527534E+00 0.1491730E+00 0.1420961E+00 0.1316886E+00 0.1181945E+00
0.1019301E+00 0.8327674E-01 0.6267205E-01 0.4060143E-01 0.1761401E-01
16 0.1256637E+01 0.3141593E+01
0.0000 0.2079 0.4067 0.5878 0.7431 0.8660 0.9511 0.9945 0.9945 0.9511
0.8660 0.7431 0.5878 0.4067 0.2079 0.0000
-1.0000 -0.9781 -0.9135 -0.8090 -0.6691 -0.5000 -0.3090 -0.1045 0.1045 0.3090
0.5000 0.6691 0.8090 0.9135 0.9871 1.0000

$STOP
/*
//
PRINTED OUTPUT
NT NPHI
2 20
XT
-0.5773503E+00 0.5773503E+00
AT
0.1000000E+01 0.1000000E+01
X
-0.9931286E+00-0.9639719E+00-0.9122344E+00-0.8391170E+00-0.7463319E+00
-0.6360537E+00-0.5108670E+00-0.3737061E+00-0.2277859E+00-0.7652652E-01
0.7652652E-01 0.2277859E+00 0.3737061E+00 0.5108670E+00 0.6360537E+00
0.7463319E+00 0.8391170E+00 0.9122344E+00 0.9639719E+00 0.9931286E+00
A
0.1761401E-01 0.4060143E-01 0.6267208E-01 0.8327675E-01 0.1019301E+00
0.1181945E+00 0.1316886E+00 0.1420961E+00 0.1491730E+00 0.1527534E+00
0.1527534E+00 0.1491730E+00 0.1420961E+00 0.1316886E+00 0.1181945E+00
0.1019301E+00 0.8327675E-01 0.6267208E-01 0.4060143E-01 0.1761401E-01
NP BK THR
16 0.1256637E+01 0.3141593E+01
RH
0.0000 0.2079 0.4067 0.5878 0.7431 0.8660 0.9511 0.9945 0.9945 0.9511
0.8660 0.7431 0.5878 0.4067 0.2079 0.0000
ZH
-1.0000 -0.9781 -0.9135 -0.8090 -0.6691 -0.5000 -0.3090 -0.1045 0.1045 0.3090
0.5000 0.6691 0.8090 0.9135 0.9871 1.0000
Y
0.2486E+01-0.2338E-02 0.4211E+00-0.4919E-02 0.6397E-01-0.8888E-02
0.5598E-01-0.1382E-01 0.4588E-01-0.1923E-01 0.4455E-01-0.2463E-01
0.3941E-01-0.2964E-01 0.3432E-01-0.3398E-01 0.2937E-01-0.3752E-01
0.2475E-01-0.4023E-01 0.2065E-01-0.4217E-01 0.1728E-01-0.4348E-01
0.1475E-01-0.4458E-01 0.1306E-01-0.4490E-01 0.9666E-03 0.9243E-01

```

-0.9622E-03-0.5803E-01-0.4589E-02-0.6791E-01-0.9512E-02-0.5939E-01
 -0.1517E-01-0.5297E-01-0.2105E-01-0.4766E-01-0.2668E-01-0.4258E-01
 -0.3164E-01-0.3741E-01-0.3578E-01-0.3226E-01-0.3903E-01-0.2732E-01
 -0.4137E-01-0.2279E-01-0.4297E-01-0.1891E-01-0.4402E-01-0.1588E-01
 -0.4484E-01-0.1362E-01-0.4470E-01-0.1244E-01

R

-0.2834E+00-0.7748E+00-0.3446E+00-0.7491E+00-0.4386E+00-0.6972E+00
 -0.5523E+00-0.6097E+00-0.6671E+00-0.4300E+00-0.7609E+00-0.3088E+00
 -0.8136E+00-0.1067E+00-0.8136E+00 0.1067E+00-0.7609E+00 0.3088E+00
 -0.6671E+00 0.4800E+00-0.5523E+00 0.6097E+00-0.4386E+00 0.6972E+00
 -0.3454E+00 0.7550E+00-0.2791E+00 0.7810E+00 0.7758E+00-0.2679E+00
 0.7267E+00-0.2959E+00 0.6295E+00-0.3378E+00 0.4891E+00-0.3684E+00
 0.3236E+00-0.3603E+00 0.1626E+00-0.2931E+00 0.4374E-01-0.1652E+00
 0.0000E+00 0.0000E+00 0.4374E-01 0.1652E+00 0.1626E+00 0.2931E+00
 0.3236E+00 0.3603E+00 0.4891E+00 0.3684E+00 0.6295E+00 0.3378E+00
 0.7281E+00 0.2922E+00 0.7778E+00 0.2620E+00

REAL JT IMAG JT MAG JT

-0.7989E+00-0.1959E+01 0.2115E+01
 -0.9480E+00-0.1930E+01 0.2151E+01
 -0.1162E+01-0.1836E+01 0.2173E+01
 -0.1408E+01-0.1658E+01 0.2175E+01
 -0.1630E+01-0.1372E+01 0.2130E+01
 -0.1764E+01-0.9802E+00 0.2018E+01
 -0.1754E+01-0.5127E+00 0.1827E+01
 -0.1569E+01-0.2428E-01 0.1569E+01
 -0.1218E+01 0.4228E+00 0.1289E+01
 -0.7475E+00 0.7772E+00 0.1078E+01
 -0.2294E+00 0.1016E+01 0.1042E+01
 0.2592E+00 0.1146E+01 0.1174E+01
 0.6569E+00 0.1210E+01 0.1377E+01
 0.9016E+00 0.1190E+01 0.1493E+01

REAL JP IMAG JP MAG JP

0.7731E+00 0.1878E+01 0.2031E+01
 0.8555E+00 0.1782E+01 0.1976E+01
 0.9995E+00 0.1602E+01 0.1888E+01
 0.1148E+01 0.1333E+01 0.1759E+01
 0.1240E+01 0.9997E+00 0.1593E+01
 0.1229E+01 0.6569E+00 0.1394E+01
 0.1106E+01 0.3761E+00 0.1168E+01
 0.9065E+00 0.2217E+00 0.9332E+00
 0.7010E+00 0.2213E+00 0.7351E+00
 0.5589E+00 0.3543E+00 0.6618E+00
 0.5201E+00 0.5631E+00 0.7665E+00
 0.5789E+00 0.7809E+00 0.9721E+00
 0.6903E+00 0.9538E+00 0.1177E+01
 0.8141E+00 0.1078E+01 0.1351E+01
 0.8998E+00 0.1148E+01 0.1459E+01

VII. THE SUBROUTINE YZ

With regard to (10), the subroutine YZ(M1, M2, NP, NPHI, NT, IN, RH, ZH, X, A, XT, AT, Y, Z) puts in Y the moment matrices Y_n defined by (86). With regard to (6) of [1], the subroutine YZ puts in Z the moment matrices Z_n defined by

$$Z_n = \begin{bmatrix} Z_n^{tt} & Z_n^{t\phi} \\ Z_n^{\phi t} & Z_n^{\phi\phi} \end{bmatrix}, \quad n = M1, M1+1, \dots M2 \quad (96)$$

The Y obtained by YZ is exactly the same as the Y obtained by the subroutine YMAT which is described and listed in Section II of Part Two. The storage arrangement of the elements of Z_n in Z is the same as the arrangement of the elements of Y_n in Y. The Z obtained by YZ is nearly the same as the Z obtained by ZMAT of [1]. The greatest difference in calculation is that, in accord with (68a), the second sum with respect to l' in (97) of [1] is replaced by the exact value of the corresponding integral with respect to t' . The purpose of the subroutine YZ is to obtain Y and Z more efficiently than by means of YMAT of the present report and ZMAT of [1]. Increased efficiency is possible because several calculations done in YMAT of the present report are repeated in ZMAT of [1].

The only output arguments of YZ are Y and Z. The rest of the arguments of YZ are input arguments. The input arguments of YZ have the same names and meanings as the input arguments of YMAT. Lines 13 and 14 put c_t and c_ϕ of (31) in CT and CP, respectively. The subroutine YZ calls the function BLOG which is described and listed in Section III of Part Two.

Minimum allocations are given by

COMPLEX Y(M3*N*N), Z(M3*N*N), GA(NPHI), GB(NPHI),
GC(NPHI), GD(NPHI), GE(NPHI), G1A(M3), G2A(M3),
G3A(M3), G1B(M3), G2B(M3), G3B(M3), G1C(M3),
G2C(M3), G3C(M3), G4A(M3), G5A(M3), G6A(M3)
G4B(M3), G5B(M3), G6B(M3)

DIMENSION RH(NP), ZH(NP), X(NPHI), A(NPHI), XT(NT)
 AT(NT), RS(NP-1), ZS(NP-1), D(NP-1), DR(NP-1)
 DZ(NP-1), DM(NP-1), C1(NPHI), C2(NPHI), C3(NPHI)
 C4(M3*NPHI), C5(M3*NPHI), C6(M3*NPHI), R2(NT),
 Z2(NT), R7(NT), Z7(NT)

Here, N and M3 are given by (87) and (88), respectively.

The elements of Y_n are calculated from (23) and (24) with $G_{m\alpha}$ evaluated as in Section III of Part One. The elements of Z_n are calculated from (48)-(51) of [1]. DO loop 11 puts ϕ_ℓ in C1, ϕ_ℓ^2 in C2, and $4 \sin^2(\frac{1}{2} \phi_\ell)$ in C3. ϕ_ℓ is given by (37) and $4 \sin^2(\frac{1}{2} \phi_\ell)$ is needed in order to evaluate (15) at $\phi = \phi_\ell$. With regard to (36), inner DO loop 29 puts

$$\pi A_\ell^{(n\phi)} \sin^2(\frac{1}{2} \phi_\ell) \cos(n\phi_\ell) \quad \text{in C4,}$$

and
$$\frac{\pi}{2} A_\ell^{(n\phi)} \cos \phi_\ell \cos(n\phi_\ell) \quad \text{in C5,}$$

$$\frac{\pi}{2} A_\ell^{(n\phi)} \sin \phi_\ell \sin(n\phi_\ell) \quad \text{in C6.}$$

The index JQ of DO loop 15 obtains q in (23) and (24) of the present report and in (48)-(51) of [1]. With regard to (22d), line 75 puts $(IN) * (\pi \Delta_q) / \rho_q$ in P2. DO loop 12 puts $k\rho'$ and kz' of (29) in R2 and Z2, respectively. DO loop 12 also accumulates the products of IN with (23a), (23b), and (23c) in P1A, P1B, and P1C, respectively. The index IP of DO loop 16 obtains p in (24) of the present report and in (48)-(51) of [1]. Lines 114-120 put kd_o of (32) in D6. If either case 1 of (30) or case 3 of (51) is obtained, branch statement 26 sends execution to statement 41. If either case 2 of (39) or case 4 of (60) is obtained, execution proceeds to line 122. Lines 122-288 calculate $G_{m\alpha}$ for case 2 and case 4. Lines 290-388 calculate $G_{m\alpha}$ for case 1 and case 3. In cases 2 and 4, the integration with respect to t' is done first. In cases 1 and 3, the integration with respect to ϕ is done first. In all cases,

$$\left. \begin{array}{l} G_{m1} \text{ is stored in } GmA(n - M1+1) \\ G_{m2} \text{ is stored in } GmB(n - M1+1) \\ G_{m3} \text{ is stored in } GmC(n - M1+1) \end{array} \right\} \begin{array}{l} m = 1, 2, 3 \\ n = M1, M1+1, \dots, M2 \end{array} \quad (97)$$

$$\left. \begin{array}{l} G_{ma} \text{ is stored in } GmA(n - M1+1) \\ G_{mb} \text{ is stored in } GmB(n - M1+1) \end{array} \right\} \begin{array}{l} m = 4, 5, 6 \\ n = M1, M1+1, \dots, M2 \end{array} \quad (98)$$

The G's in (97) are defined by (25). The G's in (98) are defined by (56)-(57) of [1].

With regard to (41), DO loop 33 puts

$$\left. \begin{array}{l} H_1(\phi_K) \text{ in } GA(K) \\ H_2(\phi_K) \text{ in } GB(K) \\ H_3(\phi_K) \text{ in } GC(K) \end{array} \right\} K = 1, 2, \dots, n_\phi$$

DO loop 33 also evaluates G_a and G_b of (71)-(72) of [1] at $\phi = \phi_K$ and puts them in $GD(K)$ and $GE(K)$, respectively. The branch statement in line 133 is based on (48). DO loop 35 uses (49) to calculate $H_\alpha(\phi_K)$ of (41). The index L of DO loop 35 obtains ℓ' in (49). DO loop 35 uses the Gaussian quadrature formulas

$$G_a = \sum_{\ell'=1}^{n_t} A_{\ell'}^{(n_t)} \frac{e^{-jkR}}{kR} \quad (99a)$$

$$G_b = \sum_{\ell'=1}^{n_t} A_{\ell'}^{(n_t)} x_{\ell'}^{(n_t)} \frac{e^{-jkR}}{kR} \quad (99b)$$

to calculate G_a and G_b of (71)-(72) of [1]. In (99),

$$R = \sqrt{(\rho' - \rho_p)^2 + (z' - z_p)^2 + 4\rho_p \rho' \sin^2(\frac{1}{2} \phi_K)} \quad (100)$$

where

$$\rho' = \rho_q + \frac{\Delta_q x_{\ell'}^{(n_t)} \sin v_q}{2} \quad (101a)$$

$$z' = z_q + \frac{x_{\ell'}^{(n_t)} \Delta_q \cos v_q}{2} \quad (101b)$$

DO loop 35 accumulates $H_1(\phi_K)$, $H_2(\phi_K)$, $H_3(\phi_K)$, G_a , and G_b in H1A, H2A, H3A, H4A, and H5A, respectively. DO loop 37 accumulates the sum on ℓ' in (44) for $\alpha = 1, 2$, and 3 in H1A, H2A, and H3A, respectively. DO loop 37 accumulates G_{a1} and G_{b1} of (79)-(80) of [1] in H4A and H5A, respectively. If $kR \leq .5$ where R is given by (100), then lines 154-159 use (89) and

$$\frac{e^{-jkR}-1}{kR} \approx kR \left(-\frac{1}{2} + \frac{(kR)^2}{24} - \frac{(kR)^4}{720} \right) + j \left(-1 + \frac{(kR)^2}{6} - \frac{(kR)^4}{120} + \frac{(kR)^6}{5040} \right) \quad (102)$$

in order to put

$$A_{\ell'}^{(n_t)} \left(G - \frac{1}{(kR)^3} - \frac{1}{2kR} \right) \quad \text{in H1B} \quad (103a)$$

and

$$A_{\ell'}^{(n_t)} \left(\frac{e^{-jkR}-1}{kR} \right) \quad \text{in H4B} \quad (103b)$$

If $kR > .5$, lines 161-165 calculate H1B and H4B directly from (103). Lines 173-199 put I_α of (47) for $\alpha = 1, 2$, and 3 in W1, W2, and W3, respectively. Lines 173-199 also put G_{a2} and G_{b2} of (87)-(88) of [1] in W4 and W5, respectively. I_1 , I_3 , and G_{a2} are even in t_o . I_2 and G_{b2} are odd in t_o . I_1 , I_2 , I_3 , G_{a2} , and G_{b2} are calculated with t_o replaced by $|t_o|$ and then the signs of I_2 and G_{b2} are changed if t_o is negative. Line 173 puts kt_o in A1.

In line 182, d^2 is compared to $10^{-5} r_{pq}^2$. If $d^2 \leq 10^{-5} r_{pq}^2$, then little confidence can be placed in the calculated value of d^2 because r_{pq}^2 and t_o^2 are very close to each other in (46b). If

$$\left. \begin{array}{l} d^2 \leq 10^{-5} r_{pq}^2 \\ |t_o| < \frac{1}{2} \Delta_q \end{array} \right\}$$

then statement 52 stops execution because an accurate value of I_1 can not be obtained. However, if

$$\left. \begin{array}{l} d^2 \leq 10^{-5} r_{pq}^2 \\ |t_o| \geq \frac{1}{2} \Delta_q \end{array} \right\}$$

then accurate calculation of I_1 may be possible. If $|t_o| - \frac{1}{2} \Delta_q$ is appreciably greater than d^2 , then the approximation (90) can be invoked. Line 184 puts the right-hand side of (90) in W4. Lines 187-190 calculate the logarithm term in (47a) according to (91). This logarithm is stored in W.

Nested DO loops 45 and 46 calculate (50) for $n = M1 + M-1$ where M is the index of DO loop 45. The index K of DO loop 46 obtains ℓ in (50). Nested DO loops 45 and 46 also calculate (91)-(93) of [1] and (91)-(93) of [1] with a replaced by b. The results of these calculations are stored according to (97) and (98). With regard to (68a), lines 271-279 put

$$\frac{-\pi}{2k^3 \rho_q^3} \sum_{\ell=1}^{n_\phi} \frac{A_\ell^{(n_\phi)}}{\sqrt{\phi_\ell^2 + \left(\frac{\Delta_q}{2\rho_q}\right)^2}} + \frac{1}{k^3 \rho_q^3} \log \left[\frac{2\rho_q \pi}{\Delta_q} + \sqrt{1 + \left(\frac{2\rho_q \pi}{\Delta_q}\right)^2} \right]$$

in D8. Line 280 puts in D9 the term which is present in G_{5a} in case 4 but not present in case 2. Replacement of the second sum with respect

to ℓ' in (97) of [1] by the exact value of the corresponding integral with respect to t' and use of (99) of [1] give

$$G_{5a} = \frac{\pi}{2} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} \cos \phi_\ell \cos(n\phi_\ell) \sum_{\ell'=1}^{n_t} A_{\ell'}^{(n_t)} \frac{e^{-jkR_{p\ell'\ell}}}{kR_{p\ell'\ell}} + D9 \quad (104)$$

where

$$D9 = - \frac{2\pi}{k\Delta_q} \sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} \log \left[\frac{\Delta_q}{2\rho_q \phi_\ell} + \sqrt{1 + \left(\frac{\Delta_q}{2\rho_q \phi_\ell}\right)^2} \right] + \frac{2}{k\rho_q} \left\{ \log \left[\frac{2\pi\rho_q}{\Delta_q} + \sqrt{1 + \left(\frac{2\pi\rho_q}{\Delta_q}\right)^2} \right] + \frac{2\pi\rho_q}{\Delta_q} \log \left[\frac{\Delta_q}{2\pi\rho_q} + \sqrt{1 + \left(\frac{\Delta_q}{2\pi\rho_q}\right)^2} \right] \right\} \quad (105)$$

Line 280 puts the right-hand side of (105) in D9. DO loop 67 adds D8 to G_{11} and D9 to G_{5a} . DO loop 67 also sets G_{21} , G_{22} , G_{23} , and G_{31} equal to zero. These four G 's are set equal to zero when $p=q$ because the exact values of the coefficients by which they are multiplied in (24) are zero then.

The index L of DO loop 13 obtains ℓ' in (35) and in (62)-(63) of [1]. With regard to (36), DO loop 17 puts $G(u, \phi_K)$ in $GA(K)$. With regard to (64)-(66) of [1], DO loop 17 puts $(e^{-jkR_{p\ell'\ell}})/(kR_{p\ell'\ell})$ in $GD(K)$. If (59) is not true, line 318 sends execution to statement 51. Lines 319-340 calculate the terms in (57) which do not involve $G(u, \phi_\ell)$ and the terms in (94) of [1] which do not involve $(e^{-jkR_{p\ell'\ell}})/(kR_{p\ell'\ell})$. DO loop 62 accumulates

$$\sum_{\ell=1}^{n_\phi} \frac{A_\ell^{(n_\phi)} \phi_\ell^2}{k^3 a_\ell^3} \quad \text{in D6,}$$

$$\sum_{\ell=1}^{n_\phi} A_\ell^{(n_\phi)} \left(\frac{1}{k^3 a_\ell^3} + \frac{1}{2ka_\ell} + \frac{\rho_p \rho' \phi_\ell^4}{8k^3 a_\ell^5} \right) \quad \text{in D7,}$$

and

$$\sum_{\ell=1}^{n_{\phi}} \frac{A_{\ell}^{(n_{\phi})}}{ka_{\ell}} \quad \text{in D9.}$$

Line 337 puts the coefficient of n in (57c) in D8. Line 338 puts the portion of (57a) which does not involve $G(u, \phi_{\ell})$ in D6. Line 339 puts in D7 the portion of (57b) which involves neither $G(u, \phi_{\ell})$ nor $(n^2 + 1)$. Line 340 puts in D9 the terms in (94) of [1] which do not involve

$(e^{-jkR_{p\ell',\ell}})/(kR_{p\ell',\ell})$. DO loop 32 accumulates G_1, G_2 , and G_3 of (36) in H1A, H2A, and H3A, respectively. DO loop 32 also accumulates G_4, G_5 , and G_6 of (64)-(66) of [1] in H4A, H5A, and H6A, respectively. The index M of outer DO loop 30 obtains $n = M + M1 - 1$. If (59) is true, lines 368-370 add to H1A, H2A, and H3A the terms in (57) which are not present in (36) and line 371 adds to H5A the terms in (94) of [1] which are not present in (65) of [1]. Lines 372-380 do the sums with respect to ℓ' in (35). Lines 381-386 do the sums with respect to ℓ' in (62)-(63) of [1].

Lines 389-509 use the previously calculated G 's of (97) and (98) to obtain the contributions (23) and (24) to the elements of the moment matrices Y_n and the contributions (48)-(51) of [1] to the moment matrices Z_n . The index M of DO loop 31 obtains $n = M + M1 - 1$. Lines 437-438 put G_{7a} and G_{7b} of (54)-(55) of [1] in H4A and H4B, respectively. The subscripts K1, K2, ..., K8 defined in lines 452-459 have the same meaning as in Table 2 on page 50 of [1]. The variables UA, UG, UB, UH, and UF incremented for $p=q$ in lines 425-429 are intended for $Y(K1), Y(K2), Y(K3), Y(K4)$, and $Y(K9)$, respectively. $Y(K9)$ is reserved for $(Y^{\phi\phi})_{n\ pq}$ of (24d).

```

001      LISTING OF THE SUBROUTINE YZ
002      THE SUBROUTINE YZ CALLS THE FUNCTION BLOG
003      SUBROUTINE YZ(M1,M2,NP,NPH,NT,IN,RH,ZH,X,A,XT,AT,Y,Z)
004      COMPLEX Y(1600),Z(1600),U,U1,U2,U3,U4,U5,U6,H1A,H2A,H3A,GA(48)
005      COMPLEX GB(48),GC(48),GD(48),GE(48),H1B,H2B,H3B,H1C,H2C,H3C,H4A
006      COMPLEX H5A,H6A,H4B,H5B,H6B,UA,UB,UC,UD,UE,UF,G1A(10),G2A(10)
007      COMPLEX G3A(10),G1B(10),G2B(10),G3B(10),G1C(10),G2C(10),G3C(10)
008      COMPLEX G4A(10),G5A(10),G6A(10),G4B(10),G5B(10),G6B(10),CMPLX
009      COMPLEX UG,UH
010      DIMENSION RH(43),ZH(43),X(48),A(48),XT(10),AT(10),RS(42),ZS(42)
011      DIMENSION D(42),DR(42),DZ(42),DM(42),C1(48),C2(48),C3(48),C4(200)
012      DIMENSION C5(200),C6(200),R2(10),Z2(10),R7(10),Z7(10)
013      CT=2.
014      CP=.1
015      DO 10 I=2,NP
016          I2=I-1
017          RS(I2)=.5*(RH(I)+RH(I2))
018          ZS(I2)=.5*(ZH(I)+ZH(I2))
019          D1=.5*(RH(I)-RH(I2))
020          D2=.5*(ZH(I)-ZH(I2))
021          D(I2)=SQRT(D1*D1+D2*D2)
022          DR(I2)=D1
023          DZ(I2)=D2
024          DM(I2)=D(I2)/RS(I2)
025      10 CONTINUE
026          M3=M2-M1+1
027          M4=M1-1
028          P12=1.570796
029          PP=9.869604
030          DO 11 K=1,NPH
031              PH=P12*(X(K)+.)
032              C1(K)=PH
033              C2(K)=PH*PH
034              SN= SIN(.5*PH)
035              C3(K)=4.*SN*SN
036              A1=P12*A(K)
037              D4=.5*A1*C3(K)
038              D5=A1*COS(PH)
039              D6=A1*SIN(PH)
040              M5=K
041              DO 29 M=1,M3
042                  PHM=(M4+M)*PH
043                  A2=COS(PHM)
044                  C4(M5)=D4*A2
045                  C5(M5)=D5*A2
046                  C6(M5)=D6*SIN(PHM)
047              M5=M5+NPH
048      29 CONTINUE
049      11 CONTINUE
050          PN1=.7853982*IN
051          PN2=8.*PN1
052          U=(0.,1.)
053          U1=.5*U
054          U2=.2*U
055          MP=NP-1
056          MT=MP-1
057          N=MT+MP
058          N2N=MT*N
059          N2=N*N
060          JN=-1-N

```

```

061      DO 15 J0=1,MP
062      KQ=2
063      IF(JQ.EQ.1) KQ=1
064      IF(JQ.EQ.MP) KQ=3
065      R1=RS(JQ)
066      Z1=ZS(JQ)
067      D1=D(JQ)
068      D2=DR(JQ)
069      D3=DZ(JQ)
070      D4=D2/R1
071      D5=D1/R1
072      SV=D2/D1
073      CV=D3/D1
074      P1=PN1*D1
075      P2=PN2*D5
076      P3=2.*D1
077      P4=2.*D4
078      P5=D4*D4
079      P6=D1*D1
080      P7=P5*D1
081      T6=CT*D1
082      T62=T6+D1
083      T62=T62*T62
084      R6=CP*R1
085      R62=R6*R6
086      P1A=0.
087      P1B=0.
088      P1C=0.
089      DO 12 L=1,NT
090      D6=XT(L)
091      R2(L)=R1+D2*D6
092      Z2(L)=Z1+D3*D6
093      D7=P1*AT(L)/R2(L)
094      D8=1.-D6
095      D9=D8*D7
096      P1A=P1A+D8*D9
097      D6=1.+D6
098      P1B=P1B+D6*D9
099      P1C=P1C+D6*D7
100      12 CONTINUE
101      DO 16 IP=1,MP
102      R3=RS(IP)
103      Z3=ZS(IP)
104      R4=R1-R3
105      Z4=Z1-Z3
106      U3=D2*U1
107      U4=D3*U1
108      DO 40 L=1,NT
109      D7=R2(L)-R3
110      D8=Z2(L)-Z3
111      R7(L)=R3*R2(L)
112      Z7(L)=D7*D7+D8*D8
113      40 CONTINUE
114      PH=R4*SV+Z4*CV
115      A1=ABS(PH)
116      A2=ABS(R4*CV-Z4*SV)
117      D6=A2
118      IF(A1.LE.D1) GO TO 26
119      D6=A1-D1
120      D6=SQRT(D6*D6+A2*A2)

```

```

121 26 IF (IP.NE.JO.AND.(R6.GT.D6.OR.T6.LE.D6)) GO TO 41
122 Z5=R4*R4+Z4*Z4
123 R5=R3*R1
124 PHM=.5*R3*SV
125 DO 33 K=1,NPHI
126 A1=C3(K)
127 RR=Z5+R5*A1
128 H1A=0.
129 H2A=0.
130 H3A=0.
131 H4A=0.
132 H5A=0.
133 IF (RR.LT.T62) GO TO 34
134 DO 35 L=1,NT
135 W=Z7(L)+R7(L)*A1
136 R=SQRT(W)
137 SN=-SIN(R)
138 CS=COS(R)
139 D6=AT(L)/R
140 H1B=D6/W*CMPLX(CS-R*SN,SN+R*CS)
141 H1A=H1B+H1A
142 H2B=XT(L)*H1B
143 H2A=H2B+H2A
144 H3A=XT(L)*H2B+H3A
145 H4B=D6*CMPLX(CS,SN)
146 H4A=H4B+H4A
147 H5A=XT(L)*H4B+H5A
148 35 CONTINUE
149 GO TO 36
150 34 DO 37 L=1,NT
151 W=Z7(L)+R7(L)*A1
152 R=SQRT(W)
153 IF (R.GT..5) GO TO 14
154 CS=R*(W*(.6944444E-2-W*.1736111E-3)-.125)
155 SN=W*(.3333333E-1-W*.1190476E-2)-.3333333
156 H1B=AT(L)*CMPLX(CS,SN)
157 CS=R*(W*(.4166667E-1-.1388889E-2*W)-.5)
158 SN=W*(W*(.1984126E-3*W-.8333333E-2)+.1666667)-1.
159 H4B=AT(L)*CMPLX(CS,SN)
160 GO TO 43
161 14 SN=-SIN(R)
162 CS=COS(R)
163 D6=AT(L)/R
164 H1B=D6*((CMPLX(CS-R*SN,SN+R*CS)-1.)/W-.3)
165 H4B=D6*CMPLX(CS-1.,SN)
166 43 H1A=H1B+H1A
167 H2B=XT(L)*H1B
168 H2A=H2B+H2A
169 H3A=XT(L)*H2B+H3A
170 H4A=H4B+H4A
171 H5A=XT(L)*H4B+H5A
172 37 CONTINUE
173 A1=PH+PHM*A1
174 A2=ABS(A1)
175 R=RR-A2*A2
176 D6=A2-D1
177 D7=A2+D1
178 D62=D6*D6
179 D72=D7*D7
180 D8=SQRT(D62+R)

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```

181      D9=SQRT(D72+R)
182      IF(R-(RR*1.E-5)) 52,52,53
183 52    IF(D6.LT.0.) STOP
184      W4=.5/D62-.5/D72
185      GO TO 54
186 53    W4=(D7/D9-D6/D8)/R
187 54    IF(D6.GE.0.) GO TO 38
188      W=ALOG((D7+D9)*(-D6+D8)/R)
189      GO TO 39
190 38    W=ALOG((D7+D9)/(D6+D8))
191 39    W1=(W4+.5*W)/D1
192      W5=A2/D1
193      W2=(.5*(D9-D8)-1./D9+1./D8)/P6-W5*W1
194      W3=(.25*(D7*D9-D6*D8)+W-R*(W4+.25*W))/P7-W5*(2.*W2+W5*W1)
195      W4=W/D1
196      W5=(D9-D8-A2*W)/P6
197      IF(A1.GE.0.) GO TO 27
198      W2=-W2
199      W5=-W5
200 27    H1A=W1+H1A
201      H2A=W2+H2A
202      H3A=W3+H3A
203      H4A=W4+H4A
204      H5A=W5+H5A
205 36    GA(K)=H1A
206      GB(K)=H2A
207      GC(K)=H3A
208      GD(K)=H4A
209      GE(K)=H5A
210 33    CONTINUE
211      K1=0
212      DO 45 M=1,M3
213      H1A=0.
214      H2A=0.
215      H3A=0.
216      H1B=0.
217      H2B=0.
218      H3B=0.
219      H1C=0.
220      H2C=0.
221      H3C=0.
222      H4A=0.
223      H5A=0.
224      H6A=0.
225      H4B=0.
226      H5B=0.
227      H6B=0.
228      DO 46 K=1,NPHI
229      K1=K1+1
230      D6=C4(K1)
231      D7=C5(K1)
232      D8=C6(K1)
233      UA=GA(K)
234      UB=GB(K)
235      UC=GC(K)
236      UD=GD(K)
237      UE=GE(K)
238      H1A=D6*UA+H1A
239      H2A=D7*UA+H2A
240      H3A=D8*UA+H3A

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241      H1B=D6*UB+H1B
242      H2B=D7*UE+H2B
243      H3B=D8*UB+H3B
244      H1C=D6*UC+H1C
245      H2C=D7*UC+H2C
246      H3C=D8*UC+H3C
247      H4A=D6*UD+H4A
248      H5A=D7*UD+H5A
249      H6A=D8*UD+H6A
250      H4B=D6*UE+H4B
251      H5B=D7*UE+H5B
252      H6B=D8*UE+H6B
253  46 CONTINUE
254      G1A(M)=H1A
255      G2A(M)=H2A
256      G3A(M)=H3A
257      G1B(M)=H1B
258      G2B(M)=H2B
259      G3B(M)=H3B
260      G1C(M)=H1C
261      G2C(M)=H2C
262      G3C(M)=H3C
263      G4A(M)=H4A
264      G5A(M)=H5A
265      G6A(M)=H6A
266      G4B(M)=H4B
267      G5B(M)=H5B
268      G6B(M)=H6B
269  45 CONTINUE
270      IF(IP.NE.JQ) GO TO 47
271      A1=D5*D5
272      D8=0.
273      D9=0.
274      DO 63 K=1,NPH
275      D8=D8+A(K)/SQRT(C2(K)+A1)
276      D9=D9+A(K)*BLOG(D5/C1(K))
277  53 CONTINUE
278      A2=3.141593/D5
279      D8=(BLOG(A2)-P(2*D8)/(R5*R1))
280      D9=2./R1*(BLOG(A2)+A2*BLOG(1./A2))-3.141593/D1*D9
281      DO 67 M=1,M3
282      G1A(M)=D8+G1A(M)
283      G2A(M)=0.
284      G2B(M)=0.
285      G2C(M)=0.
286      G3A(M)=0.
287      G5A(M)=D9+G5A(M)
288  67 CONTINUE
289      GO TO 47
290  41 DO 25 M=1,M3
291      G1A(M)=0.
292      G2A(M)=0.
293      G3A(M)=0.
294      G1B(M)=0.
295      G2B(M)=0.
296      G3B(M)=0.
297      G1C(M)=0.
298      G2C(M)=0.
299      G3C(M)=0.
300      G4A(M)=0.

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301      G5A(M)=0.
302      G6A(M)=0.
303      G4B(M)=0.
304      G5B(M)=0.
305      G6B(M)=0.
306  25  CONTINUE
307      DO 13 L=1,NT
308      R5=P7(L)
309      Z5=Z7(L)
310      DO 17 K=1,NPHI
311      W=Z5+R5*C3(K)
312      R=SQRT(W)
313      SN=-SIN(R)
314      CS=COS(R)
315      GA(K)=CMPLX(CS-R*SN,SN+R*CS)/(W*R)
316      GD(K)=CMPLX(CS,SN)/R
317  17  CONTINUE
318      (F(R62.LE.Z5) GO TO 51
319      D6=0.
320      D7=0.
321      D9=0.
322      DO 62 K=1,NPHI
323      W2=C2(K)
324      W=1./Z5+R5*W2)
325      W1=A(K)*SQRT(W)
326      D6=D6+W1*W2*W
327      D7=D7+W1*(.5+W*(1.+.(25*W*R5*W2*W2))
328      D9=D9+W1
329  62  CONTINUE
330      W1=R5/Z5
331      W2=PP*W1
332      W=SQRT(W2)
333      W3=1.+W2
334      R=SQRT(W3)
335      W4=SQRT(R5)
336      W5=ALOG(W+R)
337      D8=-P[2*D6-(W/R-W5)/(R5*W4)
338      D6=.5*D8
339      D7=((W/R*(W1-(.125+.1666667*W2)/W3)+.125*W5)/R5+.5*W5)/W4-P[2*D7
340      D9=W5/W4-P[2*D9
341  51  A1=AT(L)
342      A2=XT(L)*A1
343      A3=XT(L)*A2
344      K1=0
345      DO 30 M=1,M3
346      W=M+M4
347      H1A=0.
348      H2A=0.
349      H3A=0.
350      H4A=0.
351      H5A=0.
352      H6A=0.
353      DO 32 K=1,NPHI
354      K1=K1+1
355      H1B=GA(K)
356      W4=C4(K1)
357      W5=C5(K1)
358      W6=C6(K1)
359      H1A=W4*H1B+H1A
360      H2A=W5*H1B+H2A

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361      H3A=W6*H1B+H3A
362      H1B=GD(K)
363      H4A=W4*H1B+H4A
364      H5A=W5*H1B+H5A
365      H6A=W6*H1B+H6A
366      32 CONTINUE
367      (F(R62.LE.Z5) GO TO 44
368      H1A=D6+H1A
369      H2A=D7-(W*W+1.)*D6+H2A
370      H3A=W*D8+H3A
371      H5A=D9+H5A
372      44 G1A(M)=A1*H1A+G1A(M)
373      G2A(M)=A1*H2A+G2A(M)
374      G3A(M)=A1*H3A+G3A(M)
375      G1B(M)=A2*H1A+G1B(M)
376      G2B(M)=A2*H2A+G2B(M)
377      G3B(M)=A2*H3A+G3B(M)
378      G1C(M)=A3*H1A+G1C(M)
379      G2C(M)=A3*H2A+G2C(M)
380      G3C(M)=A3*H3A+G3C(M)
381      G4A(M)=A1*H4A+G4A(M)
382      G5A(M)=A1*H5A+G5A(M)
383      G6A(M)=A1*H6A+G6A(M)
384      G4B(M)=A2*H4A+G4B(M)
385      G5B(M)=A2*H5A+G5B(M)
386      G6B(M)=A2*H6A+G6B(M)
387      30 CONTINUE
388      13 CONTINUE
389      47 A2=D(IP)
390      A3=.5*A2
391      W1=A3*(R4*D3-Z4*D2)
392      W2=-A3*R3*D3
393      A3=DZ(IP)
394      D6=DR(IP)
395      D7=Z4*D6
396      D9=D3*D6
397      H1C=(R1*D9-D2*(R3*A3+D7))*U
398      D8=A2*D1
399      H3C=D8*U
400      H2C=Z4*H3C
401      H3C=D3*H3C
402      W3=P3*(R4*A3-D7)
403      W4=P3*(D2*A3-D9)
404      W5=P3*R1*A3
405      A1=DR(IP)
406      U5=A1*U3
407      U6=.7*U4
408      D6=-D2*A2
409      D7=D1*A1
410      A3=DH(IP)
411      JM=JN
412      DO 31 M=1,M3
413      H2A=G2A(M)
414      H1A=G1A(M)
415      H2B=G2B(M)
416      H1B=G1B(M)
417      UC=W1*H2A+W2*H1A
418      UB=W1*H2B+W2*H1B
419      UF=W3*(H2A+D4*H2B)+W4*(H2B+D4*G2C(M))+W5*(H1A+P4*H1B+P5*G1C(M))
420      UA=UC-UB

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421      UB=UC+UB
422      UG=UA
423      UH=UB
424      IF (IP.NE.JC) GO TO 48
425      UA=P1A+UA
426      UG=P1B+UG
427      UB=P1B+UB
428      UH=P1C+UH
429      UF=P2+UF
430  48 H3A=G3A(M)
431      H3B=G3B(M)
432      UC=H1C*H3A
433      UD=H1C*H3B
434      UE=H2C*(H3A+D4*H3B)+H3C*(H3B+D4*G3C(M))
435      H5A=G5A(M)
436      H5B=G5B(M)
437      H4A=G4A(M)+H5A
438      H4B=G4B(M)+H5B
439      H6A=G6A(M)
440      H6B=G6B(M)
441      H3A=U5*H5A+U6*H4A
442      H1B=U5*H5B+U6*H4B
443      H1A=H3A-H1B
444      H2A=H3A+H1B
445      H3A=-U1*H4A
446      H1B=D6*H6A
447      W=M+M4
448      A1=W*A3
449      H2B=D6*H6B-A1*H4A
450      H3B=D7*(H6A+D4*H6B)
451      H4A=W*D5*H4A
452      K1=IP+JM
453      K2=K1+1
454      K3=K1+N
455      K4=K2+N
456      K5=K2+MT
457      K6=K4+MT
458      K7=K3+N2N
459      K8=K4+N2N
460      K9=K8+MT
461      GO TO (18,20,19),KQ
462  18 Y(K6)=UC+UD
463      Z(K6)=H1B+H2B
464      IF (IP.EQ.1) GO TO 21
465      Y(K3)=Y(K3)+UB
466      Y(K7)=Y(K7)+UE
467      Z(K3)=Z(K3)+H2A-H3A
468      Z(K7)=Z(K7)+H3B-H4A
469      IF (IP.EQ.MP) GO TO 22
470  21 Y(K4)=UH
471      Y(K8)=UE
472      Z(K4)=H2A+H3A
473      Z(K8)=H3B+H4A
474      GO TO 22
475  19 Y(K5)=Y(K5)+UC-UD
476      Z(K5)=Z(K5)+H1B-H2B
477      IF (IP.EQ.1) GO TO 23
478      Y(K1)=Y(K1)+UA
479      Y(K7)=Y(K7)+UE
480      Z(K1)=Z(K1)+H1A+H3A
481      Z(K7)=Z(K7)+H3B-H4A
482      IF (IP.EQ.MP) GO TO 22
483  23 Y(K2)=Y(K2)+UG
484      Y(K8)=UE
485      Z(K2)=Z(K2)+H1A-H3A
486      Z(K8)=H3B+H4A
487      GO TO 22
488  20 Y(K5)=Y(K5)+UC-UD
489      Y(K6)=UC+UD
490      Z(K5)=Z(K5)+H1B-H2B
491      Z(K6)=H1B+H2B
492      IF (IP.EQ.1) GO TO 24
493      Y(K1)=Y(K1)+UA
494      Y(K3)=Y(K3)+UB
495      Y(K7)=Y(K7)+UE
496      Z(K1)=Z(K1)+H1A+H3A
497      Z(K3)=Z(K3)+H2A-H3A
498      Z(K7)=Z(K7)+H3B-H4A
499      IF (IP.EQ.MP) GO TO 22
500  24 Y(K2)=Y(K2)+UG
501      Y(K4)=UH
502      Y(K8)=UE
503      Z(K2)=Z(K2)+H1A-H3A
504      Z(K4)=H2A+H3A
505      Z(K8)=H3B+H4A
506  22 Y(K9)=UF
507      Z(K9)=U2*(D8*(H5A+D4*H5B)-A1*H4A)
508      JM=JM+N2
509  31 CONTINUE
510  16 CONTINUE
511      JN=JN+N
512  15 CONTINUE
513      RETURN
514      END

```

VIII. THE MAIN PROGRAM FOR BOTH THE H-FIELD AND E-FIELD SOLUTIONS

The main program for both the H-field and E-field solutions obtains the present H-field solution and the E-field solution of [1] for the electric current on a conducting body of revolution immersed in an incident plane wave. Input data are read from punched cards. These input data are the same as those for the main program for the H-field solution which is described and listed in Section VI of Part Two.

The main program for both the H-field and E-field solutions calls the subroutines YZ, PLANE, DECOMP, and SOLVE. The function subprogram BLOG is also needed because it is called by the sub-routine YZ

Minimum allocations are given by

```
COMPLEX Y(N*N), Z(N*N), RE(2*N), R(2*N), B(N), C(N)
DIMENSION XT(NT), AT(NT), X(NPHI), A(NPHI), RH(NP),
          ZH(NP), IPS(N), IPT(N)
```

where

$$N = 2*NP-3$$

The t and ϕ components of the present H-field solution for the electric current are calculated from (84) and (85). The t and ϕ components of the E-field solution of [1] for the electric current are also calculated from (84) and (85). In the present H-field solution, the coefficients I_{1p}^t and I_{1p}^ϕ in these equations are the p th elements of the vectors $\vec{I}_1^t$ and $\vec{I}_1^\phi$ which satisfy the $n=1$ equation in (10). In the E-field solution of [1], these coefficients are the elements of the vectors $\vec{I}_1^t$ and $\vec{I}_1^\phi$ which satisfy the $n=1$ equation in (6) of [1]. DO loop 28 prepares RH and ZH for use in the subroutines YZ and PLANE by multiplying them by k . With regard to (10), line 42 puts Y_1 of (86) in Y . With regard to (6) of [1], line 42 puts Z_1 of (96) in Z . Line 45 calculates IPS and changes Y . Line 46 calculates IPT and changes Z . With regard to (10), line 47 puts the excitation vectors $\vec{I}_1^t$ and $\vec{I}_1^\phi$ for the θ

polarized incident plane wave (69) in RE. With regard to (6) of [1], line 47 puts the vectors $\vec{V}_1^t$ and $-\vec{V}_1^\phi$ for the θ polarized incident plane wave in R. Line 47 also stores vectors for the ϕ polarized incident plane wave (70) further on in RE and R but these vectors are not used in the main program.

In DO loop 36, JHE = 1 obtains the present H-field solution for the electric current and JHE = 2 obtains the E-field solution of [1] for the electric current. In line 55, the output IPS and Y from the subroutine DECOMP is fed along with N and RE into the subroutine SOLVE. SOLVE puts the solution $\vec{I}_1^t$ and $\vec{I}_1^\phi$ to (10) in C. Lines 59-64 put $\vec{V}_1^t$ and $\vec{V}_1^\phi$ in B. In line 67, the output IPT and Z from the subroutine DECOMP is fed along with N and B into the subroutine SOLVE. SOLVE puts the solution $\vec{I}_1^t$ and $\vec{I}_1^\phi$ to (6) of [1] in C. The t and ϕ components of the normalized electric current are printed out under the headings JT and JP, respectively. The normalization is the same as in the main program for the H-field solution which is described in Section VI of Part Two. The sample input and output data are for the sphere examples of Figs. 2 and 4.

```

001C      LISTING OF THE MAIN PROGRAM FOR BOTH THE H-FIELD AND E-FIELD SOLUTIONS
002C      THE SUBPROGRAMS YZ,BLOG,PLANE,DECOMP, AND SOLVE ARE NEEDED
003//PGM JOB (XXXX,XXXX,1,1),'MAUTZ,JOE',REGION=200K
004// EXEC WATFIV
005//GO,SYSIN DD *
006$JOB          MAUTZ,TIME=5,PAGES=60
007      COMPLEX Y(1600),Z(1600),RE(240),R(240),B(40),C(40),U,C1
008      DIMENSION XT(10),AT(10),X(48),A(48),RH(43),ZH(43),THR(3),IPS(40)
009      DIMENSION IPT(40)
010      READ(1,15) NT,NPHI
011      15 FORMAT(2I3)
012      WRITE(3,30) NT,NPHI
013      30 FORMAT('  NT NPHI'/'1X,{3},{5})
014      READ(1,10)(XT(K),K=1,NT)
015      READ(1,10)(AT(K),K=1,NT)
016      10 FORMAT(5E14.7)
017      WRITE(3,11)(XT(K),K=1,NT)
018      WRITE(3,12)(AT(K),K=1,NT)
019      11 FORMAT('  XT'/'(1X,5E14.7))
020      12 FORMAT('  AT'/'(1X,5E14.7))
021      READ(1,10)(X(K),K=1,NPHI)
022      READ(1,10)(A(K),K=1,NPHI)
023      WRITE(3,13)(X(K),K=1,NPHI)
024      WRITE(3,14)(A(K),K=1,NPHI)
025      13 FORMAT('  X'/'(1X,5E14.7))
026      14 FORMAT('  A'/'(1X,5E14.7))
027      READ(1,16) NP,BK,THR(1)
028      16 FORMAT(13,2E14.7)
029      WRITE(3,17) NP,BK,THR(1)
030      17 FORMAT('  NP',6X,'BK',12X,'THR'/'1X,{3,2E14.7})
031      READ(1,18)(RH(I),I=1,NP)
032      READ(1,18)(ZH(I),I=1,NP)
033      18 FORMAT(10F8.4)
034      WRITE(3,19)(RH(I),I=1,NP)
035      WRITE(3,20)(ZH(I),I=1,NP)
036      19 FORMAT('  RH'/'(1X,10F8.4))
037      20 FORMAT('  ZH'/'(1X,10F8.4))
038      DO 28 J=1,NP
039      RH(J)=BK*RH(J)
040      ZH(J)=BK*ZH(J)
041      28 CONTINUE
042      CALL YZ(1,1,NP,NPHI,NT,1,RH,ZH,X,A,XT,AT,Y,Z)
043      MT=NP-2
044      N=2*MT+1
045      CALL DECOMP(N,IPS,Y)
046      CALL DECOMP(N,IPT,Z)
047      CALL PLANE(1,1,1,NP,3,NT,RH,ZH,XT,AT,THR,RE,R)
048      U=(0.,1.)
049      DO 36 JHE=1,2
050      GO TO (33,34),JHE
051      33 WRITE(3,29)(Y(J),J=1,N)
052      29 FORMAT('  Y'/'(1X,6E11.4))
053      WRITE(3,35)(RE(J),J=1,N)
054      35 FORMAT('  R'/'(1X,6E11.4))
055      CALL SOLVE(N,IPS,Y,RE,C)
056      GO TO 32
057      34 WRITE(3,31)(Z(J),J=1,N)
058      31 FORMAT('  Z'/'(1X,6E11.4))
059      DO 22 J=1,MT
060      B(J)=R(J)

```

```

061      J1=J+MT
062      B(J1)=R(J1)
063      22 CONTINUE
064      B(N)=-R(N)
065      WRITE(3,23)(B(J),J=1,N)
066      23 FORMAT(' E'/(1X,6E11.4))
067      CALL SOLVE(N, IPT, Z, B, C)
068      32 WRITE(3,21)
069      21 FORMAT('      REAL JT      (MAG JT      MAG JT')
070      DO 24 J=1, MT
071      C1=2./RH(J+1)*C(J)
072      C2=CABS(C1)
073      WRITE(3,25) C1, C2
074      25 FORMAT(1X, 3E11.4)
075      24 CONTINUE
076      WRITE(3,26)
077      26 FORMAT('      REAL JP      (MAG JP      MAG JP')
078      MP=NP-1
079      DO 27 J=1, MP
080      C1=4./(RH(J)+RH(J+1))*U*C(J+MT)
081      C2=CABS(C1)
082      WRITE(3,25) C1, C2
083      27 CONTINUE
084      36 CONTINUE
085      STOP
086      END
$ DATA
  2 20
-0.5773503E+00 0.5773503E+00
  0.1000000E+01 0.1000000E+01
-0.9931286E+00-0.9639719E+00-0.9122344E+00-0.8391170E+00-0.7463319E+00
-0.6360537E+00-0.5108670E+00-0.3737061E+00-0.2277859E+00-0.7652652E-01
  0.7652652E-01 0.2277859E+00 0.3737061E+00 0.5108670E+00 0.6360537E+00
  0.7463319E+00 0.8391170E+00 0.9122344E+00 0.9639719E+00 0.9931286E+00
  0.1761401E-01 0.4060143E-01 0.6267205E-01 0.8327674E-01 0.1019301E+00
  0.1181945E+00 0.1316886E+00 0.1420961E+00 0.1491730E+00 0.1527534E+00
  0.1527534E+00 0.1491730E+00 0.1420961E+00 0.1316886E+00 0.1181945E+00
  0.1019301E+00 0.8327674E-01 0.6267205E-01 0.4060143E-01 0.1761401E-01
  16 0.1256637E+01 0.3141593E+01
  0.0000 0.2079 0.4067 0.5878 0.7431 0.8660 0.9511 0.9945 0.9945 0.9511
  0.8660 0.7431 0.5878 0.4067 0.2079 0.0000
-1.0000 -0.9781 -0.9135 -0.8090 -0.6691 -0.5000 -0.3090 -0.1045 0.1045 0.3090
  0.5000 0.6691 0.8090 0.9135 0.9871 1.0000
$ STOP
/*
//
PRINTED OUTPUT CONSISTS OF THAT FROM THE MAIN PROGRAM FOR THE
H-FIELD SOLUTION IN SECTION VI OF PART TWO PLUS THE FOLLOWING LINES.
Z
  0.6733E-01-0.1108E+02 0.6205E-01 0.3162E+01 0.5399E-01 0.8394E+00
  0.4415E-01 0.3112E+00 0.3365E-01 0.1593E+00 0.2345E-01 0.9534E-01
  0.1437E-01 0.6434E-01 0.6919E-02 0.4879E-01 0.1242E-02 0.4132E-01
-0.2739E-02 0.3823E-01-0.5271E-02 0.3745E-01-0.6728E-02 0.3773E-01
-0.7250E-02 0.3855E-01-0.7584E-02 0.3880E-01-0.1503E+02 0.6952E-01
  0.2715E+01 0.6766E-01 0.4763E+00 0.6412E-01 0.1469E+00 0.5914E-01
  0.7427E-01 0.5315E-01 0.5232E-01 0.4656E-01 0.4501E-01 0.3977E-01
  0.4254E-01 0.3317E-01 0.4167E-01 0.2706E-01 0.4120E-01 0.2165E-01
  0.4074E-01 0.1707E-01 0.4026E-01 0.1339E-01 0.3982E-01 0.1064E-01
  0.3996E-01 0.2753E-02 0.3906E-01 0.7696E-02
B

```

-0.2754E+00-0.7539E+00-0.3120E+00-0.6811E+00-0.3507E+00-0.5623E+00
 -0.3645E+00-0.4085E+00-0.3286E+00-0.2433E+00-0.2313E+00-0.1012E+00
 -0.8351E-01-0.1840E-01 0.8351E-01-0.1840E-01 0.2313E+00-0.1012E+00
 0.3286E+00-0.2433E+00 0.3645E+00-0.4085E+00 0.3507E+00-0.5623E+00
 0.3106E+00-0.6817E+00 0.2697E+00-0.7559E+00 0.7801E+00-0.2694E+00
 0.7641E+00-0.3111E+00 0.7268E+00-0.3900E+00 0.6583E+00-0.4958E+00
 0.5505E+00-0.6128E+00 0.3995E+00-0.7201E+00 0.2107E+00-0.7956E+00
 -0.0000E+00-0.8227E+00-0.2107E+00-0.7956E+00-0.3995E+00-0.7201E+00
 -0.5505E+00-0.6128E+00-0.6583E+00-0.4958E+00-0.7268E+00-0.3900E+00
 -0.7764E+00-0.3116E+00-0.7793E+00-0.2626E+00

REAL JT IMAG JT MAG JT

-0.8238E+00-0.1971E+01 0.2137E+01
 -0.9777E+00-0.1907E+01 0.2143E+01
 -0.1186E+01-0.1802E+01 0.2158E+01
 -0.1421E+01-0.1620E+01 0.2154E+01
 -0.1629E+01-0.1337E+01 0.2108E+01
 -0.1751E+01-0.9546E+00 0.1954E+01
 -0.1733E+01-0.5013E+00 0.1804E+01
 -0.1545E+01-0.2843E-01 0.1545E+01
 -0.1196E+01 0.4045E+00 0.1263E+01
 -0.7315E+00 0.7493E+00 0.1047E+01
 -0.2198E+00 0.9839E+00 0.1008E+01
 0.2653E+00 0.1116E+01 0.1147E+01
 0.6631E+00 0.1182E+01 0.1355E+01
 0.9285E+00 0.1196E+01 0.1514E+01

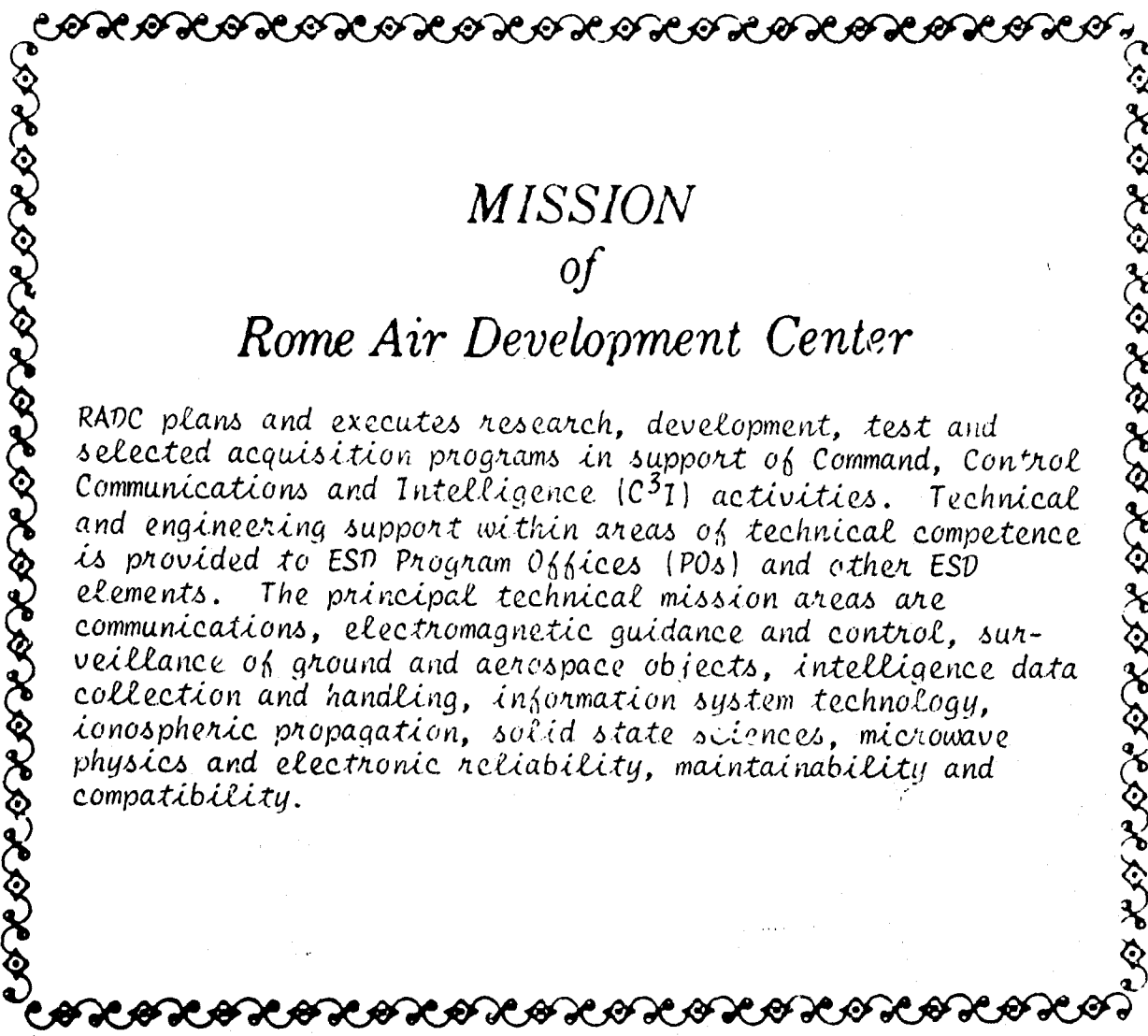
REAL JP IMAG JP MAG JP

0.7997E+00 0.1970E+01 0.2126E+01
 0.9274E+00 0.1833E+01 0.2054E+01
 0.1071E+01 0.1645E+01 0.1963E+01
 0.1221E+01 0.1362E+01 0.1829E+01
 0.1312E+01 0.1014E+01 0.1658E+01
 0.1297E+01 0.6571E+00 0.1454E+01
 0.1166E+01 0.3656E+00 0.1222E+01
 0.9561E+00 0.2058E+00 0.9780E+00
 0.7409E+00 0.2063E+00 0.7691E+00
 0.5921E+00 0.3454E+00 0.6855E+00
 0.5508E+00 0.5635E+00 0.7880E+00
 0.6120E+00 0.7924E+00 0.1001E+01
 0.7237E+00 0.9694E+00 0.1210E+01
 0.8687E+00 0.1113E+01 0.1412E+01
 0.9847E+00 0.1194E+01 0.1547E+01

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