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ERROR ANALYSIS OF HYDROGRAPHIC POSITIONING AND THE APPLICATION --ETC(U)

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ERROR ANALYSIS OF HYDROGRAPHIC POSITIONING
AND THE APPLICATION OF LEAST SQUARES

by

Ali Kaplan

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Thesis Advisor: Dudley Leath

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Repeatable accuracy of hydrographic positioning was examined in terms of the two-dimensional normal distribution function which results in an ellipitical error figure. The error ellipse was discussed, and two methods for conversion of elliptical errors to circular errors were given. These methods are "circle of equivalent probability" and "root mean square error" (d_{rms}).

Using the (d_{rms}) error concept, repeatable accuracy of ranging, azimuthal, and hyperbolic systems was evaluated, and methods were developed to draw repeatability contours for those systems.

A brief theoretical background was provided to explain the method of least squares and discuss its application to hydorgraphic survey positioning. For ranging, hyperbolic, azimuthal, sextant angle, and Global Positioning System the least squares observation equations were developed. Specific examples were constructed to demonstrate the capabilities of this data adjustment technique when applied to redundant position observations.

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Error Analysis of Hydrographic Positioning and the Application of Least Squares

by

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Submitted in partial fulfillment of the requirements for the degree of

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ABSTRACT

Repeatable accuracy of hydrographic positioning was examined in terms of the two-dimensional normal distribution function which results in an elliptical error figure. The error ellipse was discussed, and two methods for conversion of elliptical errors to circular errors were given. These methods are "circle of equivalent probability" and "root mean square error" (d_{rms}). Using the d_{rms} error concept, repeatable accuracy of ranging, azimuthal, and hyperbolic systems was evaluated, and methods were developed to draw repeatability contours for those systems.

A brief theoretical background was provided to explain the method of least squares and discuss its application to hydrographic survey positioning. For ranging, hyperbolic, azimuthal, sextant angle, and Global Positioning System the least squares observation equations were developed. Specific examples were constructed to demonstrate the capabilities of this data adjustment technique when applied to redundant position observations.

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I. INTRODUCTION

Positioning of the survey vessel is equal in importance with depth determination in the collection of hydrographic survey data. Fundamental to an understanding of the accuracy of position information is an analysis of the various errors and their sources which must be either eliminated, compensated for, or otherwise modeled. The result of this analysis is that the reliability of position data can be evaluated and used to estimate the overall accuracy of hydrographic soundings.

Once these potential error sources are understood, methods must be developed to quantify accuracy. Much research has been conducted in this area in the past. One purpose of this thesis is to collect and present useful concepts of error theory which apply directly to hydrographic survey. Simple graphical techniques were developed which can be used to produce accuracy contours as a function of the survey net geometry.

Conventional survey techniques rely on only two lines of position (LOP) to determine a positioning fix. This introduces the possibility of significant error.

In navigation, although inherently less accurate than positioning due to the techniques and systems used to determine the LOP's, three LOP's are required to produce a fix. Position is adjusted graphically by placing the fix in the center of the triangle formed by the three intersecting LOP's. This concept of taking one redundant observation can lead to significant improvement in hydrographic survey positioning data. Mathematical adjustment techniques such as the method of least squares may be used to determine the best estimate of position.

Least square adjustments are commonly performed on land survey data where redundant observations are easily made. With the advent of new positioning systems and computer technology, making redundant observations at sea is no longer impractical. The second purpose of this thesis is to explain the basic method of least squares, and to formulate examples of the least squares adjustment procedure applied to specific types of hydrographic survey systems. This data adjustment technique not only provides the best estimate of position but also may be used to determine the absolute positioning accuracy associated with each data point in a hydrographic survey.

II. REPEATABLE ACCURACY OF HYDROGRAPHIC SURVEY POSITIONS

A. TYPES OF ERRORS

It is impossible to make measurements of physical data without making errors. These measurement errors may be classified in the following manner.

1. Blunders

These are mistakes which result from misreading instruments, transposing figures, faulty computations, etc. They may be large and easily observed, or smaller and less detectable, or very small and indistinguishable in the data. Blunders are usually detected through comparing repeated measurements, careful editing, and procedural checks in the data collection process. Physical measurements will contain a constant bias if these errors are not removed from the data set.

2. Systematic Errors

Uncalibrated instruments or environmental factors, such as temperature and humidity changes which affect the performance of the measuring instruments, will induce systematic errors into the observations. The occurrence of this type of error may result in a pattern which can be recognized and mathematically modeled. The simplest pattern to model would be some observable trend in the data of constant magnitude and direction. Such a trend can easily be subtracted from the observations to remove the systematic error.

If numerous systematic errors exist, or the errors are such that they cannot be accurately modeled, then their effect on the data must be estimated by calibration. Calibration is the process of comparing the measuring instrument against a known standard. The difference between the observed and known value may be used as an estimate of the total effect of all systematic errors present. Thus, calibration provides a "corrector" which must be applied to the data set. Examples of important systematic errors in hydrographic survey positioning include instrument errors, errors in positioning control points, and variations in the propagation velocity of electromagnetic energy.

3. Random Errors

These errors result from accidental and unknown causes. Their effect cannot be removed from the observations and, therefore, must be quantified statistically. Random errors have certain characteristics which facilitate such an approach. Positive and negative errors occur with equal frequency, small errors are more probable than large errors, and extremely large errors rarely occur.

The frequency distribution of random errors can be modeled mathematically by the normal distribution function. Assuming all measurement errors are independent and random, thereby conforming to the normal distribution, measurement accuracy can be specified statistically by defining a confidence interval around the best estimate of the measured value. Procedures for computing these intervals are reviewed in Appendix A.

B. ACCURACY OF HYDROGRAPHIC POSITIONS

The achievable accuracy of a hydrographic survey positioning system is best described by defining the following terms: repeatability and predictability.

Repeatability is a measure of the accuracy with which the positioning system permits the user to return to a specific point on the surface of the earth defined in terms of the lines of position generated by the system. Included in repeatability are the effects of random errors, errors due to net geometry, and errors resulting from the angle of intersection for the two lines of position that establish a fix. Repeatable accuracy is therefore a measure of the relative accuracy of a positioning system. Unresolved biases exist in hydrographic positions due to the presence of systematic errors that have not been subtracted from the data or compensated for as a result of calibration.

Predictability is the measure of accuracy with which the system can define the location of the same point in terms of geographic (or geodetic) coordinates rather than simply the intersection of two lines of position. Thus, predictable accuracy is an absolute accuracy. Using conventional hydrographic survey techniques, predictability could be achieved only if all systematic errors were removed from the data so that only the effects of random errors, net geometry, and intersection angle remain. For example, the lattice generated by an electronic positioning system is distorted primarily as a result of the variability in the propagation velocity of electromagnetic energy. Ideally, if there was no distortion of the electronic lattice, then the accuracy of a position, corrected for any remaining systematic errors, could be quantified statistically in terms of predictable accuracy. However, since these distortions exist, the effective velocity of propagation would have to be accurately modeled throughout the survey area. Then it would be possible to subtract the effects of this systematic error and derive positions in terms of predictable geographic coordinates. Research is currently being conducted to quantify the parameters which affect propagation velocity in order to model these values for such application [Ref. 19].

A second method to achieve predictable accuracy is by making redundant observations to establish hydrographic survey positions. If three intersecting lines of position are available instead of the usual two, the resulting fix is overdetermined, and data adjustment techniques must be applied. The method of least squares is most useful in adjusting such data. Through the application of least squares adjustment techniques, the best estimate of position is found and the position's predictable accuracy is resolved. A complete discussion of this procedure is presented in Section III.

C. REPEATABLE ACCURACY

In the determination of hydrographic positions, blunders are eliminated by observing strict survey procedures, and system calibration is performed in an attempt to remove systematic errors. Because some systematic errors still remain, the accuracy of hydrographic positions must be stated in terms of repeatability.

The modeling of random errors is done by using the two-dimensional normal distribution function. When the normal distribution is applied to the positional errors, the resulting error figure is an ellipse.

1. Elliptical Errors

Hydrographic positions are determined by the intersection of two lines of position (LOP). Because of the errors in each LOP, the actual position may lie somewhere between the error limits (shown as additional arcs either side of LOP's in Figure II-1).

The intersection of the two LOP's, together with the standard errors associated with each, is drawn to an expanded scale in Figure II-2. By applying the two-dimensional normal

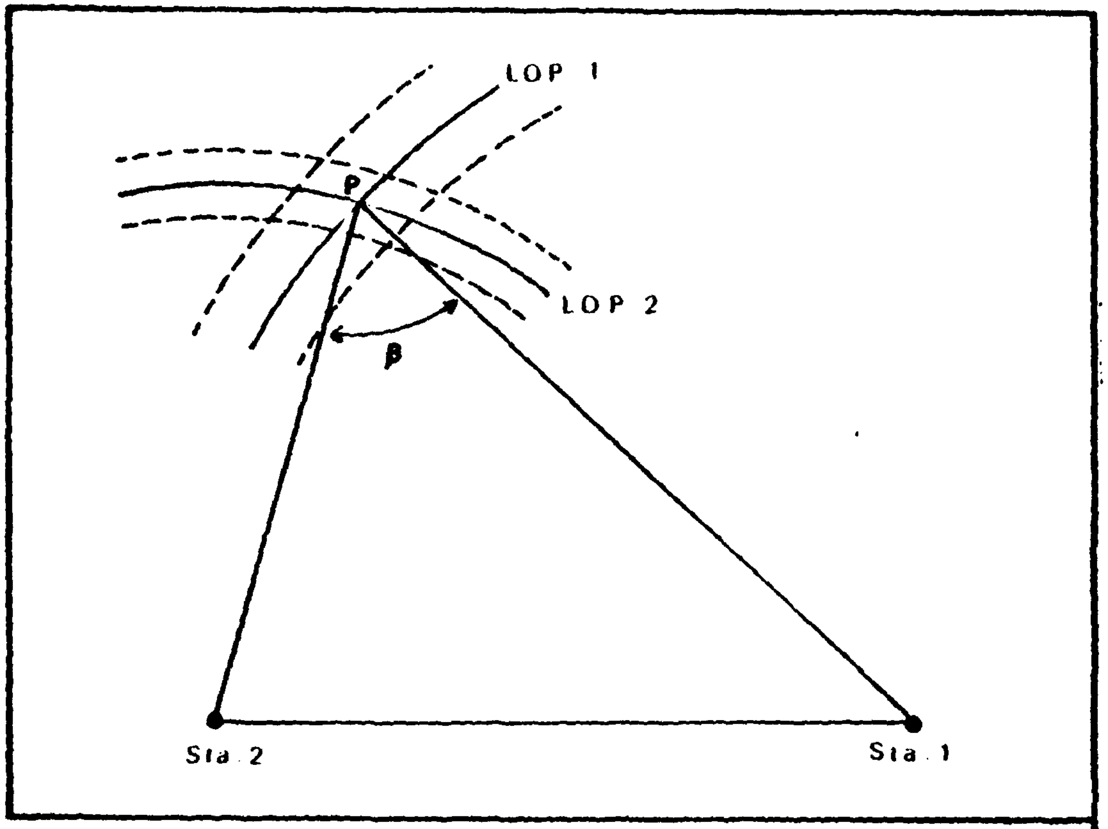

Figure II-1: Position location at the intersection of two lines of position.

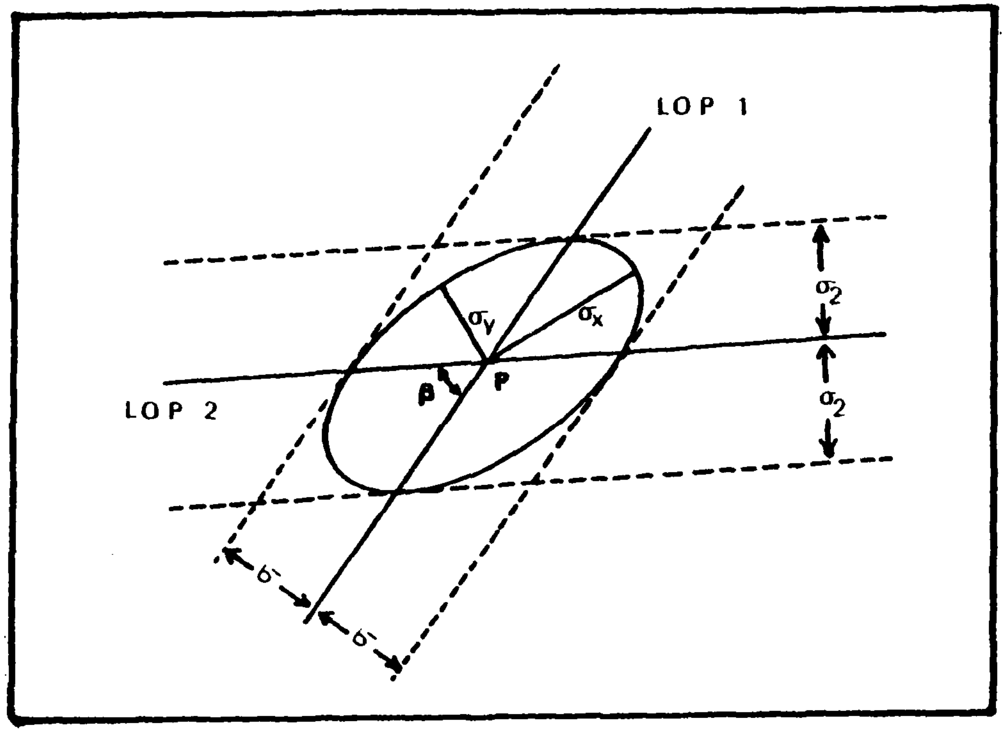

Figure II-2: Expanded view of the intersection of two lines of position and the associated error ellipse.

distribution to positional errors, it is seen that the contours of equal probability density about such an intersection are ellipses with their center at the intersection point.

For simplicity in the discussion, the following assumptions are made:

1. Only errors contributing to repeatable accuracy are considered.

2. The random errors associated with each LOP are assumed to be normally distributed.

3. The random errors in each LOP are assumed to be independent, i.e., a change in the error of one LOP has no effect upon the other.

4. The LOP's are assumed to be straight lines in the small area in the immediate vicinity of their intersection.

5. Errors of position are limited to the two-dimensional case.

As shown in Figure II-2, the general case of the intersection of two LOP's at any angle and with different values of errors associated with each LOP results in an elliptical error figure.

It is readily seen from Figure II-2 that the exact shape of the error ellipse varies with the magnitudes of both of the one-dimensional LOP errors, σ_1 and σ_2, as well as with the angle of intersection, β.

The values of the semi-major and semi-minor axes of the error ellipse (using one σ error) are given by the following equations [Refs. 4 and 5].

Semi-major axis:

$$\sigma_x^2 = \frac{1}{2\sin^2\beta}\left[(\sigma_1^2 + \sigma_2^2) + \sqrt{(\sigma_1^2+\sigma_2^2)^2 - (4\sin^2\beta)\,\sigma_1^2\sigma_2^2}\,\right]; \qquad \text{(II-1a)}$$

Semi-minor axis:

$$\sigma_y^2 = \frac{1}{2\sin^2\beta}\left[(\sigma_1^2 + \sigma_2^2) - \sqrt{(\sigma_1^2+\sigma_2^2)^2 - (4\sin^2\beta)\,\sigma_1^2\sigma_2^2}\,\right]. \qquad \text{(II-1b)}$$

Generally $\sigma_1 = \sigma_2 = \sigma$, then equations (II-1a) and (II-1b) simplify to

$$\sigma_x = \frac{\sigma}{\sqrt{2}\sin(\beta/2)} \quad \text{and} \quad \sigma_y = \frac{\sigma}{\sqrt{2}\cos(\beta/2)}. \qquad \text{(II-2)}$$

After computing the semi-major and semi-minor axes, the probability of the error ellipse is given by the distribution function

$$P(x,y) = 1 - e^{-\frac{h^2}{2}}, \qquad \text{(II-3)}$$

where

$$h^2 = \frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}$$

(x and y are the errors in the direction of σ_x and σ_y).

The solution of equation II-3 with values of h for different probabilities yields the results shown in Table II-1.

Probability (%)	h
39.35	1.0000
50.00	1.1774
63.21	1.4142
90.00	2.1460
99.00	3.0349
99.78	3.5000

Table II-1: Values of constant h.

For example, for 39.35% probability the axes of the ellipse are 1.00 σ_x and 1.00 σ_y; for 50% probability the axes are 1.1774 σ_x and 1.1774 σ_y. Figure II-3 shows the error ellipses for different values of h.

The angle, θ, between the semi-major axis of the error ellipse and the line of position which has smaller standard error is given by [Ref. 5]

$$\tan 2\theta = \frac{\sigma_1^2 \sin 2\beta}{\sigma_1^2 \cos 2\beta + \sigma_2^2} , \qquad \text{(II-4)}$$

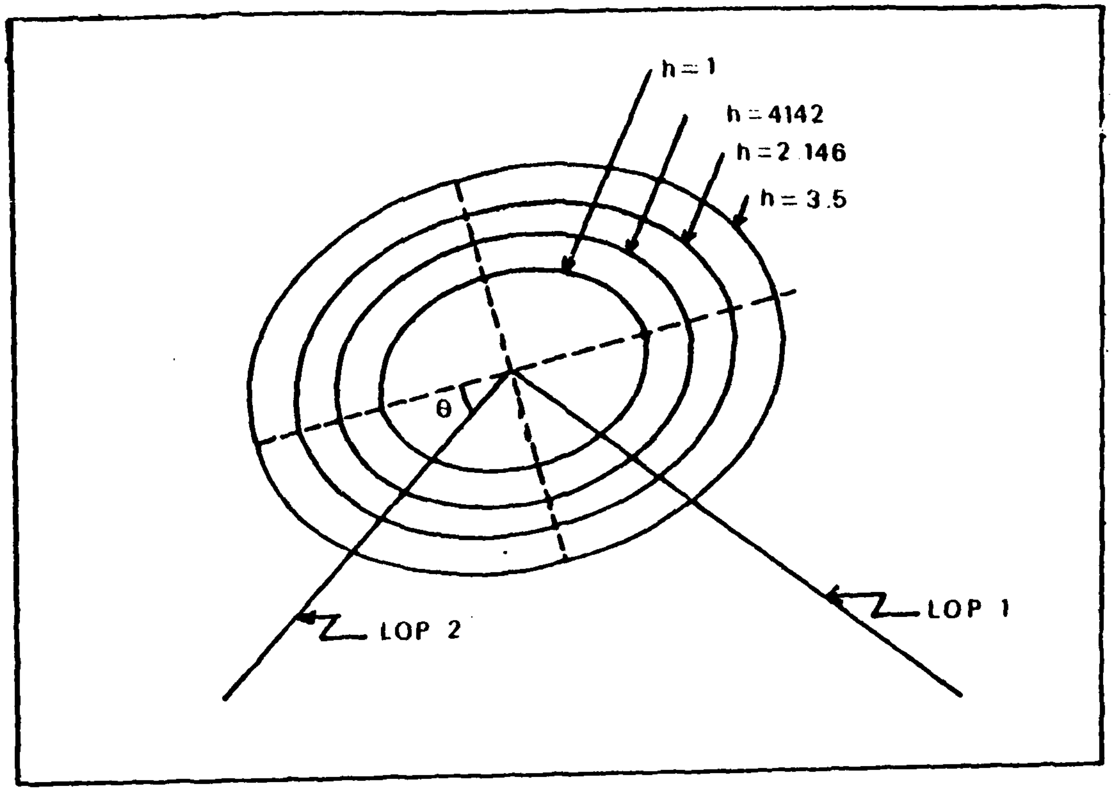

Figure II-3: Contours of equal probability areas.

where σ_1 is the smaller standard error. In the case of $\sigma_1 = \sigma_2$, equation II-4 simplifies to

$$\theta = \frac{\beta}{2} \ . \qquad \text{(II-5)}$$

The importance of the angle θ is that it specifies the orientation of the error ellipse according to the lines of position.

2. Circular Error Approximations

In general, the use of the error ellipse is complicated by the problem of axis orientation and the propagation of elliptical errors. Therefore, in order to simplify probability calculations and avoid the above problems, the elliptical errors are approximated by circular errors which are easier to use and understand. The accuracy of a hydrographic position may then be stated in terms of a circle of specified radius about the point.

Note that when the angle of intersection is a right angle and the two errors are equal, the error ellipse becomes a circle and is described by the circular normal distribution. Generally, this is not the case, and elliptical errors must be converted to circular errors. This is done by using either the circle of equivalent probability or the root-mean-square error concept.

a. Circle of Equivalent Probability

A circle of equivalent probability is obtained utilizing an existing table for the two-dimensional normal distribution (Table II-2). This table is used with the two standard errors along the semi-major and semi-minor axes of the error ellipse (Equations II-1a, II-1b or II-2). To find the radius of equivalent probability, equations II-1a, II-1b or II-2 must first be utilized to obtain the values of σ_x and σ_y. To enter the table the following ratios are needed:

$$c = \frac{\sigma_y}{\sigma_x} \quad \text{where } \sigma_x \text{ is the greater standard error}$$

and

$$K = \frac{\text{Radius of circle of equivalent probability}}{\text{Greater standard error}},$$

where K is the conversion factor needed to solve for the radius (R) of the circle of equivalent probability.

The table relates varying values of ellipticity to the radius of circles of equivalent probability. Enter the table with the computed values for c and K to determine the probability for a circle of given radius, or alternately, for a given value of probability, determine the radius of the error circle.

EXAMPLE II-1: The two standard errors of a positioning system estimated from field observations are $\sigma_1 = \sigma_2 = 6$ meters. To determine the probability of location within a circle of 10 m radius when the angle of intersection, β, is 60°, equation II-2 must be used to find σ_x and σ_y:

K \ e	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.0796557	.0443987	.0842119	.0164176	.0123875	.0099377	.0082940	.0071137	.0062299	.0055400	.0049875
0.2	.1585194	.1339783	.0844533	.0628306	.0482413	.0390193	.0327123	.0281415	.0246824	.0219757	.0198013
0.3	.2358228	.2213804	.1738300	.1318281	.1030193	.0851535	.0719102	.0621388	.0546598	.0487639	.0440025
0.4	.3108435	.3010228	.2635181	.2139084	.1742045	.1451808	.1237982	.1076237	.0950495	.0850326	.0768837
0.5	.3829249	.3755884	.3481790	.3003001	.2532953	.2152880	.1857448	.1626829	.1443941	.1296286	.1175031
0.6	.4514938	.4457708	.4255605	.3846374	.3357384	.2914882	.2548177	.2251114	.2009797	.1811783	.1647298
0.7	.5160727	.5115048	.4960883	.4633258	.4170862	.3699305	.3280302	.2925654	.2629373	.2381753	.2172965
0.8	.5762892	.5725957	.5604457	.5349387	.4941882	.4474207	.4025628	.3627122	.3283453	.2989700	.2738510
0.9	.6318707	.6288721	.6191354	.5993140	.5651564	.5213998	.4758375	.4333628	.3953279	.3620137	.3330232
1.0	.6826895	.6802325	.6723586	.6568242	.6291249	.5900953	.5461319	.5025790	.4621421	.4257[illegible]	.3934693
1.1	.7286679	.7268597	.7202682	.7079681	.6859367	.6524489	.6116316	.5687467	.5272402	.4887871	.4539256
1.2	.7698607	.7682215	.7630308	.7532175	.7359558	.7070973	.6714209	.6306168	.5863494	.5498736	.5132477
1.3	.8063990	.8050848	.8008554	.7920968	.7793550	.7567265	.7249673	.6873122	.6474304	.6079822	.5704426
1.4	.8384867	.8374049	.8340018	.8277048	.8169851	.7989288	.7720889	.7383089	.7007900	.6623035	.6216889
1.5	.8663856	.8655127	.8627728	.8577362	.8493071	.8350816	.8129287	.7833962	.7480500	.7122546	.6753475
1.6	.8904014	.8897008	.8875080	.8834914	.8768844	.8657559	.8478303	.8226246	.7917194	.7574708	.7219627
1.7	.9108691	.9103102	.9085619	.9053766	.9001746	.8915538	.8773116	.8562471	.8291137	.7977882	.7462539
1.8	.9281394	.9276964	.9263125	.9237989	.9197275	.9130680	.9019110	.8846624	.8613238	.8332175	.8021013
1.9	.9425668	.9422182	.9411296	.9391586	.9359855	.9308615	.9222277	.9083609	.8886731	.8639149	.8355255
2.0	.9544997	.9542272	.9533775	.9518415	.9493815	.9454546	.9388418	.9278799	.9115762	.8901495	.8646647
2.1	.9642712	.9640588	.9634011	.9622127	.9603170	.9573205	.9522909	.9437668	.9305013	.9122714	.8897495
2.2	.9721931	.9720304	.9715237	.9706109	.9691597	.9668845	.9631017	.9565522	.9459386	.9306821	.9110784
2.3	.9785518	.9784275	.9780408	.9773450	.9762419	.9745239	.9716934	.9667306	.9583730	.9458085	.9289946
2.4	.9836049	.9835108	.9832180	.9826918	.9818594	.9805703	.9784661	.9747495	.9682098	.9580804	.9438652
2.5	.9875807	.9875100	.9872900	.9868953	.9862720	.9853112	.9837569	.9810035	.9760522	.9679136	.9560631
2.6	.9906776	.9906249	.9904612	.9901674	.9897045	.9889934	.9878527	.9858331	.9821023	.9758089	.9659525
2.7	.9930661	.9930271	.9929062	.9926894	.9923483	.9918260	.9909944	.9895288	.9867530	.9817837	.9738786
2.8	.9948897	.9948612	.9947727	.9946141	.9943649	.9939842	.9933821	.9923249	.9902888	.9864876	.9801589
2.9	.9962684	.9962477	.9961834	.9960684	.9958878	.9956126	.9951798	.9944248	.9929482	.9900803	.9850792
3.0	.9973002	.9972853	.9972391	.9971564	.9970266	.9968294	.9965205	.9959854	.9949274	.9927925	.9888910
3.1	.9980648	.9980542	.9980212	.9979622	.9978699	.9977296	.9975109	.9971348	.9963851	.9948168	.9918113
3.2	.9986257	.9986182	.9985949	.9985533	.9984880	.9983892	.9982358	.9979733	.9974478	.9963105	.9940240
3.3	.9990332	.9990279	.9990116	.9989824	.9989368	.9988677	.9987607	.9985792	.9982147	.9974004	.9956822
3.4	.9993261	.9993225	.9993112	.9992909	.9992593	.9992115	.9991376	.9990120	.9987628	.9981868	.9969113
3.5	.9995347	.9995323	.9995245	.9995105	.9994888	.9994559	.9994053	.9993204	.9991502	.9987480	.9978125
3.6	.9996818	.9996801	.9996748	.9996653	.9996505	.9996281	.9995938	.9995364	.9994218	.9991442	.9984882
3.7	.9997844	.9997832	.9997797	.9997733	.9997633	.9997482	.9997251	.9996867	.9996102	.9994208	.9989352
3.8	.9998553	.9998545	.9998522	.9998478	.9998412	.9998311	.9998157	.9997902	.9997396	.9996119	.9992882
3.9	.9999038	.9999033	.9999018	.9998989	.9998945	.9998878	.9998776	.9998608	.9998276	.9997428	.9995020
4.0	.9999367	.9999363	.9999353	.9999334	.9999305	.9999261	.9999195	.9999085	.9998870	.9998300	.9996845
4.1	.9999587	.9999585	.9999578	.9999566	.9999547	.9999519	.9999475	.9999404	.9999268	.9998900	.9997763
4.2	.9999733	.9999732	.9999727	.9999720	.9999707	.9999689	.9999661	.9999616	.9999527	.9999292	.9998523
4.3	.9999829	.9999828	.9999826	.9999821	.9999813	.9999801	.9999783	.9999754	.9999698	.9999548	.9999034
4.4	.9999892	.9999891	.9999889	.9999886	.9999881	.9999874	.9999863	.9999845	.9999809	.9999715	.9999375
4.5	.9999932	.9999932	.9999931	.9999929	.9999925	.9999921	.9999914	.9999902	.9999881	.9999822	.9999509
4.6	.9999958	.9999957	.9999957	.9999955	.9999954	.9999951	.9999947	.9999939	.9999926	.9999889	.9999746
4.7	.9999974	.9999974	.9999973	.9999973	.9999971	.9999970	.9999967	.9999963	.9999955	.9999932	.9999840
4.8	.9999984	.9999984	.9999984	.9999983	.9999983	.9999982	.9999980	.9999977	.9999972	.9999959	.9999901
4.9	.9999990	.9999990	.9999990	.9999990	.9999990	.9999989	.9999988	.9999986	.9999983	.9999975	.9999939
5.0	.9999994	.9999994	.9999994	.9999994	.9999994	.9999993	.9999993	.9999992	.9999990	.9999985	.9999963
5.1	.9999997	.9999997	.9999997	.9999996	.9999996	.9999996	.9999996	.9999995	.9999994	.9999991	.9999978
5.2	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999997	.9999997	.9999995	.9999987
5.3	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999998	.9999997	.9999992
5.4	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999995
5.5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	.9999999	.9999999	.9999999	.9999997
5.6								1.0000000	1.0000000	.9999999	.9999998
5.7										1.0000000	.9999999
5.8											1.0000000
5.9											
6.0											

Table II-2: Circular error probabilities (Bowditch, 1977).

$$\sigma_x = \frac{\sigma}{\sqrt{2}\,\mathrm{Sin}\,\beta/2} = \frac{6}{\sqrt{2}\,\mathrm{Sin}\,60^\circ/2} = 8.48\,\mathrm{m}\,,$$

$$\sigma_y = \frac{\sigma}{\sqrt{2}\,\mathrm{Cos}\,\beta/2} = \frac{6}{\sqrt{2}\,\mathrm{Cos}\,\frac{60^\circ}{2}} = 4.9\,\mathrm{m}\,.$$

Using the ratio, $c = \frac{\sigma_y}{\sigma_x} = \frac{4.9}{8.48} \doteq .58$ and

$$K = \frac{\text{radius of circle}}{\sigma_x} = \frac{10}{8.48} \doteq 1.2\,,$$

enter Table II-2 with K = 1.2 and c = .58 ≐ .6. The probability is found to be approximately 67%. (The value in the table is .6714269.)

EXAMPLE II-2: For the system described in example II-1, the radius of the error circle with 90% probability may be determined.

First, entering Table II-2 with c = .6, for 90% probability (the closest table value is .9019110), K is found to be 1.8. The radius of the error circle is equal to K times σ_x:

$$1.8 \times 8.48 = 15.3 \text{ meters.}$$

Table II-3 is more convenient for solving problems such as in example II-2 because the table is entered by using values of c and probability, P, in order to solve for the conversion factor, K. Note that the error circles identifying the 50% probability area (circular error probable, or CEP) and 90% area (circular map accuracy standard, or CMAS) are the most frequently used probability intervals.

For constant values of σ_1 and σ_2, circular error probabilities vary as a function of the angle of intersection, β, of the lines of position. To simplify the investigation of geometrical effects, the common case of $\sigma_1 = \sigma_2 = \sigma$ will be considered. Under this condition, the equations for σ_x and σ_y simplify to equation II-2. Taking the ratio of these two values, c is found to be $c = \sigma_y/\sigma_x = \tan(\beta/2)$. Using the simplified equations, significant parameters of the error ellipse have been listed in Table II-4 as a function of the intersection angle, β, for the 50% probability interval (CEP) and in Table II-5, for the 90% probability interval. The data shows that the radius of the error circle, R, increases as the angle of intersection decreases. In the last columns, the error factor is defined as

$$\text{Error factor} = \frac{R \text{ (at any intersection angle)}}{R \text{ (at } \beta = 90^\circ)}$$

P \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
.5000	0.67449	0.68199	0.70646	0.74903	0.80785	0.87042	0.93365	0.99631	1.05766	1.11807	1.17741
.7500	1.15035	1.15473	1.16825	1.19248	1.23100	1.28534	1.35343	1.42471	1.50231	1.58377	1.66511
.9000	1.64485	1.64791	1.65731	1.67383	1.69916	1.73708	1.79153	1.86253	1.94761	2.04236	2.14597
.9500	1.95996	1.96253	1.97041	1.98420	2.00614	2.03500	2.08130	2.14506	2.23026	2.34160	2.44775
.9750	2.24140	2.24365	2.25053	2.26255	2.28073	2.30707	2.34681	2.40356	2.48494	2.58969	2.71620
.9900	2.57583	2.57778	2.58377	2.59431	2.60998	2.63257	2.66533	2.71516	2.79096	2.88744	3.03485
.9950	2.80703	2.80883	2.81432	2.82390	2.83830	2.85904	2.88654	2.93347	3.00431	3.11078	3.25525
.9975	3.02334	3.02500	3.03010	3.03896	3.05234	3.07144	3.09671	3.13949	3.20596	3.31098	3.46164
.9990	3.29053	3.29206	3.29673	3.30489	3.31715	3.33464	3.35940	3.39647	3.45606	[illegible]	3.71692

Table II-3: Factors for conversion, K, of error ellipse to circle of equivalent probability (Bowditch, 1977).

B	σ_x	σ_y	c	K	R (CEP)	Error Factor
90°	1.000σ	1.00σ	1.00	1.177	1.177σ	1.00
80°	1.100σ	.924σ	.839	1.078	1.186σ	1.01
70°	1.234σ	.865σ	.700	.996	1.228σ	1.042
60°	1.414σ	.817σ	.577	.914	1.292σ	1.090
50°	1.672σ	.782σ	.466	.847	1.420σ	1.206
45°	1.847σ	.766σ	.414	.815	1.508σ	1.281
40°	2.06σ	.753σ	.364	.783	1.620σ	1.376
30°	2.74σ	.733σ	.268	.734	2.01σ	1.710
20°	4.06σ	.718σ	.176	.700	2.85σ	2.42
10°	8.11σ	.710σ	.087	.680	5.52σ	4.69

Table II-4: For CEP, significant parameters of error ellipses when $\sigma_1 = \sigma_2$ (Bowditch, 1977).

B	C	K	R (90%)	Error Factor
90°	1.00	2.145	2.145σ	1.00
80°	.839	1.98	2.18σ	1.015
70°	.700	1.86	2.30σ	1.07
60°	.577	1.775	2.51σ	1.17
50°	.466	1.72	2.88σ	1.34
45°	.414	1.702	3.15σ	1.47
40°	.364	1.687	3.47σ	1.615
30°	.268	1.665	4.53σ	2.11
20°	.176	1.652	6.72σ	3.13
10°	.087	1.645	13.35σ	6.22

Table II-5: For 90% probability interval, significant parameters of error ellipse when $\sigma_1 = \sigma_2$ (Bowditch, 1977).

or a multiplier by which the error circle radius, R, at any intersection angle may be computed from the radius of the error circle at $\beta = 90°$. For example, from Table II-4 it is seen that at a 50° intersection angle, R is 1.206 times greater than the radius at $\beta = 90°$.

As seen in Tables II-4 and II-5, the optimum accuracy is obtained when the intersection angle, β, is 90°. It can be said that the geometric dilution of precision (GDOP) is minimum for a 90° intersection angle. Thus, the error factor defined in Tables II-4 and II-5 is commonly known as GDOP. Effects of geometric dilution are shown in Figure II-4 for CEP and 90% probability interval (CMAS). Acceptable intersection angles for LOP's used in fixing hydrographic positions usually range between the limits of 30° and 150°. As seen in Figure II-4, the radius of the 90° probability interval circle is increased by a factor of two near the acceptable limits for hydrographic fix angles. Correspondingly, positioning accuracy is decreased by a factor of two.

b. Root-Mean Square Error (d_{rms})

The root-mean-square error, d_{rms}, is defined as the square root of the sum of the squares of the error components along the major and minor axes of the error ellipse. To calculate the d_{rms} error, first equations II-1a, 1b or II-2 are utilized to obtain the values of σ_x and σ_y. Then the definition of d_{rms} is used,

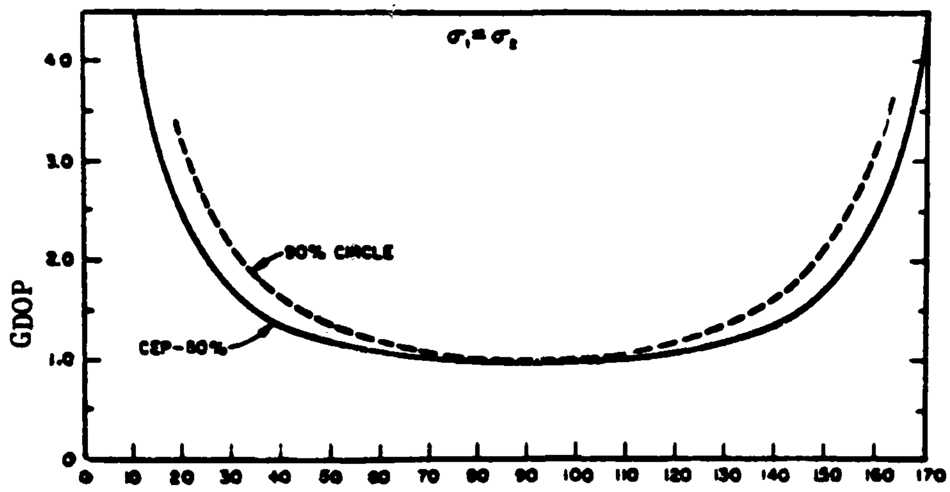

Figure II-4: Geometric dilution of precision for CEP and 90% probability interval (Bowditch, 1977).

$$d_{rms} = \sqrt{a^2 + b^2} = \sqrt{\sigma_x^2 + \sigma_y^2}\ , \qquad \text{(II-6)}$$

where $a = \sigma_x$ is the semi-major axis of the error ellipse and $b = \sigma_y$ is the semi-minor axis.

Alternately, formulas II-1a and II-1b are substituted into the definition of d_{rms} error (equation II-6) and a more useful form of d_{rms} is obtained in terms of σ_1, σ_2 and the angle of intersection, β:

$$d_{rms} = \frac{1}{\sin\beta}\sqrt{\sigma_1^2 + \sigma_2^2}\ . \qquad \text{(II-7)}$$

Figure II-5 illustrates the definition of d_{rms} error.

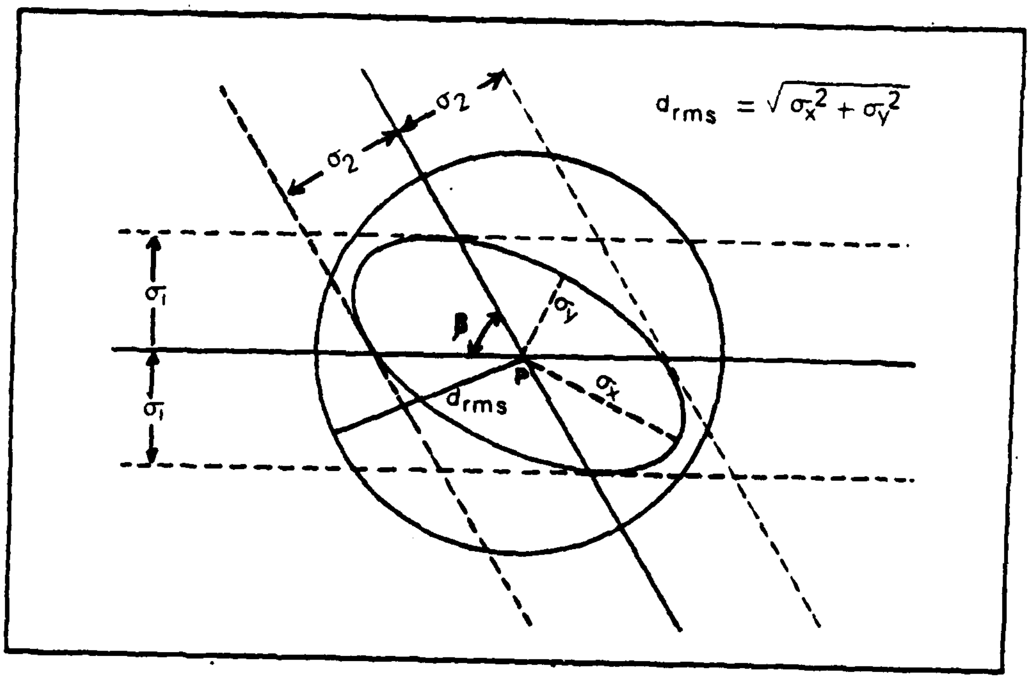

Figure II-5: Illustration of root mean square error.

One d_{rms} is defined as the radius of the error circle obtained using one σ_x and one σ_y as the semi-major and semi-minor axes of the error ellipse. Two d_{rms} is defined as the radius of the error circle obtained using two times the σ_x and σ_y values.

The value of d_{rms} does not correspond to a fixed probability interval for given values of σ_1 and σ_2. It corresponds to a fixed probability interval only when $\beta = 90°$ and $\sigma_1 = \sigma_2$ so that the resulting probability figure is a circle. In the elliptical cases, the probability associated with a fixed value of d_{rms} varies as a function of the eccentricity of the error ellipse. This can easily be seen with an example using Table II-2.

First, consider $\sigma_x = 15$ m, $\sigma_y = 10$ m,

$$d_{rms} = \sqrt{(15)^2 + (10)^2} = 18\text{m},$$

$$c = \frac{\sigma_y}{\sigma_x} = .666 \quad \text{and} \quad K = \frac{\text{radius of error circle}}{\sigma_x} = \frac{18}{15} = 1.2 .$$

For c = .666 table values must be interpolated. Enter Table II-2 with c = .6 and c = .7 for K = 1.2. The corresponding table values are found to be .6714269 and .6306168. Thus the probability of 18 m d_{rms} is found to be 64.78%.

Secondly, consider $\sigma_x = 17$ m, $\sigma_y = 6$ m,

$$d_{rms} = \sqrt{(17)^2 + (6)^2} = 18\text{ m},$$

$$c = 6/17 = .353 \quad \text{and} \quad K = \frac{18}{17} = 1.059 .$$

The interpolated probability from Table II-2 is 67.4%. As seen above for the two cases, d_{rms} errors are equal but c values (eccentricity) are different. As a result the corresponding probabilities are 64.78% and 67.4%.

Table II-6 shows the variations in probability associated with the values of 1 d_{rms} and 2 d_{rms} as a function of eccentricity (σ_y/σ_x), and Figure II-6 shows the same information graphically for 1 d_{rms} error.

$\frac{\sigma_y}{\sigma_x}$	Probability $1\ d_{rms}$	$2\ d_{rms}$
0	.683	.954
.1	.682	.955
.2	.682	.957
.3	.676	.961
.4	.671	.966
.5	.662	.969
.6	.650	.973
.7	.641	.977
.8	.635	.980
.9	.632	.981
1.0	.632	.982

Table II-6: Variations in probability as a function of eccentricity (Bowditch, 1977).

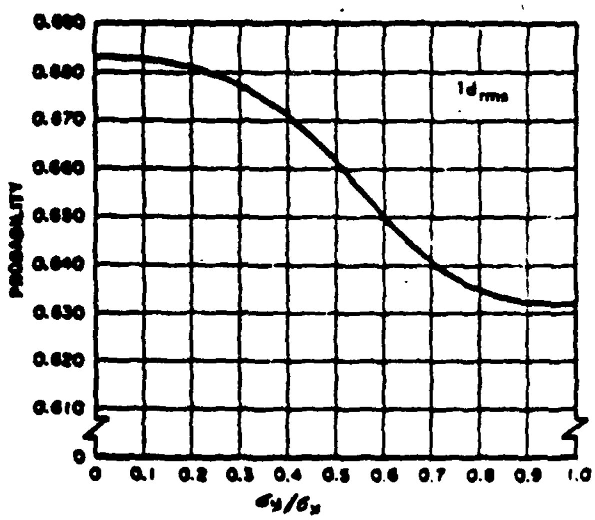

Figure II-6: Variation in d_{rms} with ellipticity ($1\ d_{rms}$) (Bowditch, 1977).

As seen in Table II-6, the probability that the position will be within the 1 d_{rms} error circle ranges from 68.3% when $\sigma_y = 0.0$ to 63.2% when $\sigma_y = \sigma_x$ and two d_{rms} ranges from 95.4% to 98.2%, respectively.

In Equation II-7, the d_{rms} error was given assuming the errors in each line of position are independent. If the measurement of line of position #1 is related to measurement of line of position #2, then there is correlation between σ_1 and σ_2; e.g., σ_1 is dependent on σ_2, or a change in σ_1 produces a corresponding change in σ_2. In this case, the equation for the root mean square position error is given as

$$d_{rms} = \frac{1}{\sin\beta}\sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2\cos\beta}\,, \qquad \text{(II-8)}$$

where ρ is the correlation coefficient between σ_1 and σ_2. Two different derivations of this equation are presented in the following papers: Bigelow (1963) and Heinzen (1977) [Refs. 2 and 10].

In summary, root mean square error is easy to obtain mathematically, and it yields relative values of accuracy which are normally understood. Therefore, in subsequent sections, d_{rms} will be used to explain the repeatable accuracies of hydrographic positioning systems.

D. REPEATABLE ACCURACY OF HYDROGRAPHIC POSITIONING SYSTEMS

1. Ranging Systems

In ranging systems, the lines of positions are drawn as circles centered about each control station. The repeatability of this type of system is a function of the intersection angle, β, and the random errors associated with each line of position.

The two ranges are independent of each other. Therefore, the correlation coefficient, ρ, is zero, and d_{rms} is given by Equation II-7 which is repeated here :

$$d_{rms} = \frac{1}{\sin\beta}\sqrt{\sigma_1^2 + \sigma_2^2} \; . \qquad \text{(III-9)}$$

Usually, the standard errors of the two shore stations are equal. The system standard error, σ_s, of a time measuring positioning system is given as

$$\sigma_1 = \sigma_2 = \sigma_s \; .$$

The system standard error, σ_s, of a phase comparison positioning system is computed as a fraction of the lane width so that $\sigma_1 = \sigma_2 = \sigma w = \sigma_s$, where σ is the standard error of range in fractions of a lane (i.e., $\sigma = .1$ lanes) and w is lane width. Then Equation II-9 reduces to

$$d_{rms} = \frac{\sqrt{2}\,\sigma_s}{\sin\beta} \quad , \tag{II-10}$$

where σ_s is the system standard error.

As seen from the above formula, the d_{rms} is smallest at a 90° intersection angle and becomes large as β approaches 0° or 180°.

2. Hyperbolic Systems

As in ranging systems, the repeatable accuracy of hyperbolic systems is a function of intersection angle and random errors. Because landwidth is not constant for hyperbolic systems, the change in lane width must also be quantified. As the user moves away from the base line between the master and a slave unit, the lane becomes wider due to the divergence of the hyperbolic LOP's [Ref. 8]. This divergence is expressed as an expansion factor, E :

$$E_i = 1/\sin\,(\theta_i/2) \, ,$$

where θ_i is the angle between the radius vectors from the position at **p** to the master and the respective slave station (Figure II-7). Then the standard error of one line of position at **p** is

$$\sigma_i = \sigma_w E_i = \frac{\sigma_w}{\sin\frac{\theta_i}{2}} \quad , \tag{a}$$

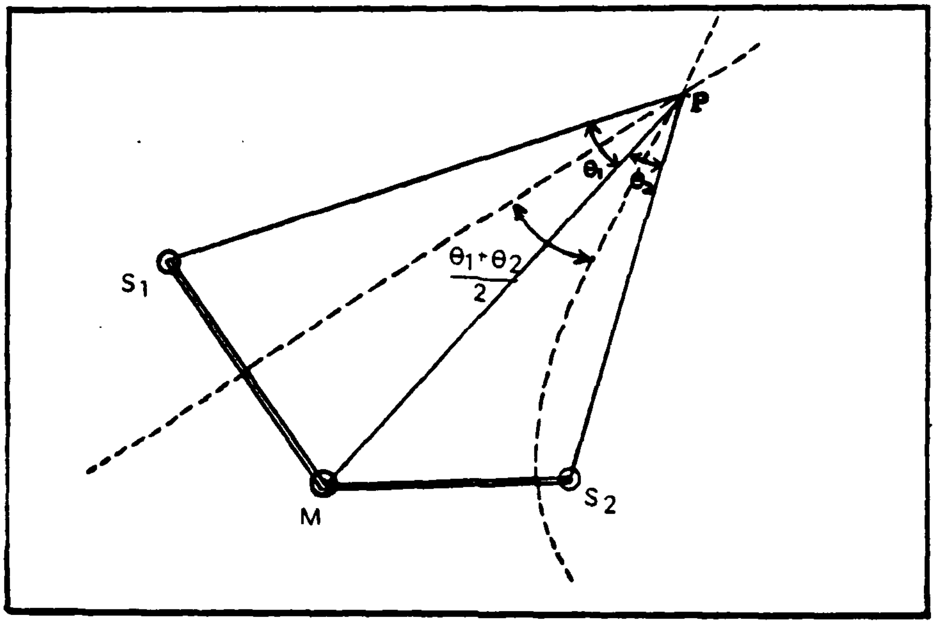

Figure II-7: A hyperbolic triad

where σ is standard error in the base line in fractions of a line, w is lane width and E expansion factor.

The hyperbolic LOP's bisect the angle between the radius vectors from p to master station and the respective slave station. Therefore, the angle of intersection, β, is

$$\beta = \frac{\theta_1 + \theta_2}{2} . \tag{b}$$

Substituting equations (a) and (b) into II-8, d_{rms} becomes

$$d_{rms} = \frac{\sigma w}{\sin\left(\frac{\theta_1+\theta_2}{2}\right)} \sqrt{\frac{1}{\sin^2\frac{\theta_1}{2}} + \frac{1}{\sin^2\frac{\theta_2}{2}} + \frac{2\rho\cos(\theta_1+\theta_2/2)}{\sin\frac{\theta_1}{2}\cdot\sin\frac{\theta_2}{2}}} . \tag{II-11}$$

In a triad (three-station net) one range is common to both lines of position. Therefore, the correlation coefficient is not zero. Bigelow (1963) [Ref. 2] assumes a value for the correlation coefficient, ρ, of 0.33 while Swanson (1963) [Ref. 14] gets $\rho = 0.4$. Since the determination of this value is based on observations comprising a statistical sample, the most conservative value of d_{rms} may be obtained by using $\rho = 0.4$.

3. Azimuthal Systems

In an azimuthal system, whether it is optical or electronic, the lines of position are radial vectors eminating from each of the shore stations. The repeatability of such systems is dependent upon the angular resolution of the system, and the angle of intersection of the radial vectors.

The errors of position depend on:

(1) the distance, r, along the radial,

(2) the angular resolution, a, in degrees,

(3) the angle of intersection, β.

The angular error may be expressed as an arc distance perpendicular to the respective radial at p as

$$\sigma_i = \frac{r_i \cdot a}{57.296}, \tag{a}$$

where r is the distance along the radial, a is angular resolution, and 57.296 is conversion factor from degrees to radians.

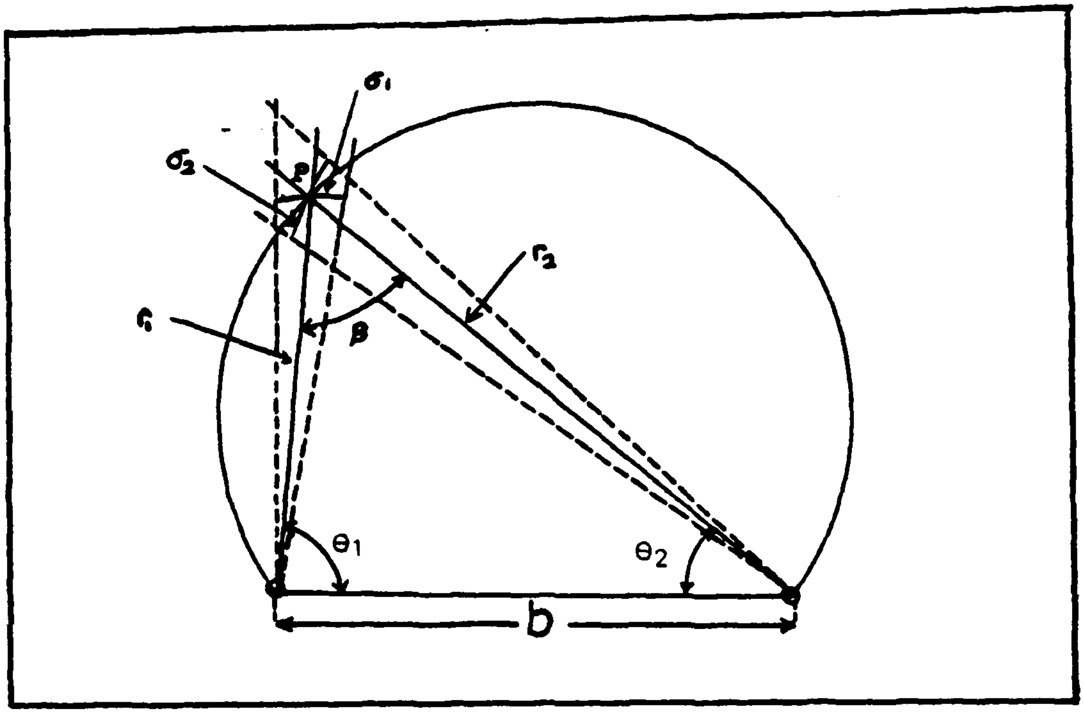

Figure II-8: Azimuthal System Repeatability

The two shore stations are independent; therefore, the correlation coefficient, ρ, is zero.

Substituting Equation (a) into II-7,

$$d_{rms} = \frac{a}{57.296} \cdot \frac{1}{\sin\beta} \sqrt{r_1^2 + r_2^2} \ . \qquad \text{(II-12)}$$

Applying the sine law to the triangle shown in Figure II-8, it is seen that

$$\frac{r_1}{\sin\theta_2} = \frac{r_2}{\sin\theta_1} = \frac{b}{\sin[180-(\theta_1+\theta_2)]} \ ,$$

where $\sin[180° - (\theta_1 + \theta_2)] = \sin\beta = \sin(\theta_1 + \theta_2)$ and b = baseline distance. And Equation II-12 may be written as

$$d_{rms} = \frac{a \cdot b}{57.296} \cdot \frac{1}{\sin^2(\theta_1+\theta_2)} \sqrt{\sin^2\theta_1 + \sin^2\theta_2} \quad . \qquad \text{(II-13)}$$

Bigelow (1963) [Ref. 2] approximates Equation II-13 by letting

$$r_1 = r_2 = \frac{b}{2 \sin \beta/2} \quad .$$

Then Equation II-13 reduces to

$$d_{rms} = \frac{a \cdot b}{\sqrt{2} \times 57.296} \cdot \frac{1}{\sin\beta \cdot \sin\beta/2} \qquad \text{(II-14)}$$

Equation II-14 is the approximate form of Equation II-13. However, Equation II-14 is easier to compute and the error introduced is negligible. For $a = .03°$, $b = 8000$ meters, $\theta_1 = 80°$, $\theta_2 = 30°$, comparing the equations II-13 and II-14:

using II-13

$$d_{rms} = \frac{(.03)(8000)}{57.296} \cdot \frac{1}{\sin^2(80°+30°)} \cdot \sqrt{\sin^2 80° + \sin^2 30°} = 5.2\text{m} ,$$

using II-14

$$d_{rms} = \frac{(.03)(8000)}{\sqrt{2} \times 57.296} \cdot \frac{1}{\sin 70° \sin 70°/2} = 5.5\text{ m} ,$$

it is seen that the difference between Equation II-13 and Equation II-14 is negligible.

4. Sextant Angle Positions (Three Point Fixes)

The evaluation of repeatability for sextant angle positions is difficult. The mathematics involved in the computation are quite complex. Thus, repeatability of sextant angle positions is more easily evaluated by a graphical analysis. For the development of an analytical solution, see Heinzen (1977) [Ref. 10].

As will be seen in later sections, it is much easier to derive the accuracy of sextant angle positions by applying the method of least squares.

E. REPEATABILITY CONTOURS

Using the root mean square error concept, one can construct a family of curves to display convenient values of d_{rms} in terms of the system geometry.

1. Ranging Systems

For ranging systems, the d_{rms} is given by Equation II-10 as

$$d_{rms} = \frac{\sqrt{2}\,\sigma_s}{\sin \beta} \qquad \text{(II-10)}$$

Note that the intersection angle, β, is the only controlling geometric factor of d_{rms}. Figure II-9 shows an example of suitable geometry for a ranging system. Mathematically, it can be proven that the intersection angle, β, is equal to the angle formed by radius vectors from **p** to the slaves and also the angles S_1OD and S_2OD. The locus of points having a constant d_{rms} and constant β describes a circle of radius $r = b/2 \sin \beta$ with the two shore stations as points on the circle.

The distance, e, along the perpendicular bisector of the line connecting two shore stations to the center of the circle is

$$e = \frac{b}{2 \tan \beta} \qquad \text{(II-15)}$$

(since, from Figure II-9, $\tan \beta = \frac{b/2}{e}$), where b is the distance between shore stations S_1 and S_2.

Using 2 σ_s error (approximately 95% probability interval), Equation II-10 may be written in the following form

$$\frac{d_{rms}}{\sigma_s} = \frac{2\sqrt{2}}{\sin \beta} .$$

Writing Equation II-15 as

$$\frac{e}{b} = \frac{1}{2 \tan \beta} ,$$

Figure II-10 was constructed to show the relationship of d_{rms}/σ_s and e/b as a function of intersection angle, β. Using this graph, selected contours of constant d_{rms} may be drawn as in Figure II-11. First, plot the location

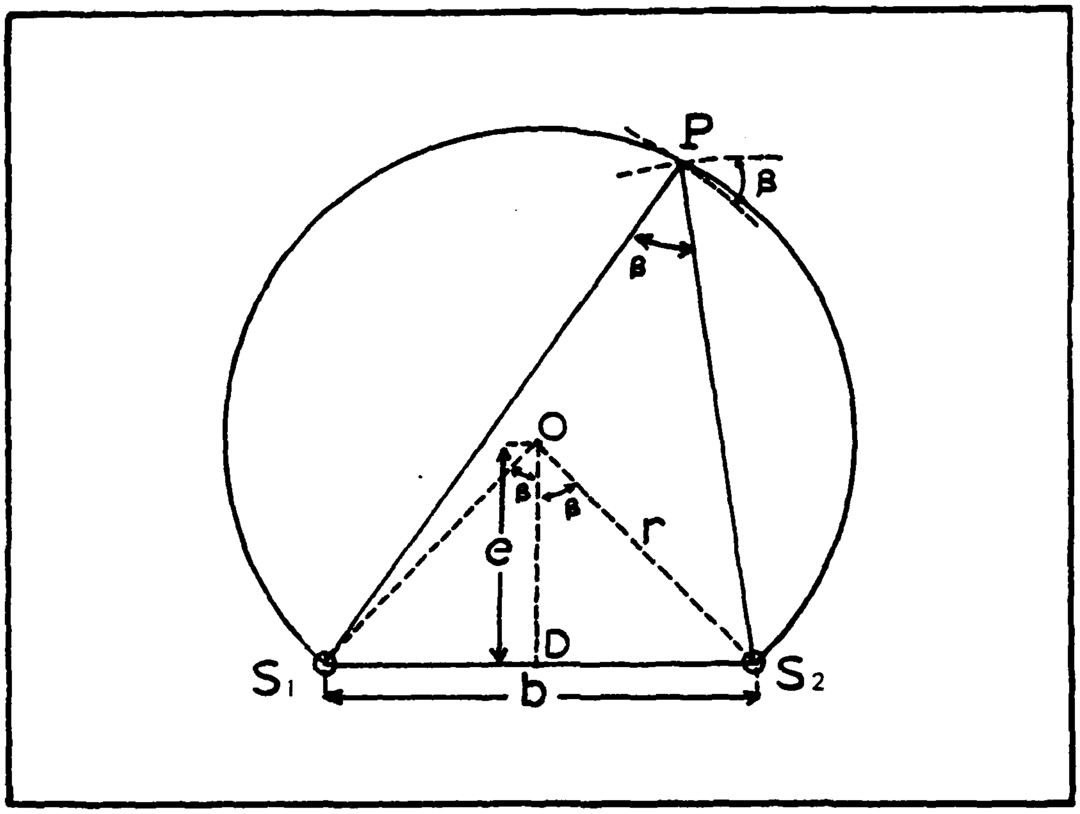

Figure II-9: Ranging system geometry.

of the two shore stations at a convenient scale. Draw a perpendicular bisector to the line joining them. Using Figure II-10 determine the values of e/b for the desired d_{rms} contours. From the known value of b, determine distance e for each contour. Lay off distance e along the perpendicular bisector to define the center, O, of the desired constant d_{rms} circle. The radius of the selected contour is the distance from the center, point O, to the shore stations.

Example II-3: A phase comparison range-range positioning system has standard error σ_s = .01 w (lane width). It operates at 2 Mhz frequency. The distance, b, between two shore stations is 20,000 m.

$$\text{Lane width, } w = \frac{v}{2f} = \frac{300{,}000}{2 \times 2000} = 75 \text{ m},$$

$$\sigma_s = \sigma w = 75 \times .01 = .75 \text{ m}.$$

For the 2 m d_{rms} contour, d_{rms}/σ_s = 2/.75 = 2.66.

Enter Figure II-10 with d_{rms}/σ_s = 2.66 which intersect the d_{rms}/σ_s curve at 85°. Follow the 85° line vertically to

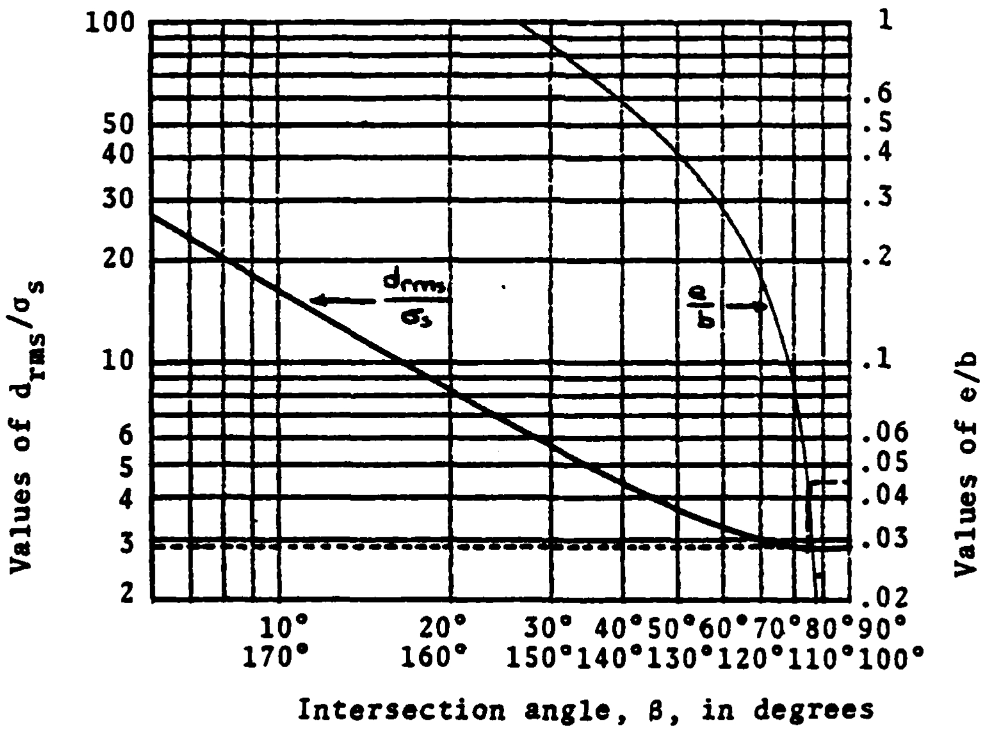

Figure II-10: For ranging systems, the graph of the d_{rms}/σ_s and e/b. (For enlarged figure, see Appendix B.)

the e/b curve. It intersects at e/b = .043. For this specific pair where b = 20,000 m,

$$e = b\,(.043) = 860 \text{ m}$$

Using the described technique, the 2m d_{rms} contour can be drawn, and the result is shown in Figure II-11. Thus between the 2m d_{rms} contour for $\beta = 85°$ and the 2m d_{rms} contour for $\beta = 95°$, the d_{rms} error for the described system will be $\leq$ 2m. 95% of the time.

Note that when the angle of intersection, $\beta = 90°$, d_{rms} error is minimum. Therefore, as β increases toward 180° or decreases toward 0°, d_{rms} becomes larger. Because the tangent of the angles greater than 90° is negative, e values will be negative as well. Thus, the center of the constant d_{rms} circles, for angles greater than 90°, will be on opposite side of the baseline. As shown in Figure II-11, d_{rms} error increases as the baseline is approached. Contours for d_{rms} values of 3, 4 and 5 meters may be constructed by following the procedures outlined above.

2. Azimuthal Systems

For azimuthal systems, d_{rms} error is given by Equation II-14 as

$$d_{rms} = \frac{a \cdot b}{\sqrt{2} \times 57.296} \cdot \frac{1}{\sin\beta \sin\beta/2} ,$$

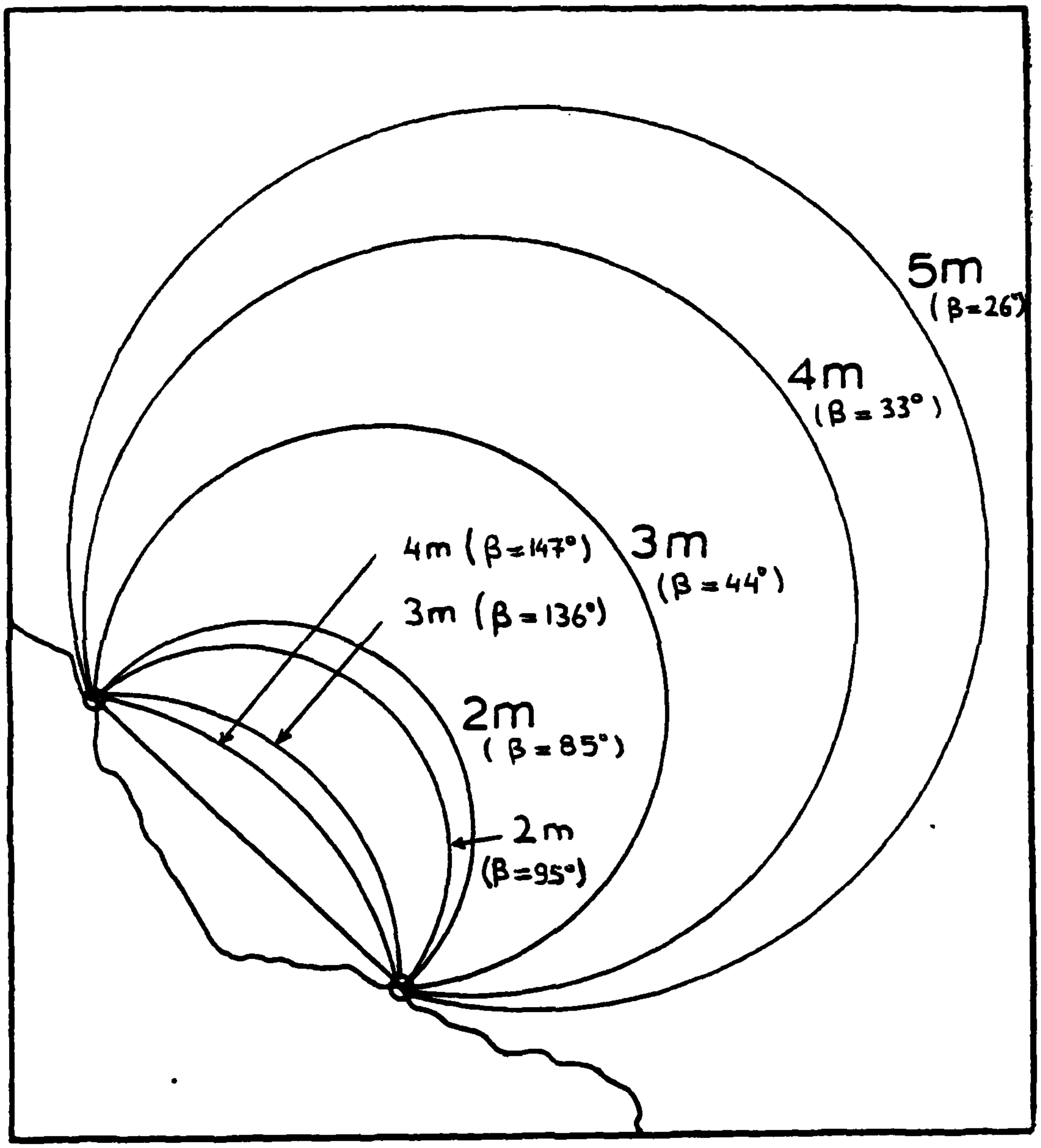

Figure II-11: Repeatability contours of a ranging pair.

where a is the angular resolution, measured in degrees, b is the distance between two azimuth stations and β is the intersection angle of radial vectors which is defined by the equation $\beta = 180° - (\theta_1 + \theta_2)$ (Figure II-8). As with ranging systems, the intersection angle, β, is the only geometric factor contributing to d_{rms}. Constant error contours are obtained in a similar fashion.

Writing the Equation II-14 with 2a error (approximately 95% probability interval) as

$$\frac{d_{rms}}{a \cdot b} = \frac{2}{\sqrt{2}\,(57.296)} \cdot \frac{1}{\sin\beta \cdot \sin\beta/2} \quad ,$$

where a = angular resolution and b = baseline distance. Since $e/b = 1/(2 \tan \beta)$, a graph is drawn showing $d_{rms}/a \cdot b$ and e/b as a function of intersection angle, β (Figure II-12).

Figure II-12 provides a convenient means to obtain the values of e distance for a selected d_{rms} if a and b are known.

Example II-4: The distance, b, between two azimuth shore stations is 2,000 m. Angular resolution of the station is $a_1 = a_2 = .01°$.

For the 2 meters d_{rms} contour

$$\frac{d_{rms}}{a \cdot b} = \frac{2}{.01 \times 2000} = .1 \ .$$

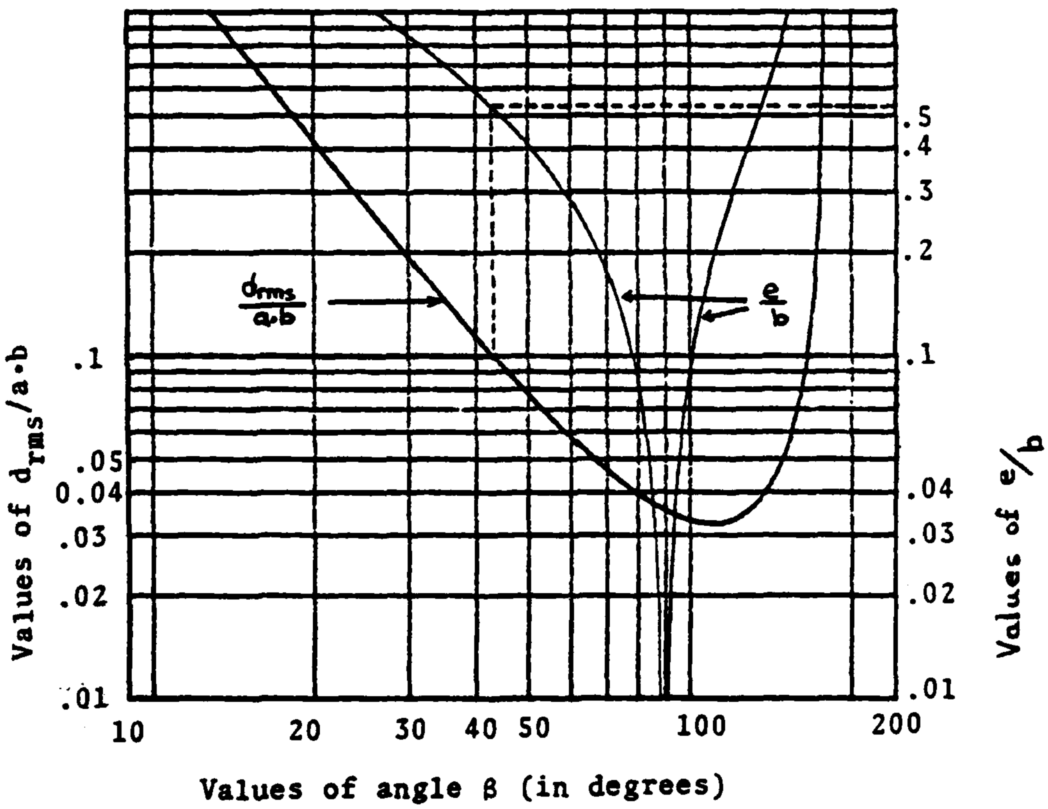

Figure II-12: For azimuthal systems, the graph of $\frac{d_{rms}}{a \cdot b}$ and $\frac{e}{b}$

(For enlarged figure, see Appendix B.)

Enter Figure II-12 with $d_{rms}/a \cdot b = .1$ which intersects the $d_{rms}/a \cdot b$ curve at 44°. Follow 44° line vertically to the e/b curve, which intersects at $e/b = .52$. For this pair where $b = 2{,}000$ m, $e = 2{,}000 \times .52 = 1040$ m. Using the technique as described for ranging systems, the 2m d_{rms} error contour may be drawn (Figure II-13). Other contours are computed in same manner.

Note that when $\beta > 90°$, the tangent value is negative and the center of constant d_{rms} circle will be on the opposite side of the baseline. For azimuthal systems, the minimum d_{rms} error is found at $\beta = 109°$.

3. Hyperbolic Systems

The root mean square error for hyperbolic systems is given by Equation II-11 as

$$d_{rms} = \frac{\sigma w}{\sin\beta}\sqrt{\frac{1}{\sin^2\frac{\theta_1}{2}} + \frac{1}{\sin^2\frac{\theta_2}{2}} + \frac{2\rho\cos\beta}{\sin\frac{\theta_1}{2}\cdot\sin\frac{\theta_2}{2}}}\ , \qquad \text{(II-11)}$$

where σ is the standard error along the baseline between the master and respective slave station in fractions of a lane, w is the lane width and β is the intersection angle which is equal to

$$\beta = \frac{\theta_1 + \theta_2}{2} \quad \text{(Figure II-7).}$$

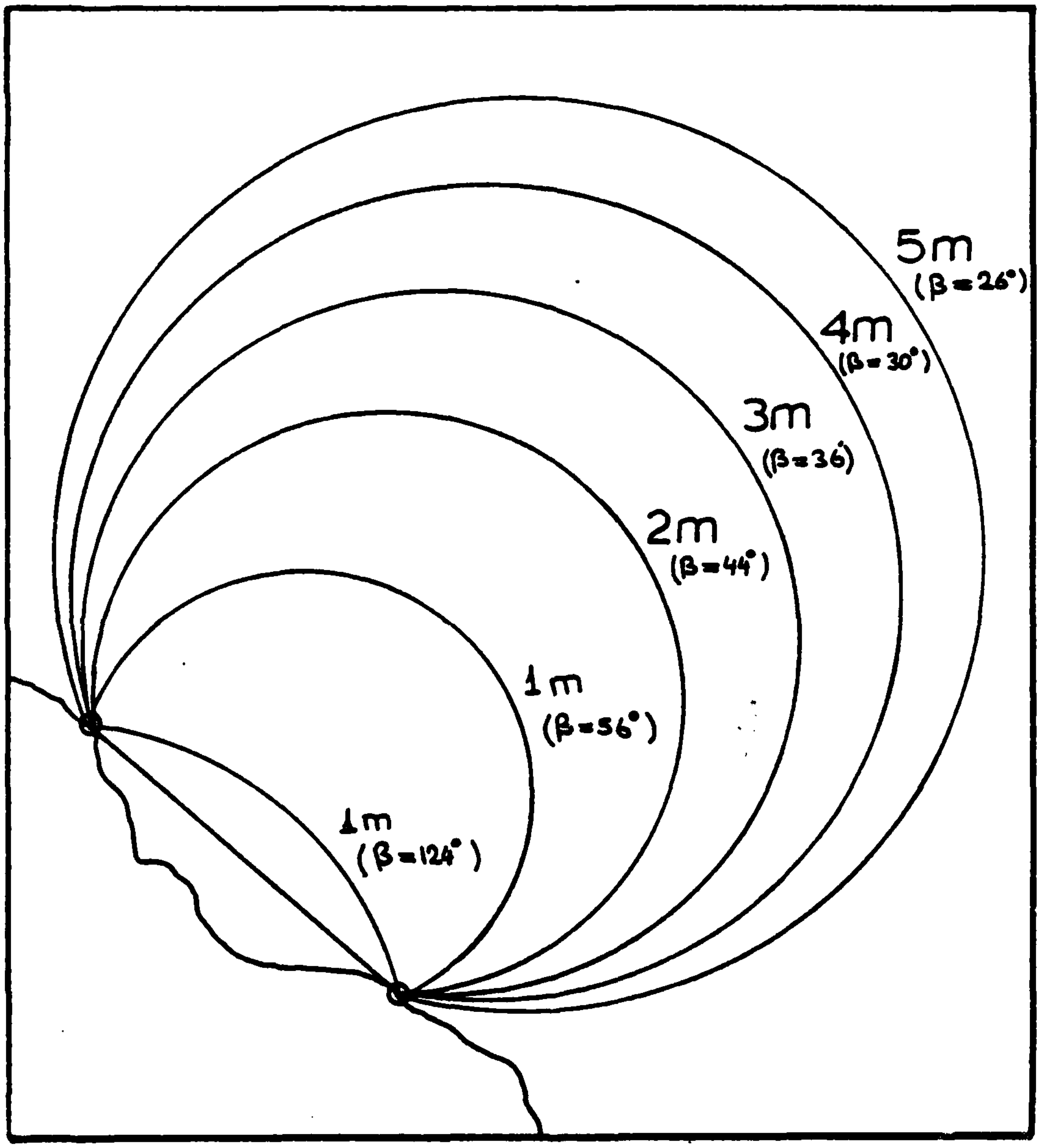

Figure II-13: Repeatability contours of an azimuthal system.

Equation II-11 is written with 2σ error as

$$\frac{d_{rms}}{\sigma w} = \frac{2}{\sin\beta}\sqrt{\frac{1}{\sin^2\frac{\theta_1}{2}} + \frac{1}{\sin^2\frac{\theta_2}{2}} + \frac{2(.4)\cos\beta}{\sin\frac{\theta_1}{2}\cdot\sin\frac{\theta_2}{2}}}\ ,$$

where the correlation coefficient, ρ, was taken as .4 [Ref. 14]. Figure II-14 was produced to show the $d_{rms}/\sigma w$ values as a function of the angle subtended by the two slave stations, i.e., 2β.

A parameter, p, defined as

$$p = \frac{\theta_1+\theta_2}{\theta_1} = \frac{2\beta}{\theta_1} \quad \text{or} \quad p = \frac{\theta_1+\theta_2}{\theta_2} = \frac{2\beta}{\theta_2}\ .$$

The parameter, p, is computed with the smaller of the two angles, σ_1 or σ_2, in the denominator. Thus when p = 2 the master station is positioned on the bisector of the angle subtended by two slave stations, p = 3 places master station on one of the two trisectors, and so on (Table II-7).

Knowing the angle subtended by the two slave stations at a particular point, Figure II-14 may be used to develop contours of constant d_{rms}. First determine the $d_{rms}/\sigma w$ ratio for a selected d_{rms}. Enter Figure II-14, for several values of parameter p, and read the corresponding values of angle 2β. Using the relation between 2β and θ_1 (θ_2) (Table II-7), plot these angles, 2β and θ_1 (θ_2), on a conveniently scaled chart

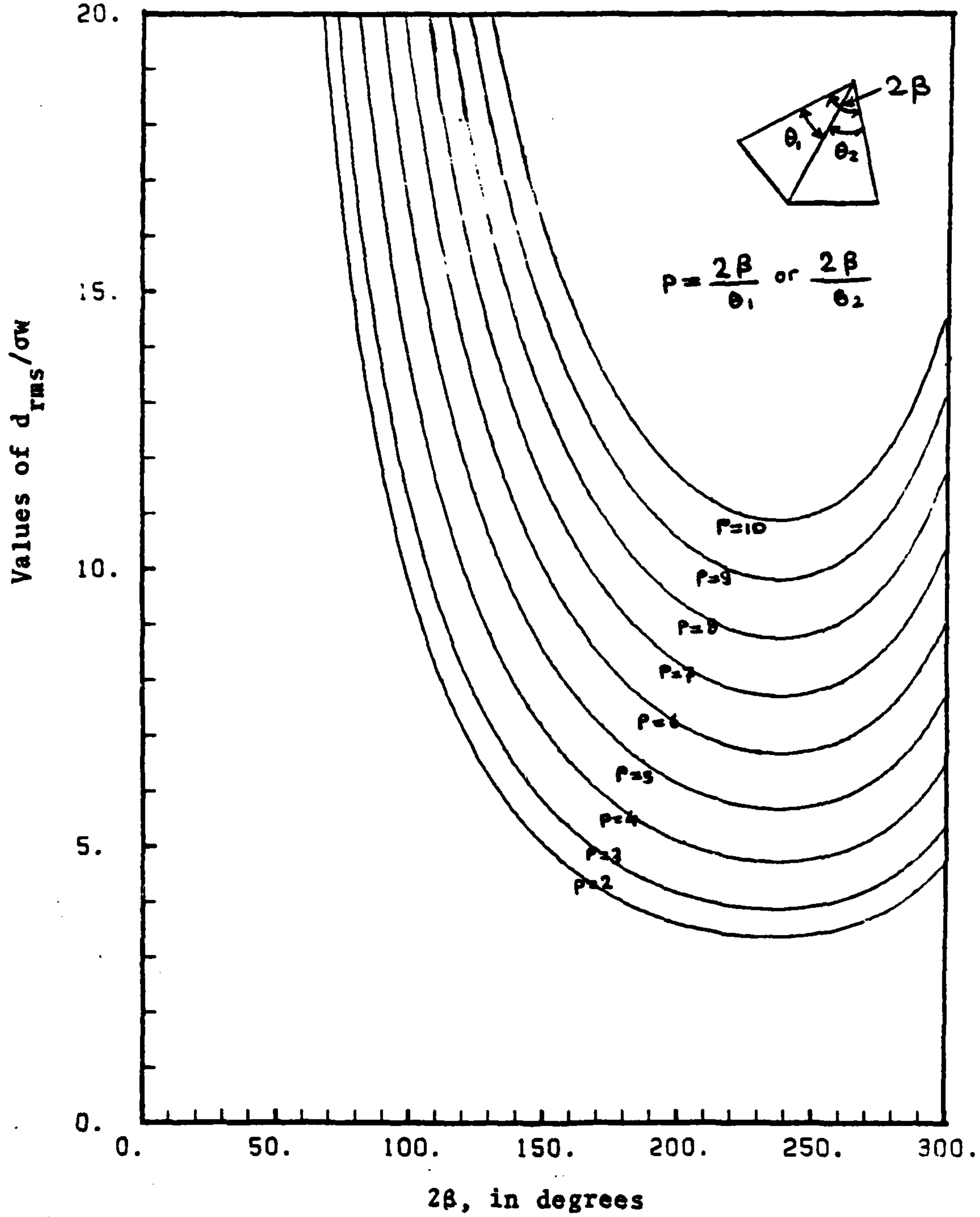

Figure II-14: In hyperbolic systems, for several values of p, $d_{rms}/\sigma w$ curves.

p	θ_1 or θ_2
2	$(2\beta)/2$
3	$(2\beta)/3$
4	$(2\beta)/4$
5	$(2\beta)/5$
6	$(2\beta)/6$
7	$(2\beta)/7$
8	$(2\beta)/8$
9	$(2\beta)/9$
10	$(2\beta)/10$

Table II-7: Relation between 2β and θ_1 or θ_2.

with a three arm protractor. Interpolating between the points, draw the d_{rms} contour. The curve thus determined defines the location of a selected d_{rms} contour for the specific conditions of triad configuration.

Example II-5: A hyperbolic system has standard error, σ, equal to .01 lanes along the base line. It operates at a frequency of 2 Mhz. Triad configuration is as seen in Figure II-15 :

$$\text{lane width, } w = \frac{v}{2f} = \frac{300{,}000}{2 \times 2{,}000} = 75 \text{ m}$$

$$\sigma w = 75 \text{ m} \times .01 = .75 \text{ m} .$$

For the 4 m d_{rms} contour, $d_{rms}/\sigma w = 4/.75 = 5.32$. Enter Figure II-14 with $d_{rms}/\sigma w = 5.32$. For several values of p, read the corresponding values of angle 2β. Determine the values of angles θ_1 or θ_2 according to Table II-7. For the 4m contour, these values are shown in Table II-8. Using a three arm protractor, the points defining the 4m d_{rms} contour may be plotted. The other contours are drawn in a similar manner (Figure II-15).

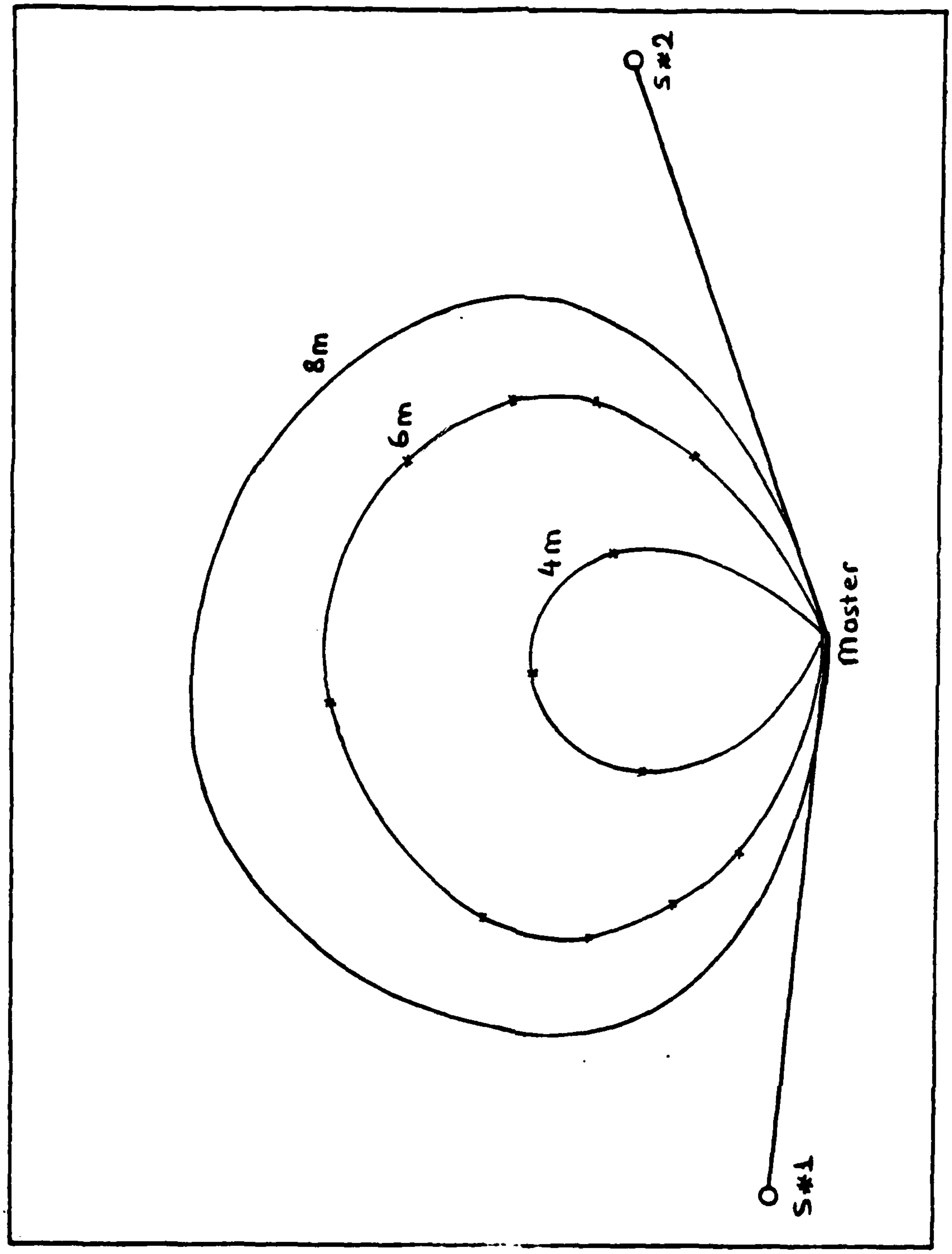

Figure II-15: Repeatability contours of a hyperbolic system (σ = .01 lane width and f = 2 mhz)

p = 2	p = 3		p = 4	
$2\beta = 146°$	$2\beta = 164°$	$2\beta = 298°$	$2\beta = 194°$	$2\beta = 278°$
$\theta_1(\theta_2) = 73°$	$\theta_1(\theta_2) = 55°$	$\theta_1(\theta_2) = 99°$	$\theta_1(\theta_2) = 48.5°$	$\theta_1(\theta_2) = 69.5°$
$\theta_2(\theta_1) = 73°$	$\theta_2(\theta_1) = \log°$	$\theta_2(\theta_1) = 199°$	$\theta_2(\theta_1) = 145.5°$	$\theta_2(\theta_1) = 208.5°$

Table II-8: The angles 2β and $\theta_1(\theta_2)$ defining the 4m d_{rms} contour for several values of p.

III. APPLICATION OF LEAST SQUARES TO HYDROGRAPHIC SURVEY POSITIONS

A. THE PRINCIPLE OF LEAST SQUARES

Given a set of unknown parameters to be computed from measured physical quantities such as distance or azimuth, the least squares method provides a mathematical procedure by which the best values for the unknown parameters may be obtained.

Equations must be written to define the relationship between the observed and the unknown parameters. If the number of equations that can be written is equal to the number of unknowns, then a unique solution may be computed. However, no statement can be made about the accuracy of the solution. In the least square method, the number of equations must be greater than the number of unknowns. As a result of this over-determined solution, the best values for the unknown parameters are estimated.

This computational procedure is referred to as a least squares adjustment. In application, corrections are computed and applied to observed quantities and these quantities are then said to be adjusted.

For a given set of equations, the fundamental condition of the least square technique requires that the sum of the squares of the residuals be minimized. A residual is defined as the difference between an oberved value of a quantity and the arithmetical mean value of that quantity obtained from a number of observations. If the arithmetical mean value is stated by $\bar{x}_i$ and observed value by x_i, the residual, v, is expressed as

$$v = x_i - \bar{x}_i \,. \qquad \text{(III-1)}$$

Suppose a set of observations were taken having residuals $v_1, v_2, v_3 \ldots\ldots, v_n$. Then in equation form, the fundamental condition of least squares is expressed as

$$\sum_{i=1}^{n} (v_i)^2 = (v_1)^2 + (v_2)^2 + (v_3)^2 + \ldots\ldots + (v_n)^2 = \text{minimum}\,, \qquad \text{(III-2)}$$

or in matrix form: $V^T V$ = minimum.

1. Weighted Observations

In general, some of the observed values may be more precise, and, therefore, entitled to have greater influence upon the result. Observations are assigned values called weights corresponding to their quality or worth.

The assignment of weights to observed values is largely a matter of judgment. For example, if one set of measurements of a distance was made with four repetitions and another was made with eight repetitions, the mean of the second set of observations may be given twice the weight of the first set. Or, when measuring angles in azimuth angle positions, the atmosphere may be so unsteady during one observation that the observer arbitrarily assigns a weight of one half.

As a general rule, if a standard error, σ, has been computed for a set of observations, then weights are usually estimated according to the equation

$$w_i = \frac{k^2}{\sigma_i^2} , \tag{III-3}$$

where w_i is the weight of the ith observed quantity, σ_i is the standard error of that observation, and k^2 is any number which has the same value for all observations. Equation III-3 states that weights are inversely proportional to the square of the standard error. Usually, the weight corresponding to the least accurate measurement is assigned a value of 1 (a unit weight). Then the value of k can be found and the other weights computed accordingly.

For example, consider the standard error for two observations where $\sigma_1 = 3$, and $\sigma_2 = 1.5$. Assigning $w_1 = 1$,

$$1 = \frac{k^2}{(3)^2} \longrightarrow k^2 = 9 ,$$

$$w_2 = \frac{k^2}{\sigma_2^2} = \frac{9}{(1.5)^2} = 4 ,$$

it is found that the second observation has a weight of 4 relative to the first observation.

If measured values are to be weighted and used in a least squares adjustment, then the condition is that the sum of the weight times their corresponding squared residuals must be minimized,

$$\sum_{i=1}^{n} w_i (v_i)^2 = w_1 v_1^2 + w_2 v_2^2 + \cdots\cdots + w_n v_n^2 = \text{minimum}, \qquad \text{(III-4)}$$

or in the matrix form, $V^T W V = \text{minimum}$.

2. Method of Least Squares Adjustment

In the "observation equations" method of least squares adjustment, the observed quantities are related to the desired unknown quantities through *formulas or functions* which are called observation equations.

One observation equation is written for each measurement, and it is assumed that observations are independent of each other. In order to solve for the best value of each unknown parameter, at least one redundant observation equation must be written. That is, the number of observations must be greater than the unknowns.

Observation equations may be linear or higher order functions. Linear observation equations can be written in general as follows:

$$
\begin{array}{l}
a_1x + b_1y + c_1z + \cdots + k_1 = G_1 \\
a_2x + b_2y + c_2z + \cdots + k_2 = G_2 \\
\vdots \\
a_nx + b_ny + c_nz + \cdots + k_n = G_n ,
\end{array} \qquad \text{(III-5)}
$$

where a's, b's, c's, etc. are coefficients of unknowns x, y, z, etc. and the k's are constants.

Because the observations (G_1, $G_2 \ldots . G_n$) are not free from random errors, each G_i must be corrected by a residual value, v_i, in order to obtain a mathematically correct equation system. Thus,

$$
\begin{array}{l}
a_1x + b_1y + c_1z + \cdots + k_1 = G_1 + v_1 \\
a_2x + b_2y + c_2z + \cdots + k_2 = G_2 + v_2 \\
\vdots \\
a_nx + b_ny + c_nz + \cdots + k_n = G_n + v_n .
\end{array} \qquad \text{(III-6)}
$$

Introducing a new notation $\ell_1 = G_1 - k_1$, $\ell_2 = G_2 - k_2$, etc., the following equation is obtained:

$$
\begin{array}{l}
a_1 x + b_1 y + c_1 z + \cdots\cdots - \ell_1 = v_1 \\
a_2 x + b_2 y + c_2 z + \cdots\cdots - \ell_2 = v_2 \\
\vdots \\
a_n x + b_n y + c_n z + \cdots\cdots - \ell_n = v_n ,
\end{array}
\qquad \text{(III-7)}
$$

or in the matrix form, $V = AX - L$ (III-8)

This equation is called the observation equation or observation equation matrix, where

$$
{}_nV_1 = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad {}_nA_m = \begin{bmatrix} a_1 , & b_1 , & c_1 & \cdots & m_1 \\ a_2 , & b_2 , & c_2 & \cdots & m_2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_n , & b_n , & c_n & \cdots & m_n \end{bmatrix}
$$

$$
{}_nX_1 = \begin{bmatrix} x \\ y \\ z \\ \vdots \end{bmatrix} \qquad {}_nL_1 = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \vdots \\ \ell_n \end{bmatrix} .
$$

In the above matrices, the subscript n denotes the number of observations and m denotes the number of unknowns.

For a group of equally weighted observations, recall that the following condition must be enforced in order to perform a least square adjustment:

$$\sum_{i=1}^{n} (v_i)^2 = \text{minimum} ,$$

or in the matrix form,

$$V^T V = \text{minimum} .$$

Substituting the value for the V matrix from the observation Equation III-8 where V = AX - L ,

$$\begin{aligned} V^T V &= (AX-L)^T (AX-L) \\ &= [(AX)^T - L^T](AX-L) \qquad\qquad [\,(AX)^T = X^T A^T\,] \\ &= (X^T A^T - L^T)(AX-L) \qquad\qquad (\text{from matrix algebra}) \\ &= X^T A^T A X - X^T A^T L - L^T A X + L^T L \end{aligned}$$

and from matrix algebra, $L^T A X = X^T A^T L$,

then

$$V^T V = X^T A^T A X - 2\, X^T A^T L + L^T L .$$

The minimum of this function can be found by taking the partial derivatives of the function with respect to each unknown or with respect to the X matrix (which contains all of the unknowns) and equating it to zero, i.e.:

$$\frac{\partial}{\partial X}(V^T V) = 2A^T A X - 2A^T L = 0.$$

Dividing by 2, the following result is obtained:

$$A^T A X - A^T L = 0. \qquad (\text{III-9})$$

This is called the normal equation. In conventional notation, the normal equation (III-9) becomes

$$\begin{array}{ccccccc}
[aa]x & + [ab]y & + [ac]z & + \ldots\ldots & [a\ell] & = 0 \\
[ba]x & + [bb]y & + [bc]z & + \ldots\ldots & [b\ell] & = 0 \\
[ca]x & + [cb]y & + [cc]z & + \ldots\ldots & [c\ell] & = 0 \\
\vdots & \vdots & \vdots & & \vdots & \\
[na]x & + [nb]y & + [nc]z & + \ldots\ldots & [n\ell] & = 0,
\end{array}$$

where the symbol [ ] denotes the sum of the products, i.e., $[aa] = a_1a_1 + a_2a_2 + a_3a_3 + \ldots\ldots + a_na_n$, $[ba] = b_1a_1 + b_2a_2 + b_ea_3 + \ldots\ldots + b_na_n$.

In Equation III-9, A^TA is the matrix of normal equation coefficients of the unknowns. Multiplying Equation III-9 by $(A^TA)^{-1}$ and reducing, the solution is obtained.

$$(A^TA)^{-1}(A^TA)X - (A^TA)^{-1}A^TL = 0,$$

$$X = (A^TA)^{-1}A^TL. \qquad \text{(III-10)}$$

Equation III-10 is the basic least squares matrix equation for equally weighted observations. The matrix X consist of best values for the unknowns x, y, z, etc.

For a system of weighted observations the fundamental condition is

$$\sum_{i=1}^{n} w_i (v_i)^2 = \text{minimum},$$

or in the matrix form , $V^TWV = \text{minimum}.$

The normal equation matrix is derived similarly to the unweighted case.

$$A^TWAX - A^TWL = 0, \qquad \text{(III-11)}$$

or in conventional notation,

$$
\begin{aligned}
[waa]x + [wab] + \cdots\cdots - [wa\ell] &= 0 \\
[wba]x + [wbb] + \cdots\cdots - [wb\ell] &= 0 \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad & \vdots \\
[wna]x + [wnb] + \cdots\cdots - [wn\ell] &= 0 .
\end{aligned}
$$

In Equation III-11 the matrices are identical to those of the equally weighted equation, with the addition of the matrix, W, which is a diagonal nxn matrix.

In detail, W becomes

$$
W = \begin{bmatrix} w_1 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 \\ 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & w_4 \end{bmatrix}, \tag{III-12}
$$

where according to Equation III-3,

$$
w_1 = \frac{k^2}{\sigma_1^2}, \quad w_2 = \frac{k^2}{\sigma_2^2}, \quad w_3 = \frac{k^2}{\sigma_3^2}, \; \ldots\ldots \text{etc.}
$$

The best values of unknowns are obtained by solving Equation III-11 as

$$X = (A^T W A)^{-1} A^T W L. \tag{III-13}$$

From the combination of Equations III-8 and III-9 or III-8 and III-11, it is seen that

$$A^T(V+L) - A^T L = 0 \qquad \text{or}$$

$$A^T W(V+L) - A^T W L = 0.$$

Therefore,

$$A^T V = 0 \qquad \text{or} \qquad A^T W V = 0. \tag{III-14}$$

Equation III-14 can be used as a check on the computation.

Example III-1[1]: As an elementary example illustrating the method of least squares adjustment by the observation equation method, consider the following equally weighted observations:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 10 \\ x_1 - 2x_2 + 3x_3 &= 5 \\ 7x_1 + x_2 - 2x_3 &= 3 \\ -x_1 - x_2 - x_3 &= -6. \end{aligned}$$

[1]The numerical values of this example problem were taken from Ref. 17, page 517.

These four equations relate the three unknowns x_1, x_2 and x_3 to the observations.

By including residuals, the equations may be rewritten as observation equations as follows:

$$\begin{aligned}
2x_1 + 3x_2 + x_3 &= 10 + v_1 \\
x_1 - 2x_2 + 3x_3 &= 5 + v_2 \\
7x_1 + x_2 - 2x_3 &= 3 + v_3 \\
-x_1 - x_2 - x_3 &= -6 + v_4 ,
\end{aligned}$$

or in matrix form,

$$ {}_4V_1 = {}_4A_3\,{}_3X_1 - {}_4L_1 \, ,$$

where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 7 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad L = \begin{bmatrix} 10 \\ 5 \\ 3 \\ -6 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

The normal equation is $A^T A X - A^T L = 0$;

$$A^T A = \begin{bmatrix} 2 & 1 & 7 & -1 \\ 3 & -2 & 1 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 7 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 55 & 12 & -8 \\ 12 & 15 & -4 \\ -8 & -4 & 15 \end{bmatrix},$$

$$A^T L = \begin{bmatrix} 2 & 1 & 7 & -1 \\ 3 & -2 & 1 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 52 \\ 29 \\ 25 \end{bmatrix},$$

$$A^T A X - A^T L = \begin{bmatrix} 55 & 12 & -8 \\ 12 & 15 & -4 \\ -8 & -4 & 15 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} - \begin{bmatrix} 52 \\ 29 \\ 25 \end{bmatrix} = 0.$$

And the solution is $X = (A^T A)^{-1} A^T L$;

$$(A^T A)^{-1} = \begin{bmatrix} .022859 & -.016187 & .0078748 \\ -.016187 & .083233 & .0135623 \\ .007875 & .0135623 & .074483 \end{bmatrix}$$

$$X = \begin{bmatrix} .022859 & -.016187 & .0078748 \\ -.016187 & .083233 & .0135623 \\ .007875 & .0135623 & .074483 \end{bmatrix} \begin{bmatrix} 52 \\ 29 \\ 25 \end{bmatrix},$$

$$X = \begin{bmatrix} .9161 \\ 1.91109 \\ 2.66488 \end{bmatrix}.$$

Thus the best values for the unknown parameters x_1, x_2 and x_3 are $x_1 = .9161$, $x_2 = 1.91109$ and $x_3 = 2.66488$.

This computation was performed by requiring that V^TV = minimum. Thus, when the best values are used in the equation $V = AX - L$, the resulting minimized residuals can be found. If the minimized residuals are applied to the observations then the observations are said to be adjusted.

$$V = AX - L = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 7 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} .9161 \\ 1.91109 \\ 2.6649 \end{bmatrix} - \begin{bmatrix} 10 \\ 5 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} .23035 \\ .08856 \\ -.0597 \\ .50793 \end{bmatrix},$$

$$V = \begin{bmatrix} .23035 \\ .08856 \\ -.0597 \\ .50793 \end{bmatrix}.$$

Then adjusted observations are $G_1 = 10.23035$, $G_2 = 5.08856$, $G_3 = 2.9403$ and $G_4 = -5.49207$.

Computational check: A^TV must be equal to zero according to Equation III-14.

$$A^TV = \begin{bmatrix} 2 & 1 & 7 & -1 \\ 3 & -2 & 1 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} .23035 \\ .08856 \\ -.0597 \\ .50793 \end{bmatrix} = \begin{bmatrix} .000 \\ .000 \\ .000 \end{bmatrix}.$$

According to the theory of probability, the above values of x_1, x_2 and x_3 have the highest probability of occurrence.

Example III-2: Suppose the constant terms 10, 5, 3, and -6 of the observation equations of Example III-1 represent measurements having relative weights of 1, 2, 2, and 3, respectively. Using weighted least squares, best values for x_1, x_2 and x_3 will be calculated.

The observation equations in Example III-1 were

$$2x_1 + 3x_2 + x_3 = 10 + v_1$$

$$x_1 - 2x_2 + 3x_3 = 5 + v_2$$

$$7x_1 + x_2 - 2x_3 = 3 + v_3$$

$$-x_1 - x_2 - x_3 = -6 + v_4 ,$$

or in the matrix form,

$$V = AX - L ,$$

where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 7 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, L = \begin{bmatrix} 10 \\ 5 \\ 3 \\ -6 \end{bmatrix}, \quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

The normal equation for weighted observations is

$$A^TWAX - A^TWL = 0,$$

where weight matrix W is a diagonal matrix of weights as follows:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

$$A^TWA = \begin{bmatrix} 2 & 1 & 7 & -1 \\ 3 & -2 & 1 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 7 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 107 & 19 & -17 \\ 19 & 22 & -10 \\ -17 & -10 & 30 \end{bmatrix},$$

$$A^TWL = \begin{bmatrix} 2 & 1 & 7 & -1 \\ 3 & -2 & 1 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 90 \\ 34 \\ 46 \end{bmatrix},$$

and

$$A^TWAX - A^TWL = \begin{bmatrix} 107 & 19 & -17 \\ 19 & 22 & -10 \\ -17 & -10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 90 \\ 34 \\ 46 \end{bmatrix} = 0.$$

The solution is $X = (A^TWA)^{-1} A^TWL$;

$$X = \begin{bmatrix} .0113839 & -.0081314 & .00374 \\ -.0081314 & .0593795 & .015185 \\ .00374 & .0151854 & .0405147 \end{bmatrix} \begin{bmatrix} 90 \\ 34 \\ 46 \end{bmatrix},$$

$$X = \begin{bmatrix} .9201 \\ 1.9856 \\ 2.7166 \end{bmatrix},$$

or $x_1 = .9201$, $x_2 = 1.9856$ and $x_3 = 2.7166$.

Residuals are found using Equation III-8:

$$V = AX - L = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ 7 & 1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} .9201 \\ 1.9856 \\ 2.7166 \end{bmatrix} - \begin{bmatrix} 10 \\ 5 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} .5136 \\ .0987 \\ -.0069 \\ .3777 \end{bmatrix}.$$

Computational check: $A^T W V$ must be equal to zero.

$$A^T W V = \begin{bmatrix} 2 & 1 & 7 & -1 \\ 3 & -2 & 1 & -1 \\ 1 & 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} .5136 \\ .0987 \\ -.0069 \\ .3777 \end{bmatrix} = \begin{bmatrix} .005 \\ .000 \\ .000 \end{bmatrix}.$$

3. Higher Order Functions

The observation equations presented by Equation III-8 are linear equations. If this relationship is nonlinear, thus defined by a higher order function, then the observation equations must be linearized in order to apply the least square adjustment method.

Defining the general observation equation as $G = f(x, y)$, where f represents a non-linear function. The function must be linearized by Taylor series expansion or by some other method. The best values of x and y can be regarded as the sum of an approximate value x_o, y_o and a small correction Δx, Δy. Therefore $x = x_o + \Delta x$ and $y = y_o + \Delta y$ and the above function is written in the following form

$$G = f(x_o + \Delta x, y_o + \Delta y).$$

Using Taylor series expansion the observation equations may be linearized.

$$G = f(x_o, y_o) + \frac{\partial f}{\partial x_o}\Delta x + \frac{\partial f}{\partial y_o}\Delta y + \text{Higher order terms.} \qquad \text{(III-14)}$$

The higher order terms in the series are neglected and only the zero and first order terms are maintained.

After linearization, the observation equations become

$$v_1 = \frac{\partial f_1(x,y)}{\partial x_o}\Delta x + \frac{\partial f_1(x,y)}{\partial y_o}\Delta y + f_1(x_o, y_o) - G_1$$

$$v_2 = \frac{\partial f_2(x,y)}{\partial x_o}\Delta x + \frac{\partial f_2(x,y)}{\partial y_o}\Delta y + f_2(x_o, y_o) - G_2 \qquad \text{(III-15)}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$v_n = \frac{\partial f_n(x,y)}{\partial x_o}\Delta x + \frac{\partial f_n(x,y)}{\partial y_o}\Delta y + f_n(x_o, y_o) - G_n ,$$

or in the matrix form, V = AX - L, where

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_o}, & \frac{\partial f_1}{\partial y_o} \\ \frac{\partial f_2}{\partial x_o}, & \frac{\partial f_2}{\partial y_o} \\ \vdots & \vdots \\ \frac{\partial f_n}{\partial x_o}, & \frac{\partial f_n}{\partial y_o} \end{bmatrix} \qquad X = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \qquad V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}.$$

$$L = \begin{bmatrix} G_1 - f_1(x_o, y_o) \\ G_2 - f_2(x_o, y_o) \\ \vdots \\ G_n - f_n(x_o, y_o) \end{bmatrix}$$

The remainder of the least square procedure is the same as indicated by Equations III-9, III-10 or III-11, III-13.

In the linearization process, the higher order terms were neglected. For this assumption to be valid, Δx and Δy should be small so that their products in the series expansion approach zero ($\Delta x \cdot \Delta y \cong 0$). This can be achieved only if the values of x_o and y_o are very close to the values of x and y. Therefore, x_o and y_o must be precomputed, or the original assumed x_o and y_o must be improved by successive iterations until the adjusted observations equal the measured values.

4. Equations for the Precision of Adjusted Quantities

After calculating the best values of the unknowns, or X matrix, the V matrix, or the adjustments to the observations, can be computed from the observation equation which is $V = AX - L$, whether the observations are weighted or not.

Using the V matrix, the standard error of an observation of unit weight is given by the following equations [Ref. 17]:

for unweighted observations,

$$\sigma_0 = \sqrt{\frac{\sum v_i^2}{n-m}} = \sqrt{\frac{V^T V}{n-m}} \; ; \qquad \text{(III-16a)}$$

for weighted observations,

$$\sigma_0 = \sqrt{\frac{\sum w_i v_i^2}{n-m}} = \sqrt{\frac{V^T W V}{n-m}} \; ; \qquad \text{(III-16b)}$$

where

σ_0 is the standard error of an observation which has unit weight,

n is the number of observations,

m is the number of unknowns.

Standard errors of the best values for the unknowns are then given by the following equation:

$$\sigma_i = \sigma_o \sqrt{q_{ii}}\,, \tag{III-17}$$

where

σ_i is the standard error of the ith adjusted quantity, e.g., the quantity in the ith row of the X matrix,

σ_o is the standard error of unit weight as found by Equation III-16a or III-16b,

q_{ii} is an element of, for unweighted case, $(A^TA)^{-1}$ or, for weighted case, $(A^TWA)^{-1}$ matrix.

If the $(A^TA)^{-1}$ or $(A^TWA)^{-1}$ matrices are written in detailed form as

$$(A^TA)^{-1} \text{ or } (A^TWA)^{-1} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \cdots \\ q_{21} & q_{22} & q_{23} \cdots \\ q_{31} & q_{32} & q_{33} \cdots \\ \vdots & \vdots & \end{bmatrix},$$

then the standard errors of the best values of the individual adjusted quantities are:

$$\sigma_1 = \sigma_o \sqrt{q_{11}}\,,$$

$$\sigma_2 = \sigma_0 \sqrt{q_{22}},$$

$$\sigma_3 = \sigma_0 \sqrt{q_{33}},$$

Example III-3: The standard errors of the best values for x_1, x_2 and x_3 in Example III-1.

Standard error of unit weight for unweighted observation is

$$\sigma_0 = \sqrt{\frac{V^T V}{n-m}} .$$

$$V^T V = \begin{bmatrix} .23035 & .08856 & -.00597 & .50793 \end{bmatrix} \begin{bmatrix} .23035 \\ .08856 \\ -.00597 \\ .50793 \end{bmatrix},$$

$$V^T V = .319 ,$$

$$\sigma_0 = \sqrt{\frac{V^T V}{n-m}} = \sqrt{\frac{.319}{4-3}} = .565 .$$

The standard errors of the best values are given by Equation III-17 as $\sigma_i = \sqrt{q_{ii}}$.

For unweighted observations, q_{ii}'s are the elements of $(A^TA)^{-1}$ which was calculated in Example III-1.

$$(A^TA)^{-1} = \begin{bmatrix} .022859 & -.0161872 & .007875 \\ -.016187 & .083233 & .0135623 \\ .007875 & .0135623 & .074483 \end{bmatrix},$$

$$\sigma_{x_1} = \sigma_0\sqrt{q_{11}} = .565\sqrt{.022859} = .085,$$
$$\sigma_{x_2} = \sigma_0\sqrt{q_{22}} = .565\sqrt{.083233} = .163,$$
$$\sigma_{x_3} = \sigma_0\sqrt{q_{33}} = .565\sqrt{.074483} = .154.$$

In the $(A^TA)^{-1}$ matrix, off diagonal terms are used to find the covariances of unknowns. Covariance σ_{12} is equal to

$$\sigma_{12} = \sigma_0\sqrt{q_{12}}.$$

From covariances, the correlation coefficients of variables are obtained. Correlation coefficient ρ_{12} is given as

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1\sigma_2}.$$

The interpretation of the standard errors computed above is that there is a 68% probability that the adjusted values for x_1, x_2 and x_3 are within ±.085, ±.163 and ±.154 of their true values, respectively.

Example III-4: The standard errors of x_1, x_2 and x_3 in Example III-2.

The standard error of unit weight for weighted observations is

$$\sigma_o = \sqrt{\frac{V^T W V}{n-m}} .$$

$$V^T W V = \begin{bmatrix} .5136 & .0987 & -.0069 & .3777 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} .5136 \\ .0987 \\ -.0069 \\ .3777 \end{bmatrix} .$$

$$V^T W V = .7114 ,$$

$$\sigma_o = \sqrt{\frac{V^T W V}{n-m}} = \sqrt{\frac{.7114}{4-3}} = .843 .$$

The standard errors of best values are given as

$$\sigma_i = \sigma_0 \sqrt{q_{ii}}.$$

For weighted observations, q_{ii}'s are the elements of $(A^TWA)^{-1}$ which has been already calculated in Example III-2.

$$(A^TWA)^{-1} = \begin{bmatrix} .0113839 & -.0081314 & .00374 \\ -.0081314 & .0593795 & .0151854 \\ .00374 & .0151854 & .040514 \end{bmatrix},$$

$$\sigma_{x_1} = \sigma_0 \sqrt{q_{11}} = .843 \sqrt{.0113839} = .0899,$$

$$\sigma_{x_2} = \sigma_0 \sqrt{q_{22}} = .843 \sqrt{.0593795} = .2054,$$

$$\sigma_{x_3} = \sigma_0 \sqrt{q_{33}} = .843 \sqrt{.040514} = .1697.$$

B. APPLICATION OF LEAST SQUARES TO HYDROGRAPHIC POSITIONING SYSTEMS

If redundant data are available, the least square adjustment method may be used to compute the coordinates of hydrographic survey positions. Observation equations may be written for various types of survey methods. By expressing these equations in matrix notation and using successive approximations of the unknowns, the best values for the coordinates of survey positions may be determined. The predictable accuracy of these best values may also be found. Thus, redundant observations, coupled with mathematical data adjustment techniques, produce a viable method of system calibration for hydrographic survey data. This method of calibration is referred to as auto calubration.

1. Azimuth Angle Positions

The working range of azimuthal systems is limited to line of sight distances, i.e., 5-15 nautical miles, depending upon the height of the observing instrument.

Because of this range limit, the Universal Transverse Mercator (UTM), or other plane coordinate systems, may be used. Let

y_1, y_2, y_3 = Northings of the shore stations 1, 2 and 3, respectively,

x_1, x_2, x_3 = Eastings of the shore stations 1, 2 and 3, respectively,

P = The position of the survey vessel.

Then, the azimuth (from north) of the survey vessel from shore stations can be written in terms of coordinates as

$$A_{1P} = \tan^{-1} \frac{y_P - y_1}{x_P - x_1}, \quad A_{2P} = \tan^{-1} \frac{y_P - y_2}{x_P - x_2}, \quad A_{3P} = \tan^{-1} \frac{y_P - y_3}{x_P - x_3}.$$

In these equations, x_p and y_p are the best estimate of the survey vessel coordinates which are to be determined.

These equations are non-linear. Thus, in order to form observation equations, they must be linearized. Letting $x_p = x_o + \Delta x$ and $y_p = y_o + \Delta y$, where x_o and y_o are the approximate coordinates of the vessel's position, and using Taylor series expansion for linearization,

$$f(x,y) = f(x_o, y_o) + \frac{\partial f}{\partial x_o} \Delta x + \frac{\partial f}{\partial y_o} \Delta y + \cdots\cdots,$$

where $$f(x_o, y_o) = A_{io} = \tan^{-1} \frac{y_o - y_i}{x_o - x_i}.$$

The partial derivatives are

$$\frac{\partial f}{\partial x_o} = -\frac{y_o - y_i}{S_{io}^2}, \quad \frac{\partial f}{\partial y_o} = \frac{x_o - x_i}{S_{io}^2}.$$

The observation equations now may be written in the following detailed form:

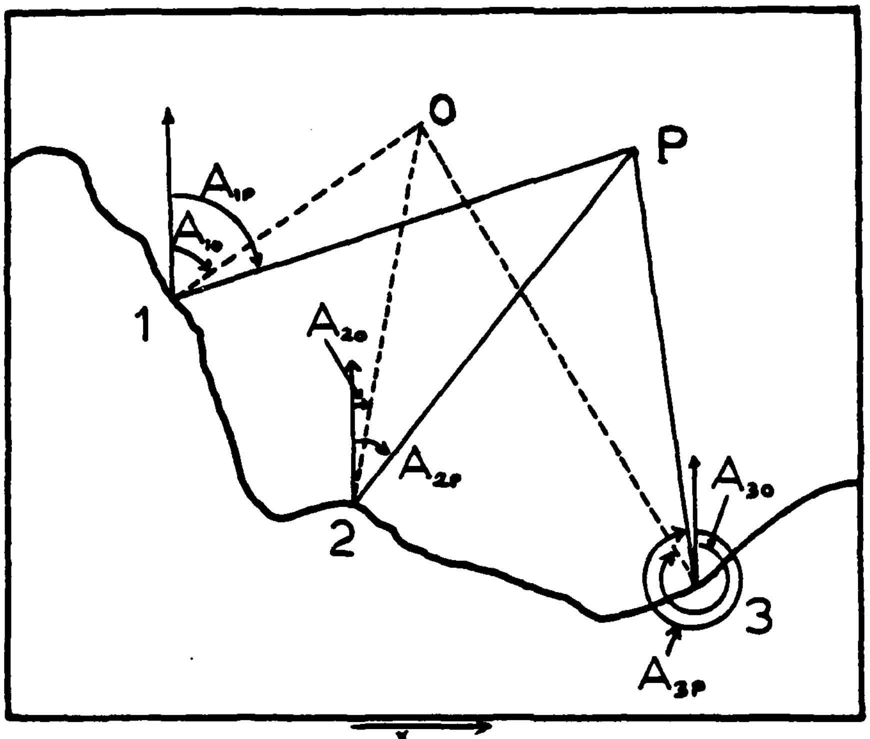

Figure III-1. Azimuth angle positions.

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \rho \begin{bmatrix} -\dfrac{y_o - y_1}{S_{1o}^2} & \dfrac{x_o - x_1}{S_{1o}^2} \\ -\dfrac{y_o - y_2}{S_{2o}^2} & \dfrac{x_o - x_2}{S_{2o}^2} \\ -\dfrac{y_o - y_3}{S_{3o}^2} & \dfrac{x_o - x_3}{S_{3o}^2} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} A_{1p} - A_{1o} \\ A_{2p} - A_{2o} \\ A_{3p} - A_{3o} \end{bmatrix},$$

where

$$A_{io} = \tan^{-1} \frac{y_o - y_i}{x_o - x_i},$$

and A_{1p}, A_{2p}, A_{3p}

are the measured azimuths, and $\varrho = 57.2958$, the conversion factor from radians to degrees.

Having obtained the observation equation, the normal equation may be formed and solved by following the procedures outlined in Section II.A.

Using the computed values of Δx and Δy, new trial point coordinates may be formed as follows:

$$x = x_o + \Delta x,$$

$$y = y_o + \Delta y.$$

The values are substituted in the observation equation for the initial x_o, y_o coordinates. The least square solution is iterated until the Δx and Δy values approach zero.

Example III-5: Referring to Figure III-2, the coordinates of the shore stations are

	Luces (#1)	Mussel (#2)	MB4 (#3)
x	4,055,042.7 m	4,053,453.2 m	4,053,917.2 m
y	595,794.5 m	597,967.8 m	603,425.2 m

The standard errors for the azimuth observations are $\sigma_1 = .02°$, $\sigma_2 = .024°$ and $\sigma_3 = .018°$. The following angles were measured:

P - Luces - Mussel = $\alpha_1 = 50.^\circ164$

P - Mussel - Luces = $\alpha_2 = 99.^\circ360$

P - MB4 - Mussel = $\alpha_3 = 47.^\circ865$

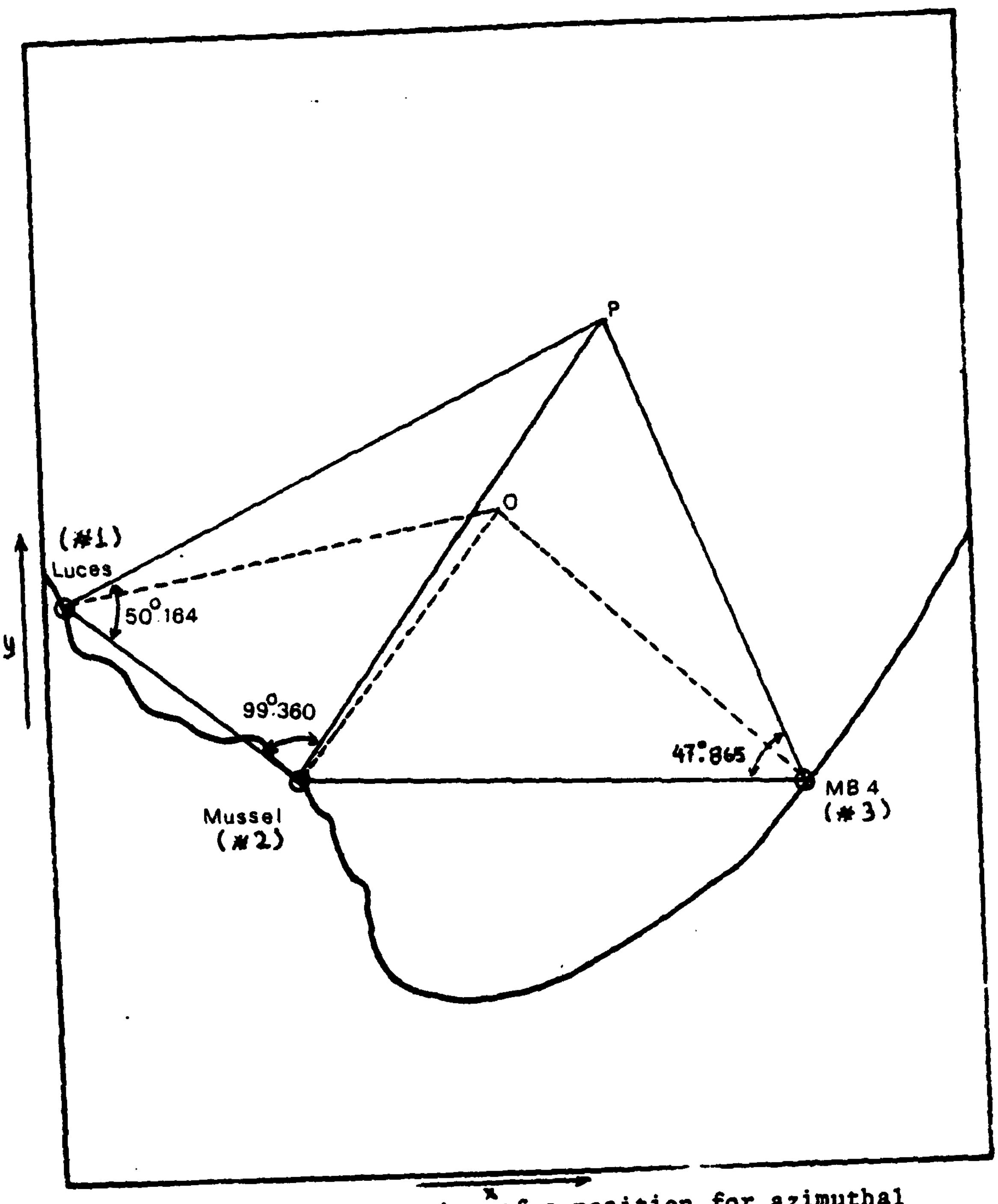

Figure III-2: Determination of a position for azimuthal systems using the least square method.

The least square method will be used to determine the best values for the coordinates of the survey vessel.

Given: $A_{12} = 126\overset{\circ}{.}180$ | Given: $A_{21} = 306\overset{\circ}{.}180$

Measured: $\alpha_1 = 50\overset{\circ}{.}164$ | Measured: $\alpha_2 = 99\overset{\circ}{.}360$

$A_{1p} = 76\overset{\circ}{.}016$ | $A_{2p} = 45\overset{\circ}{.}540$

Given: $A_{32} = 265\overset{\circ}{.}140$

Measured: $\alpha_3 = 47\overset{\circ}{.}865$

$A_{3p} = 313\overset{\circ}{.}005$

First, assume $x_o = 4{,}055{,}000$

$y_o = 600{,}000$

The observation equation, V = AX - L, is then

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -.01362 & -.000138 \\ -.01785 & .013587 \\ .015208 & .004807 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} -14.566 \\ -7.183 \\ 25.462 \end{bmatrix}.$$

Using the standard errors of each station, a weight matrix is formed:

$$W = \begin{bmatrix} k^2/\sigma_1^2 & 0 & 0 \\ 0 & k^2/\sigma_2^2 & 0 \\ 0 & 0 & k^2/\sigma_2^2 \end{bmatrix}.$$

Let k = 0.024, then

$$W = \begin{bmatrix} 1.44 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.78 \end{bmatrix}.$$

Using the A, L and W matrices, the components of the normal equation, $A^TWAX - A^TWL = 0$, are written

$$\begin{bmatrix} .0009974 & -.0001097 \\ -.0001097 & .0002257 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} 1.1032 \\ .1232 \end{bmatrix} = 0.$$

The solution, $X = (A^TWA)^{-1}(A^TWL)$, is then

$$X = \begin{bmatrix} 1059.2 & 514.6 \\ 514.6 & 4679.4 \end{bmatrix} \begin{bmatrix} 1.1032 \\ .1232 \end{bmatrix} = \begin{bmatrix} 1231.8 \\ 1144.0 \end{bmatrix}.$$

Now, the new trial point coordinates are found.

$$x = x_o + \Delta x = 4,055,000 + 1231.8 = 4,056,231.8,$$

$$y = y_o + \Delta y = 600,000 + 1144.0 = 601,144.0.$$

Using new trial point coordinates, solutions are repeated until Δx and Δy vanish. Table III-1 shows the data for other trial Δx and Δy values and the new trial point coordinates.

The best estimate of the coordinates for the survey vessel is

$$x_P = 4,056,302.9$$

$$y_P = 600,867.4$$

2. Sextant Angle Positions

In sextant angle positions, similar to the resection problem in geodetic work, the measured quantities are the included angles at the sounding vessel between the shore stations as shown in Figure III-3.

Iteration No.	Change in Coordinates Δx	Δy	Coordinates x	y
0			4,055,000	600,000
1	1231.8	1144.0	4,056,231.8	601,144.0
2	63.4	-173.4	4,056,295.2	600,970.6
3	9.0	-101.9	4,056,304.2	600,867.8
4	-1.3	-.4	4,056,302.9	600,867.4

Table III-1: For least squares solution, successive iterations applied to azimuth angle positions.

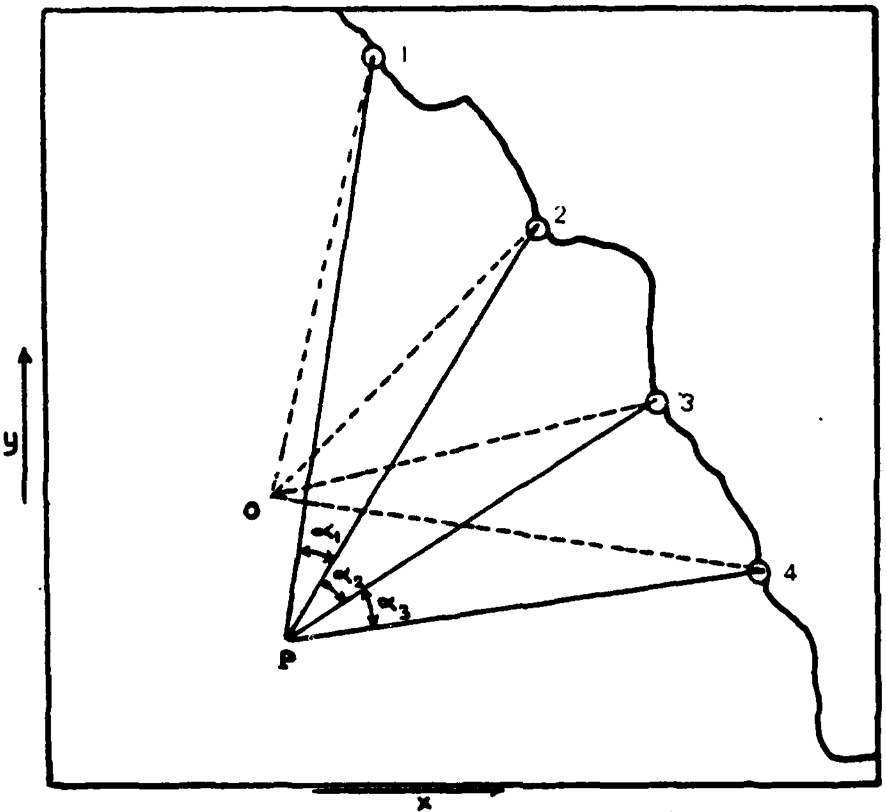

Figure III-3: Sextant Angle Positions

Plane coordinates are again used because of the visual range limitation.

It can be seen from Figure III-3 that

$$A_{p2} - A_{p1} = \alpha_1 ,$$

$$A_{p3} - A_{p2} = \alpha_2 ,$$

$$A_{p4} - A_{p3} = \alpha_3 ;$$

or in terms of the coordinates,

$$\tan^{-1}\frac{y_2-y_p}{x_2-x_p} - \tan^{-1}\frac{y_1-y_p}{x_1-x_p} = \alpha_1,$$

$$\tan^{-1}\frac{y_3-y_p}{x_3-x_p} - \tan^{-1}\frac{y_2-y_p}{x_2-x_p} = \alpha_2,$$

$$\tan^{-1}\frac{y_4-y_p}{x_4-x_p} - \tan^{-1}\frac{y_3-y_p}{x_3-x_p} = \alpha_3.$$

Letting $x_p = x_o + \Delta x$ and $y_p = y_o + \Delta y$ and linearizing with Taylor series expansion,

$$f(x,y) = f(x_o, y_o) + \frac{\partial f}{\partial x_o}\Delta x + \frac{\partial f}{\partial y_o}\Delta y + \cdots\cdots,$$

where

$$\frac{\partial f_1}{\partial x_o} = \frac{y_2-y_o}{S_{2o}^2} - \frac{y_1-y_o}{S_{1o}^2},$$

$$\frac{\partial f_1}{\partial y_o} = \frac{x_1-x_o}{S_{1o}^2} - \frac{x_2-x_o}{S_{2o}^2}.$$

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The observation equations may then be expressed as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \rho \begin{bmatrix} \frac{y_2 - y_o}{S^2_{2o}} - \frac{y_1 - y_o}{S^2_{1o}}, & \frac{x_1 - x_o}{S^2_{1o}} - \frac{x_2 - x_o}{S^2_{2o}} \\ \frac{y_3 - y_o}{S^2_{3o}} - \frac{y_2 - y_o}{S^2_{2o}}, & \frac{x_2 - x_o}{S^2_{2o}} - \frac{x_3 - x_o}{S^2_{3o}} \\ \frac{y_4 - y_o}{S^2_{4o}} - \frac{y_3 - y_o}{S^2_{3o}}, & \frac{x_3 - x_o}{S^2_{3o}} - \frac{x_4 - x_o}{S^2_{4o}} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} \alpha_1 - (A_{o2} - A_{o1}) \\ \alpha_2 - (A_{o3} - A_{o2}) \\ \alpha_3 - (A_{o4} - A_{o3}) \end{bmatrix}$$

where

ρ = 57.2958 is the conversion factor from radians to degrees,

A_{o1}, A_{o2}, A_{o3}, A_{o4} are the computed azimuths of lines o1, o2, o3, o4 using trial point coordinates x_o, y_o.

Once establishing the observation equations, the solution is found as

$$X = (A^T A)^{-1} A^T L \ .$$

The process is repeated until Δx and Δy become very small.

Example III-6: Referring to Figure III-4, the coordinates of the shore station are

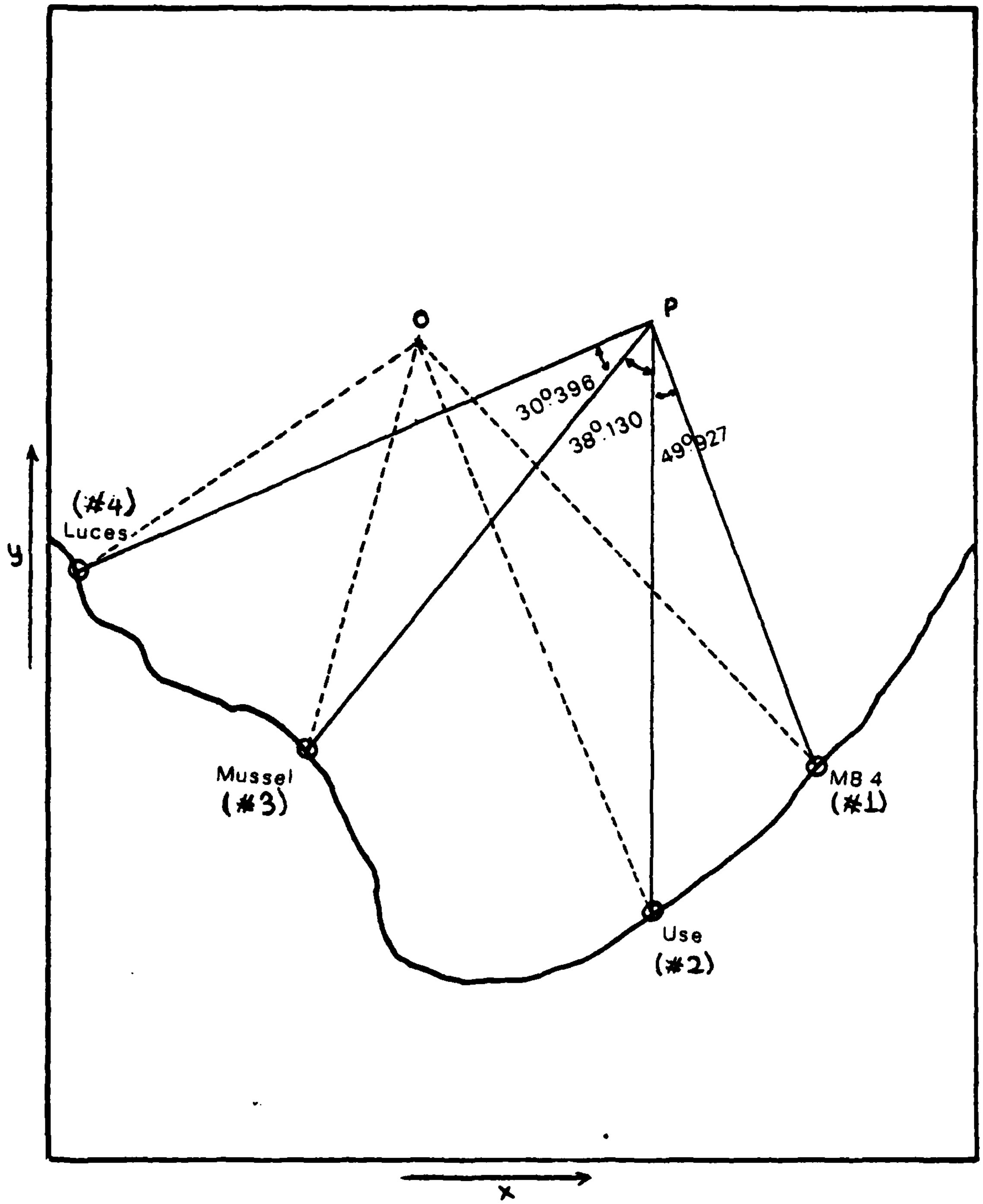


Figure III-4: Determination of a position for sextant angle fixes using least squares adjustment.

	MB4 (#1)	Use (#2)	Mussel (#3)	Luces (#4)
x	4,053,917.2	4,051,216.9	4,053,453.2	4,055,042.7
y	603,425.2	600,372.0	597,967.8	595,794.5

Measured angles are

MB4 - P - Use = 49°927,

Use-P - Mussel = 38°130,

Mussel - P - Luces = 30.396.

The least square method will be used to determine the best values for the coordinates of the survey vessel. Let the first assumed position be x_o = 4,057,000 and y_o = 599,000. Using the first approximate position,

$$A_{01} = \tan^{-1} \frac{y_1 - y_o}{x_1 - x_o},$$

A_{01} = 124°862, A_{02} = 165°712, A_{03} = 196°235 and A_{04} = 238°591 are obtained.

The observation equation is

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -.006366 & .003834 \\ -.006687 & .004984 \\ -.008683 & -.006942 \end{bmatrix} \begin{bmatrix} \Delta x \\ \\ \Delta y \end{bmatrix} - \begin{bmatrix} 9°.077 \\ 7°.607 \\ -12°.160 \end{bmatrix},$$

and normal equation, $A^T AX - A^T L = 0$, is

$$\begin{bmatrix} .0001606 & .0000025 \\ .0000025 & .0000877 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} -.00307 \\ .15714 \end{bmatrix} = 0 .$$

The solution, $X = (A^T A)^{-1} A^T L$, is

$$X = \begin{bmatrix} 6229.4 & -1776 \\ -177.6 & 11407.6 \end{bmatrix} \begin{bmatrix} -.00307 \\ .15714 \end{bmatrix} = \begin{bmatrix} -47.0 \\ 1739.1 \end{bmatrix} .$$

Then, the new trial point coordinates are

$x = x_o + \Delta x = 4{,}057{,}000 + (-47) = 4{,}056{,}953$,

$y = y_o + \Delta y = 599{,}000 + 1793.1 = 600{,}793.1$.

Using new trial point coordinates, the above steps are repeated until Δx and Δy values become vanishingly small. For every trial Δx and Δy value, new trial points coordinates are tabulated in Table III-2.

The best values for the coordinates of the sounding vessel are

$x_p = 4{,}056{,}512.3$,

$y_p = 600{,}864.5$.

Iteration No.	Change in Coordinates		Coordinates	
	Δ_x	Δ_y	x	y
0	--	--	4,057,000	599,000
1	-47.0	1793.1	4,056,953	600,793.1
2	-481.0	119.7	4,056,472	600,912.8
3	-14.0	-26.3	4,056,458	600,886.5
4	53.8	-21.0	4,056,511.8	600,865.5
5	.5	-1.0	4,056,512.3	600,864.5

Table III-2: For least squares solution, successive iterations applied to sextant angle positions

3. Range-Range Positions

In range-range positioning, when the distances are short (i.e., line of sight type equipment, less than 20 nautical miles), a plane coordinate system may be used. Let

x_1, x_2, x_3 = Eastings of the shore stations #1, 2, 3

y_1, y_2, y_3 = Northings of the shore stations #1, 2, 3

x_p = Easting of the survey vessel

y_p = Northing of the survey vessel

Then, the distance between the ith shore station and the survey vessel, in plane coordinates, is

$$S_{ip} = \sqrt{(x_p - x_i)^2 + (y_p - y_i)^2}\,, \qquad (i = 1, 2, 3).$$

This function is non-linear and has to be linearized. Introducing approximate coordinates (x_o, y_o) for x_p and y_p, then

$$x_p = x_o = \Delta x\,,$$

$$y_o = y_o + \Delta y\,.$$

Using Taylor series for linearization, the result becomes

$$S_{ip} = f(x_o, y_o) + \frac{\partial f}{\partial x_o}\Delta x + \frac{\partial f}{\partial y_o}\Delta y + \cdots\cdots .$$

After linearization, the observation equation is

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} \dfrac{x_o - x_1}{S_{1o}} , & \dfrac{y_o - y_1}{S_{1o}} \\ \dfrac{x_o - x_2}{S_{2o}} , & \dfrac{y_o - y_2}{S_{2o}} \\ \dfrac{x_o - x_3}{S_{3o}} , & \dfrac{y_o - y_3}{S_{3o}} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} S_{1p} - S_{1o} \\ S_{2p} - S_{2o} \\ S_{3p} - S_{3o} \end{bmatrix},$$

or in the matrix form, $V = AX - L$,

where:

S_{1p}, S_{2p}, S_{3p} are the measured distances,

S_{1o}, S_{2o}, S_{3o} are the computed distances using x_o and y_o.

In ranging systems, when the distances are long, coordinate computations must be carried out on the appropriate ellipsoid using rigorous geodetic formulas. Let

ϕ_o, λ_o = Computed geographical coordinates, latitude and longitude, of the survey vessel,

ϕ_i, λ_i = Latitude and longitude of the ith shore station,

A_{io} = Azimuth from ith shore station to approximated position O,

A_{oi} = Azimuth from O to ith shore station,

S_{oi} = Distance between O and ith shore station.

Although the computed observations must utilize rigorous geodetic solutions, the differential equations of the observations may be approximated using spherical trigonometry [Ref. 3]:

$$dS_{oi} = \sin 1'' \left[-R_o \cos A_{oi}\, d\phi_o - R_i \cos A_{io}\, d\phi_i + N_i \cos\phi_i \sin A_{io} (d\lambda_o - d\lambda_i) \right],$$

where $d\phi$ and $d\lambda$ are in seconds of arc, dS in meters. Sin 1" is the conversion factor from seconds to radians. R_o and R_i are the radius of curvature in the plane of meridian at point o and ith shore station, respectively, defined as [Ref. 18]

$$R = \frac{a(1-e^2)}{(1-e^2 \sin^2\phi)^{3/2}},$$

where a is the semi major axis of the datum ellipsoid, and e^2 is the eccentricity of the datum ellipsoid. N_o and N_i are the radius of curvature in the plane of prime vertical at point o and at ith shore station, respectively, defined as [Ref. 18]

$$N = \frac{a}{(1-e^2 \sin^2\phi)^{1/2}}.$$

ϕ_o is the latitude of point o and ϕ_i is the latitude of the ith shore station.

The partial derivatives of computed observations with respect to parameters are

$$\frac{\partial S_{oi}}{\partial \phi_o} = -\sin 1'' R_o \cos A_{oi},$$

$$\frac{\partial S_{oi}}{\partial \lambda_o} = \sin 1'' N_i \cos \phi_i \sin A_{io}.$$

Then, observation equation is

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \sin 1'' \begin{bmatrix} -R_o \cos A_{o1}, & N_1 \cos\phi_1 \sin A_{1o} \\ -R_o \cos A_{o2}, & N_2 \cos\phi_2 \sin A_{2o} \\ -R_o \cos A_{o3}, & N_3 \cos\phi_3 \sin A_{3o} \end{bmatrix} \begin{bmatrix} \Delta\phi \\ \Delta\lambda \end{bmatrix} - \begin{bmatrix} S_{p1} - S_{o1} \\ S_{p2} - S_{o2} \\ S_{p3} - S_{o3} \end{bmatrix},$$

where S_{o1}, S_{o2}, S_{o3} are computed distances using inverse distance and azimuth formulas (these formulas could be found in any geodesy text); and S_{p1}, S_{p2}, S_{p3} are measured distances between point **p** and the respective shore station.

After forming the observation equation, the normal equation is found and solved as in previous examples. This process is repeated until $\Delta\phi$ and $\Delta\lambda$ become smaller than the resolution of the positioning system.

Example III-7: Referring to Figure III-5, the coordinates of the shore stations are

	Luces (#1)	Mussel (#2)	MB4 (#3)
x	4,055,042.7	4,053,453.2	4,053,917.2
y	595,794.5	597,967.8	603,425.2

Using the least squares procedure, best values of coordinates of the vessel may be found. Let the first assumed position x_o = 4,056,000 m and y_o = 598,000 m. Measured distances are ρ - LUCES = 4350 m, ρ - MUSSEL = 4506 m, and ρ - MB4 = 5267 m. For the first approximate position of the vessel, the observation equation is written:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} .999 & .0126 \\ .398 & .917 \\ .358 & -.933 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} 1803.0 \\ 2101.7 \\ -544.3 \end{bmatrix}.$$

Normal equation, $A^TAX - A^TL = 0$, is

$$\begin{bmatrix} 1.287 & .043 \\ .043 & 1.713 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} - \begin{bmatrix} 2834.8 \\ 1442.7 \end{bmatrix} = 0.$$

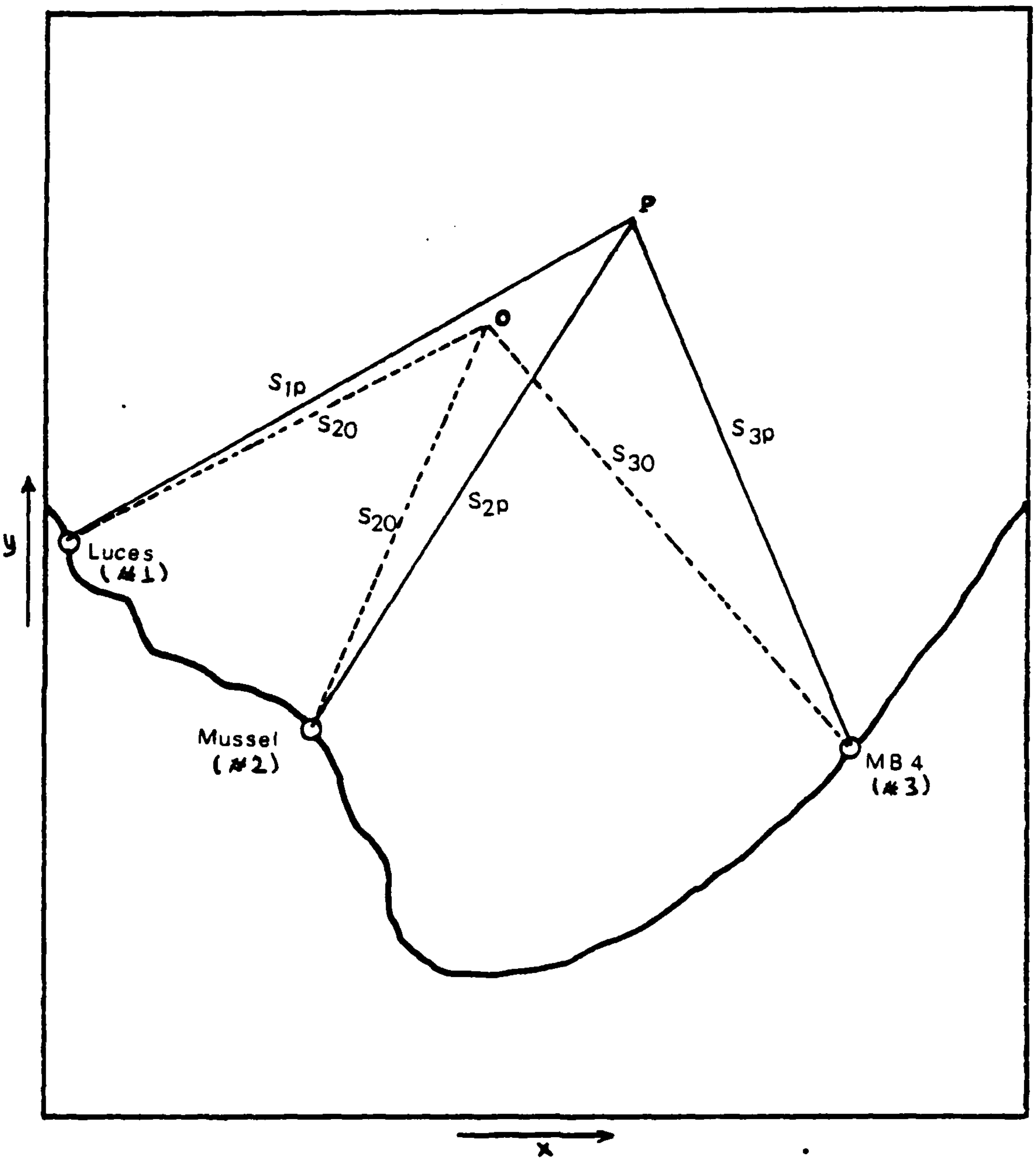


Figure III-5: Determination of a position for range-range systems using least squares adjustment.

And solution, $X = (A^T A)^{-1} A^T L$, is

$$X = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} .777 & -.019 \\ -.019 & .584 \end{bmatrix} \begin{bmatrix} 2834.8 \\ 1442.7 \end{bmatrix} = \begin{bmatrix} 2183.2 \\ 787.4 \end{bmatrix}.$$

Then, new trial point coordinates are:

$x = x_o + \Delta x = 4{,}056{,}000 + 2183.2 = 4{,}058{,}183.2$

$y = y_o + \Delta y = 598{,}000 + 787.4 = 598{,}787.4$

Using the new trial point coordinates, the above steps are repeated until Δx and Δy values become smaller than the system resolution. For every trial, the change in coordinates and the coordinates of new trial points are tabulated in Table III-3.

The best values for the coordinates of the sounding vessel are

$$x_p = 4{,}057{,}501.2$$

$$y_p = 599{,}567.7$$

The standard errors in the northing and easting may also be calculated:

$V = AX - L$ (for A and L, Last iteration values are used),

$$V = \begin{bmatrix} .930 & .367 \\ .546 & .337 \\ .680 & -.333 \end{bmatrix} \begin{bmatrix} -.8 \\ .7 \end{bmatrix} - \begin{bmatrix} -3.2 \\ 2.7 \\ .5 \end{bmatrix} = \begin{bmatrix} 2.7 \\ 2.6 \\ -1.5 \end{bmatrix};$$

Iteration No.	Change in Coordinates		Coordinates	
	Δx	Δy	x	y
0	--	--	4,056,000	598,000
1	2183.2	787.4	4,058,183.2	598,787.4
2	-597.6	860.5	4,057,585.6	599,647.9
3	-83.6	-80.9	4,057,502	599,567.0
4	-.8	.7	4,057,501.2	599,567.7

Table III-3: For least squares solution, successive iterations applied to range-range positions.

$$V^T V = \begin{bmatrix} 2.7 & 2.6 & -1.5 \end{bmatrix} \begin{bmatrix} 2.7 \\ 2.6 \\ -1.5 \end{bmatrix} = 16.3;$$

$$\sigma_o = \sqrt{\frac{V^T V}{n-m}} = \sqrt{\frac{16.3}{3-2}} = 4.04;$$

$$\sigma_i = \sigma_o \sqrt{q_{ii}},$$

where q_{ii}'s are the elements of $(A^T A)^{-1}$ which has been calculated as

$$(A^T A)^{-1} = \begin{bmatrix} .64 & -.14 \\ -.14 & .76 \end{bmatrix},$$

$$\sigma_x = \sigma_o \sqrt{q_{11}} = 4.04 \sqrt{.64} = 3.23 \text{ m},$$

$$\sigma_y = \sigma_o \sqrt{q_{22}} = 4.04 \sqrt{.76} = 3.52 \text{m},$$

4. <u>Hyperbolic Positioning Systems</u>

Hyperbolic positioning systems measure the difference in distance from a vessel to the two shore stations. In Figure III-6, station number 2 is the master station, and

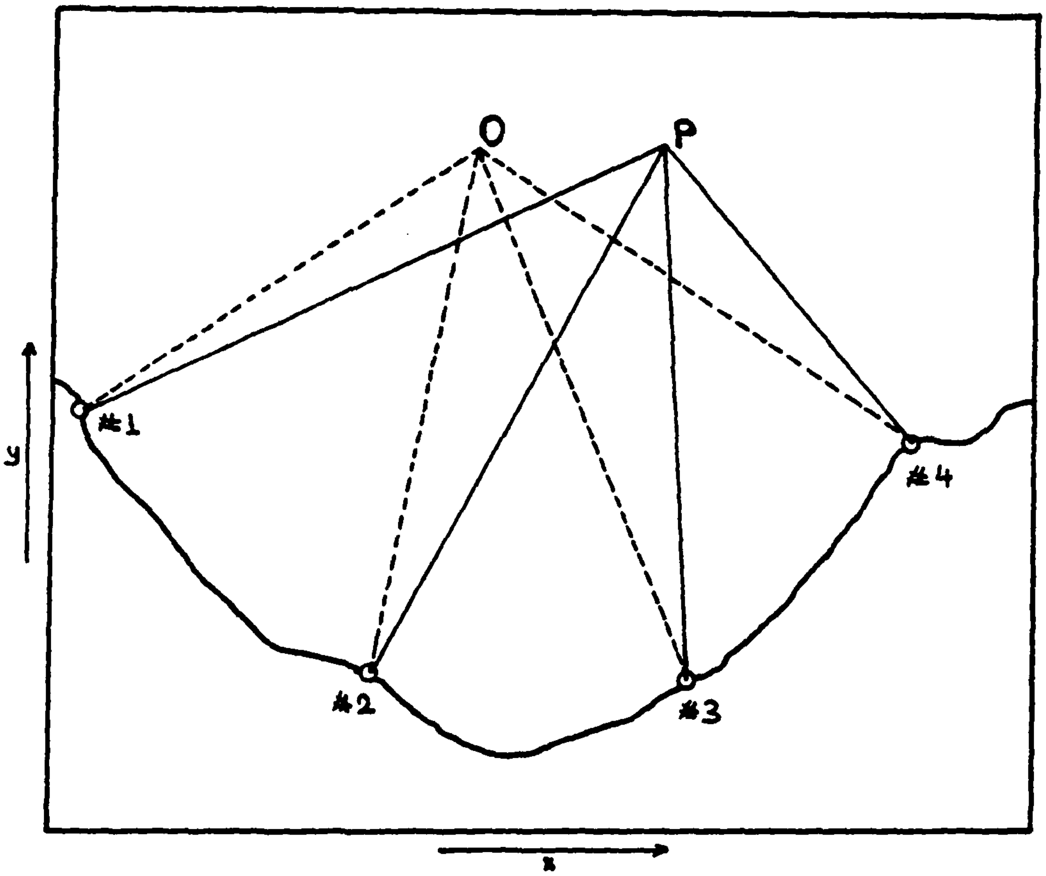


Figure III-6: Determination of a position for hyperbolic systems using least squares adjustment method.

numbers 1, 3 and 4 are slaves. Point **p** is the vessel's position, and its coordinates are designated as ϕp and λp. Point **O** is the first approximate position, with ϕ_O and λ_O representing its coordinates.

The differential equations of the computed distances to each station, S_{Oi}, may be written as [Ref. 3]:

$$ds_{oi} = \sin 1'' [-R_o \cos A_{oi}\, d\phi_o - R_i \cos A_{io}\, d\phi_i$$

$$+ N_i \cos\phi_i \sin A_{io} (d\lambda_o - d\lambda_i)]$$

where i represents the shore station number.

A_{io} represents the azimuth from the ith shore station to approximate position 0.

R_O is the radius of curvature in the plane of meridian at point **O** (as defined in Section B.3.).

N_i is the radius of curvature in the plane prime vertical at the ith shore station (as defined in Section B.3.).

The partial derivatives of the function with respect to ϕ_O and λ_O are

$$\frac{\partial S_{oi}}{\partial \phi_o} = -\sin 1'' \cos A_{oi},$$

$$\frac{\partial S_{oi}}{\partial \lambda_o} = \sin 1'' \cos\phi_i \sin A_{io}.$$

The range difference between the distance from the vessel to the master and the distance from the vessel to the respective slave station is expressed in the equations below.

$$\frac{\partial s_{o2}}{\partial \phi_o} - \frac{\partial s_{oi}}{\partial \phi_o} = \sin 1'' R_o (\cos A_{oi} - \cos A_{o2})$$

$$\frac{\partial s_{o2}}{\partial \lambda_o} - \frac{\partial s_{oi}}{\partial \lambda_o} = \sin 1'' N_i \cos\phi_i (\sin A_{2o} - \sin A_{io})$$

Note that station number 2 is the master station, and the range difference is stated in terms of the partial derivatives.

With this information, the observation equation is written as:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \sin 1'' \begin{bmatrix} R_o (\cos A_{o1} - \cos A_{o2}), & N_1 \cos\phi_1 (\sin A_{2o} - \sin A_{1o}) \\ R_o (\cos A_{o3} - \cos A_{o2} & , N_2 \cos\phi_3 (\sin A_{2o} - \sin A_{3o}) \\ R_o (\cos A_{o4} - \cos A_{o2} & , N_3 \cos\phi_4 (\sin A_{2o} - \sin A_{4o} \end{bmatrix}$$

$$\begin{bmatrix} \Delta\phi \\ \Delta\lambda \end{bmatrix} - \begin{bmatrix} (S_{p2} - S_{p1}) - (S_{o2} - S_{o1}) \\ (S_{p2} - S_{p3}) - (S_{o2} - S_{o3}) \\ (S_{p2} - S_{p4}) - (S_{o2} - S_{o4}) \end{bmatrix},$$

where:

$A_{01}, A_{02},\ldots$ are computed azimuths from approximate position 0 to shore station 1, 2,... .

$A_{10}, A_{20},\ldots$ are computed azimuths from shore station number 1, 2,... to approximate position 0.

$(S_{p2} - S_{p1}), (S_{p2} - S_{p3}), \ldots$ are measured range differences.

$(S_{02} - S_{01}), (S_{02} - S_{03}), \ldots$ are computed range differences.

Sin 1" is conversion factor from second to radian.

After writing the observation equation, the normal equation is solved and the best estimate of the coordinate values is found as previously discussed.

The process is iterated until $\Delta\phi$ and $\Delta\lambda$ become smaller than the standard error of the specific hyperbolic system being used.

5. Global Positioning System (GPS)

Global Positioning System fixes are obtained utilizing the computed distances from the position of GSP satellites to a GPS receiver. The receiver measures the arrival of a timing pulse from every satellite within acquisition range. The transmit time of each pulse is encoded in the received signal. Thus, distance is computed using the one way travel time between each satellite and the receiver multiplied by the propagation velocity of electromagnetic energy. Three such satellite to receiver ranges may then be applied to solve for the coordinates of the receiver.

Using three satellites to determine a fix results in a unique solution for the position coordinates (x, y, z). However, significant error may be induced due to drift in the receiver clock. This additional unknown, receiver clock bias (E), may be resolved by processing four satellite ranges.

For position fixing at sea, it is likely that the z coordinate may be input as a known value based on a given antenna height above sea level. Thus, the number of unknowns will be reduced to three. By using four or more satellites, redundant observations are then available so that the data can be adjusted by the method of least squares.

Introducing the following variables, observation equations may be written in a straightforward manner:

R_1, R_2, R_3 ,... = Measured distances from receiver to satellites S_1, S_2, S_3, ..

(x_1, y_1, z_1), (x_2, y_2, z_2), ... = Known positions of satellites S_1, S_2, S_3,...

x, y, z = Unknown position of the observer, **p**.

E = Receiver clock bias (unknown).

Then, the basic equations are

$$R_1 = E + \sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}$$

$$R_2 = E + \sqrt{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2}$$

$$R_3 = E + \sqrt{(x-x_3)^2 + (y-y_3) + (z-z_3)^2}$$

$$\vdots$$

$$R_n = E + \sqrt{(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2} \; .$$

Here, the ranges R_1, R_2, R_3, ... , R_n include the actual satellite to receiver distance plus some offset due to receiver clock error. In the above equations, the satellite positions are known, and the four unknowns are the user position (x, y, z) and user clock error.

Since the observation equations are non-linear, the Taylor series must be applied to form equations suitable for use with the method of least squares. Let

$$x = x_o + \Delta x \qquad z = z_o + \Delta z$$

$$y = y_o + \Delta y \qquad E = E_o + \Delta E \, .$$

Using Taylor series,

$$R_{ip} = E_o + \sqrt{(x_o-x_i)^2 + (y_o-y_i)^2 + (z_o-z_i)^2} + \frac{x_o-x_i}{R_{io}} \Delta x$$

$$\frac{y_o-y_i}{R_{io}} \Delta y + \frac{z_o-z_i}{R_{io}} \Delta z + \Delta E$$

where R_{ip} is the distance between a satellite and the user position, p.

R_{io} is the distance between a satellite and approximated user position.

Observation equations are written in the following detailed matrix form:

$$\begin{bmatrix} V_1 \\ \\ V_2 \\ \vdots \\ \\ V_n \end{bmatrix} = \begin{bmatrix} \frac{x_o - x_1}{R_{1o}}, & \frac{y_o - y_1}{R_{1o}}, & \frac{z_o - z_1}{R_{1o}} & 1 \\ \frac{x_o - x_2}{R_{2o}}, & \frac{y_o - y_2}{R_{2o}}, & \frac{z_o - z_2}{R_{2o}} & 1 \\ \vdots & \vdots & \vdots & \\ \frac{x_o - x_n}{R_{no}}, & \frac{y_o - y_n}{R_{no}}, & \frac{z_o - z_n}{R_{no}} & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta E \end{bmatrix} - \begin{bmatrix} R_{1p} - R_{1o} \\ R_{2p} - R_{2o} \\ \vdots \\ R_{np} - R_{no} \end{bmatrix},$$

where

R_{1p}, R_{2p} ..... are the measured distances,

R_{1o}, R_{2o} ..... are computed ranges from the formula

$$R_{io} = E_o + \sqrt{(x_o - x_i)^2 + (y_o - y_i)^2 + (z_o - z_i)^2}.$$

From this matrix, the normal equation may be formed and solved as previously discussed. The process is repeated until the values of Δx, Δy, Δz and ΔE approach zero.

Example III-8: At 0800 Zulu, May 1, 1980, a satellite fix was taken using a GPS receiver aboard USNS ACANIA in Monterey Bay. The measured distances between the satellites and the receiver were

$S_{p1} = 20,640,380.8$ m

$S_{p2} = 20,357,184.1$ m

$S_{p3} = 23,287,346.8$ m

$S_{p4} = 21,699,908.4$ m

$S_{p5} = 25,416,133.6$ m

The satellite coordinates were

S #1	S #2	S #3
$x_1 = 6,097,294.4$	$x_2 = 1,819,274.3$	$x_3 = 9,268,094.7$
$y_1 = -4,364,543.9$	$y_2 = -2,240,846.4$	$y_3 = 13,290,138.0$
$z_1 = 22,658,876.2$	$z_2 = 23,721,192.7$	$z_3 = 13,622,934.4$

S #4	S #5
$x_4 = -8,198,461.9$	$x_5 = -21,419,309.7$
$y_4 = -18,813,603.1$	$y_5 = 12,865,351.8$
$z_4 = 19,040,626.8$	$z_5 = 4,832,143.1$

Applying the method of least squares to determine the user position, first assume:

x_o = -2,640,000

y_o = -4,235,000

z_o = 3,960,000

E_o = 10,000.

For the first iteration, the observation equation is written as

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix} = \begin{bmatrix} -.4231 & .00627 & -.9055 & 1 \\ -.2198 & -.0979 & -.9703 & 1 \\ -.5111 & -.7522 & -.4147 & 1 \\ .2560 & .6715 & -.6946 & 1 \\ .73856 & -.6726 & -.0343 & 1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta E \end{bmatrix} - \begin{bmatrix} -9509.4 \\ -8810.4 \\ -10106.5 \\ -9342.0 \\ -7349.8 \end{bmatrix},$$

and the normal equation, $A^TAX - A^TL = 0$,

$$\begin{bmatrix} 1.099 & .0783 & .6043 & -.1584 \\ .0783 & 1.4787 & -.0421 & -.8449 \\ .6043 & -.0421 & 2.147 & -3.109 \\ -.1584 & -.8449 & -3.019 & 5.0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta E \end{bmatrix} - \begin{bmatrix} 3296.9 \\ 7075.3 \\ 28091.7 \\ -45118.1 \end{bmatrix} = 0.$$

The solution, $X = (A^T A)^{-1} A^T L$, is

$$X = \begin{bmatrix} 3040.9 \\ -1646.6 \\ -2705.2 \\ -10840.9 \end{bmatrix}.$$

New trial point coordinates are

x_o = -2,636,959.1 $\qquad$ z_o = 3,957,294.8

y_o = -4,236,646.6 $\qquad$ E_o = -840.9 .

Using new trial point coordinates, the above steps are repeated until Δx, Δy, Δz and ΔE values become vanishingly small. For other trials (iterations), the Δx, Δy, Δz and ΔE values and new trial points coordinates are tabulated in Table III-4.

The final user coordinates are

x = -2,636,937.1

y = -4,236,666.2

z = 3,957,250.8 .

And receiver clock bias is E = -869.8 m.

	Change in Coordinates				Coordinates			
	Δx	Δy	Δz	ΔE	X	Y	Z	E
0	--	--	--	--	-2640000	-4235000	3960000	10000
1	3040.9	-1646.6	-2705.2	-10840.9	-2636959.1	-4236646.6	3957294.8	-849.9
2	93.9	-256.7	-295.0	-259.4	-2636865.2	-4236903.3	3956999.8	-1100.3
3	-73.8	238.4	253.3	232.8	-2636939.0	-4236664.9	3957253.1	-867.5
4	-56.1	33.5	13.1	-2.5	-2656995.1	-4236631.4	39572662	-870.0
5	56.2	-33.5	-13.5	1.8	-2636938.9	-4236664.9	3957252.7	-868.2
6	1.8	-1.3	-1.9	-1.6	-2636937.1	-4236666.2	3957250.8	-869.8

Table III-4: For least squares solution, successive iterations applied to GPS fixes.

C. USE OF THE ERROR ELLIPSE IN ANALYZING THE ACCURACY OF HYDROGRAPHIC POSITIONS

In the least square adjustment process, the positional errors are found in the direction of the x and y (ϕ and λ) coordinate axes. These σ_x and σ_y values indicate the expected displacement of the fix in the direction of the coordinate axes, but they do not necessarily define the maximum and minimum errors associated with the axes of error ellipse (Figure III-7].

Maximum and minimum standard errors are found by defining the orientation of the error ellipse in terms of the x,y coordinate system. Let the coordinate system defining the semi-major and semi-minor axes of the error ellipse be u and v as indicated in Figure III-8.

The following relationship exists between the ellipse (u and v) and the ground (x and y) coordinate system.

$$\begin{aligned} u &= x \sin\theta + y \cos\theta \\ v &= x \cos\theta - y \sin\theta \end{aligned} \qquad \text{(III-18)}$$

In these transformation equations, the angle θ is the rotational angle between the y and u axes (measured clockwise from y axis to the u axis).

The lengths of the semi-major and semi-minor axes are given by:

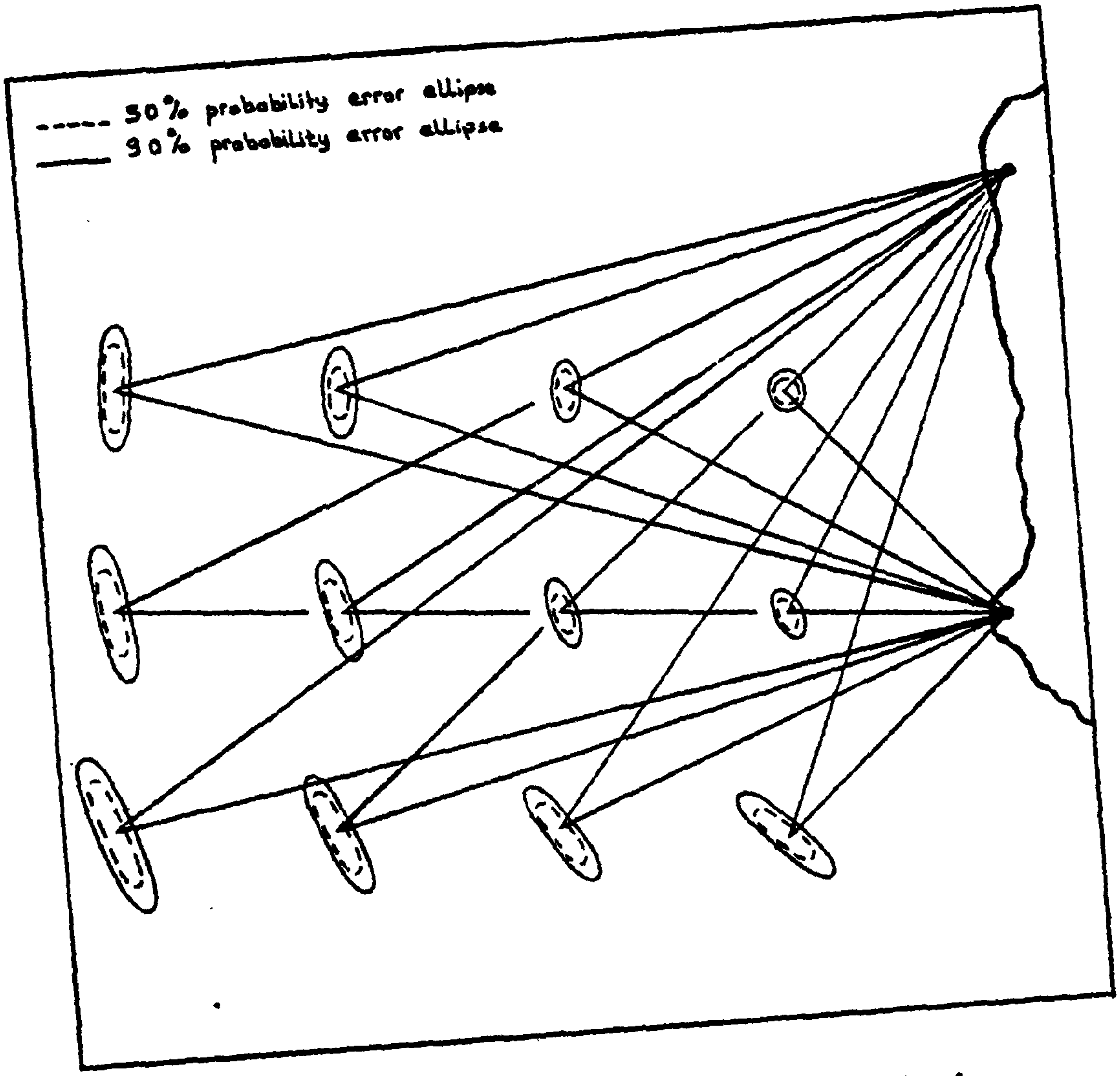


Figure III-7: Error ellipses formed at the determined positions (D. B. Thomson and D. E. Wells, 1977).

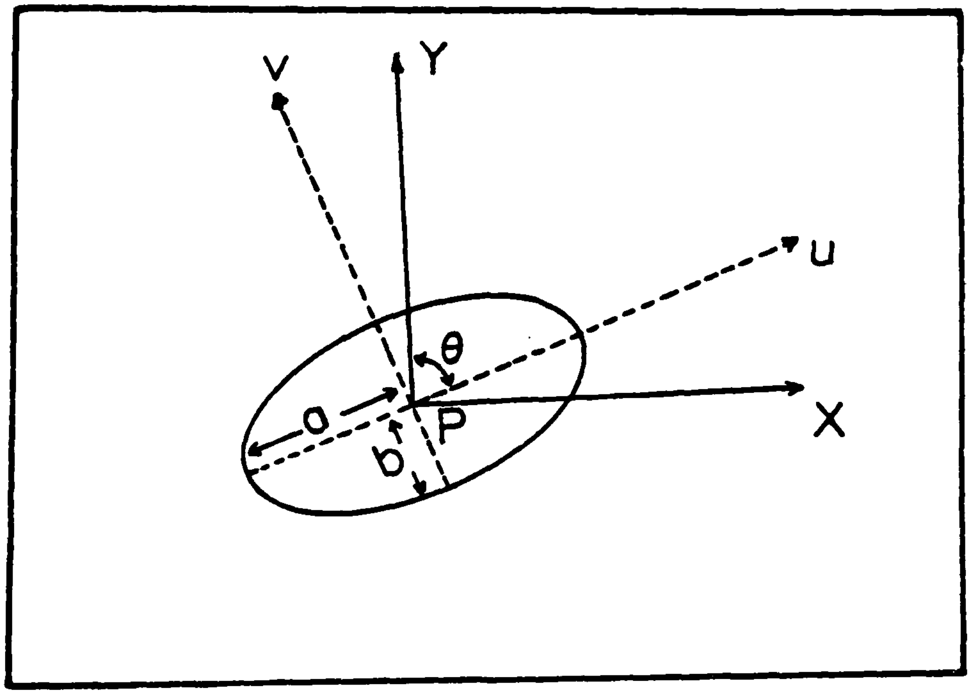


Figure III-8: Error Ellipse.

$$\sigma_u = \sigma_o \sqrt{q_{uu}}$$
$$\sigma_v = \sigma_o \sqrt{q_{vv}} \; . \qquad \text{(III-19)}$$

In above equations, σ_o, the standard error of unit weight, is known from the least square adjustment of point P, q_{uu} and q_{vv} are given by[1]:

---

[1]For detailed derivation of these equations see Ref. 16, pages 181-183.

$$q_{uu} = \frac{1}{2}(q_{xx} + q_{yy}) + \frac{1}{2}(q_{xx} - q_{yy})\cos 2\theta + q_{xy}\sin 2\theta \qquad \text{(III-20)}$$

$$q_{vv} = \frac{1}{2}(q_{xx} + q_{yy}) + \frac{1}{2}(q_{xx} - q_{yy})\cos 2\theta - q_{xy}\sin 2\theta, \qquad \text{(III-21)}$$

where q_{xx}, q_{yy} and q_{xy} are the elements of the $(A^TA)^{-1}$ (for unweighted observations) or $(A^TWA)^{-1}$ (for weighted observations), i.e.,

$$(A^TA)^{-1} \text{ or } (A^TWA)^{-1} = \begin{bmatrix} q_{xx} & q_{xy} \\ q_{yx} & q_{yy} \end{bmatrix}.$$

Equation III-20 reaches its extreme value, and q_{uu} is maximum, when

$$\tan 2\theta = -\frac{2q_{xy}}{q_{xx} - q_{yy}}. \qquad \text{(III-22)}$$

Inserting Equation III-22 into III-20 and III-21, and defining D as

$$D = [(q_{xx} - q_{yy})^2 + 4(q_{xy})^2]^{1/2}, \qquad \text{(III-23)}$$

q_{uu} and q_{vv} may be written as

$$q_{uu} = \frac{1}{2}(q_{xx} + q_{yy} + D) \tag{III-24}$$

$$q_{vv} = \frac{1}{2}(q_{xx} + q_{yy} - D); \tag{III-25}$$

and from Equation III-19, the semi-major and semi-minor axes are

$$a^2 = \frac{1}{2}\sigma_0^2(q_{xx} + q_{yy} + D) \tag{III-26}$$

$$b^2 = \frac{1}{2}\sigma_0^2(q_{xx} + q_{yy} - D). \tag{III-27}$$

Using these expressions, the error ellipse can be constructed at any point whose coordinates were determined by least square adjustment if the $(A^TA)^{-1}$ or $(A^TWA)^{-1}$ matrixes are known.

Example III-9: In order to determine the error ellipse parameters for the range-range example problem (example III-7), recall that $(A^TA)^{-1}$ was determined as

$$(A^TA)^{-1} = \begin{bmatrix} .64 & -.14 \\ -.14 & .76 \end{bmatrix},$$

and the standard error of unit weight was $\sigma_o = 4.04$.

The semi-major and semi-minor axes of the error ellipse, according to Equations III-26 and III-27, are found by first solving Equation III-23:

$$D = [(q_{xx} - q_{yy})^2 + 4(q_{xy})^2]^{1/2}$$

$$D = [(.64 - .76)^2 + 4(-.14)^2]^{1/2} = .305.$$

Then,

$$a^2 = \frac{1}{2}\sigma_o^2(q_{xx} + q_{yy} + D)$$

$$a^2 = \frac{1}{2}(4.04)^2(.64 + .76 + .305) = 13.9$$

$$a = 3.73\ m$$

$$b^2 = \frac{1}{2}\sigma_o^2(q_{xx} + q_{yy} - D)$$

$$b^2 = \frac{1}{2}(4.04)^2(.64 + .76 - .305) = 8.94$$

$$b = 2.99\ m.$$

The semi-major axis is 3.73 m, and semi-minor axis is 2.99 m. According to Equation III-22, the angle θ is found:

$$\tan 2\theta = -\frac{2q_{xy}}{q_{xx}-q_{yy}} = -\frac{2(-.14)}{(.64-.76)}$$

$2\theta = 113^\circ.2$

$\theta = 56^\circ.6$ which defines the orientation of the error ellipse.

1. <u>Some Characteristics of the Error Ellipse</u>

A number of important properties of the error ellipse can be obtained by analyzing the equations given in the previous section. The existence of the error ellipse points out an important fact that the accuracy of the location of a point in question is not the same in every direction. An analysis of equations III-26 and III-27 demonstrates that the formulas for the semi-major and semi-minor axes are composed of two parts: the standard error of unit weight, which defines the scale of the error ellipse, and the elements of the $(A^TA)^{-1}$ or $(A^TWA)^{-1}$ matrix, which define its shape.

To reduce the error ellipse into a circle, where the accuracy of position is equal in every direction, the following condition must be met:

$$a/b = 1 .$$

According to Equations III-26 and III-27 this is possible only if $D = 0$:

$$D = [(q_{xx}-q_{yy})^2 + 4(q_{xy})^2]^{1/2} = 0 ,$$

which means th

$$q_{xx} = q_{yy} \text{ and } q_{xy} = 0,$$

and, according to Equation III-17,

$$\sigma_x = \sigma_y .$$

Another important characteristic is that the sum of the squares of the standard errors in x and y directions is invariant to the rotation of the coordinate system, or

$$a^2 + b^2 = \sigma_u^2 + \sigma_v^2 = \sigma_x^2 + \sigma_y^2 . \qquad \text{(III-28)}$$

Equation III-28 leads to the concept of root mean square error, d_{rms}, as follows :

$$d_{rms} = \sqrt{\sigma_x^2 + \sigma_y^2} = \sqrt{\sigma_u^2 + \sigma_v^2} = \sqrt{a^2 + b^2} \qquad \text{(III-29)}$$

or, from Equations III-26 and III-27.

$$d_{rms} = \sigma_o \sqrt{q_{xx} + q_{yy}} .$$

IV. CONCLUSION

Conventional survey systems will provide the primary means of hydrographic positioning for several years to come. Thus, the concepts of d_{rms} and the graphical approach to developing error contours are very useful tools in survey planning and execution.

Survey planners must exercise care in establishing the position of navigation aids. The resultant net geometry determines the accuracy, and thus the d_{rms} error, of the fix positions. Accuracy requirements for the collection of hydrographic survey data greatly limit the size of the effective survey area. Through careful planning, the number of navigation aid shore stations can be minimized while still meeting position accuracy requirements for the survey.

Currently, more research is needed to determine the environmental factors which govern variations in the propagation velocity of electromagnetic energy. If this important parameter could be more accurately modeled throughout the survey area, the effects of systematic errors due to these velocity variations could be greatly minimized.

As shown in this paper, methods exist today by which the accuracy of survey positions can be greatly improved through the use of redundant observations and data adjustment techniques.

The method of least squares adjustment provides a best estimate of position, plus the size and orientation of the error ellipse associated with that position, for every point determined by the survey system. In addition, the error ellipse quantifies the predictable accuracy (as shown in Figure II-7) of each position as compared to the repeatable accuracy available from conventional survey methods.

The application of these techniques will become more widespread when the Global Positioning System is fully operational. Observations of position from any number of navigation and positioning systems (GPS, LORAN, hydro positioning systems, etc.) can be combined in a least square solution. Observation equations may be written and weights can be assigned as a function of accuracy for each system.

In preparation for these future improvements, hydrographers must work to understand and implement the concepts discussed in this paper. Data must be processed by computer. Therefore, programs need to be written which can perform the iterative least squares adjustment on the appropriate observation equations. Errors must be analyzed to assess the improvement in accuracy resulting from redundant observations. Additionally, standard hydrographic survey procedures need to be reviewed to determine if more efficient methods may be adopted when redundant observations and data adjustment techniques are used. For example, the method of least squares may be programmed into onboard computers so that data may be adjusted and evaluated in real time. Alternately, redundant data may be collected and recorded for later processing ashore.

The technology is available today to employ data adjustment methods in hydrography. This technology must be analyzed and adapted to match the systems and requirements unique to hydrographic survey.

ANALYSIS OF RANDOM ERRORS

APPENDIX A

(The information given in this section was taken directly from References 1 and 20.)

ONE DIMENSIONAL ERRORS

An error in a measurement is the difference between the "true" value of a quantity and the measured or derived value. The "true" value can never really be determined because of instrument limitations and human fallibility. In determining the value of a quantity, only one measurement may be necessary when an approximate value is sufficient. If, on the other hand, the quantity is important enough to require a more precise value, repeated measurements are made. Variations will exist between the values obtained from several measurements. Applying the theory of the normal distribution to these measurements, the "best" value for the quantity is the mean or average of all the observed values. The differences between the mean and the observed values are the apparent errors or residuals which are used to derive a statement of precision for the measuring process. When the residuals are randomly distributed about the mean, the precision of the measurement is expressed by a single term, the standard error, which is commonly designated by the Greek letter "sigma" (σ). For a one dimensional normal distribution, this value is computed by squaring all the residual errors (v), adding the squared values, dividing by the number of errors less one (if n independent direct measurements are taken of the same quantity, then the first measurement establishes a value for the unknown and all additional measurements, (n-1) in number, are redundant), and taking the square root:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n-1}} ,$$

where v_i is the residual defined by the equation $v_i = x_i - \bar{x}$,

x_i : observed value

$\bar{x}$: mean value

n : the number of observations.

The normal distribution itself is represented by the function:

$$p(v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma^2}} .$$

The normal distribution curve and the meaning of the standard errors are illustrated in Figure A-1. The central vertical axis, p(v), represents the probability of zero error with positive errors plotted to the right and negative errors to the left. The height of the curve above a particular point on the horizontal axis is proportional to the probability of an error of that amount.

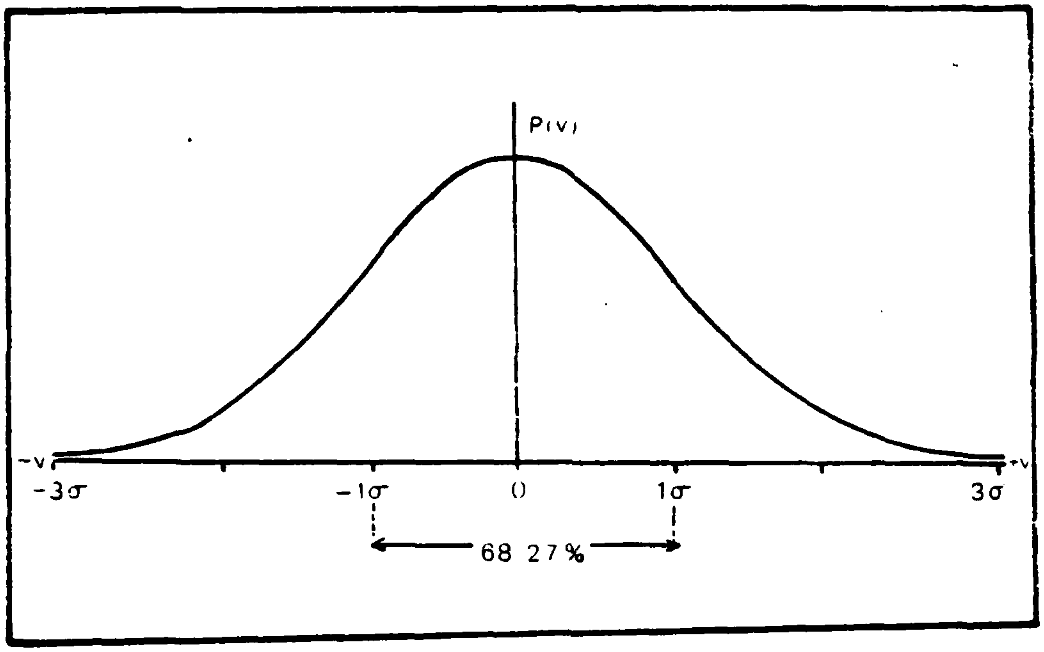


Figure A1: One dimensional Normal Distribution Curve

It can be observed from the normal distribution curve that the total area under the curve is equal to unity. Also, the area under the curve between any two values of v_1 and v_2 is equal to the probability of an error occurring between these limits. So, to find the probability of an error between v_1 and v_2, p(v) has to be integrated between v_1 and v_2. The area under the curve between the limits of $v_1 = -\sigma$ and $v_2 = +\sigma$ is 68.27% of the total area under the curve. This means that there is 68.27% probability that errors in any further measurements made under the same conditions will not exceed the standard error, σ, with a 68.27% probability. The standard error does not indicate the probability that an error of a certain size will occur; it only indicates that 68.27% of the errors will fall within the specified limits of plus or minus one sigma.

If other probability levels are desired, the appropriate conversion factor may be found in Table A1. For example, for 95% probability, σ should be multiplied by a linear error conversion factor of 2.

Probability, %	Linear error conversion factor
50	.6745
68.27	1.000
90	1.6449
95	2.000
99.7	3.000

Table A1: Linear error conversion factors for several probability levels.

TWO-DIMENSIONAL ERRORS

A two-dimensional error is the error in a quantity defined by two random variables. For example, consider the position of a point referred to x and y axes. Each observation of the x and y coordinates may contain the errors v_x and v_y.

If the errors are random and independent, each error has a probability density distribution of

$$p(v_x) = \frac{1}{\sigma_x\sqrt{2\pi}}\, e^{-\frac{v_x^2}{2\sigma_x^2}} \quad \text{and} \quad p(v_y) = \frac{1}{\sigma_y\sqrt{2\pi}}\, e^{-\frac{v_y^2}{2\sigma_y^2}}$$

"The probability of two events occurring simultaneously is equal to the product of their individual probabilities" [Ref. 1]. Applying this rule, the two-dimensional probability density function becomes:

$$p(v_x, v_y) = \frac{1}{2\pi\sigma_x\sigma_y}\, e^{-\frac{1}{2}\left(\frac{v_x^2}{\sigma_x^2}+\frac{v_y^2}{\sigma_y^2}\right)} \; ;$$

rearranging terms,

$$p(v_x, v_y)\, 2\pi\,\sigma_x\sigma_y = e^{-\frac{1}{2}\left(\frac{v_x^2}{\sigma_x^2}+\frac{v_y^2}{\sigma_y^2}\right)} \; ;$$

taking the logarithm,

$$(-2)\ln\left[p(v_x, v_y)\, 2\pi\,\sigma_x\,\sigma_y\right] = \frac{v_x^2}{\sigma_x^2} + \frac{v_y^2}{\sigma_y^2} \, .$$

For given values of $p(v_x, v_y)$ [physical meaning of $p(v_x, v_y)$ is that the probability that two random variables v_x and v_y take values in the interval $\pm v_x$ and $\pm v_y$], the left side of equation is a constant, k_2, then

$$k^2 = \frac{v_x^2}{\sigma_x^2} + \frac{v_y^2}{\sigma_y^2} .$$

For several values of $p(v_x, v_y)$, a family of equal probability density ellipses are formed with axes $k\sigma_x$ and $k\sigma_y$ (Figure A2).

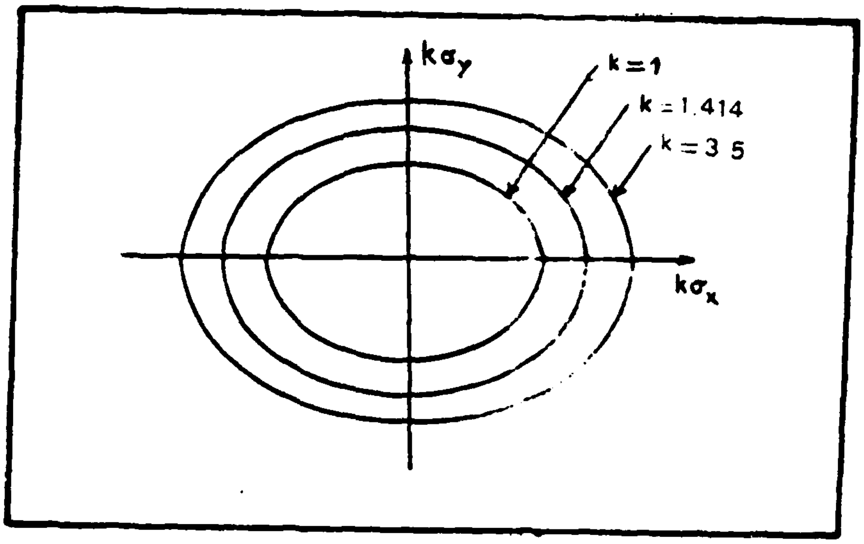


Figure A2: Equal probability density ellipses.

In general, when the two errors are correlated, i.e., a change in the one error has some effect upon the other, the probability density function, $p(v_x, v_y)$, becomes

$$p(v_x, v_y) = \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2 - \sigma_{xy}^2}}\, e^{-\left[\frac{\sigma_x^2\sigma_y^2}{2(\sigma_x^2\sigma_y^2-\sigma_{xy}^2)}\left(\frac{v_x^2}{\sigma_x^2} - 2\sigma_{xy}\frac{v_x v_y}{\sigma_x^2\sigma_y^2} + \frac{v_y^2}{\sigma_y^2}\right)\right]}$$

Then, the equation of constant probability density ellipses (Figure A3) is

$$k^2 = \frac{1}{(1-\rho^2)}\left[\frac{v_x^2}{\sigma_x^2} - 2\rho\frac{v_x v_y}{\sigma_x\sigma_y} + \frac{v_y^2}{\sigma_y^2}\right]$$

where ρ = correlation coefficient of v_x and v_y and is given by

$$\rho = \frac{\sigma_{xy}}{\sigma_x\sigma_y}$$

The probability density function integrated over a certain region becomes the probability distribution function which yields the probability that v_x and v_y will occur simultaneously within that region, or:

$$P(v_x, v_y) = \iint p(v_x, v_y)\, dv_x\, dv_y$$

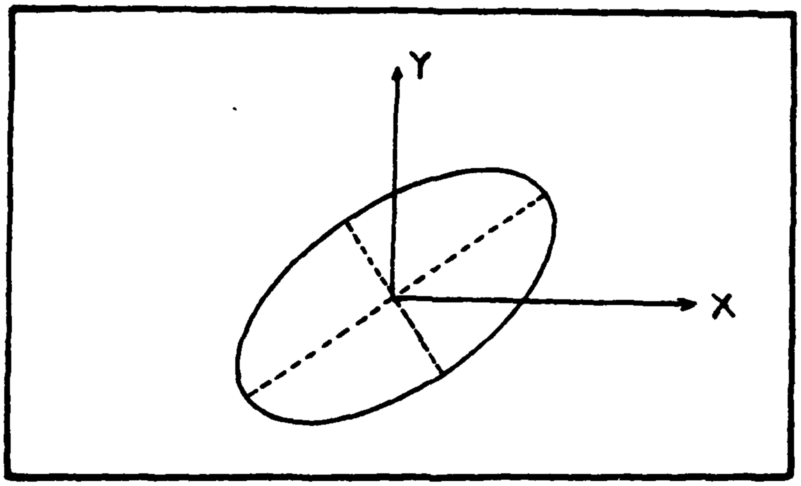
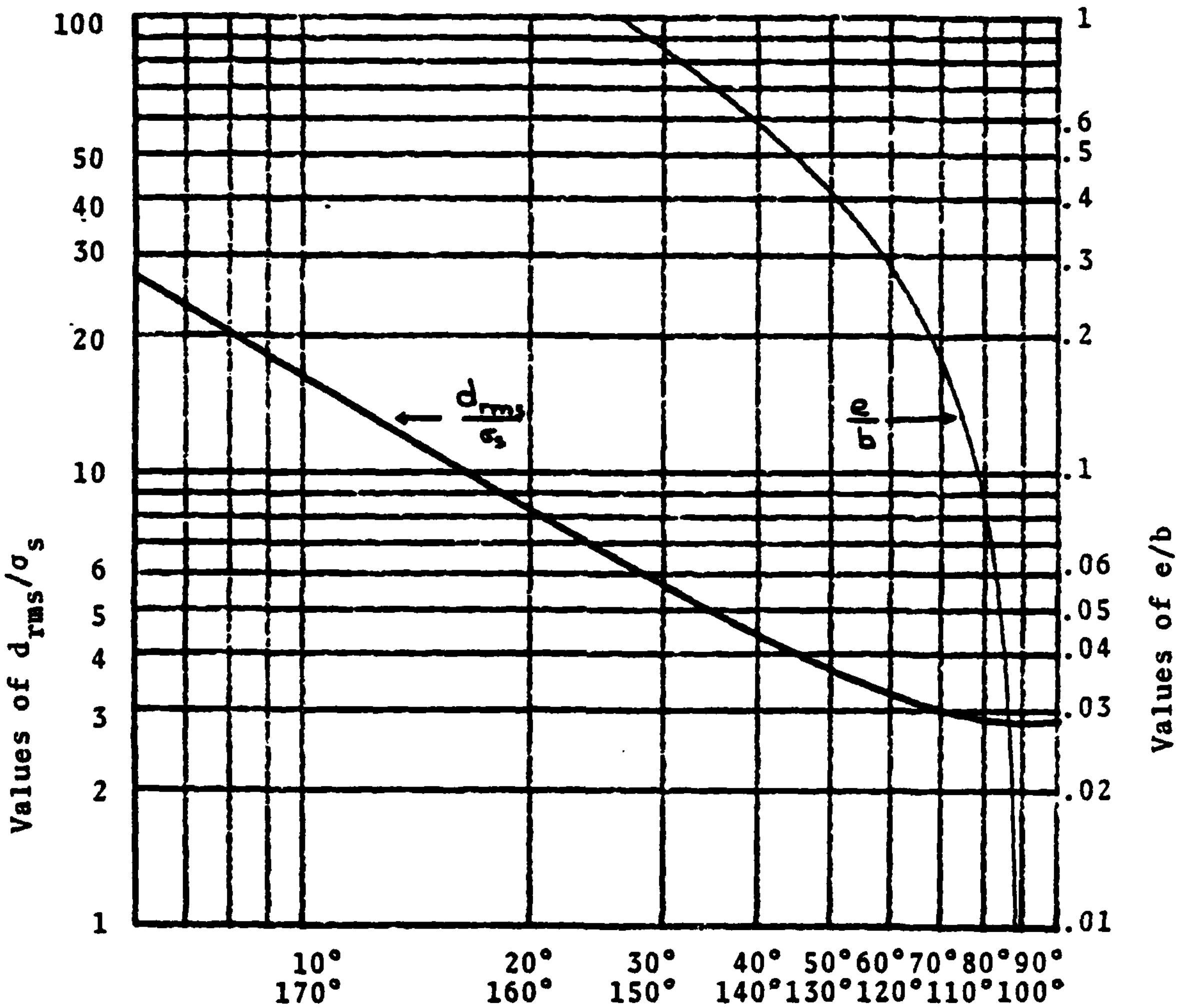


Figure A3: Constant probability density ellipse for correlated errors.

APPENDIX B

USEFUL GRAPHS FOR THE DETERMINATION OF REPEATABILITY CONTOURS



Intersection angle, B, in degrees.

Figure A1: For ranging systems, the graph of the d_{rms}/σ_s and e/b.

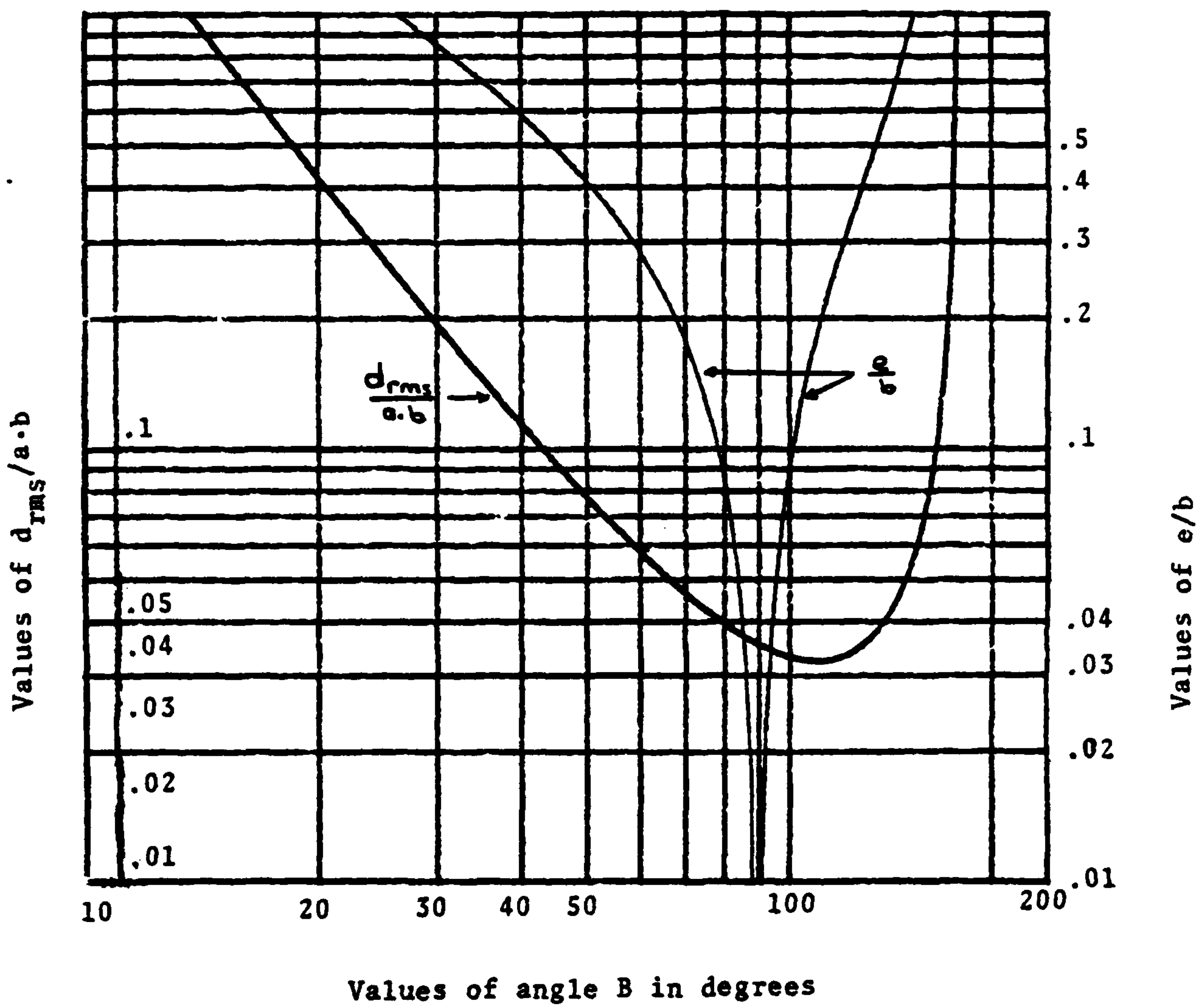


Figure A2: For azimuthal systems, the graph of $d_{rms}/a \cdot b$ and e/b.

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