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THE WARING DISTRIBUTION.

By

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A CHARACTERIZATION OF THE WARING DISTRIBUTION.

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SUMMARY. In this note the Waring distribution is characterized by the following property: For a positive integer-valued ramdom variable X, $P(x=r) = p_r$, $r=1,2,\ldots$, and with a finite mean μ define two new random variables Y and Z by

$$P(Y=r) = q_r = \left(\sum_{k=r+1}^{\infty} p_k + ap_r\right)/(\mu+a) \ , \ r=0,1,2,\ldots,$$

$$P(Z=r) = q_r' = (r+b)p_r/(\mu+b) \ , \ r=1,2,\ldots,$$

where $a \geq 0$ and b are constants with $b-a+1>0$. Then Z and Y truncated at 0 have the same distribution if and only if X has a Waring distribution.

Let X be a positive integer-valued random variable (r.v.) with

$$P(X=r) = p_r, \ r=1,2,\ldots, \tag{1}$$

Let X have a finite mean μ. One can define two new classes of r.v!s Y and Z by

$$P(Y=r) = q_r = \left(\sum_{k=r+1}^{\infty} p_k + ap_r\right)/(\mu+a) \ , \ r=0,1,\ldots, \tag{2}$$

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$$P(Z=r) = q_r = (r+b)p_r/(\mu+b) \ , \ r=1,2,\ldots, \qquad (3)$$

where $a \geq 0$ and $b > -1$ are constants. Distributions of the type (2) have been defined and studied in the literature. See Johnson and Kotz (1969, p.261). It is natural to ask: For what distributions X, Y and Z are essentially the same r.v.'s? Since Y takes the values 0,1,..., while Z the values 1,2,..., it is necessary to truncate Y at 0, and then it turns out that the above property is a unique property of the Waring distribution. For our purpose we define the Waring distribution of Irwin (1963) by

$$P(W=r) = (\lambda-c)c^{[r-1]}/\lambda^{[r]} \ , \ r=1,2,\ldots, \qquad (4)$$

where

$$\lambda - c > 1 \ , \ c > 0;$$

and

$$c^{[r]} = c(c+1)\ldots(c+r-1) \ , \ r=1,2,\ldots, \ c^{[0]} = 1.$$

We are now ready to characterize the Waring distribution (4):

Theorem: Let X be a positive integer-valued random variable given by (1) with a finite mean μ. Define Y and Z by (2) and (3) respectively for some constants $a \geq 0$, b satisfying $b-a+1>0$. Then Y truncated at 0 has the same distribution as Z if and only if X has the Waring distribution (4).

<u>Proof</u> : "Only if" part: Let q_r and q_r' be given by (2) and (3) for some $a \geq 0$ and $b-a+1>0$ with $q_r' = (1-q_0)^{-1} q_r, r=1,2,\ldots,$

Then substituting for the p's in (2) from (3) we have

$$q_r = \{(\mu+b)/(\mu+a)\}\{\sum_{k=r+1}^{\infty} q_k'/(k+b) + aq_r'/(r+b)\}, \quad r=1,2,\ldots, \tag{5}$$

$$= (\mu+a)^{-1}, \quad r=0,$$

From (5) we obtain

$$q_r - q_{r+1} = \left[q_{r+1}'/(r+1+b) + a\{q_r'/(r+b) - q'_{r+1}/(r+1+b)\}\right](\mu+b)/(\mu+a)$$

$$r=1,2,\ldots,$$

from which we get the recurring relation

$$q_{r+1}' = \{(r+b-ax)(r+b+1)\} q_r' / \{(r+b+1+x-ax)(r+b)\}, \tag{6}$$

where

$$x = (\mu+b)/\{(\mu+a)(1-q_0)\} = (\mu+b)/(\mu+a-1) \tag{7}$$

Equation (6) yields

$$q_{r+1}' = \frac{(r+b-ax)\ldots(1+b-ax)(r+1+b)}{(r+1+b+x-ax)\ldots(2+b+x-ax)(1+b)} q_1' \tag{8}$$

$$r=1,2,\ldots$$

However, from (5) and the assumed relation $q_r' = (1-q_0)^{-1} q_1$ it follows that

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$$q_1' = q_0 \,(1+b)/\{(1-q_0)(1+b+x-ax)\} \tag{9}$$

Finally from (3), (5)-(9) we obtain

$$p_r = (1+b-ax)^{[r-1]}/(1+b+x-ax)^{[r]}, \quad r=1,2,\ldots \tag{10}$$

which is Waring (4) with

$$c = (1+b-ax) \text{ and } \lambda = (1+b+x-ax).$$

It is easily seen from (5)-(7) and the assumption $b-a+1>0$ that $c = (\mu-1)(b-a+1)/(\mu+a-1)>0$, and since $x>1$ we also have $\lambda-c=x>1$.

"If" part: Let the p's be given by (4). Then using the Waring expansion $(\lambda-c)^{-1} = \sum_{r=0}^{\infty} c^{[r]}/\lambda^{[r+1]}$, which is valid for $\lambda>c>0$, we obtain

$$\sum_{k=r+1}^{\infty} p_k = (c+r-1)p_r/(\lambda-c)$$

from which and the fact that $\mu=c/(\lambda-c-1)+1$ one could easily verify that $q_r' = (1-q_0)^{-1}q_r$ where q_r, q_r' are given by (2) and (3) with $a = (b-c+1)/(\lambda-c)$, b arbitrary but $b-c+1>0$. Since $\lambda-c$ is assumed to be greater than 1, we have for the above a and b that $a<b-c+1$ or $0<c<b-a+1$. **This completes the** proof of the theorem.

Note that our theorem covers the characterization of the Yule distribution: In the "Only if" part the Yule distribution corresponds to the case $a=b=0$ and in the "if" part to $c=1$.

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There is, in fact, a certain connection to the characterization of the Yule distribution given by Krishnaji (1970). Krishnaji showed that X has the Yule distribution

$$P(x=r) = (d-1)r!/\{(d+1)(d+2)\ldots(d+r)\},\ r=1,2,\ldots \quad (11)$$

where $d-1>0$ if and only if X and the greatest integer in UX trucated at 0 have the same distribution. Here U is a uniform r.v. on (0,1) independently distributed of X. Now if in the "Only if" part of our theorem we take a=b=0, (5) will reduce to

$$q_r = \sum_{k=r+1}^{\infty} q_k'/k\ ,\ r=0,1,\ldots$$

which is $P([UZ]=r)$ as has been shown by Krishnaji. Here U is uniform on (0,1) distributed independently of Z (and hence of X) and $[x]$ denotes the greatest integer in x. Thus Krishnaji's result as applied to Z and $[UZ]$ yields that Z has the Yule distribution (11), and X in turn has the Yule distribution (4) with c=1. Of course, our "Only if" part gives the same result.

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