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Tensors and Differential Geometry Applied to Analytic and Numerical Coordinate Generation

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by
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The two main objectives of this monograph are, (1) to present and collect at one place some important classical results and concepts from the theories of tensor analysis and differential geometry, and, (2) to use the presented results in devising differntial models for genrating coordinates in arbitrarily bounded regions. Though most of the discussions on tensors and differential geometry are in the context of curvilinear coordinate generation, the first two parts can profitably be used for applied problems in various branches of engineering both by students and researchers. The last part of

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the monograph is concerned with the development of two methods, based on differential equations, for the generation of coordinates. The selected models are based on elloptic partial differential equations which can be solved on a computer to provide smooth differentiable coordinate curves in the regions of interest.

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FORWARD

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Tensors and Differential Geometry Applied

to

Analytic and Numerical Coordinate Generation†

by

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Abstract

The two main objectives of this monograph are, (i) to present and collect at one place some important classical results and concepts from the theories of tensor analysis and differential geometry, and, (ii) to use the presented results in devising differential models for generating coordinates in arbitrarily bounded regions. Though most of the discussions on tensors and differential geometry are in the context of curvilinear coordinate generation, the first two parts can profitably be used for applied problems in various branches of engineering both by students and researchers. The last part of the monograph is concerned with the development of two methods, based on differential equations, for the generation of coordinates. The selected models are based on elliptic partial differential equations which can be solved on a computer to provide smooth differentiable coordinate curves in the regions of interest.

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†Research supported in part by the Grant AFOSR No. 80-0185.

*Professor

Introduction

The purpose of this monograph is to present the theories of basic tensor analysis and of the differential geometry of surfaces for the purpose of formulating problems of coordinate generation in regions bounded by arbitrary curves or surfaces. Since the writing of the first memoir on the subject of tensor analysis by Ricci and Levi-Civita [1] in 1901 some very significant developments in the theory of tensor analysis have taken place, though, the major applications of the subject have only been confined to the general theory of relativity and to the continuum mechanics. In this monograph an attempt has been made to utilize the theories of classical tensor analysis and differential geometry of surfaces in developing new methods for the generation of coordinates in arbitrary regions. Only those results of tensor theoretic and differential geometric significance have been explained which are needed in the development of the subject in a fruitful manner. However, it turns out that for a better understanding and a sound conceptual orientation some basic ideas, by the way of definitions and notations, have also to be introduced. Though this elementary exposition forms a small part of the total effort, and is explained much better in the references given below, nevertheless, its inclusion imparts a sort of continuity to the whole presentation.

Almost all the material explained in Parts I and II of this monograph is available in the standard texts, such as, Levi-Civita [2], Weatherburn [3], McConnell [4], Eisenhart [5], [6], Tolman [7], Graustein [8], Synge and Schild [9], Brand [10], Spain [11], Truesdell and Toupin [12], Struik [13], Sokolnikoff [14], Willmore [15], O'Neill [16], and Kreyszig [17], [18], on the classical topics in tensors and differential geometry. Some other texts and monographs which can be used with advantage are Aris [19], Borisenko and Tarapov [20], Stoker [21], Spivak [22], do Carmo [23], Flügge [24], Howard [25], and Eiseman [26].

Part III of this monograph is the culmination of the ideas developed in Parts I and II. Specifically §§2 and 8 of Part I and §§2 and 3 of Part II provide the necessary material for the development of new methods of coordinate generation. It is the hope of the author that the material of Part III will form a framework for further research in the area of mesh generation for physical problems, based on partial differential equations.

Part I

Fundamental Concepts and Basic Tensor Forms

§1. Preliminaries.

In this section we summarize some elementary operations on vectors and tensors with the assumption that an Euclidean space is available in which a set of rectangular Cartesian coordinates has been introduced. Further, to maintain a sort of continuity of exposition with the rest of the sections, we also clarify the nomenclature of some commonly used terms. For further details the reader is referred to References [4], [11] and [14].

In this report the vectors and tensors will be denoted by using the symbol ~ under and above a letter, respectively. Thus, the vectors are denoted as $\underset{\sim}{u}$, $\underset{\sim}{\phi}$, etc., and the tensors as $\tilde{T}$, $\tilde{\tau}$, etc.

A rectangular Cartesian system of coordinates in a three-dimensional Euclidean space will usually be denoted by x_i $(i = 1,2,3)$, or, occasionally as x,y,z. The orientation of axes will always be assumed to be right-handed.

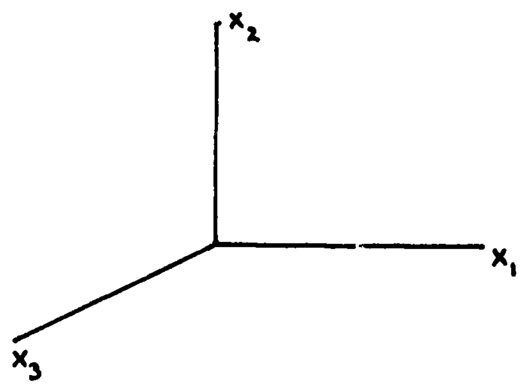


Figure 1.

The basis of a rectangular Cartesian coordinate system will be denoted by a system of constant unit vectors $\underset{\sim}{e}_i$ $(i = 1,2,3)$. The components of a vector $\underset{\sim}{v}$ with respect to a rectangular Cartesian system will be denoted by v_i $(i = 1,2,3)$.

§1.1. Summation Convention on Cartesian Components.

A repeated index on quantities either appearing as a single entity or as products will imply summation. Thus

$$a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1a}$$

$$a_i T_{ij} = a_1 T_{1j} + a_2 T_{2j} + a_3 T_{3j} \tag{1b}$$

$$T_{ii} = T_{11} + T_{22} + T_{33} \tag{1c}$$

while no summation is implied in

$$T_{ij} + T_{ji} \tag{2}$$

§1.2. Vector Multiplications Using Cartesian Components.

(i) Scalar or dot product:

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = ab \cos\theta \tag{3a}$$

$$= a_i b_i \tag{3b}$$

where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and a, b are the magnitudes of the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively. Obviously

$$a = |\underset{\sim}{a}| = \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{a}} = \sqrt{a_i a_i} \tag{4}$$

(ii) Cross product:

$$\underset{\sim}{v} = \underset{\sim}{a} \times \underset{\sim}{b}$$

$$= (ab \sin\theta)\underset{\sim}{n} \qquad (5)$$

where $\underset{\sim}{n}$ is the unit vector normal to the plane containing $\underset{\sim}{a}$ and $\underset{\sim}{b}$. The i-th component of $\underset{\sim}{v}$ is then

$$v_i = e_{ijk} a_j b_k \qquad (6)$$

where e_{ijk} is the permutation symbol†. The permutation symbol has the value +1 if i,j,k are taken in a right-handed cyclic permutations of 1,2,3; the value -1 if i,j,k are in the cyclic permutations of 1,3,2. Thus

$$e_{123} = 1 \ , \ e_{321} = -1 \ , \ e_{112} = 0, \text{ etc.}$$

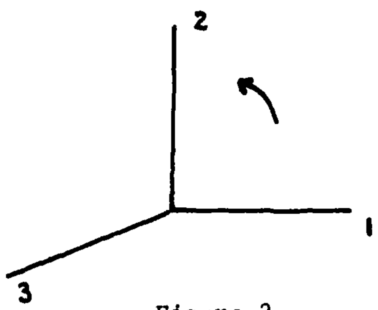


Figure 2.

(iii) Scalar triple product:

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (7)$$

†Also sometimes written as e^{ijk}.

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{b} \cdot (\underset{\sim}{c} \times \underset{\sim}{a}) = \underset{\sim}{c} \cdot (\underset{\sim}{a} \times \underset{\sim}{b}) \tag{8}$$

(iv) Vector triple product:

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \times \underset{\sim}{c} = \underset{\sim}{b}(\underset{\sim}{a} \cdot \underset{\sim}{c}) - \underset{\sim}{a}(\underset{\sim}{b} \cdot \underset{\sim}{c}) \tag{9}$$

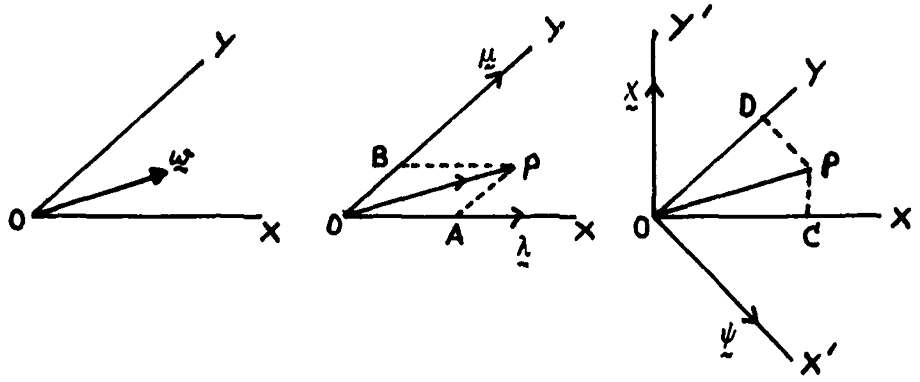
$$\underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c}) = \underset{\sim}{b}(\underset{\sim}{a} \cdot \underset{\sim}{c}) - \underset{\sim}{c}(\underset{\sim}{a} \cdot \underset{\sim}{b}) \tag{10}$$

(v) Lagrange identity:

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot (\underset{\sim}{c} \times \underset{\sim}{d}) = (\underset{\sim}{a} \cdot \underset{\sim}{c})(\underset{\sim}{b} \cdot \underset{\sim}{d}) - (\underset{\sim}{a} \cdot \underset{\sim}{d})(\underset{\sim}{b} \cdot \underset{\sim}{c}) \tag{11}$$

§1.3. Placement of Indices (Covariant and Contravariant).

For simplicity consider a two-dimensional rectilinear but skew coordinate system in a plane as shown in Fig. 3a.



Figures 3a, b, c.

Let a vector $\underset{\sim}{w}$ emanate from the point O. We now decide not to use any subscripted variables on the components of $\underset{\sim}{w}$ since we want to develop a consistent method of index notation. Obviously there are two ways to write the vector $\underset{\sim}{w}$ in a linear form.

(i) Parallel projection:

Let $\underset{\sim}{\lambda}$ and $\underset{\sim}{\mu}$ be a basis for the coordinate axes Ox and Oy respectively as shown in Fig. 3b. From the tip of $\underset{\sim}{w}$ draw lines parallel to OX and OY to have a parallelogram OAPB. Thus

$$\underset{\sim}{w} = \underset{\sim}{\lambda}p + \underset{\sim}{\mu}q \tag{12}$$

where p and q are the components of $\underset{\sim}{w}$ with respect to the basis $(\underset{\sim}{\lambda},\underset{\sim}{\mu})$. To find the lengths OA and OB, we introduce unit vectors

$$\underset{\sim}{u} = \underset{\sim}{\lambda}/|\underset{\sim}{\lambda}| \ , \ \underset{\sim}{v} = \underset{\sim}{\mu}/|\underset{\sim}{\mu}| \ .$$

Then

$$\underset{\sim}{w} = \underset{\sim}{u}|\underset{\sim}{\lambda}|p + \underset{\sim}{v}|\underset{\sim}{\mu}|q \tag{13}$$

so that $|\underset{\sim}{\lambda}|p$ and $|\underset{\sim}{\mu}|q$ are the respective parallel projections of $\underset{\sim}{w}$ on the coordinate axes.

(ii) Orthogonal projection:

Another method of writing $\underset{\sim}{w}$ in a linear form is to draw perpendicular lines PD and PC on the coordinate axes as shown in Fig. 3c. We now draw lines OX' and OY' parallel respectively to PD and PC. Obviously the axes OX' and OY' are perpendicular to OY and OX respectively. Let $\underset{\sim}{\psi}$ and $\underset{\sim}{\chi}$ be the basis for this new coordinate system. Then

$$\underset{\sim}{w} = \underset{\sim}{\psi}R + \underset{\sim}{\lambda}S \tag{14}$$

But since

$$\underset{\sim}{\lambda} \cdot \underset{\sim}{\chi} = 0 \ , \ \underset{\sim}{\mu} \cdot \underset{\sim}{\psi} = 0$$

so that writing

$$r = \underset{\sim}{w} \cdot \underset{\sim}{\lambda} \ , \ s = \underset{\sim}{w} \cdot \underset{\sim}{\mu}$$

we obtain

$$\underset{\sim}{w} = \frac{\underset{\sim}{\psi}}{\underset{\sim}{\lambda} \cdot \underset{\sim}{\psi}} r + \frac{\underset{\sim}{\chi}}{\underset{\sim}{\mu} \cdot \underset{\sim}{\chi}} s \tag{15}$$

Because of the two possible linear representations of the same vector $\underset{\sim}{w}$, viz. (13) and (15), it is important to introduce a new system of labeling. It is a standard convention to write

$$\underset{\sim}{\lambda} = \underset{\sim}{a}_1 \ , \ \underset{\sim}{\mu} = \underset{\sim}{a}_2$$

$$\frac{\underset{\sim}{\psi}}{\underset{\sim}{\lambda} \cdot \underset{\sim}{\psi}} = \underset{\sim}{a}^1 \ , \ \frac{\underset{\sim}{\chi}}{\underset{\sim}{\mu} \cdot \underset{\sim}{\chi}} = \underset{\sim}{a}^2$$

so that

$$\underset{\sim}{a}_1 \cdot \underset{\sim}{a}^1 = 1 \ , \ \underset{\sim}{a}_2 \cdot \underset{\sim}{a}^2 = 1 \ , \ \underset{\sim}{a}_1 \cdot \underset{\sim}{a}^2 = \underset{\sim}{a}_2 \cdot \underset{\sim}{a}^1 = 0$$

$$p = w^1 \ , \ q = w^2$$

$$r = w_1 \ , \ s = w_2$$

Thus (13) and (15) can be written as

$$\underset{\sim}{w} = \underset{\sim}{a}_1 w^1 + \underset{\sim}{a}_2 w^2 \tag{16}$$

$$\underset{\sim}{w} = \underset{\sim}{a}^1 w_1 + \underset{\sim}{a}^2 w_2 \tag{17}$$

The quantities w_i are called the covariant components, and w^i are called the contravariant components of the same vector $\underset{\sim}{w}$. Similarly the vectors $\underset{\sim}{a}_i$ and $\underset{\sim}{a}^i$ are respectively the covariant and contravariant base vectors. It is easy to conclude that if the axes OX and OY are orthogonal, thus forming a rectangular Cartesian coordinate system, then there is no distinction between the covariant and contravariant components.

§1.4. Dyads.

An indefinite product of two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ written as

$$\tilde{\phi} = \underset{\sim}{a}\underset{\sim}{b} \tag{18}$$

is called a dyad. Some authors put the symbol $\otimes$ between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. It is instructive to view dyads as operators since their utility lies in the area of operations with vectors or other dyads. In Cartesian coordinates we can also write

$$\tilde{\phi} = a_i b_j \underset{\sim}{e}_i \underset{\sim}{e}_j \tag{19}$$

The dyad $\underset{\sim}{b}\underset{\sim}{a}$ is the transpose of $\tilde{\phi}$ written as $\tilde{\phi}^*$. Thus

$$\tilde{\phi}^* = \underset{\sim}{b}\underset{\sim}{a} \tag{20}$$

so that

$$\tilde{\phi}^* = a_i b_j \underset{\sim}{e}_j \underset{\sim}{e}_i \tag{21}$$

§1.4.1. Operations with Dyads.

The scalar product of a dyad with a vector $\underset{\sim}{u}$ is a vector. That is,

$$\underset{\sim}{v} = \tilde{\phi} \cdot \underset{\sim}{u} \tag{22}$$

In general the vector $\underset{\sim}{w}$ obtained by pre-multiplication

$$\underset{\sim}{w} = \underset{\sim}{u} \cdot \tilde{\phi} \tag{23}$$

is different from $\underset{\sim}{v}$. However, it is easy to verify that

$$\underset{\sim}{v} = \tilde{\phi} \cdot \underset{\sim}{u} = \underset{\sim}{u} \cdot \tilde{\phi}^* \tag{24}$$

For two arbitrary vectors $\underset{\sim}{v}$ and $\underset{\sim}{w}$, we have the result

$$\underset{\sim}{w} \cdot (\tilde{\phi} \cdot \underset{\sim}{v}) = \underset{\sim}{v} \cdot (\tilde{\phi}^* \cdot \underset{\sim}{w}) \tag{25}$$

The unit dyad is defined as

$$\tilde{I} = \underset{\sim}{e}_i \underset{\sim}{e}_i \tag{26}$$

since its dot product with a vector

$$\underset{\sim}{v} = v_j \underset{\sim}{e}_j$$

is again $\underset{\sim}{v}$, viz.,

$$\underset{\sim}{v} = \tilde{I} \cdot \underset{\sim}{v} \tag{27}$$

The scalar product of two dyads

$$\tilde{\phi} = \underset{\sim}{a}\underset{\sim}{b} \ , \ \tilde{\psi} = \underset{\sim}{c}\underset{\sim}{d}$$

is

$$\tilde{\phi} \cdot \tilde{\psi} = (\underset{\sim}{a}\underset{\sim}{b}) \cdot (\underset{\sim}{c}\underset{\sim}{d}) = (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a}\underset{\sim}{d} \tag{28}$$

The double scalar or inner product is <u>defined</u> as

$$\tilde{\phi} : \tilde{\psi} = (\underset{\sim}{a}\underset{\sim}{b}) : (\underset{\sim}{c}\underset{\sim}{d})$$

$$= (\underset{\sim}{a} \cdot \underset{\sim}{c})(\underset{\sim}{b} \cdot \underset{\sim}{d}) \tag{29}$$

§1.5. Curvilinear Coordinates.

General curvilinear coordinates introduced either in Euclidean or non-Euclidean spaces (cf. §2) will always be denoted by x^i. As stated earlier, the rectangular Cartesian coordinates will be denoted by x_i. The general coordinates also form a right-handed system.

In general coordinates x^i, a repeated lower and upper index on quantities either appearing as a single entity or as products will always imply summation. Thus

$$A^i_{ij} = A^1_{1j} + A^2_{2j} + A^3_{3j}$$

$$A^i_j a_i = A^1_j a_1 + A^2_j a_2 + A^3_j a_3$$

but no summation is implied in the expression

$$A^i_j + A^j_i \text{ on either i or j.}$$

All quantities, with the exception of x_i and x^i, with subscripts are termed covariant components, while all with superscripts are termed contravariant components. It is customary in all the standard works cited before to call v_i and v^i as covariant and contravariant vectors respectively. Similar is the case with tensors too. However, it is clear that they are the respective components of the same entity $\underset{\sim}{v}$ or $\tilde{T}$. Occasionally it is helpful to write the entity form such as

$$\underset{\sim}{v} = v^i \underset{\sim}{a}_i = v_j \underset{\sim}{a}^j \tag{30}$$

$$\tilde{T} = T^{ij} \underset{\sim}{a}_i \underset{\sim}{a}_j = T_{k\ell} \underset{\sim}{a}^k \underset{\sim}{a}^\ell \tag{31}$$

where as described before, $\underset{\sim}{a}^i$ are the contravariant base vectors, or the reciprocal basis to $\underset{\sim}{a}_i$. The two bases are related as

$$\underset{\sim}{a}^i \cdot \underset{\sim}{a}_j = \delta^i_j \tag{32}$$

where δ^i_j is the mixed Kronecker delta defined as

$$\delta^i_j = 1 \quad \text{if } i = j$$

$$= 0 \quad \text{if } i \neq j \;.$$

§1.5.1. Various Representations in Terms of $\underset{\sim}{a}_i$ and $\underset{\sim}{a}^i$.

All quantities which follow certain transformation of coordinate rules are called tensors. Thus scalars and vectors are also tensors of orders† zero and one respectively. However, it is customary to name the quantities of orders greater than or equal to two as tensors. The order of a tensor is determined by the total number of free indices used in

†Also called ranks.

the description of its components. Thus T^{i}_{jk} is a third order tensor; covariant of order two and contravariant of order one. The total number of components of a tensor $\tilde{T}$ are given by N^m, where N is the space dimension and m the order of the tensor. Thus T^{i}_{jk} has 27 components in a three-dimensional space.

The dyadic representation of the unit tensor $\tilde{I}$ (also called the Idem tensor) in general coordinates is

$$\tilde{I} = \underset{\sim}{a}_i \underset{\sim}{a}^i \tag{33}$$

In eq. (30), v^i and v_j are the contravariant and covariant components of a vector $\underset{\sim}{v}$, while in eq. (31) T^{ij} are the contravariant components of a second order tensor $\tilde{T}$. The covariant components of $\tilde{T}$ are given by

$$\tilde{T} = T_{ij} \underset{\sim}{a}^i \underset{\sim}{a}^j \tag{34}$$

A tensor is said to be symmetric if

$$T_{ij} = T_{ji}$$

and (35)

$$T^{ij} = T^{ji}$$

In entity form symmetry implies

$$\tilde{T} = \tilde{T}^* .$$

A tensor is said to be antisymmetric if

$$T_{ij} = -T_{ji}$$

and

$$T^{ij} = -T^{ji} \tag{36}$$

or,

$$\tilde{T} = -\tilde{T}^{*} .$$

Note that the transpose of the representation (34) is

$$\tilde{T}^{*} = T_{ij}\underset{\sim}{a}^{j}\underset{\sim}{a}^{i}$$

and of (31) is

$$\tilde{T}^{*} = T^{ij}\underset{\sim}{a}_{j}\underset{\sim}{a}_{i} . \tag{37}$$

1.6. Differential Operations in Curvilinear Coordinates.

The continuity and differentiability of vector and tensor functions in general coordinates follow the same rules as those by functions of real variables in multivariate calculus. Thus, let $\phi(x^{i})$ be a scalar function of general coordinates. Then its first differential is given by

$$d\phi = \frac{\partial\phi}{\partial x^{i}}\, dx^{i} \tag{38}$$

where dx^{i} are the differentials of the coordinates x^{i}. Also since dx^{i} are the contravariant components of the displacement vector $d\underset{\sim}{r}$ (cf. §2.), we have

$$d\underset{\sim}{r} = \underset{\sim}{a}_{i}dx^{i} \tag{39}$$

Scalar multiplication with $\underset{\sim}{a}^j$ on both sides of (39) and a use of eq. (32) gives

$$dx^i = \underset{\sim}{a}^i \cdot d\underset{\sim}{r} \tag{40}$$

Thus

$$d\phi = \left(\frac{\partial\phi}{\partial x^i}\,\underset{\sim}{a}^i\right) \cdot d\underset{\sim}{r} \tag{41}$$

Equation (41) defines the operator $\underset{\sim}{\nabla}$ or grad as

$$d\phi = (\text{grad}\ \phi) \cdot d\underset{\sim}{r} \tag{42}$$

where

$$\text{grad} = \underset{\sim}{\nabla} = \underset{\sim}{a}^i \frac{\partial}{\partial x^i} \tag{43}$$

The divergence (div or $\underset{\sim}{\nabla}\ \cdot$) of a vector function is given by

$$\text{div}\ \underset{\sim}{v} = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \underset{\sim}{a}^i \cdot \frac{\partial \underset{\sim}{v}}{\partial x^i} \tag{44}$$

and the curl or $\underset{\sim}{\nabla}\times$ as

$$\text{curl}\ \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} = \underset{\sim}{a}^i \times \frac{\partial \underset{\sim}{v}}{\partial x^i} \tag{45}$$

1.6.1. Gradient of Vectors and Divergence of Tensors.

Let $\underset{\sim}{v}$ be a vector function of x^i, then

$$d\underset{\sim}{v} = \frac{\partial \underset{\sim}{v}}{\partial x^i}\, dx^i \tag{46a}$$

$$= dx^i \frac{\partial \underset{\sim}{v}}{\partial x^i} \tag{46b}$$

Using (40) in (46a) and (46b), we get

$$d\underset{\sim}{v} = \frac{\partial \underset{\sim}{v}}{\partial x^i} \underset{\sim}{a}^i \cdot d\underset{\sim}{r} = \left(\frac{\partial \underset{\sim}{v}}{\partial x^i} \underset{\sim}{a}^i\right) \cdot d\underset{\sim}{r} \tag{47a}$$

$$= (d\underset{\sim}{r} \cdot \underset{\sim}{a}^i)\frac{\partial \underset{\sim}{v}}{\partial x^i} = d\underset{\sim}{r} \cdot \left(\underset{\sim}{a}^i \frac{\partial \underset{\sim}{v}}{\partial x^i}\right) \tag{47b}$$

Thus there is a <u>duality</u> in the representation of grad $\underset{\sim}{v}$. It can be either represented as

$$\frac{\partial \underset{\sim}{v}}{\partial x^i} \underset{\sim}{a}^i \quad \text{or} \quad \underset{\sim}{a}^i \frac{\partial \underset{\sim}{v}}{\partial x^i} .$$

In this report, we take the first representation to represent grad $\underset{\sim}{v}$, i.e.,

$$\text{grad } \underset{\sim}{v} = \frac{\partial \underset{\sim}{v}}{\partial x^i} \underset{\sim}{a}^i \tag{48}$$

then its conjugate is

$$(\text{grad } \underset{\sim}{v})^* = \underset{\sim}{a}^i \frac{\partial \underset{\sim}{v}}{\partial x^i} \tag{49}$$

As is obvious from eqs. (47), we have the identity

$$d\underset{\sim}{v} = (\text{grad } \underset{\sim}{v}) \cdot d\underset{\sim}{r} = d\underset{\sim}{r} \cdot (\text{grad } \underset{\sim}{v})^* \tag{50}$$

In the same manner, we define the gradient of a tensor as

$$\text{grad } \tilde{T} = \frac{\partial \tilde{T}}{\partial x^i} \underset{\sim}{a}^i . \tag{51}$$

The divergence of a tensor is then the trace of (51), that is

$$\mathrm{div}\ \tilde{T} = \frac{\partial \tilde{T}}{\partial x^i} \cdot \underset{\sim}{a}^i \tag{52}$$

Below we list some important vector and tensor formulae involving vectors and tensors under the operations of grad, divergence and curl, [27].

(i) For two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, the divergence of the dyad $\underset{\sim}{u}\underset{\sim}{v}$ is

$$\mathrm{div}(\underset{\sim}{u}\underset{\sim}{v}) = (\mathrm{grad}\ \underset{\sim}{u}) \cdot \underset{\sim}{v} + (\mathrm{div}\ \underset{\sim}{v})\underset{\sim}{u} \tag{53}$$

(ii) If f is a scalar, then

$$\mathrm{div}(\tilde{I}f) = \mathrm{grad}\ f \tag{54}$$

(iii) The Laplacian of a vector $\underset{\sim}{u}$ is

$$\nabla^2\underset{\sim}{u} = \mathrm{div}(\mathrm{grad}\ \underset{\sim}{u}) = \mathrm{grad}(\mathrm{div}\ \underset{\sim}{u}) - \mathrm{curl}(\mathrm{curl}\ \underset{\sim}{u}) \tag{55}$$

(iv) $(\mathrm{grad}\ \underset{\sim}{u}) \cdot \underset{\sim}{u} = \mathrm{grad}(\tfrac{1}{2}|\underset{\sim}{u}|^2) + (\mathrm{curl}\ \underset{\sim}{u}) \times \underset{\sim}{u}$ (56)

(v) For two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$

$$(\mathrm{curl}\ \underset{\sim}{u}) \times \underset{\sim}{v} = [\mathrm{grad}\ \underset{\sim}{u} - (\mathrm{grad}\ \underset{\sim}{u})^*] \cdot \underset{\sim}{v} \tag{57}$$

(vi) $$\mathrm{div}(\mathrm{grad}\ \underset{\sim}{u})^* = \mathrm{grad}(\mathrm{div}\ \underset{\sim}{u}) \tag{58}$$

(vii) $\mathrm{grad}(\underset{\sim}{u} \cdot \underset{\sim}{v}) = (\mathrm{grad}\ \underset{\sim}{u})^* \cdot \underset{\sim}{v} + (\mathrm{grad}\ \underset{\sim}{v})^* \cdot \underset{\sim}{u}$ (59a)

(viii) $$\mathrm{curl}(\underset{\sim}{u} \times \underset{\sim}{v}) = \mathrm{div}(\underset{\sim}{u}\underset{\sim}{v} - \underset{\sim}{v}\underset{\sim}{u}) \tag{59b}$$

(ix) $\operatorname{div}[\operatorname{grad} \underset{\sim}{u} - (\operatorname{grad} \underset{\sim}{u})^*] = -\operatorname{curl}(\operatorname{curl} \underset{\sim}{u})$ (60)

(x) $$\tilde{I} : \operatorname{grad} \underset{\sim}{u} = \operatorname{div} \underset{\sim}{u} \tag{61}$$

(xi) For a tensor $\tilde{T}$ and a vector $\underset{\sim}{u}$,

$$\operatorname{div}(\tilde{T} \cdot \underset{\sim}{u}) = (\operatorname{div} \tilde{T}^*) \cdot \underset{\sim}{u} + \tilde{T}^* : (\operatorname{grad} \underset{\sim}{u}) \tag{62}$$

(xii) If $\tilde{T}$ is a symmetric tensor and $\underset{\sim}{r}$ is the position vector, then

$$\operatorname{div}(\underset{\sim}{r} \times \tilde{T}) = \underset{\sim}{r} \times (\operatorname{div} \tilde{T}) \tag{63}$$

(xiii) Let $\tilde{\Omega}$ be a skewsymmetric tensor in a three-dimensional space, then

$$\begin{gathered} \Omega_{11} = 0 \;,\; \Omega_{22} = 0 \;,\; \Omega_{33} = 0 \\ \Omega_{12} = -\Omega_{21} \;,\; \Omega_{13} = -\Omega_{31} \;,\; \Omega_{23} = -\Omega_{32} \end{gathered} \tag{64}$$

With $\tilde{\Omega}$ we can associate a vector $\underset{\sim}{w}$, such that

(xiv) $$\operatorname{div} \tilde{\Omega} = -\operatorname{curl} \underset{\sim}{w} \tag{65}$$

(xv) $$\operatorname{div} \tilde{\Omega}^* = \operatorname{curl} \underset{\sim}{w} \tag{66}$$

For an arbitrary vector $\underset{\sim}{v}$

(xvi) $$\tilde{\Omega} \cdot \underset{\sim}{v} = \underset{\sim}{w} \times \underset{\sim}{v} \tag{67}$$

(xvii) $$\tilde{\Omega} : (\operatorname{grad} \underset{\sim}{v}) = \underset{\sim}{w} \cdot (\operatorname{curl} \underset{\sim}{v}) \tag{68}$$

For an arbitrary tensor $\tilde{T}$

(xviii) $$\underset{\sim}{w} \times \tilde{T} = \tilde{\Omega} \cdot \tilde{T} \tag{69}$$

§2. Euclidean and Riemannian Spaces†.

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†Refer to §8 of Part I for a complete discussion on spaces.

Spaces of various kinds, abstract as well as perceivable, are needed to analyze mathematically the basic nature of practically all problems in engineering science. The most widely studied is the Euclidean space E^N of dimension N. We shall usually be interested in E^3 space, though most of the results are immediately extensible to any value of N. The most important property of an Euclidean space is that in this space rectangular Cartesian coordinates can always be introduced on a <u>global scale</u>. As an aid to form some intuitive ideas about spaces, it is worthwhile to realize that the two-dimensional space formed by the surface of a sphere is non-Euclidean since rectangular Cartesian coordinates cannot be introduced in it on a global scale.

In rectangular Cartesian coordinates the position vector $\underset{\sim}{r}$ of a point from the origin is obviously given by

$$\underset{\sim}{r} = \underset{\sim}{e}_i x_i = \underset{\sim}{e}_1 x_1 + \underset{\sim}{e}_2 x_2 + \underset{\sim}{e}_3 x_3 \ . \tag{70}$$

This type of global expression for the position vector $\underset{\sim}{r}$ is not available in terms of general coordinates x^i either in the Euclidean or non-Euclidean spaces. Nevertheless, the infinitesimal vector $d\underset{\sim}{r}$, which is the directed segment between two infinitesimally close points, is fundamental to all geometric considerations. According to Lanczos [28], the line element ds (s is the arc length) which is the magnitude of $d\underset{\sim}{r}$, viz.,

$$ds = |d\underset{\sim}{r}|$$

is the fountainhead of metrical geometry. To fix ideas, let in E^3 a Cartesian coordinate system has already been introduced. Then the infinitesimal vector $d\underset{\sim}{r}$ is given by

$$d\underset{\sim}{r} = \underset{\sim}{e}_k dx^k$$

where for the purpose of further comparison we have written $x_k = x^k$. The magnitude ds is then given by

$$(ds)^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = \underset{\sim}{e}_k \cdot \underset{\sim}{e}_\ell dx^k dx^\ell$$

$$= \delta_{k\ell} dx^k dx^\ell \tag{71}$$

where $\delta_{k\ell}$ is the Kronecker delta,

$$\delta_{k\ell} = 1 \quad \text{if} \quad k = \ell$$

$$= 0 \quad \text{if } k \neq \ell \ .$$

In E^3 we now introduce a curvilinear coordinate system x^i. The infinitesimal vector $d\underset{\sim}{r}$ is then a function of x^i, so that

$$d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial x^i} dx^i = \underset{\sim}{a}_i dx^i \tag{72}$$

and

$$(ds)^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = (\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j) dx^i dx^j \ .$$

Writing

$$\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j = g_{ij} \tag{73}$$

we obtain

$$(ds)^2 = g_{ij}dx^i dx^j \tag{74}$$

The coefficients g_{ij} are functions of x^i and are called the fundamental metric coefficients of the chosen space. The chosen space is still Euclidean and fundamentally its metric coefficients are the constants δ_{ij} but because of the introduction of a curvilinear coordinate system the g_{ij} are expressed as functions of x^i. The introduction of a curvilinear coordinate system in E^3 does not change the nature of space†.

It is obvious from (73) that g_{ij} is symmetric, i.e.,

$$g_{ij} = g_{ji} \tag{75}$$

and using the condition that ds is an invariant, we can equate (71) and (74). This equation immediately yields the expressions for the g_{ij} in terms of the derivatives of the Cartesian coordinates with respect to the curvilinear coordinates.

The name "Euclidean" for a space is due to the fact that in this space the five axioms and five postulates of Euclid are assumed to be true. Some important consequences on the basis of these axioms and postulates are summarized below.

(i) The Pythagorean theorem for right triangles can be proved both in the infinitesimal and global regions.

(ii) Possibility of introducing rectangular Cartesian coordinates both in the infinitesimal and global regions.

(iii) Global parallelism. That is, a vector in E^3 or E^n can be

†According to Lanczos [28] p. 236, this understanding was the motive force in overcoming the difficulties which Einstein faced in the year 1914 while working on the theory of general relativity.

displaced parallel t‹ itself on any space curve without a change in magnitu Thus in an Euclidean space, a parallel field of vectors is constant in magnitude and direction.

Now though the metric equation (74) has been obtained by introducing curvilinear coordinates in an Euclidean space, we have only used the results (i) and (ii), implicitly through Eq. (72), on an infinitesimal basis. Further, no where in the derivation of (74) the result (iii) on global parallelism has been used. Following Riemann, we now take (74) as the one and only axiom of a geometry in which the functions g_{ij} are arbitrary but continuous and at least twice continuously differentiable functions of the coordinates x^i. Because of the general nature of such g_{ij}'s this geometry will be non-Euclidean. However, the possibility of introducing a rectangular Cartesian system locally in this general space in an infinitesimal region still exists. These assertions have been proved in §8.2. Spaces in which the Euclidean background has been deleted and the formula for the metric is as given in (74) are called Riemannian.

The purpose of the preceding two paragraphs has been to bring out the subtle differences between the Euclidean and non-Euclidean spaces. It so happens that a majority of analytical constructions (such as the g_{ij} considered before and obtained essentially from Euclidean considerations) can immediately be interpreted in the sense of a Riemannian space. This technique eliminates some of the abstractness surrounding the tensor theory and allows us to obtain all the essential formulae of Riemannian geometry while essentially remaining in the Euclidean space. This is the scheme for further development of the subject in this report. It must be realized that most of the Riemannian constructions are analytic by nature since the human mind is not capable of imagining a curved surface of dimensions greater than two.

§3. Fundamental Tensor Structures and Transformation Laws.

The fundamental metric tensor g_{ij} was earlier defined through the use of the base vectors as

$$g_{ij} = \underset{\sim}{a}_i \cdot \underset{\sim}{a}_j \ . \qquad (76)$$

If the contravariant base vectors $\underset{\sim}{a}^i$ are multiplied scalarly then we define the new second order components

$$g^{ij} = \underset{\sim}{a}^i \cdot \underset{\sim}{a}^j \qquad (77)$$

To find the relation between g_{ij} and g^{ij}, we first write an arbitrary vector $\underset{\sim}{A}$ both in the covariant and contravariant components

$$\underset{\sim}{A} = A_i \underset{\sim}{a}^i \ , \ \underset{\sim}{A} = A^j \underset{\sim}{a}_j$$

so that

$$A_i \underset{\sim}{a}^i = A^j \underset{\sim}{a}_j \qquad (78)$$

Taking the dot product of both sides of (78) by $\underset{\sim}{a}_k$ and using (32), we get

$$A_k = g_{jk} A^j \qquad (79)$$

Similarly, taking the dot product with $\underset{\sim}{a}^k$, we get

$$A^k = g^{ik} A_i \qquad (80)$$

Solving the set of equations (79) for A^k, we have

$$A^k = \frac{A_i G^{ik}}{g} \tag{81}$$

where

$$g = \det(g_{ij}) \tag{82a}$$

$$= g_{ik}G^{ik} \quad \text{(by Kramer's expansion)} \tag{82b}$$

and G^{ik} is the cofactor of g_{ik} in the determinant g. Comparing (80) and (81), the required relation is

$$g^{ik} = \frac{G^{ik}}{g} = \frac{\text{cofactor of } g_{ik} \text{ in } g}{g} \tag{83}$$

The tensor components g^{ik} are called the <u>conjugates</u> of the metric components g_{ik}. Equation (83) can also be expressed as

$$g^{ij} = (g_{rs}g_{\ell t} - g_{rt}g_{\ell s})/g \tag{84}$$

where the groups (i,r,ℓ) and (j,s,t) separately have their indices in the cyclic permutations of 1,2,3, in this order. Obviously g^{ij} are also symmetric in i,j.

Having defined g^{ij}, we find from Eqs. (79) and (80) that if the contravariant components of a vector are known then the covariant components can be obtained, and vice versa. In (79) the index has been lowered and in (80) the index has been raised. These operations are called <u>lowering</u> or <u>raising</u> an index respectively through g_{ij} and g^{ij}.

§3.1. Relations Between the Base Vectors.

From eq. (32) we have the fundamental relation between the covariant and contravariant base vectors, which is

$$\underset{\sim}{a}_i \cdot \underset{\sim}{a}^j = \delta_i^j \tag{85}$$

Equation (85) shows that one vector from the reciprocal basis $\underset{\sim}{a}^i$ is orthogonal to two vectors from the basis $\underset{\sim}{a}_i$. Thus, for example, $\underset{\sim}{a}_2 \times \underset{\sim}{a}_3$ must be parallel to $\underset{\sim}{a}^1$, so that

$$\underset{\sim}{a}^1 = p(\underset{\sim}{a}_2 \times \underset{\sim}{a}_3) \tag{86a}$$

where p is a scalar function of the coordinates. Further, since $\underset{\sim}{a}^1 \cdot \underset{\sim}{a}_1 = 1$, we have

$$\frac{1}{p} = \underset{\sim}{a}_1 \cdot (\underset{\sim}{a}_2 \times \underset{\sim}{a}_3) .$$

Using the vector formula given in (8), we also have

$$\frac{1}{p} = \underset{\sim}{a}_1 \cdot (\underset{\sim}{a}_2 \times \underset{\sim}{a}_3) = \underset{\sim}{a}_2 \cdot (\underset{\sim}{a}_3 \times \underset{\sim}{a}_1) = \underset{\sim}{a}_3 \cdot (\underset{\sim}{a}_1 \times \underset{\sim}{a}_2) . \tag{86b}$$

All the possible forms such as in (86a) can therefore be written as

$$\underset{\sim}{a}^i = p(\underset{\sim}{a}_j \times \underset{\sim}{a}_k) \tag{87}$$

where i, j, k are in the cyclic permutations of 1, 2, 3, in this order.

Similar considerations show that

$$\underset{\sim}{a}_\ell = q(\underset{\sim}{a}^m \times \underset{\sim}{a}^n) \tag{88}$$

where ℓ, m, n are in the cyclic permutations of 1, 2, 3, in this order. From (87) and (88) using (11), we get

$$\delta^i_\ell = pq(\delta^m_j\delta^n_k - \delta^n_j\delta^m_k) \tag{89}$$

so that

$$pq = 1 \ .$$

Further, on the basis of the result (84) it is easy to show, using (87) and (11), that

$$p = \frac{1}{\sqrt{g}} \ , \ q = \sqrt{g} \ .$$

Having obtained the values of p and q, we can rewrite (87) and (88) in the following useful forms

$$\underset{\sim}{a}_j \times \underset{\sim}{a}_k = \sqrt{g}\ e_{ijk}\underset{\sim}{a}^i \tag{90a}$$

$$\underset{\sim}{a}^j \times \underset{\sim}{a}^k = \frac{1}{\sqrt{g}}\ e^{ijk}\underset{\sim}{a}_i \tag{90b}$$

Note that from (90a,b) we also have

$$\underset{\sim}{a}^i = \frac{1}{2\sqrt{g}}\ e^{ijk}(\underset{\sim}{a}_j \times \underset{\sim}{a}_k) \tag{90c}$$

$$\underset{\sim}{a}_i = \frac{\sqrt{g}}{2}\ e_{ijk}(\underset{\sim}{a}^j \times \underset{\sim}{a}^k) \tag{90d}$$

where e^{ijk} is also a permutation symbol written in contravariant form so as to be consistent with the summation convention.

We now use the rule of lowering and raising an index to base vectors. It is obvious that

$$\underset{\sim}{a}_i = g_{ik}\underset{\sim}{a}^k \tag{91}$$

$$\underset{\sim}{a}^j = g^{jk}\underset{\sim}{a}_k \tag{92}$$

If eq. (91) is rewritten using different indices as

$$\underset{\sim}{a}_j = g_{j\ell}\underset{\sim}{a}^\ell \tag{93}$$

then the dot product of (91) and (93) gives

$$g_{ik}g_{j\ell}g^{k\ell} = g_{ij}$$

$$= g_{ik}\delta^k_j$$

From this we obtain the important result

$$g_{j\ell}g^{k\ell} = \delta^k_j \tag{94}$$

In (33), the idem tensor $\tilde{I}$ referred to general coordinates was defined. We can also write (33) as

$$\tilde{I} = \delta^i_j\underset{\sim}{a}_i\underset{\sim}{a}^j \tag{95}$$

Using (91) and (92), the other two representations are

$$\tilde{I} = g_{ij}\underset{\sim}{a}^i\underset{\sim}{a}^j \tag{96}$$

$$= g^{ij}\underset{\sim}{a}_i\underset{\sim}{a}_j \tag{97}$$

The use of base vectors also allows us to write vectors and tensors in the entity forms. The choice of a particular form of components is usually dictated by the user according to his needs. For writing tensors, the following forms can be used.

(i) All components in contravariant form.

(ii) All components in covariant form.

(iii) All components in mixed covariant-contravariant form.

In eqs. (30) and (31) we expressed a vector and a second order tensor in component forms. Thus, for a vector

$$\underset{\sim}{v} = v^i \underset{\sim}{a}_i \tag{98a}$$

$$= v_j \underset{\sim}{a}^j \tag{98b}$$

For a tensor of the second order,

$$\tilde{T} = T^{ij} \underset{\sim}{a}_i \underset{\sim}{a}_j \tag{99a}$$

$$= T_{ij} \underset{\sim}{a}^i \underset{\sim}{a}^j \tag{99b}$$

$$= T^i_{\cdot j} \underset{\sim}{a}_i \underset{\sim}{a}^j \tag{99c}$$

$$= T_i^{\cdot j} \underset{\sim}{a}^i \underset{\sim}{a}_j \tag{99d}$$

Using (92) for $\underset{\sim}{a}^i$ in (99b) and then equating with (99c) we get

$$T^i_{\cdot j} = g^{in} T_{nj} \tag{100}$$

Similarly, using (91) for $\underset{\sim}{a}_i$ in (99a) and equating with (99d), we get

$$T_i^{\cdot j} = g_{im} T^{mj} \tag{101}$$

The dot placed before an index indicates which index has been raised or lowered.

Tensors of various orders can be written by using as many base vectors as the order of the tensor. When there is no confusion in recognizing which symbol has been raised or lowered, we may suspend the use of dots. Thus

$$T^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots i_q}$$

are the mixed components of a tensor of (p + q)th order; covariant of order q and contravariant of order p.

§3.2. Transformation Laws for Vectors and Tensors.

We now consider the transformation laws for vectors and tensors under a change of the coordinate system x^i to $\bar{x}^i$. Implicitly there is a functional relation between the two coordinate systems, viz., the coordinates $\bar{x}^i$ are functions of the coordinates x^i. Thus

$$\bar{x}^i = \phi^i(x^j) \tag{102}$$

We assume that the mapping or transformation (102) is nonsingular so that the functions ϕ^i are continuously differentiable and their functional determinant (i.e., Jacobian)

$$\bar{J} = \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \tag{103}$$

is no where vanishing. This implies that the functions (102) can be inverted to have

$$x^i = \psi^i(\bar{x}^j) \tag{104}$$

(i) <u>Vectors</u>:

The simplest but fundamental is the vector $d\underset{\sim}{r}$. When the coordinate system is x^i, then as noted earlier

$$d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial x^i} dx^i \tag{105a}$$

$$= \underset{\sim}{a}_i dx^i \tag{105b}$$

On change of coordinates from x^i to $\bar{x}^i$, the same vector $d\underset{\sim}{r}$ can be written as

$$d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial \bar{x}^i} d\bar{x}^i \tag{106a}$$

$$= \bar{\underset{\sim}{a}}_i d\bar{x}^i \tag{106b}$$

Obviously

$$\underset{\sim}{a}_j dx^j = \bar{\underset{\sim}{a}}_i d\bar{x}^i \tag{107}$$

By the chain rule of differentiation, we have

$$dx^j = \frac{\partial x^j}{\partial \bar{x}^i} d\bar{x}^i \tag{108a}$$

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j \tag{108b}$$

Using (108a,b) in (107), we obtain

$$\underset{\sim}{a}_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{\underset{\sim}{a}}_i \tag{109a}$$

$$\bar{a}_i = \frac{\partial x^j}{\partial \bar{x}^i} a_j \qquad (109b)$$

A study of (108a) and (109a) or (108b) and (109b) is revealing. It suggests that the set of quantities dx^i and set of vectors $\underset{\sim}{a}_j$ follow different transformation laws on changing from x^i to $\bar{x}^i$ or vice versa. Quantities which transform in the manner of (108) are called the contravariant components while those which transform in the manner of (109) are called the covariant components. Thus $d\underset{\sim}{r}$ is a vector whose contravariant components are dx^i, while the vectors $\underset{\sim}{a}_i$ are called the covariant base vectors. (This was the reason for denoting the coordinates as x^i and base vectors as $\underset{\sim}{a}_i$). Another simple vector which has covariant components in a natural way is the gradient of a scalar. If f is a scalar, then its first partial derivatives with respect to x^i are $\frac{\partial f}{\partial x^i}$. On changing the coordinates to $\bar{x}^i$, we have

$$\frac{\partial f}{\partial x^j} = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial f}{\partial \bar{x}^i} \qquad (110)$$

which is exactly of the form (109a). Thus the first partial derivatives of a scalar form the covariant components of the vector grad f.

Based on the above deductions we now state the transformation laws for any vector $\underset{\sim}{A}$.

The contravariant components A^i of the vector $\underset{\sim}{A}$ change to $\bar{A}^i$ on a change of coordinates from x^i to $\bar{x}^i$ according to the laws

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \qquad (111a)$$

$$A^j = \frac{\partial x^j}{\partial \bar{x}^i} \bar{A}^i \qquad (111b)$$

Similarly, the covariant components A_i transform according to the laws

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j \qquad (112a)$$

$$A_j = \frac{\partial \bar{x}^i}{\partial x^j} \bar{A}_i \qquad (112b)$$

It must be noted that the transformation laws for vectors are linear. That is, the vector components in the new system are linear functions of the vector components in the old system.

(ii) Tensors:

Consider a second order tensor $\tilde{T}$. Because of the tensor invariance, we again have

$$\tilde{T} = T^{ij} \underset{\sim}{a}_i \underset{\sim}{a}_j \qquad (113a)$$

$$= \bar{T}^{k\ell} \bar{\underset{\sim}{a}}_k \bar{\underset{\sim}{a}}_\ell \qquad (113b)$$

Using the transformation law of base vectors (109a) in (113a), we easily get

$$\bar{T}^{k\ell} = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^\ell}{\partial x^j} T^{ij} \qquad (114)$$

Similarly

$$T^{ij} = \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial x^j}{\partial \bar{x}^\ell} \bar{T}^{k\ell} \qquad (115)$$

Equations (114) and (115) are the transformation laws for the contravariant components of a tensor

For the covariant components we again have from the invariance condition

$$\tilde{T} = T_{ij}\tilde{a}^i\tilde{a}^j = \overline{T}_{k\ell}\tilde{\overline{a}}^k\tilde{\overline{a}}^\ell$$

Taking the dot product with $\overline{\tilde{a}}_p$, we get

$$T_{ij}\tilde{a}^i\tilde{a}^j \cdot \overline{\tilde{a}}_p = \overline{T}_{k\ell}\tilde{a}^k\delta^\ell_p$$

But

$$\overline{\tilde{a}}_p = \tilde{a}_m \frac{\partial x^m}{\partial \overline{x}^p}$$

so that

$$T_{im}\frac{\partial x^m}{\partial \overline{x}^p}\tilde{a}^i = \overline{T}_{kp}\tilde{a}^k$$

Taking another dot product with $\overline{\tilde{a}}_n$, we obtain

$$\overline{T}_{np} = \frac{\partial x^s}{\partial \overline{x}^n}\frac{\partial x^m}{\partial \overline{x}^p} T_{sm} \tag{116a}$$

Similarly

$$T_{sm} = \frac{\partial \overline{x}^n}{\partial x^s}\frac{\partial \overline{x}^p}{\partial x^m}\overline{T}_{np} \tag{116b}$$

Equations (116) are the transformation laws for the covariant components of a tensor.

Following the same procedure, we list the transformation laws for the mixed components. Starting from

$$\tilde{T} = T^{i}_{\cdot j}\underset{\sim}{a}_{i}\underset{\sim}{a}^{j} = \bar{T}^{k}_{\cdot \ell}\underset{\sim}{\bar{a}}_{k}\underset{\sim}{\bar{a}}^{\ell}$$

we get

$$\bar{T}^{k}_{\cdot p} = \frac{\partial x^{m}}{\partial \bar{x}^{p}} \frac{\partial \bar{x}^{k}}{\partial x^{s}} T^{s}_{\cdot m} \tag{117a}$$

$$T^{s}_{\cdot m} = \frac{\partial \bar{x}^{p}}{\partial x^{m}} \frac{\partial x^{s}}{\partial \bar{x}^{k}} \bar{T}^{k}_{\cdot p} \tag{117b}$$

Similarly, it can be proved that the transformation law for the components $T_{j}^{\cdot i}$ follow the same rules as given in (117).

Generally, we then have

$$\bar{T}^{\ell_1\ell_2\cdots\ell_p}_{m_1m_2\cdots m_q} = \left(\frac{\partial \bar{x}^{\ell_1}}{\partial x^{i_1}} \cdots \frac{\partial \bar{x}^{\ell_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial \bar{x}^{m_1}} \cdots \frac{\partial x^{j_q}}{\partial \bar{x}^{m_q}}\right) T^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q} \tag{118}$$

<u>Metric Tensor</u>:

Because of the special status of the metric components g_{ij}, and g^{ij}, we consider them in detail. Looking from the point of view of the definitions

$$g^{ij} = \underset{\sim}{a}^{i} \cdot \underset{\sim}{a}^{j} \ , \ g_{ij} = \underset{\sim}{a}_{i} \cdot \underset{\sim}{a}_{j}$$

we immediately conclude that they are symmetric in i and j. If base vectors are not brought into picture and the g_{ij}'s are assumed to be functions defining a metric in a Riemannian space then we can use the formula

$$(ds)^2 = g_{ij}dx^i dx^j$$

to write

$$(ds)^2 = \frac{1}{2}(g_{ij} + g_{ji})dx^i dx^j + \frac{1}{2}(g_{ij} - g_{ji})dx^i dx^j \tag{119}$$

By direct expansion, we can show that the last term in (119) is zero, proving the symmetry of g_{ij}. Similarly g^{ij} is also symmetric.

The components g_{ij} are covariant, while g^{ij} are the contravariant components of the metric tensor. Thus, the transformation laws for them are

$$\bar{g}_{np} = \frac{\partial x^s}{\partial \bar{x}^n}\frac{\partial x^m}{\partial \bar{x}^p} g_{sm} \tag{120}$$

$$\bar{g}^{np} = \frac{\partial \bar{x}^n}{\partial x^s}\frac{\partial \bar{x}^p}{\partial x^m} g^{sm} \tag{121}$$

All the preceding transformation laws are linear. We therefore list the following important conclusions regarding the nature of tensors.

(I) A tensor equation or expression has the property that it can be obtained in any legitimate reference system, i.e., $J \neq 0$. If it is correct in any one reference system then it must remain correct in any other legitimate reference system. The above property is due to the linearity of the transformation laws, since any component from the old system is a linear function of the components from the new system.

(II) If all the tensor components vanish in any reference system then they remain zero in any other system.

(III) Because of the linearity of transformation laws, a symmetric tensor remains symmetric on coordinate transformation. Thus symmetry is an absolute property.

§3.3. Algebraic Properties of Tensors.

1. The components of two tensors of the same order and structure* can be added and subtracted according to the usual arithmetical rules. If A_{ij} and B_{ij} are the covariant components of the tensors $\tilde{A}$ and $\tilde{B}$, then on addition or subtraction we generate a new tensor $\tilde{C}$ whose covariant components are

$$C_{ij} = A_{ij} \pm B_{ij} \tag{122}$$

2. The <u>outer product</u> of two tensors of any order or structure is obtained by arithmetical multiplication to produce new tensors. For example, let A_{ℓ}^{ijk} be a fourth order tensor, and let B_{pq}^{m} be a third order tensor. Then the outer product is the new tensor

$$C_{\ell pq}^{ijkm} = A_{\ell}^{ijk} B_{pq}^{m} \tag{123}$$

which is a seventh order tensor, contravariant of order four and covariant of order three.

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*Variance.

3. The <u>inner product</u> of two tensors is obtained by equating <u>one</u> index of the first with one index of opposite variance of the second, and, then summing over this index. Thus in (123) if we set $\ell = m$ and sum over m, we obtain

$$C^{ijk}_{pq} = A^{ijk}_{m} B^{m}_{pq} \tag{124}$$

The resulting tensor is only of the fifth order.

§4. Differentiation of Vectors and Tensors.

One begins to feel the power of the method of tensor analysis after the differential aspects of tensors have been completed. In this connection we first consider the partial derivatives of the base vectors.

From the definition of base vectors $\underset{\sim}{a}_i$, we first note the following result.

$$\frac{\partial \underset{\sim}{a}_i}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{\partial \underset{\sim}{r}}{\partial x^i}\right) = \frac{\partial}{\partial x^i}\left(\frac{\partial \underset{\sim}{r}}{\partial x^k}\right)$$

$$= \frac{\partial \underset{\sim}{a}_k}{\partial x^i} \tag{125}$$

for any values of i and k.

We now select any three indices, say i,j,k, and consider the following three equations

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial}{\partial x^k}(\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j) \tag{126}$$

$$\frac{\partial g_{jk}}{\partial x^i} = \frac{\partial}{\partial x^i}(\underset{\sim}{a}_j \cdot \underset{\sim}{a}_k) \tag{127}$$

$$\frac{\partial g_{ik}}{\partial x^j} = \frac{\partial}{\partial x^j}(\underset{\sim}{a}_i \cdot \underset{\sim}{a}_k) \tag{128}$$

Adding (127) and (128), subtracting (126) from it and using (125), we get

$$\frac{\partial \underset{\sim}{a}_i}{\partial x^j} \cdot \underset{\sim}{a}_k = [ij,k] \tag{129}$$

where

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{130}$$

The quantities [ij,k] defined in (130) are called the Christoffel symbols of the <u>first kind</u>. These quantities are symmetric in i and j but they are not tensors.

Equation (129) implies

$$\frac{\partial \underset{\sim}{a}_i}{\partial x^j} = [ij,k]\underset{\sim}{a}^k \tag{131}$$

Taking the dot product both sides by $\underset{\sim}{a}^\ell$, we then obtain

$$\frac{\partial \underset{\sim}{a}_i}{\partial x^j} \cdot \underset{\sim}{a}^\ell = \Gamma^\ell_{ij} \tag{132}$$

where

$$\Gamma^{\ell}_{ij} = g^{k\ell}[ij,k] \tag{133}$$

are called the Christoffel symbols of the <u>second kind</u>. These quantities are also symmetric in the lower two indices i and j, but are not tensors.

Equation (132) implies

$$\frac{\partial \underset{\sim}{a}_i}{\partial x^j} = \Gamma^{\ell}_{ij}\underset{\sim}{a}_{\ell} \tag{134}$$

Equation (134) expresses the first partial derivatives of the covariant base vectors in terms of the derivatives of g_{ij}.

Having established the preceding definitions and results, we now consider the partial derivatives of an arbitrary vector $\underset{\sim}{A}$.

Let $\underset{\sim}{A}$ be an arbitrary vector, and we express it in terms of its contravariant components A^i. Thus

$$\underset{\sim}{A} = A^i\underset{\sim}{a}_i \tag{135}$$

Differentiating with respect to x^k, we get

$$\frac{\partial \underset{\sim}{A}}{\partial x^k} = \frac{\partial A^i}{\partial x^k}\underset{\sim}{a}_i + A^i\frac{\partial \underset{\sim}{a}_i}{\partial x^k}$$

On using (134) and adjusting the dummy indices, we get

$$\frac{\partial \underset{\sim}{A}}{\partial x^k} = \left(\frac{\partial A^i}{\partial x^k} + \Gamma^i_{jk}A^j\right)\underset{\sim}{a}_i$$

We use a special notation for the terms in parentheses,

$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{jk} A^j \tag{136}$$

which is called the covariant derivative of the contravariant components A^i. Thus

$$\frac{\partial \underset{\sim}{A}}{\partial x^k} = A^i_{,k} \underset{\sim}{a}_i \tag{137}$$

To find the covariant derivative of the covariant components A_i of the vector $\underset{\sim}{A}$, we start differentia ing

$$\underset{\sim}{A} = A_i \underset{\sim}{a}^i$$

so that

$$\frac{\partial \underset{\sim}{A}}{\partial x^k} = \frac{\partial A_i}{\partial x^k} \underset{\sim}{a}^i + A_i \frac{\partial \underset{\sim}{a}^i}{\partial x^k} \tag{138}$$

To obtain $\frac{\partial \underset{\sim}{a}^i}{\partial x^k}$, we differentiate the relation

$$\underset{\sim}{a}^i \cdot \underset{\sim}{a}_j = \delta^i_j$$

yielding

$$\frac{\partial \underset{\sim}{a}^i}{\partial x^k} \cdot \underset{\sim}{a}_j + \underset{\sim}{a}^i \cdot \frac{\partial \underset{\sim}{a}_j}{\partial x^k} = 0 \tag{139}$$

Using (134) in (139), we get

$$\frac{\partial \underset{\sim}{a}^i}{\partial x^k} \cdot \underset{\sim}{a}_j = -\Gamma^i_{jk}$$

which implies

$$\frac{\partial \underset{\sim}{a}^i}{\partial x^k} = -\Gamma^i_{jk}\underset{\sim}{a}^j \tag{140}$$

Using (140) in (138), the derivative becomes

$$\frac{\partial \underset{\sim}{A}}{\partial x^k} = A_{i,k}\underset{\sim}{a}^i \tag{141}$$

where

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j_{ik}A_j \tag{142}$$

is called the covariant derivative of the covariant components A_i.

The comma notation will always imply *the operation of covariant* differentiation. The name "covariant" for this type of differentiation is due to the fact that the differentiated component gains one covariant index with each application. For example, in (142), the covariance of the components is now of the second order due to the covariant differentiation. Similarly, $A^i_{,k}$ is a mixed tensor, contravariant of order one, and covariant of order one due to differentiation.

Following the method described in §3.2, we can introduce a transformation from x^i to $\bar{x}^i$ in (136) and (142) to have

$$\bar{A}^i_{,n} = \frac{\partial x^j}{\partial \bar{x}^n}\frac{\partial \bar{x}^i}{\partial x^k}A^k_{,j} \tag{143}$$

and

$$\bar{A}_{i,\ell} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial x^n}{\partial \bar{x}^\ell} A_{j,n} \tag{144}$$

where

$$\bar{A}^i_{,n} = \frac{\partial \bar{A}^i}{\partial \bar{x}^n} + \bar{\Gamma}^i_{rn} \bar{A}^r \tag{145}$$

$$\bar{A}_{i,\ell} = \frac{\partial \bar{A}_i}{\partial \bar{x}^\ell} - \bar{\Gamma}^r_{i\ell} \bar{A}_r \tag{146}$$

A bar on the quantities in (145) and (146) denotes values in the new coordinate system $\bar{x}^i$. The transformation equations (143) and (144) prove that $A^i_{,k}$ and $A_{i,k}$ are tensors of the stated structures.

An important point to be noted is that on comparison of (136) with (145) and of (142) with (146), there is found no change in the forms of the covariant derivatives in changing the coordinates from x^i to $\bar{x}^i$. Thus there is no preference of one coordinate system over any other as far as the covariant differentiation is concerned. Also in the case of rectangular Cartesian coordinates, since the Christoffel symbols are zero, the covariant differentiation reduces to partial differentiation. Because of these properties, the covariant differentiation is also called "absolute differentiation."

In obtaining the partial derivatives of a tensor as an entity, we again encounter the covariant derivatives of the components in which the tensor has been expressed. Thus, for a second order tensor written in contravariant components, we have

$$\tilde{T} = T^{ij} \underset{\sim}{a}_i \underset{\sim}{a}_j$$

Thus

$$\frac{\partial \tilde{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k} \underset{\sim}{a}_i \underset{\sim}{a}_j + T^{ij}\left(\frac{\partial \underset{\sim}{a}_i}{\partial x^k} \underset{\sim}{a}_j + \underset{\sim}{a}_i \frac{\partial \underset{\sim}{a}_j}{\partial x^k}\right)$$

Using (134) and adjusting the dummy indices, we get

$$\frac{\partial \tilde{T}}{\partial x^k} = T^{ij}_{,k} \underset{\sim}{a}_i \underset{\sim}{a}_j \tag{147}$$

Similarly

$$\frac{\partial \tilde{T}}{\partial x^k} = T_{ij,k} \underset{\sim}{a}^i \underset{\sim}{a}^j \tag{148}$$

$$= T^{i}_{\cdot j,k} \underset{\sim}{a}_i \underset{\sim}{a}^j \tag{149}$$

$$= T_{j,k}^{\cdot i} \underset{\sim}{a}^j \underset{\sim}{a}_i \tag{150}$$

where the covariant derivatives are

$$T^{ij}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + \Gamma^{i}_{\ell k} T^{\ell j} + \Gamma^{j}_{\ell k} T^{i\ell} \tag{151}$$

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^{\ell}_{ik} T_{\ell j} - \Gamma^{\ell}_{jk} T_{i\ell} \tag{152}$$

$$T^{i}_{\cdot j,k} = \frac{\partial T^{i}_{\cdot j}}{\partial x^k} + \Gamma^{i}_{\ell k} T^{\ell}_{\cdot j} - \Gamma^{\ell}_{jk} T^{i}_{\cdot \ell} \tag{153}$$

$$T_{j,k}^{\cdot i} = \frac{\partial T_{j}^{\cdot i}}{\partial x^k} + \Gamma^{i}_{\ell k} T_{j}^{\cdot \ell} - \Gamma^{\ell}_{jk} T_{\ell}^{\cdot i} \tag{154}$$

Ricci's Theorem: The covariant derivatives of the metric tensor g_{ij}, g^{ij}, or δ^i_j are identically zero.

This theorem can be proved by replacing T by g in (151) and (152) and using the expression for the Christoffel symbols given in (133). Thus

$$g^{ij}_{,k} = 0 \ , \ g_{ij,k} = 0 \ , \ \delta^i_{j,k} = 0 \tag{155}$$

and the metric coefficients behave like constants under covariant differentiation. Because of this property, e.g.,

$$(g^{ij}T_{mn})_{,k} = g^{ij}T_{mn,k} \tag{156a}$$

From (155), we have

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^\ell_{ik}g_{\ell j} + \Gamma^\ell_{jk}g_{i\ell} \tag{156b}$$

$$\frac{\partial g^{ij}}{\partial x^k} = -\Gamma^i_{\ell k}g^{\ell j} - \Gamma^j_{\ell k}g^{i\ell} \tag{156c}$$

§5. Christoffel Symbols: Their Properties and Transformation Laws.

The definitions of the Christoffel symbols of the first and second kinds have already been given in (130) and (133) respectively. It must be restated that these symbols are not the components of any tensor. The transformation laws considered in this section will prove this assertion.

In taking the divergences of vectors and tensors, a contracted Christoffel symbol of the form Γ^i_{ij} appears. To find its value, we use equation (82b),

$$g = g_{\ell m} G^{\ell m}$$

where $G^{\ell m}$ is the cofactor of $g_{\ell m}$ in the determinant g. Thus

$$\frac{\partial g}{\partial x^j} = G^{\ell m} \frac{\partial g_{\ell m}}{\partial x^j} \tag{157}$$

and since

$$G^{\ell m} = g g^{\ell m} \tag{158}$$

we get

$$\frac{\partial g}{\partial x^j} = g g^{\ell m} \frac{\partial g_{\ell m}}{\partial x^j} \tag{159}$$

Now in (133) setting $\ell = i$, and summing over i while using the property that g_{ij} is symmetric, we get

$$\Gamma^i_{ij} = \frac{1}{2} g^{ik} \frac{\partial g_{ik}}{\partial x^j}$$

On using (159), we have

$$\Gamma^i_{ij} = \frac{1}{2g} \frac{\partial g}{\partial x^j} \tag{160a}$$

$$= \frac{\partial}{\partial x^j}(\ell n\sqrt{g}) \tag{160b}$$

§5.1. Transformation Laws for Christoffel Symbols.

Let x^i and $\bar{x}^i$ be two general coordinate systems. We assume that x^i

and $\overline{x}^i$ are functionally related and that the Jacobian of the transformation is not zero. Recall from (120) and (121) that the metric components transform as

$$\overline{g}_{ij} = g_{k\ell} \frac{\partial x^k}{\partial \overline{x}^i} \frac{\partial x^\ell}{\partial \overline{x}^j} \tag{161}$$

$$\overline{g}^{ij} = g^{k\ell} \frac{\partial \overline{x}^i}{\partial x^k} \frac{\partial \overline{x}^j}{\partial x^\ell} \tag{162}$$

From (130) and (133), the Christoffel symbols of the first and second kinds respectively for the coordinates $\overline{x}^i$ are

$$\overline{[ij,k]} = \frac{1}{2}\left(\frac{\partial \overline{g}_{ik}}{\partial \overline{x}^j} + \frac{\partial \overline{g}_{jk}}{\partial \overline{x}^i} - \frac{\partial \overline{g}_{ij}}{\partial \overline{x}^k}\right) \tag{163}$$

$$\overline{\Gamma}^\ell_{ij} = \overline{g}^{k\ell}\,\overline{[ij,k]} \tag{164}$$

If we now use (161) in (163) and perform the indicated differentiations, we get

$$\overline{[\ell m,n]} = [ij,k] \frac{\partial x^i}{\partial \overline{x}^\ell} \frac{\partial x^j}{\partial \overline{x}^m} \frac{\partial x^k}{\partial \overline{x}^n} + g_{ij} \frac{\partial x^i}{\partial \overline{x}^n} \frac{\partial^2 x^j}{\partial \overline{x}^\ell \partial \overline{x}^m} \tag{165}$$

Inner multiplication by $\overline{g}^{np}$ (given in (162)), gives

$$\overline{\Gamma}^p_{\ell m} = \Gamma^s_{ij} \frac{\partial \overline{x}^p}{\partial x^s} \frac{\partial x^i}{\partial \overline{x}^\ell} \frac{\partial x^j}{\partial \overline{x}^m} + \frac{\partial \overline{x}^p}{\partial x^j} \frac{\partial^2 x^j}{\partial \overline{x}^\ell \partial \overline{x}^m} \tag{166}$$

Equations (165) and (166) are the transformation laws for the Christoffel symbols. Because of the appearance of the second derivatives of the coordinates on the right of eqs. (165) and (166), the symbols do not transform like the components of any tensor. This proves that the Christoffel symbols are not tensors.

A formula expressing the second derivatives of coordinates can be obtained from (166). On taking the inner multiplication of (166) by $\frac{\partial x^r}{\partial \bar{x}^p}$, we get

$$\frac{\partial^2 x^r}{\partial \bar{x}^\ell \partial \bar{x}^m} = \bar{\Gamma}^p_{\ell m} \frac{\partial x^r}{\partial \bar{x}^p} - \Gamma^r_{ij} \frac{\partial x^i}{\partial \bar{x}^\ell} \frac{\partial x^j}{\partial \bar{x}^m} \tag{167}$$

§5.1.1. Formulae: Cartesian to Curvilinear and Vice Versa.

All the preceding formulae are applicable for any space and for any two general coordinate systems. In engineering applications, we usually transform from a rectangular Cartesian to a curvilinear and vice versa. We consider two cases.

(i) x^i are Cartesian and $\bar{x}^i$ curvilinear.

(ii) $\bar{x}^i$ are Cartesian and x^i curvilinear.

<u>Case (i)</u>:

If x^i are Cartesian, we denote x^i as decided earlier, by x_i. For this case

$$g_{k\ell} = g^{k\ell} = \delta_{k\ell}$$

so that (168)

$$\Gamma^i_{jk} \equiv 0$$

For brevity of notation, we denote $\bar{x}^i = \xi^i$, and also remove the over bar from the quantities in ξ^i. Thus

$$g_{ij} = \frac{\partial x_k}{\partial \xi^i} \frac{\partial x_k}{\partial \xi^j} \quad \text{(sum on k)} \tag{169}$$

$$g^{ij} = \frac{\partial \xi^i}{\partial x_k} \frac{\partial \xi^j}{\partial x_k} \tag{170}$$

To find the partial derivatives of the curvilinear coordinates with respect to the Cartesian, take the inner multiplication of (170) with $\frac{\partial x_r}{\partial \xi^j}$. Thus

$$\frac{\partial \xi^i}{\partial x_r} = g^{ij} \frac{\partial x_r}{\partial \xi^j} \tag{171}$$

Recall, from (94) that

$$g_{ij} g^{mj} = \delta^m_i \tag{172}$$

From (167), the second partial derivatives are given by

$$\frac{\partial^2 x_r}{\partial \xi^\ell \partial \xi^m} = \Gamma^p_{\ell m} \frac{\partial x_r}{\partial \xi^p} \tag{173}$$

Inner multiplication of (173) with $\frac{\partial \xi^n}{\partial x_r}$ yields the formula for the Christoffel symbols in terms of the second derivatives of the Cartesian with respect to the curvilinear coordinates

$$\Gamma^r_{ij} = \frac{\partial \xi^r}{\partial x_s} \frac{\partial^2 x_s}{\partial \xi^i \partial \xi^j} \tag{174a}$$

$$= g^{rt} \frac{\partial x_s}{\partial \xi^t} \frac{\partial^2 x_s}{\partial \xi^i \partial \xi^j} \quad \text{(sum on s and t)} \tag{174b}$$

As an application of the preceding results in a two-dimensional plane in which the Cartesian coordinates are $x_1 = x$, $y_1 = y$ and the introduced curvilinear coordinates are $\xi^1 = \xi$, $\xi^2 = \eta$, we have the following formulae. In all the formulae given below, a variable subscript denotes a partial derivative.

$$g_{11} = x_\xi^2 + y_\xi^2 \ , \ g_{12} = x_\xi x_\eta + y_\xi y_\eta \ , \ g_{22} = x_\eta^2 + y_\eta^2$$

$$g^{11} = \xi_x^2 + \xi_y^2 \ , \ g^{12} = \xi_x \eta_x + \xi_y \eta_y \ , \ g^{22} = \eta_x^2 + \eta_y^2 \tag{175}$$

$$g^{11} = \frac{g_{22}}{g} \ , \ g^{12} = \frac{-g_{12}}{g} \ , \ g^{22} = \frac{g_{11}}{g}$$

where

$$g = g_{11}g_{22} - (g_{12})^2 = (x_\xi y_\eta - x_\eta y_\xi)^2 \tag{176a}$$

$$= (\xi_x \eta_y - \eta_x \xi_y)^{-2} \tag{176b}$$

$$\xi_x = y_\eta/\sqrt{g} \ , \ \xi_y = -x_\eta/\sqrt{g} \ , \ \eta_x = -y_\xi/\sqrt{g} \ , \ \eta_y = x_\xi/\sqrt{g} \tag{177}$$

$$\Gamma_{11}^1 = [g_{22} \frac{\partial g_{11}}{\partial \xi} + g_{12}(\frac{\partial g_{11}}{\partial \eta} - 2 \frac{\partial g_{12}}{\partial \xi})]/2g \tag{178a}$$

$$= (y_\eta x_{\xi\xi} - x_\eta y_{\xi\xi})/\sqrt{g} \tag{178b}$$

$$\Gamma_{22}^2 = [g_{11} \frac{\partial g_{22}}{\partial \eta} + g_{12}(\frac{\partial g_{22}}{\partial \xi} - 2 \frac{\partial g_{12}}{\partial \eta})]/2g \tag{179a}$$

$$= (x_\xi y_{\eta\eta} - y_\xi x_{\eta\eta})/\sqrt{g} \tag{179b}$$

$$\Gamma^1_{22} = [g_{22}(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) - g_{12}\frac{\partial g_{22}}{\partial \eta}]/2g \tag{180a}$$

$$= (y_\eta x_{\eta\eta} - x_\eta y_{\eta\eta})/\sqrt{g} \tag{180b}$$

$$\Gamma^2_{11} = [g_{11}(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) - g_{12}\frac{\partial g_{11}}{\partial \xi}]/2g \tag{181a}$$

$$= (x_\xi y_{\xi\xi} - y_\xi x_{\xi\xi})/\sqrt{g} \tag{181b}$$

$$\Gamma^1_{12} = \Gamma^1_{21}$$

$$= (g_{22}\frac{\partial g_{11}}{\partial \eta} - g_{12}\frac{\partial g_{22}}{\partial \xi})/2g \tag{182a}$$

$$= (y_\eta x_{\xi\eta} - x_\eta y_{\xi\eta})/\sqrt{g} \tag{182b}$$

$$\Gamma^2_{12} = \Gamma^2_{21}$$

$$= (g_{11}\frac{\partial g_{22}}{\partial \xi} - g_{12}\frac{\partial g_{11}}{\partial \eta})/2g \tag{183a}$$

$$= (x_\xi y_{\xi\eta} - y_\xi x_{\xi\eta})/\sqrt{g} \tag{183b}$$

Also

$$\Gamma^1_{11} + \Gamma^2_{12} = \frac{1}{2g}\frac{\partial g}{\partial \xi} \tag{184a}$$

$$\Gamma^1_{12} + \Gamma^2_{22} = \frac{1}{2g}\frac{\partial g}{\partial \eta} \tag{184b}$$

Case (ii):

If we treat $\bar{x}^i$ as the Cartesian coordinates, then we denote them as x_i. Also writing $x^i = \xi^i$ in (167) we get

$$\frac{\partial^2 \xi^r}{\partial x_\ell \partial x_m} = -\Gamma^r_{ij} \frac{\partial \xi^i}{\partial x_\ell} \frac{\partial \xi^j}{\partial x_m} \tag{185}$$

On using (171), we can write (185) as

$$\frac{\partial^2 \xi^r}{\partial x_\ell \partial x_m} = -\Gamma^r_{ij} g^{jq} \frac{\partial \xi^i}{\partial x_\ell} \frac{\partial x_m}{\partial \xi^q} \tag{186a}$$

or

$$= -\Gamma^r_{ij} g^{ip} g^{jq} \frac{\partial x_\ell}{\partial \xi^p} \frac{\partial x_m}{\partial \xi^q} \tag{186b}$$

Equation (186b) expresses the second partial derivatives of general coordinates in terms of the first partial derivatives of the Cartesian with respect to the general coordinates. Equation (186a) is suitable for obtaining the Laplacian of the general coordinates. For, on contracting the indices ℓ and m, viz., setting ℓ = m and performing the sum on m, we get

$$\begin{aligned} \nabla^2 \xi^r &= -\Gamma^r_{ij} g^{jq} \delta^i_q \\ &= -g^{ij} \Gamma^r_{ij} \end{aligned} \tag{187}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_m \partial x_m} \quad \text{(sum on m)}.$$

Thus in two dimensions, writing $x^1 = x$, $x^2 = y$, $\xi^1 = \xi$, $\xi^2 = \eta$ we have

$$\nabla^2\xi = (2g_{12}\Gamma^1_{12} - g_{11}\Gamma^1_{22} - g_{22}\Gamma^1_{11})/g \tag{188}$$

$$\nabla^2\eta = (2g_{12}\Gamma^2_{12} - g_{11}\Gamma^2_{22} - g_{22}\Gamma^2_{11})/g \tag{189}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ .$$

A second order differential operator defined as

$$D = g_{22}\frac{\partial^2}{\partial\xi^2} - 2g_{12}\frac{\partial^2}{\partial\xi\partial\eta} + g_{11}\frac{\partial^2}{\partial\eta^2} \tag{190}$$

and the use of eqs. (178)-(181) yields another form of the Laplacians,

$$\nabla^2\xi = (x_\eta Dy - y_\eta Dx)/g^{3/2} \tag{191}$$

$$\nabla^2\eta = (y_\xi Dx - x_\xi Dy)/g^{3/2} \tag{192}$$

§6. Gradient, Divergence, Curl, and Laplacian

(i) Scalars:

There are two types of scalar quantities. One is called an absolute scalar or an invariant, while the other is called a scalar density.

Any function of the coordinates x^i is called an absolute scalar if on coordinate transformation from x^i to $\bar{x}^i$ the value of ϕ does not change. Thus

$$\phi(x^1,x^2,x^3) = \bar{\phi}(\bar{x}^1,\bar{x}^2,\bar{x}^3) \tag{193}$$

There are scalars which on coordinate transformation do not transform like (193). As an example the function g, viz.,

$$g = \det(g_{ij})$$

is not an absolute scalar. On coordinate transformation

$$\bar{g} = \det(\bar{g}_{ij})$$

On actual substitution of (161) in the above determinant and by expansion, we obtain

$$\bar{g} = (J)^2 g \tag{194}$$

Similarly

$$g = (\bar{J})^2 \bar{g} \tag{195}$$

where

$$J = \det\left(\frac{\partial x^i}{\partial \bar{x}^j}\right) \quad , \quad \bar{J} = \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \tag{196}$$

Thus g or $\sqrt{g}$ is not an absolute scalar, its value in some other coordinate system is given by $J\sqrt{g}$.

Multiplying the absolute scalars, vectors, or tensors, by $\sqrt{g}$ we get the corresponding densities.

In §1.6 we have already defined the operator $\underset{\sim}{\nabla}$ or grad as

$$\text{grad} = \underset{\sim}{\nabla} = \underset{\sim}{a}^i \frac{\partial}{\partial x^i}$$

If ϕ is an absolute scalar, then grad ϕ is a vector given as

$$\text{grad}\ \phi = \underset{\sim}{a}^i \frac{\partial \phi}{\partial x^i}$$

so that the covariant components of grad ϕ are

$$(\text{grad}\ \phi)_i = \frac{\partial \phi}{\partial x^i} = \phi_{,i} \tag{198}$$

Using the method of raising an index (cf. §3), the contravariant components of grad ϕ are

$$(\text{grad}\ \phi)^i = g^{ik} \frac{\partial \phi}{\partial x^k} \tag{199}$$

(ii) Vectors:

The divergence of a vector $\underset{\sim}{v}$ was defined in (44) as

$$\text{div}\ \underset{\sim}{v} = \underset{\sim}{a}^i \cdot \frac{\partial \underset{\sim}{v}}{\partial x^i}$$

On using (137), we get

$$\text{div}\ \underset{\sim}{v} = v^j_{,i}(\underset{\sim}{a}^i \cdot \underset{\sim}{a}_j)$$

$$= v^j_{,i}\delta^i_j = v^i_{,i} \tag{200}$$

From (136)

$$v^i_{,i} = \frac{\partial v^i}{\partial x^i} + \Gamma^i_{ij} v^j$$

so that on using (160b),

$$\operatorname{div} \underset{\sim}{v} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g}\, v^i) \tag{201}$$

which is a scalar. Another form is obtained by using (141), which gives

$$\operatorname{div} \underset{\sim}{v} = A_{i,k} \underset{\sim}{a}^i \cdot \underset{\sim}{a}^k$$

$$= g^{ik} A_{i,k} \; . \tag{202}$$

The gradient of a vector $\underset{\sim}{v}$ appears quite often in fluid and solid mechanics. In §1.6.1, we decided to choose the definition of grad $\underset{\sim}{v}$ as

$$\operatorname{grad} \underset{\sim}{v} = \frac{\partial \underset{\sim}{v}}{\partial x^i} \underset{\sim}{a}^i \tag{203}$$

and that of its conjugate as

$$(\operatorname{grad} \underset{\sim}{v})^* = \underset{\sim}{a}^i \frac{\partial \underset{\sim}{v}}{\partial x^i} \tag{204}$$

Using the expressions for $\frac{\partial \underset{\sim}{v}}{\partial x^i}$ from §4 we can write it in the following forms

$$\operatorname{grad} \underset{\sim}{v} = v^i_{,k} \underset{\sim}{a}_i \underset{\sim}{a}^k \text{ , mixed components.} \tag{205a}$$

$$= g_{ij} v^i_{,k} \underset{\sim}{a}^j \underset{\sim}{a}^k \text{ , covariant components.} \tag{205b}$$

$$= v_{i,k} \underset{\sim}{a}^i \underset{\sim}{a}^k \text{ , covariant components.} \tag{205c}$$

$$= g^{kj} v^i_{,k} \underset{\sim}{a}_i \underset{\sim}{a}_j \text{ , contravariant components.} \tag{205d}$$

In the same manner (grad $\underset{\sim}{v}$)* can be written. For example, one useful representation is

$$(\text{grad } \underset{\sim}{v})^* = v^i_{,k} \underset{\sim}{a}^k \underset{\sim}{a}_i \tag{206}$$

In mechanics, we sometimes need the inner products of grad $\underset{\sim}{v}$. Using the definition (29), we easily obtain

$$(\text{grad } \underset{\sim}{v}) : (\text{grad } \underset{\sim}{v}) = v^i_{,k} v^k_{,i} \tag{207}$$

$$(\text{grad } \underset{\sim}{v}) : (\text{grad } \underset{\sim}{v})^* = g_{im} g^{kn} v^i_{,k} v^m_{,n} \tag{208}$$

where both are scalars.

The curl of a vector $\underset{\sim}{v}$ is defined in the usual way.

$$\text{curl } \underset{\sim}{v} = \underset{\sim}{a}^j \times \frac{\partial \underset{\sim}{v}}{\partial x^j}$$

$$= v_{k,j} (\underset{\sim}{a}^j \times \underset{\sim}{a}^k)$$

Using (90b), we have

$$\text{curl } \underset{\sim}{v} = \frac{1}{\sqrt{g}} e^{ijk} v_{k,j} \underset{\sim}{a}_i \tag{209}$$

Thus the contravariant components of curl $\underset{\sim}{v}$ are given by

$$(\text{curl } \underset{\sim}{v})^i = \frac{1}{\sqrt{g}} \left(\frac{\partial v_k}{\partial x^j} - \frac{\partial v_j}{\partial x^k} \right) \tag{210}$$

where i, j, k are in the cyclic permutations of 1, 2, 3, in this order.

(iii) Tensors:

The divergence of a tensor of second order has been defined as (cf. eq. (52))

$$\text{div}\ \tilde{T} = \frac{\partial \tilde{T}}{\partial x^k} \cdot \underset{\sim}{a}^k$$

Using the derivatives given in (147)-(149) we obtain

$$\text{div}\ \tilde{T} = T^{ik}_{,k}\underset{\sim}{a}_i \tag{211a}$$

$$= g^{jk}T_{ij,k}\underset{\sim}{a}^i \tag{211b}$$

$$= g^{jk}T^{i}_{\cdot j,k}\underset{\sim}{a}_i \tag{211c}$$

Thus div $\tilde{T}$ is a vector whose contravariant components are given by (211a, c) and the covariant components are given by (211b). The operation of divergence thus reduces the order of the tensor by one.

For a divergence-free tensor

$$T^{ik}_{,k} = 0 \tag{212a}$$

or,

$$g^{jk}T_{ij,k} = 0 \tag{212b}$$

or

$$g^{jk}T^{i}_{\cdot j,k} = 0 \tag{212c}$$

If a tensor is such that

$$T^{i}_{\cdot j,i} = 0 \tag{212d}$$

then it is called a covariant divergence-free tensor.

(iv) Laplacian of a scalar:

The Laplacian of an absolute scalar ϕ can now be obtained by first using the formula for the covariant derivative of covariant components of a vector, viz., (142), to have

$$(\phi_{,i})_{,j} = \frac{\partial^2\phi}{\partial x^i \partial x^j} - \Gamma^r_{ij}\frac{\partial\phi}{\partial x^r} \tag{213}$$

Since $\Gamma^r_{ij} = \Gamma^r_{ji}$, hence

$$(\phi_{,i})_{,j} = (\phi_{,j})_{,i}$$

that is, the covariant differentiation of absolute scalars is commutative.

Having obtained the covariant derivative, we now obtain the Laplacian using (202),

$$\nabla^2\phi = \operatorname{div}(\operatorname{grad}\phi) = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \Gamma^r_{ij}\frac{\partial\phi}{\partial x^r}\right) \tag{214}$$

It is easy to verify that if ϕ is a curvilinear coordinate, x^m, then

$$\nabla^2 x^m = -g^{ij}\Gamma^m_{ij}$$

which was obtained earlier by another method (cf. (187)). Similarly, if ϕ is a Cartesian coordinate, x_n, then

$$g^{ij}\left(\frac{\partial^2 x_n}{\partial x^i \partial x^j} - \Gamma^r_{ij} \frac{\partial x_n}{\partial x^r}\right) = 0$$

or

$$g^{ij} \frac{\partial^2 x_n}{\partial x^i \partial x^j} + \frac{\partial x_n}{\partial x^r} \nabla^2 x^r = 0 \tag{215}$$

As an example, consider the Cartesian coordinates $x_1 = x$, $x_2 = y$ in a plane, and let $x^1 = \xi$, $x^2 = \eta$ be the curvilinear coordinates. Then introducing the operator D defined in (190), the equations for x and y as dependent variables are

$$\begin{aligned} Dx &= -g(x_\xi \nabla^2 \xi + x_\eta \nabla^2 \eta) \\ Dy &= -g(y_\xi \nabla^2 \xi + y_\eta \nabla^2 \eta) \end{aligned} \tag{216}$$

Equations (216) have been used in Ref. [29] to compute the coordinates for arbitrary shaped two-dimensional bodies.

§7. Miscellaneous Derivations.

In this section we consider a few derivations which are used in the study of geometry and mechanics.

(i) Intrinsic derivative:

Let $x^k = x^k(t)$ be the parametric equations of a space curve with t as a parameter. A vector function $\underset{\sim}{u}$ of position will then also be a function of t on this curve, viz.,

$$\underset{\sim}{u} = \underset{\sim}{u}(x^k(t))$$

The intrinsic derivative, also called the total or substantive derivative, of $\underset{\sim}{u}$ with respect to t is defined as $\frac{d\underset{\sim}{u}}{dt}$. Writing

$$\underset{\sim}{u} = u^i \underset{\sim}{a}_i$$

we get

$$\frac{d\underset{\sim}{u}}{dt} = \frac{du^i}{dt} \underset{\sim}{a}_i + u^i \frac{d\underset{\sim}{a}_i}{dt}$$

$$= \frac{du^i}{dt} \underset{\sim}{a}_i + u^i \frac{\partial \underset{\sim}{a}_i}{\partial x^j} \frac{dx^j}{dt}$$

$$= \left(\frac{du^i}{dt} + u^r \Gamma^i_{rj} \frac{dx^j}{dt}\right) \underset{\sim}{a}_i$$

$$= \frac{\delta u^i}{\delta t} \underset{\sim}{a}_i$$

The quantity

$$\frac{\delta u^i}{\delta t} = \frac{du^i}{dt} + u^r \Gamma^i_{rj} \frac{dx^j}{dt} \tag{217}$$

is called the intrinsic derivative. We can also write (217) as

$$\frac{\delta u^i}{\delta t} = \frac{\partial u^i}{\partial x^j}\frac{dx^j}{dt} + u^r \Gamma^i_{rj}\frac{dx^j}{dt}$$

$$= u^i_{,j}\frac{dx^j}{dt} \tag{218}$$

As an example, in fluid mechanics, the velocity vector $\underset{\sim}{u}$ is defined as

$$\underset{\sim}{u} = \frac{d\underset{\sim}{r}}{dt}$$

where t is the absolute time. Since $d\underset{\sim}{r} = \underset{\sim}{a}_i dx^i$, so that

$$\underset{\sim}{u} = \underset{\sim}{a}_i \frac{dx^i}{dt}$$

$$= \underset{\sim}{a}_i u^i$$

Thus $u^i = \frac{dx^i}{dt}$ are the contravariant components of $\underset{\sim}{u}$. The components u^i can also be explicit functions of t beside being implicit, but $\underset{\sim}{a}_i$ are only implicit functions of t. Thus

$$\frac{d\underset{\sim}{u}}{dt} = \frac{du^i}{dt}\,\underset{\sim}{a}_i + u^i\,\frac{d\underset{\sim}{a}_i}{dt}$$

$$= \left(\frac{\partial u^i}{\partial t} + u^j\,\frac{\partial u^i}{\partial x^j}\right)\underset{\sim}{a}_i + u^i\Gamma^r_{ij}u^j\underset{\sim}{a}_r$$

$$= \left(\frac{\partial u^i}{\partial t} + u^j u^i_{,j}\right)\underset{\sim}{a}_i \tag{219}$$

which is the well known substantive derivative defining the acceleration vector.

The intrinsic derivative of a tensor of any structure can be found by using the method followed in obtaining (218). Thus for a tensor $\tilde{T}$,

$$\frac{d\tilde{T}}{dt} = \frac{\delta}{\delta t}\,T^{u_1u_2\cdots u_s}_{r_1r_2\cdots r_p}\,\underset{\sim}{a}_{u_1}\cdots\underset{\sim}{a}_{u_s}\,\underset{\sim}{a}^{r_1}\cdots\underset{\sim}{a}^{r_p}$$

Higher order intrinsic derivatives can be obtained in a straight forward manner. Thus

$$\frac{\delta^2 u^i}{\delta t^2} = \frac{\delta}{\delta t}\left(\frac{\delta u^i}{\delta t}\right) = \left(u^i_{,j}\,\frac{dx^j}{dt}\right)_{,k}\frac{dx^k}{dt}\ .$$

Intrinsic differentiation in general is not commutative. Other uses of intrinsic derivative are in the definitions of a parallel field of vectors and of the geodesic curves in space.

A field of vectors $\underset{\sim}{A}$ along a curve $x^i = x^i(t)$ and in any space are called parallel if at all points of this curve where

$$\underset{\sim}{A} = \underset{\sim}{A}(x^i(t))$$

we have

$$\frac{d\underset{\sim}{A}}{dt} = 0 \tag{220a}$$

Thus in an Euclidean space, the meaning of eq. (220a) is that the components of $\underset{\sim}{A}$ referred to the rectangular Cartesian coordinates are constants.

For arbitrary coordinates and in any space, a field of vectors is called parallel when

$$\frac{d\underset{\sim}{A}}{dt} = \frac{\delta A^i}{\delta t} \underset{\sim}{a}_i = 0$$

Thus, the parallel field of vectors satisfy the equation

$$\frac{\delta A^i}{\delta t} = \frac{dA^i}{dt} + A^r \Gamma^i_{rj} \frac{dx^j}{dt} = 0 \tag{220b}$$

Equation (220b) forms a system of N equations in an N-dimensional space and can be solved by specifying A^i at an initial point $t = t_0$, [15].

The geodesics or the geodesic curves of a space are the curves along which the distance between two points is minimal. Let s be the arc length along a curve, then we define the unit tangent vector field $\underset{\sim}{t}(s)$ whose contravariant components are given by

$$\underset{\sim}{t}(s) = \frac{dx^i}{ds} \underset{\sim}{a}_i$$

The field $\underset{\sim}{t}(s)$ is said to be a tangent vector field on a geodesic when

$$\frac{d\underset{\sim}{t}}{ds} = 0 \tag{221a}$$

Using again the definition (217), we obtain the equations for geodesics as

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{rj} \frac{dx^r}{ds} \frac{dx^j}{ds} = 0 \tag{221b}$$

Thus the geodesics are the solution of the second order equations (221b).

(ii) <u>Magnitude of a vector</u>:

The magnitude of a vector $\underset{\sim}{u}$ is a scalar. This magnitude is obtained by taking the scalar product of $\underset{\sim}{u}$ with itself. Thus

$$(u)^2 = \underset{\sim}{u} \cdot \underset{\sim}{u}$$

$$= (\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j) u^i u^j$$

$$= g_{ij} u^i u^j \tag{222a}$$

Also

$$(u)^2 = g^{ij} u_i u_j \tag{222b}$$

$$= u^j u_j \tag{222c}$$

In the same manner, the magnitude of grad ϕ denoted as $|\text{grad } \phi|$ is given by

$$|\text{grad } \phi|^2 = g^{ik} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} \tag{223}$$

In two dimensions, writing $x^1 = \xi$, $x^2 = \eta$, we have

$$|\mathrm{grad}\ \phi|^2 = [g_{22}(\phi_\xi)^2 - 2g_{12}\phi_\xi\phi_\eta + g_{11}(\phi_\eta)^2]/g \tag{224}$$

(iii) <u>Angle between two vectors</u>:

The angle θ between the two vectors

$$\underset{\sim}{u} = u^i \underset{\sim}{a}_i$$

and

$$\underset{\sim}{v} = v^j \underset{\sim}{a}_j$$

is given by

$$\cos\theta = g_{ij}u^i v^j / \sqrt{(u^k u_k)(v^\ell v_\ell)} \tag{225}$$

The two vectors are orthogonal if

$$g_{ij}u^i v^j = 0 \ .$$

The angle between any two coordinate curves at a point is given by the base vectors corresponding to these curves. Thus the angle θ_{ij} between the curves x^i and x^j is given by

$$\cos\theta_{ij} = (\underset{\sim}{a}_i \cdot \underset{\sim}{a}_j)/\sqrt{|\underset{\sim}{a}_i||\underset{\sim}{a}_j|}$$

$$= g_{ij}/\sqrt{g_{ii}g_{jj}} \tag{226}$$

where, since i and j are fixed numbers, there is no implicit summation on repeated indices. If x^i and x^j are orthogonal, then $g_{ij} = 0$ for $i \neq j$.

(iv) Cross product of vectors:

For the cross product of two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$, the use of eqs. (90) yields the result

$$\underset{\sim}{u} \times \underset{\sim}{v} = \sqrt{g}\, e_{ijk} u^j v^k \underset{\sim}{a}^i \tag{227a}$$

$$= \frac{1}{\sqrt{g}} e^{ijk} u_j v_k \underset{\sim}{a}_i \tag{227b}$$

giving the covariant and contravariant components respectively.

(v) Physical components of a vector:

In a three-dimensional space if all the coordinates are orthogonal, then as noted in (226)

$$g_{ij} = 0 \quad \text{for} \quad i \neq j$$

and the non-zero terms are g_{11}, g_{22}, g_{33}. It is customary to use the notation

$$\begin{aligned} h_1^2 &= g_{11} = \frac{1}{g^{11}} \\ h_2^2 &= g_{22} = \frac{1}{g^{22}} \\ h_3^2 &= g_{33} = \frac{1}{g^{33}} \end{aligned} \tag{228}$$

and

$$g = g_{11} g_{22} g_{33} = (h_1 h_2 h_3)^2 \tag{229}$$

The covariant and the contravariant components of a vector $\underset{\sim}{v}$ referred to the orthogonal coordinates are then related as

$$v_1 = h_1^2 v^1 \ , \ v_2 = h_2^2 v^2 \ , \ v_3 = h_3^2 v_3 \tag{230}$$

The physical components of the vector $\underset{\sim}{v}$ are the orthogonal projections of the vector on the coordinate axes. Denoting these components by V_i (subscript i is just a label), we get

$$\begin{aligned} V_1 &= h_1 v^1 = v_1/h_1 \\ V_2 &= h_2 v^2 = v_2/h_2 \\ V_3 &= h_3 v^3 = v_3/h_3 \end{aligned} \tag{231}$$

The magnitude of the velocity vector $\underset{\sim}{v}$ is then simply

$$\begin{aligned} |\underset{\sim}{v}|^2 &= V_1{}^2 + V_2{}^2 + V_3{}^2 \\ &= v_1 v^1 + v_2 v^2 + v_3 v^3 \end{aligned} \tag{232}$$

(vi) <u>Arc lengths, elements of area and volume</u>:

In any coordinate system (orthogonal or non-orthogonal) the arc lengths are easily obtained by the metric equation

$$(ds)^2 = g_{ij} dx^i dx^j$$

Thus for i, j, and k as fixed numbers, we have

$$(ds)_{\substack{x^j = \text{const.} \\ x^k = \text{const.}}} = \sqrt{g_{ii}}\, dx^i \quad \text{(no summation)} \tag{233}$$

is an arc length along the x^i curve.

Similarly denoting the element of area on which the curve x^i = const. as $d\sigma_i$, we have

$$d\sigma_1 = |\underset{\sim}{a}_2 dx^2 \times \underset{\sim}{a}_3 dx^3|$$

$$= [g_{22}g_{33} - (g_{23})^2]^{1/2} dx^2 dx^3 \tag{234a}$$

$$d\sigma_2 = |\underset{\sim}{a}_3 dx^3 \times \underset{\sim}{a}_1 dx^1|$$

$$= [g_{11}g_{33} - (g_{13})^2]^{1/2} dx^1 dx^3 \tag{234b}$$

$$d\sigma_3 = |\underset{\sim}{a}_1 dx^1 \times \underset{\sim}{a}_2 dx^2|$$

$$= [g_{11}g_{22} - (g_{12})^2]^{1/2} dx^1 dx^2 \tag{234c}$$

The element of volume is

$$dV = \underset{\sim}{a}_1 \cdot (\underset{\sim}{a}_2 \times \underset{\sim}{a}_3) dx^1 dx^2 dx^3$$

$$= \sqrt{g}\, dx^1 dx^2 dx^3 \ . \tag{235}$$

§8. The Curvature Tensor and Its Implications.

Questions regarding the nature of spaces have been raised and discussed, mostly by philosophers, at different stages of human civilization.

A definitive philosophic work on this subject was published by Immanuel Kant in the "Critique of Pure Reason" in 1787. Despite a work of such brilliance, the description of space remained shrouded in mystery and abstract formalisms. Scientific answers to the questions regarding space started emerging after the works of Gauss and Riemann in the first half of the nineteenth century. In this section we shall try to define a space and its structure through analytic constructions as simply as possible. The material of this section supplements the discussions of §2.

After gaining a working knowledge of basic tensor rules and particularly after having the metric equation (74) at our disposal, we now pose the following simple problem. "Is it possible to devise a coordinate system $x^1 = \xi$, $x^2 = \eta$ in a two-dimensional plane such that the element of length between two infinitesimally close points be given by the metric

$$(ds)^2 = (d\xi)^2 + (\cos^2\xi)(d\eta)^2 \quad ?\text{"} \tag{236}$$

In essence, the problem is to find whether in a two-dimensional plane can we have $g_{11} = 1$ and $g_{22} = \cos^2\xi$?

The answer to the above question is that we can never introduce the above metric in a plane. In fact, as we shall see later, this metric suits the surface of a sphere which is a curved two-dimensional space. Recall that in a two-dimensional plane we can introduce Cartesian, and plenty of orthogonal, or non-orthogonal curvilinear coordinate systems. Each chosen coordinate system yields a specific set of the functions g_{ij}.

For example in a plane:

$g_{11} = 1$, $g_{12} = 0$, $g_{22} = 1$, for Cartesian coordinates.

$g_{11} = 1$, $g_{12} = 0$, $g_{22} = (\xi)^2$, for polar coordinates.

$g_{11} = 1$, $g_{12} = -2\cos\alpha$, $g_{22} = 1$, for oblique rectilinear coordinates with α as the included angle between the coordinates. We can go on adding to the above list, but the g_{ij}'s of (236) are forbidden. These considerations suggest that there must be a condition or a set of conditions on the g_{ij}'s which must be satisfied in each specific space. To get started on this problem, we proceed as follows.

Let $\underset{\sim}{A}$ be an arbitrary vector and x^i a coordinate system in our chosen space whose structure we wish to study. We have the result from (141) that the partial derivatives of the entity $\underset{\sim}{A}$ can be expressed in terms of the covariant derivatives as

$$\frac{\partial \underset{\sim}{A}}{\partial x^n} = A_{j,n}\underset{\sim}{a}^j$$

Differentiating partially once more with respect to x^m and using (140), we obtain

$$\frac{\partial^2 \underset{\sim}{A}}{\partial x^m \partial x^n} = \left(\frac{\partial A_{j,n}}{\partial x^m} - \Gamma^p_{mj}A_{p,n}\right)\underset{\sim}{a}^j$$

$$= (A_{j,n})_{,m}\underset{\sim}{a}^j + \Gamma^{\ell}_{mn}A_{j,\ell}\underset{\sim}{a}^j \tag{237}$$

where a comma, as before, denotes covariant differentiation.

Proceeding again from

$$\frac{\partial \underset{\sim}{A}}{\partial x^m} = A_{j,m} \underset{\sim}{a}^j$$

we obtain

$$\frac{\partial^2 \underset{\sim}{A}}{\partial x^n \partial x^m} = (A_{j,m})_{,n} \underset{\sim}{a}^j + \Gamma^{\ell}_{mn} A_{j,\ell} \underset{\sim}{a}^j \tag{238}$$

Subtracting (238) from (237), we get

$$\frac{\partial^2 \underset{\sim}{A}}{\partial x^m \partial x^n} - \frac{\partial^2 \underset{\sim}{A}}{\partial x^n \partial x^m} = R^{\ell}_{\cdot jnm} A_{\ell} \underset{\sim}{a}^j \tag{239}$$

where

$$R^{\ell}_{\cdot jnm} = \frac{\partial}{\partial x^n} \Gamma^{\ell}_{jm} - \frac{\partial}{\partial x^m} \Gamma^{\ell}_{jn} + \Gamma^{\ell}_{ns} \Gamma^{s}_{jm} - \Gamma^{\ell}_{ms} \Gamma^{s}_{jn} \tag{240}$$

It is a direct algebraic problem to show, using eq. (152), that

$$(A_{j,n})_{,m} - (A_{j,m})_{,n} = R^{\ell}_{\cdot jnm} A_{\ell} \tag{241}$$

The structure of the quantities $R^{\ell}_{\cdot jnm}$ shows that they are the components of a fourth order tensor, covariant of order three and contravariant of order one. This tensor is known as the Riemann-Christoffel tensor. It is formed of Γ^{i}_{jk} and their first partial derivatives. In turn we may state that the Riemann-Christoffel tensor is formed purely of the metric coefficients and their first and second partial derivatives.

From (241) we conclude that the covariant differentiation in a space is commutative provided that

$$R^{\ell}_{\cdot jnm} = 0 \tag{242}$$

for all values of its indices and for all coordinate systems introduced in the chosen space.

Suppose in the chosen space it is possible for us to introduce a set of rectangular Cartesian coordinates on a global scale. The metric tensor components are then the Kronecker deltas δ_{ij} whose values are either one or zero. Thus their partial derivatives and so also all the Christoffel symbols Γ^{i}_{jk} are identically zero. The vanishing of all the Christoffel symbols makes (240) zero and so eq. (242) is satisfied. The vanishing of a tensor (here $R^{\ell}_{\cdot jnm}$) in one coordinate system means that all its components should remain zero in any other coordinate system introduced in the same space. (Refer to the three properties of a tensor expressions in §3.2; listed after eq. (121)). It must be noted that when the coordinates are not rectangular Cartesian then all the g_{ij} and also the Γ^{i}_{jk} are functions of the coordinates. Nevertheless, eq. (242) will still remain valid. Spaces in which eq. (242) remains valid are called Euclidean. Such spaces are also called flat because as will be seen shortly, the tensor $R^{\ell}_{\cdot jnm}$ determines the curvature of the curved space. Spaces for which eq. (242) is not satisfied are called Riemannian or non-Euclidean.

It is now obvious why pure reasoning fails to provide a classification of spaces. The idea of a curved space is implicit not only in the values of the metric g_{ij} but also in their distributions (derivatives). Admittedly, the whole burden of our results depends on one axiom, viz., the axiom of the Riemannian metric, eq. (74). However, various physical experiences such as Einstein's theory of gravitation, and the consistency of the derived results forces one to accept the validity of the axiom of Riemann.

§8.1. Algebra of the Curvature Tensor.

From (240) it is obvious that

$$R^{\ell}_{\cdot jnm} = -R^{\ell}_{\cdot jmn} \tag{243a}$$

$$R^{\ell}_{\cdot jnm} + R^{\ell}_{\cdot nmj} + R^{\ell}_{\cdot mjn} = 0 \tag{243b}$$

and

$$R^{\ell}_{\cdot \ell mn} = 0 \tag{243c}$$

A fourth order tensor is now formed by contracting the upper index as

$$R_{rjnp} = g_{r\ell}R^{\ell}_{\cdot jnp} \tag{244}$$

The tensor R_{rjnp} is called the covariant Riemann curvature tensor. It can be represented in the following three alternative forms, [11].

$$\begin{aligned} R_{rjnp} = {} & \frac{\partial}{\partial x^n}[g_{r\ell}\Gamma^{\ell}_{jp}] - \frac{\partial}{\partial x^p}[g_{r\ell}\Gamma^{\ell}_{jn}] - \Gamma^{\ell}_{jp}\frac{\partial g_{r\ell}}{\partial x^n} \\ & + \Gamma^{\ell}_{jn}\frac{\partial g_{r\ell}}{\partial x^p} + g_{r\ell}\Gamma^{\ell}_{ns}\Gamma^{s}_{jp} - g_{r\ell}\Gamma^{\ell}_{ps}\Gamma^{s}_{jn} \end{aligned} \tag{245}$$

$$R_{rjnp} = \frac{\partial}{\partial x^n}[jp,r] - \frac{\partial}{\partial x^p}[jn,r] + \Gamma^{\ell}_{jn}[rp,\ell] - \Gamma^{\ell}_{jp}[rn,\ell] \tag{246}$$

$$R_{rjnp} = \frac{1}{2}\left(\frac{\partial^2 g_{rp}}{\partial x^j \partial x^n} + \frac{\partial^2 g_{jn}}{\partial x^r \partial x^p} - \frac{\partial^2 g_{rn}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^r \partial x^n}\right)$$

$$+ g^{ts}([jn,s][rp,t] - [jp,s][rn,t]) \tag{247}$$

where $[ij,k]$ and Γ^i_{jk} are the Christoffel symbols of the first and second kinds respectively as defined in (130) and (133).

From (247) it is obvious that

$$R_{rjnp} = -R_{jrnp} \tag{248a}$$

$$R_{rjnp} = -R_{rjpn} \tag{248b}$$

$$R_{rjnp} = R_{nprj} \tag{248c}$$

and

$$R_{rjnp} + R_{rnpj} + R_{rpjn} = 0 \tag{248d}$$

so that

$$R_{jrnp} + R_{jprn} + R_{jnpr} = 0 \tag{248e}$$

From (248) we also note that

$$R_{rjnp} = 0 \quad \text{if} \quad r = j \quad \text{or,} \quad n = p \tag{249}$$

Thus apart from sign, the only non-vanishing components are of the form

$$R_{rjrj} \text{ , } R_{rjrp} \text{ , } R_{rjnp} \tag{250}$$

where r, j, n and p are distinct from one another. The total number of distinct components for a space of dimension N are

$$\frac{N^2}{12}(N^2 - 1) \text{ .}$$

Thus the curvature tensor has only one component in two-dimensional space, six in a three-dimensional space, twenty in a four-dimensional space, and so on. It can be seen immediately that in a two-dimensional space, the component is R_{1212}, while in a three-dimensional space the components are

$$R_{1212} \text{ , } R_{1313} \text{ , } R_{2323}\text{, } R_{1213} \text{ , } R_{1232} \text{ , } R_{1323} \tag{251}$$

(Refer to Part III, § 2 for an expanded form of the equations for a three-dimensional space.)

It should be noted that for a flat space

$$R_{rjnp} = 0 \tag{252}$$

for all values of the indices r, j, n and p.

§8.2. The Possibility of Local Cartesian Coordinates in a Riemannian Space.

We are now in a position to investigate further the curved nature of Riemannian spaces. In this section we will show that in a Riemannian space it is possible to introduce a coordinate transformation in which all the metric coefficients are constants and all the Christoffel symbols are zero locally, (cf. §2).

Let x^j be a curvilinear coordinate system in a curved space, viz., not all the components of R_{ijkm} are zero. We now introduce a coordinate transformation from x^j to $\bar{x}^j$ at a fixed point of the space denoted by subscript 0, as

$$\bar{x}^j = x^j - x_0^j + \frac{1}{2}(\Gamma^j_{rs})_0(x^r - x_0^r)(x^s - x_0^s) \tag{253}$$

Thus

$$\frac{\partial \bar{x}^j}{\partial x^p} = \delta^j_p + (\Gamma^j_{pr})_0(x^r - x_0^r) \tag{254}$$

and

$$\delta^j_m = \frac{\partial x^j}{\partial \bar{x}^m} + (\Gamma^j_{pr})_0(x^r - x_0^r)\frac{\partial x^p}{\partial \bar{x}^m} \tag{255}$$

Differentiating (255) with respect to $\bar{x}^\ell$, we get

$$0 = \frac{\partial^2 x^j}{\partial \bar{x}^\ell \partial \bar{x}^m} + (\Gamma^j_{pr})_0 \frac{\partial x^r}{\partial \bar{x}^\ell}\frac{\partial x^p}{\partial \bar{x}^m} + (\Gamma^j_{pr})_0(x^r - x_0^r)\frac{\partial^2 x^p}{\partial \bar{x}^\ell \partial \bar{x}^m}$$

Thus

$$\left(\frac{\partial^2 x^j}{\partial \bar{x}^\ell \partial \bar{x}^m}\right)_0 = -(\Gamma^j_{pr})_0\ \delta^r_\ell \delta^p_m = -(\Gamma^j_{\ell m})_0 \tag{256}$$

Evaluating (166) at x_0^i, i.e., $\bar{x}^i \to x_0^i$, and using (256), we get

$$(\bar{\Gamma}^p_{\ell m})_0 = (\Gamma^s_{ij})_0\ \delta^p_s \delta^i_\ell \delta^j_m - (\Gamma^j_{\ell m})_0\ \delta^p_j = 0 \tag{257}$$

This proves that the Christoffel symbols at the point x_0^i in the new coordinates are zero.

Now using (161), we find that

$$(\bar{g}_{ij})_0 = (g_{ij})_0 \tag{258}$$

Further differentiating (161) with respect to $\bar{x}^k$ and then using the expression for the derivative of g_{ij} from (156b), we get

$$\left(\frac{\partial \bar{g}_{ij}}{\partial \bar{x}^k}\right)_0 = 0 \tag{259}$$

The properties (257) and (259) are peculiar only to Cartesian coordinates; hence the stated result. It must be restated that the preceding results, both for g_{ij} and Γ^i_{jk}, are applicable <u>only locally</u> and not on a global scale. These results also show that the basic nature of a space cannot be guessed simply by g_{ij} and Γ^i_{jk} but by the derivatives of Γ^i_{jk}.

The coordinate system $\bar{x}^j$ discussed above is also called a geodesic polar coordinate system.

§8.3. Ricci's Tensor.

A contraction of ℓ and m in the tensor $R^{\ell}_{\cdot jnm}$ yields a tensor of the second order which is called Ricci's tensor. Ricci's tensor of opposite sign will be obtained if ℓ and n are contracted. Thus Ricci's tensor is

$$R_{jn} = R^{\ell}_{\cdot jn\ell} = g^{\ell s} R_{sjn\ell} \tag{260}$$

In expanded form it can be represented in the following two forms:

$$R_{jn} = \frac{\partial^2}{\partial x^j \partial x^n}(\ell n\sqrt{g}) - \frac{\partial}{\partial x^\ell}\Gamma^\ell_{jn} + \Gamma^\ell_{ns}\Gamma^s_{j\ell} - \Gamma^s_{jn}\frac{\partial}{\partial x^s}(\ell n\sqrt{g}) \tag{261}$$

From (240)

$$R_{jn} = R^\ell_{\cdot jn\ell}$$

$$= \frac{\partial}{\partial x^n}\Gamma^\ell_{j\ell} - \frac{\partial}{\partial x^\ell}\Gamma^\ell_{jn} + \Gamma^\ell_{ns}\Gamma^s_{j\ell} - \Gamma^\ell_{\ell s}\Gamma^s_{jn} \tag{262}$$

From (261) it is obvious that the tensor R_{jn} is symmetric,

$$R_{jn} = R_{nj} \tag{263}$$

since

$$\Gamma^i_{jk} = \Gamma^i_{kj} \quad .$$

The tensor representation (262) is of much importance in the Einstein theory of relativity because it is symmetric and has as many components as the metric tensor g_{ij}. A scalar R can be obtained by the inner multiplication of g^{jn} and R_{jn}, viz.,

$$R = g^{jn}R_{jn} \tag{264}$$

and is called the <u>curvature invariant</u>.

§8.4. Bianchi's Identity.

If we differentiate a second order tensor

$$\tilde{T} = T_{ij}\underset{\sim}{a}^i\underset{\sim}{a}^j$$

and find the partial derivatives

$$\frac{\partial\tilde{T}}{\partial x^m \partial x^n}, \quad \frac{\partial^2\tilde{T}}{\partial x^n \partial x^m}$$

then subtracting the two, we obtain

$$(T_{ik,m})_{,n} - (T_{ik,n})_{,m} = -R^{\ell}_{\cdot inm}T_{\ell k} - R^{\ell}_{\cdot knm}T_{i\ell} \tag{265}$$

We now take the covariant derivative of eq. (241) both sides and using the notation

$$T_{ij} = A_{i,j}$$

we write three equations by cyclic permutation as

$$(T_{jm,n})_{,r} - (T_{jn,m})_{,r} = -R^{\ell}_{\cdot jnm,r}A_{\ell} - R^{\ell}_{\cdot jnm}T_{\ell r} \tag{266a}$$

$$(T_{jr,m})_{,n} - (T_{jm,r})_{,n} = -R^{\ell}_{\cdot jmr,n}A_{\ell} - R^{\ell}_{\cdot jmr}T_{\ell n} \tag{266b}$$

$$(T_{jn,r})_{,m} - (T_{jr,n})_{,m} = -R^{\ell}_{\cdot jrn,m}A_{\ell} - R^{\ell}_{\cdot jrn}T_{\ell m} \tag{266c}$$

Adding eqs. (266), using (265) and (243b), we get

$$R^{\ell}_{\cdot jnp,r} + R^{\ell}_{\cdot jpr,n} + R^{\ell}_{\cdot jrn,p} = 0 \tag{267}$$

This is the first form of the Bianchi's identity. The second form can be obtained by using (244) and using the fact that the metric coefficients g^{ij} behave like constants under covariant differentiation. This form is

$$R_{mjnp,r} + R_{mjpr,n} + R_{mjrn,p} = 0 \tag{268}$$

§8.4.1. A Divergence-Free Tensor.

The use of Ricci's tensor and the Bianchi's identity produces an important tensor. Inner multiplication of (268) with $g^{mp}g^{jn}$ and use of (248) and (260) yields

$$(g^{jn}R_{jn})_{,r} - (g^{jn}R_{jr})_{,n} - (g^{mp}R_{mr})_{,p} = 0 \tag{269}$$

The first term under covariant differentiation in (269) is the curvature invariant R defined in (264), so that

$$R_{,r} - 2(g^{jn}R_{jr})_{,n} = 0 \tag{270}$$

If we now introduce a mixed tensor

$$G^{n}_{\cdot r} = g^{jn}R_{jr} - \frac{1}{2}\,\delta^{n}_{r}R \tag{271}$$

then eq. (270) implies that

$$G^{n}_{\cdot r,n} = 0 \tag{272}$$

That is, the covariant divergence of the mixed tensor is zero.

In place of the mixed tensor, we can have a contravariant symmetric tensor by first writing

$$R_{jr} = g_{jp}g_{rq}R^{pq}$$

Thus (270) becomes

$$R_{,r} - 2g_{rq}R^{nq}_{,n} = 0$$

Inner multiplication by $g^{r\ell}$, and because of symmetry of R^{nq}, we get

$$E^{ij}_{,j} = 0 \tag{273}$$

where

$$E^{ij} = R^{ij} - \frac{1}{2}g^{ij}R \; . \tag{274}$$

Equations (272) and (273) state that the tensor components defined in (271) and (274) are divergence-free. The tensor E^{ij} is symmetric and is called the "energy-momentum tensor." Both eqs. (272) and (273) state a conservation law of much importance in physics. Note that the covariant components of the energy-momentum tensor can be obtained from (274) by the usual rule of lowering an index. Thus

$$E_{ij} = R_{ij} - \frac{1}{2}g_{ij}R \; . \tag{275}$$

§9. The Geometry of the Event-Space.

An event-space is a coordinate space in which the time variable is also taken as one of the coordinates so as to have a space-time continuum in which physical events occur. All the tensor theoretic results obtained so far are obviously applicable in this space.

The geometry of the event-space has all along been important to the theory of relativity. However, it is the opinion of this author that all mechanics, whether relativistic or non-relativistic, should be treated at least in the start as a unified subject†. The rigid classifications of relativistic and non-relativistic mechanics deprives one from a correct understanding of the mechanics and of the geometry associated with it. In this section we first briefly state the basic postulates of mechanics and go on to explore some of the consequences from a geometrical standpoint.

An inertial frame of reference is precisely defined by Newton's first law of motion. However, we can rephrase it as: An inertial frame is a coordinate frame with respect to which bodies, under the absence of external forces, move with zero acceleration.

The two basic postulates of mechanics are:

(I) All physical laws are form-invariant when transformed between inertial frames.

(II) Light travels isotropically and with a constant finite speed $c(= 2.998 \times 10^{10}$ cm/sec) in all inertial frames.

In this section we shall use the Greek suffixes for index values ranging from 1 to 4 and Latin index values from 1 to 3.

Let there be two inertial frames in which the coordinates are denoted as x^α, $\bar{x}^\alpha$. The first three coordinates are the rectangular Cartesian and the fourth is the time. Thus, for example

$$x^1 = x \ , \ x^2 = y \ , \ x^3 = z \ , \ x^4 = t \ .$$

†The inspiration for the work of this section is due to a paper by G. C. McVittie [30].

Let the second inertial frame move along x^1 with a constant velocity V with respect to the first. A general coordinate transformation between x^α and $\overline{x}^\alpha$ can be written as four equations

$$\overline{x}^\beta = \phi^\beta(x^1,x^2,x^3,x^4) \tag{276}$$

If now the two postulates, and specifically of the isotropic propagation and constancy of light, are used then as shown by Tolman [31] the only possible transformation forms for ϕ^β in (276) are

$$\overline{x}^1 = k(x^1 - Vx^4) \tag{277a}$$

$$\overline{x}^2 = x^2 \tag{277b}$$

$$\overline{x}^3 = x^3 \tag{277c}$$

$$\overline{x}^4 = k(x^4 - \frac{Vx^1}{c^2}) \tag{277d}$$

where

$$k = (1 - \frac{V^2}{c^2})^{-1/2}$$

The mapping (277) is called the Lorentz transformation. It is immediately seen from (277), that when V is very small in comparison to c, then $k \to 1$ and

$$\overline{x}^1 = x^1 - Vx^4 \tag{278a}$$

$$\overline{x}^2 = x^2 \tag{278b}$$

$$\overline{x}^3 = x^3 \tag{278c}$$

$$\bar{x}^4 = x^4 \qquad (278d)$$

which is called a Newtonian or Galilean transformation.

As a check on the form-invariancy of a physical law, we can take Newton's second law of motion for a body of constant mass m_0 and acted upon by a force system F^i,

$$F^i = \frac{d}{dx^4}\left(m_0 \frac{dx^i}{dx^4}\right) \ , \ i = 1,2,3 \qquad (279)$$

It can be easily verified that under the transformation (278) the law takes the form

$$\bar{F}^i = \frac{d}{d\bar{x}^4}\left(m_0 \frac{d\bar{x}^i}{d\bar{x}^4}\right) \ , \ i = 1,2,3 \qquad (280)$$

so that the form is preserved under a Galilean transformation. If we repeat the same procedure using the Lorentz transformation then we find that (279) cannot be transformed to the form (280).

We now consider the invariant nature of the element of length for an Euclidean four-dimensional space. Recall from eq. (71) that the element of length in E^4 will be

$$(ds)^2 = \delta_{\alpha\beta} dx^\alpha dx^\beta$$

This metric does not remain invariant either for the transformation (277) or (278). Thus the Euclidean metric is completely unsuitable for describing a physical phenomena in an event space. For the Lorentz transformation, it can be shown that the metric

$$(ds)^2 = (cdx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{281}$$

transforms to

$$(ds)^2 = (cd\bar{x}^4)^2 - (d\bar{x}^1)^2 - (d\bar{x}^2)^2 - (d\bar{x}^3)^2 \tag{282}$$

so that the metric (281) is form-invariant under Lorentz transformation. We can write (281) in the form of eq. (74),

$$(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

so that

$$g_{11} = -1 \ , \ g_{22} = -1 \ , \ g_{33} = -1 \ , \ g_{44} = c^2$$

$$g_{\alpha\beta} = 0 \quad \text{if} \quad \alpha \neq \beta$$

Thus, for this metric

$$g = -c^2$$

and consequently in all the tensor formulae g must be replaced by -g.

To complete the consequences of (281), we first write it in the form

$$(d\sigma)^2 = \left(\frac{ds}{c}\right)^2$$

$$= (dx^4)^2 - \frac{1}{c^2}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \tag{284}$$

It must be realized that the element ds in (281) is <u>not</u> the distance between two closely spaced points. Thus dσ is an interval which tends to an interval of time only when c is considered to be infinite rather than finite as stated in the second postulate of mechanics.

Now writing

$$\left(\frac{dx^1}{dx^4}\right)^2 + \left(\frac{dx^2}{dx^4}\right)^2 + \left(\frac{dx^3}{dx^4}\right)^2 = (u)^2$$

$$= u^2$$

in (284) to have

$$\frac{dx^4}{d\sigma} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \tag{285}$$

Note that in the Newtonian mechanics $c \to \infty$ and $dx^4 = d\sigma$, and this is all we get out of the metric (284).

A four dimensional Minkowski momentum vector is now defined by the components

$$m_0 \frac{dx^\alpha}{d\sigma}, \ \alpha = 1,2,3,4 \tag{286}$$

where m_0 is the mass of the body when at rest, viz. $u = 0$. The fourth component of (286) is defined as the mass m of the body in motion. That is

$$m_0 \frac{dx^4}{d\sigma} = m$$

which on using (285) gives the well known relativistic mass

$$m = m_0\left(1 - \frac{u^2}{c^2}\right)^{-1/2} \tag{287}$$

The other three components are

$$m_0 \frac{dx^i}{d\sigma} = m_0 \frac{dx^i}{dx^4}\frac{dx^4}{d\sigma}$$

$$= m \frac{dx^i}{dx^4} , \quad i = 1,2,3 \tag{288}$$

which are the Newtonian momentum components.

The four-dimensional Minkowski force vector has components F^α,

$$F^\alpha = m_0 \frac{d^2x^\alpha}{d\sigma^2}$$

$$= \frac{d}{d\sigma}(m_0 \frac{dx^\alpha}{d\sigma})$$

$$= (1 - \frac{u^2}{c^2})^{-1/2} \frac{d}{dx^4}(m_0 \frac{dx^\alpha}{d\sigma}) \tag{289}$$

Thus

$$F^i = (1 - \frac{u^2}{c^2})^{-1/2} \frac{d}{dx^4}(m \frac{dx^i}{dx^4}) \tag{290}$$

and

$$F^4 = (1 - \frac{u^2}{c^2})^{-1/2} \frac{dm}{dx^4}$$

$$= \frac{1}{c^2}(1 - \frac{u^2}{c^2})^{-1/2} \frac{dE}{dx^4} \tag{291}$$

where from (287)

$$E = mc^2 = m_0c^2 + \frac{1}{2} m_0u^2 + \cdots \tag{292}$$

This short discussion on the fundamentals of the special theory establishes the connection between the relativistic and non-relativistic mechanics and more importantly brings out the structure of the metric needed to describe a space-time continuum.

In the case of general relativity, Einstein proposed the principle of covariance which states that the physical laws under a general transformation of coordinates are form-invariant. This principle thus sweeps away the privileged position of inertial frames as embodied in the two principles of mechanics. For the description of general relativity theory, the Riemannian metric, eq. (74), is used in its most general form with the metric coefficients g_{ij} as related to the distribution of matter. For details refer to [7], [31], etc.

§9.1. Newtonian Mechanics Using the Principles of Special Relativity.

Newtonian mechanics with reference to an inertial frame of reference is described by the three spatial Cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ at each absolute instant of time $x^4 = t$. Thus time is not affected by the motion and remains the same for all coordinate systems. In essence time is not a coordinate any more but is a parameter which describes the transformation of a three-dimensional Euclidean space into itself with the passage of time. The geometry of this event-space is then simply defined by an Euclidean E^3 metric in either the Cartesian or any general three-dimensional coordinate system with t or x^4 as a parameter.

If we attempt to describe the motion of a mass point as a collection of four numbers (x^1,x^2,x^3,x^4) in a four-dimensional manifold then we have to use the metric of the special relativity given in (284), with the option of performing a general coordinate transformation from the Cartesian to curvilinear while keeping x^4 the same, as described by McVittie [30].

Let us introduce a transformation of spatial coordinates to a curvilinear system ξ^α as

$$x^r = \phi^r(\xi^\alpha) \;, \; r = 1,2,3; \; \alpha = 1,2,3,4$$
$$x^4 = \xi^4 \tag{293}$$

Thus

$$dx^r = \frac{\partial x^r}{\partial \xi^\alpha} d\xi^\alpha \tag{294a}$$

$$dx^4 = d\xi^4 \tag{294b}$$

Writing (284) in the form

$$(d\sigma)^2 = (dx^4)^2 - \frac{1}{c^2}\delta_{ij}dx^i dx^j \tag{295}$$

and using (294), we obtain

$$(d\sigma)^2 = g_{\alpha\beta}d\xi^\alpha d\xi^\beta \tag{296}$$

where

$$g_{\alpha\beta} = \delta_{\alpha 4}\delta_{\beta 4} - \frac{1}{c^2}\gamma_{\alpha\beta} \tag{297a}$$

and

$$\left.\begin{aligned}\gamma_{44} &= \delta_{ij}\,\frac{\partial x^i}{\partial \xi^4}\,\frac{\partial x^j}{\partial \xi^4} \\ \gamma_{p4} = \gamma_{4p} &= \delta_{ij}\,\frac{\partial x^i}{\partial \xi^p}\,\frac{\partial x^j}{\partial \xi^4} \\ \gamma_{pq} = \gamma_{qp} &= \delta_{ij}\,\frac{\partial x^i}{\partial \xi^p}\,\frac{\partial x^j}{\partial \xi^q}\end{aligned}\right\} \qquad (297b)$$

It must be recalled that the Greek indices range from 1 to 4, while the Latin indices range from 1 to 3. Obviously, as defined in (297b), the metric (296) has symmetric coefficients, viz.,

$$g_{\alpha\beta} = g_{\beta\alpha} \ .$$

Let

$$g = \det(g_{\alpha\beta}) \ , \ \Delta_0 = \det(\gamma_{\alpha\beta})$$

and

$$\Delta = \gamma_{11}\gamma_{22}\gamma_{33} + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{11}\gamma_{23}^2 - \gamma_{22}\gamma_{13}^2 - \gamma_{33}\gamma_{12}^2 \qquad (298)$$

then

$$g = \frac{-\Delta}{c^6} + \frac{\Delta_0}{c^8} \qquad (299a)$$

and

$$\sqrt{-g} = \frac{\sqrt{\Delta}}{c^3}\sqrt{1 - \frac{\Delta_0}{c^2\Delta}} \tag{299b}$$

In Newtonian mechanics we deal with velocities which are much smaller than c. Thus $x^r(\xi^\alpha)$ should be such that $\gamma_{\mu\nu}$ do not contain a factor of c^2. Hence g_{44} is of the order of one, while the remaining $g_{\mu\nu}$ are of the order of $\frac{1}{c^2}$.

Let $\underset{\sim}{v}$ be a four component velocity vector, then according to the special theory of relativity its c(::ravariant components are

$$u^\alpha = \frac{dx^\alpha}{d\sigma} = [1 - \frac{(u)^2}{c^2}]^{-1/2} \frac{dx^\alpha}{dx^4}$$

For u << c, neglecting terms of order $\frac{1}{c^2}$, we get

$$u^\alpha = \frac{dx^\alpha}{dx^4} \tag{300}$$

Thus, for Newtonian mechanics

$$u^4 = 1 \ , \ u^i = \frac{dx^i}{dx^4} \ , \ i = 1,2,3 \ . \tag{301}$$

In the transformed coordinates defined by

$$x^i = x^i(\xi^\alpha)$$

we have the components as

$$v^4 = 1 \ , \ v^i = \frac{d\xi^i}{d\xi^4} \tag{302}$$

The divergence of $\underset{\sim}{v}$ in special relativity is, (eq. (201)),

$$\operatorname{div} \underset{\sim}{v} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \xi^{\alpha}} (\sqrt{-g}\; v^{\alpha}) \tag{303}$$

Using (299b) and neglecting terms of order $\frac{1}{c^2}$, we get

$$\operatorname{div} \underset{\sim}{v} = \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial \xi^{\alpha}} (\sqrt{\Delta}\; v^{\alpha}) \tag{304}$$

with $v^4 = 1$.

From eq. (211a) the divergence of a contravariant tensor in special relativity (four space) will be

$$T^{\mu\nu}_{\ ,\nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \xi^{\nu}} (\sqrt{-g}\; T^{\mu\nu}) + \Gamma^{\mu}_{\sigma\nu} T^{\sigma\nu} \tag{305}$$

Again using (299b) and neglecting terms of order $\frac{1}{c^2}$, we get

$$T^{\mu\nu}_{\ ,\nu} = \frac{1}{\sqrt{\Delta}} \frac{\partial}{\partial \xi^{\alpha}} (\sqrt{\Delta}\; T^{\mu\alpha}) + \Gamma^{\mu}_{\sigma\alpha} T^{\sigma\alpha} \tag{306}$$

§9.1.1. Application to the Navier-Stokes Equations.

As an application of the preceding approximations of the special relativity to the Newtonian mechanics, we consider the transformation of the complete† Navier-Stokes system of equations to time dependent coordinates‡.

The Navier-Stokes system of equations for a viscous compressible fluid in the invariant vector form is

†McVittie [30] has considered only the inviscid equations.

‡A derivation without using the special relativity has been obtained by the present author, [32].

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho \underset{\sim}{v}) = 0 \tag{307}$$

$$\frac{\partial}{\partial t}(\rho \underset{\sim}{v}) + \operatorname{div} \tilde{\tau} = 0 \tag{308}$$

where ρ is the density, $\tilde{\tau}$ is the stress tensor, and the div operator is the spatial three-dimensional divergence.

We now define a four-dimensional energy-momentum tensor

$$T^{\alpha\beta} = \rho v^{\alpha} v^{\beta} + \tau^{\alpha\beta} \tag{309}$$

such that

$$\begin{aligned} v^4 &= 1 \\ T^{44} &= \rho \\ T^{i4} = T^{4i} &= \rho v^i \\ T^{ij} &= T^{ji} \end{aligned} \tag{310}$$

where i and j vary from 1 to 3. Using (304) and (306) we can write both (307) and (308) as a single equation

$$T^{\alpha\beta}_{,\beta} = 0 \tag{311}$$

As before, a comma in (311) denotes covariant differentiation; α = 1, 2, 3 correspond to the three equations of motion (308) and α = 4 corresponds to the equation of continuity (307).

Earlier, in §8.4.1 the subject of divergence-free tensors was discussed and an equation exactly similar to (311) was obtained for curved geometries, eq. (273). These similarities tell us a lot about the connection of geometry with mechanics and the physical laws.

Part II

The Geometry of Curves and Surfaces

§1. Theory of Curves

In this section we shall consider only those parts of the theory of curves in space which are needed in the theory of surface. All our considerations will be confined to an Euclidean E^3 in which the space curves in parametric form are defined by

$$\underset{\sim}{r} = \underset{\sim}{r}(t) \tag{1}$$

where t is a parameter which takes values in a certain interval $a \leq t \leq b$.

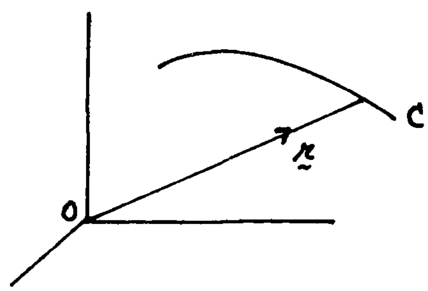


Figure 4.

It is assumed that the real vector function $\underset{\sim}{r}(t)$ is $p \geq 1$ times continuously differentiable for every value of t in the specified interval, and at least one component of the first derivative

$$\underset{\sim}{r}' = \frac{d\underset{\sim}{r}}{dt} \tag{2}$$

is different from zero. Note that the parameter t can be expressed in terms of any other parameter, say τ, provided that $\frac{d\tau}{dt} \neq 0$.

(i) Tangent vector:

Choose s, the arc length along the curve as a parameter. Let $\underset{\sim}{r}(s)$ and $\underset{\sim}{r}(s+h)$ be nearby points on the curve. Then the limit

$$\underset{\sim}{t}(s) = \lim_{h \to 0} \frac{\underset{\sim}{r}(s + h) - \underset{\sim}{r}(s)}{h}$$

$$= \frac{d\underset{\sim}{r}}{ds} \qquad (3)$$

is the unit tangent vector at the point s on the curve. Note that

$$|\underset{\sim}{t}| = \frac{|d\underset{\sim}{r}|}{ds} = 1 \; .$$

If s is replaced by another parameter t, then

$$\underset{\sim}{t} = \frac{d\underset{\sim}{r}}{dt}\frac{dt}{ds} = \underset{\sim}{r}'/|\underset{\sim}{r}'| \qquad (4)$$

A straight line in the direction of $\underset{\sim}{t}$ from the point s on the curve is the tangent line to the curve.

(ii) Principal normal:

Since

$$\underset{\sim}{t} \cdot \underset{\sim}{t} = 1$$

hence by differentiation

$$\underset{\sim}{t} \cdot \frac{d\underset{\sim}{t}}{ds} = 0$$

so that the vector $\frac{d\underset{\sim}{t}}{ds}$ is orthogonal to $\underset{\sim}{t}$ and is called the curvature vector. We shall denote it as $\underset{\sim}{k}$,

$$\hat{\underset{\sim}{k}} = \frac{d\underset{\sim}{t}}{ds} \qquad (5)$$

The unit principal normal vector is then defined as

$$\underset{\sim}{p} = \hat{\underset{\sim}{k}}/|\hat{\underset{\sim}{k}}| \qquad (6)$$

The magnitude

$$k(s) = |\hat{\underset{\sim}{k}}| \ , \ \rho(s) = 1/k \qquad (7)$$

is the curvature of the curve and ρ is the radius of curvature. The principal normal is directed toward the center of curvature of the curve at that point.

(iii) Normal plane:

The totality of all vectors which are bound at a point of the curve and which are orthogonal to the unit tangent vector at that point lie in a plane. This plane is called the normal plane.

(iv) Osculating plane:

Choose any three nearby points on a space curve through which a plane can pass. Let the equation of this plane be written in the current variable $\underset{\sim}{r}$ as

$$\underset{\sim}{r} \cdot \underset{\sim}{a} = c \qquad (8)$$

where $\underset{\sim}{a}$ is perpendicular to the plane. We now define a function f(u) of the parameter u,

$$f(u) = \underset{\sim}{r} \cdot \underset{\sim}{a} - c \qquad (9)$$

Let $\underset{\sim}{x}_i$ be the point on the curve where the parameter has the value u_i. The three points chosen on the curve are denoted as u_1, u_2, u_3 such that they satisfy eq. (9), i.e.,

$$f(u_1) = 0 \ , \ f(u_2) = 0 \ , \ f(u_3) = 0 \tag{10}$$

Hence according to the Rolle's theorem

$$\left.\begin{aligned} f'(\xi_1) = 0 \ , \ u_1 < \xi_1 < u_2 \\ f'(\xi_2) = 0 \ , \ u_2 < \xi_2 < u_3 \end{aligned}\right\} \tag{11}$$

Because of eqs. (11) we can again apply Rolle's theorem to the function $f'(u)$ in the interval $\xi_1 \le u \le \xi_2$, so that

$$f''(\xi_3) = 0 \ , \ \xi_1 < \xi_3 < \xi_2 \tag{12}$$

As the two points u_2 and u_3 approach in the limit to u_1, we have

$$u_2, u_3, \xi_1, \xi_2, \xi_3 \to u_1$$

and equations (10), (11) and (12) yield

$$f(u_1) = 0 \ , \ f'(u_1) = 0 \ , \ f''(u_1) = 0 \tag{13a}$$

or,

$$\underset{\sim}{x}_1 \cdot \underset{\sim}{a} = c \ , \ \underset{\sim}{t} \cdot \underset{\sim}{a} = 0 \ , \ \hat{\underset{\sim}{k}} \cdot \underset{\sim}{a} = 0 \tag{13b}$$

Combining equations (8) and (13b), we get the equation of a plane at $\underset{\sim}{x}_1$,

$$\underset{\sim}{r} = \underset{\sim}{x}_1 + \lambda \underset{\sim}{t} + \mu \underset{\sim}{p} \tag{14}$$

where λ and μ are scalar parameters. This plane is called the osculating plane, and as shown by (14) it is spanned by the unit tangent and the unit principal normal vectors.

(v) <u>Binormal vector</u>:

A unit vector $\mathbf{b}(s)$ which is orthogonal to both $\mathbf{t}$ and $\mathbf{p}$ is called the binormal vector. Its orientation is fixed by taking $\mathbf{t}$, $\mathbf{p}$, $\mathbf{b}$ to form a right-handed triad as shown in Fig. 5.

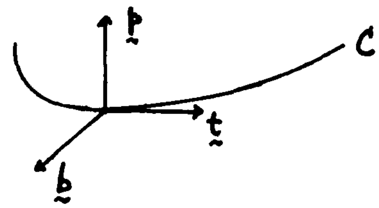


Figure 5.

Thus

$$\mathbf{b} = \mathbf{t} \times \mathbf{p} \tag{15}$$

Note that for plane curves the binormal $\mathbf{b}$ is the constant unit vector normal to the plane, and the principal normal is the usual normal to the curve directed toward the center of curvature at that point.

The twisted curves in space have their binormals as functions of s. Because of twisting a new quantity called torsion appears, which is obtained as follows.

Consider the obvious equations

$$\mathbf{b} \cdot \mathbf{b} = 1 \; , \; \mathbf{b} \cdot \mathbf{t} = 0 \tag{16}$$

Differentiating each equation with respect to s, we obtain

$$\underset{\sim}{b} \cdot \frac{d\underset{\sim}{b}}{ds} = 0 \tag{17a}$$

$$\underset{\sim}{b} \cdot \frac{d\underset{\sim}{t}}{ds} + \frac{d\underset{\sim}{b}}{ds} \cdot \underset{\sim}{t} = 0 \tag{17b}$$

Thus

$$\frac{d\underset{\sim}{b}}{ds} \cdot \underset{\sim}{t} = -k\underset{\sim}{b} \cdot \underset{\sim}{p}$$

$$= 0 \tag{17c}$$

From (17a,c) we find that $\frac{d\underset{\sim}{b}}{ds}$ is a vector which is orthogonal to both $\underset{\sim}{t}$ and $\underset{\sim}{b}$. Thus $\frac{d\underset{\sim}{b}}{ds}$ lies along the principal normal,

$$\frac{d\underset{\sim}{b}}{ds} = \pm\tau\underset{\sim}{p}$$

To decide about the sign we take the cross product of $\underset{\sim}{b}$ with $\frac{d\underset{\sim}{b}}{ds}$ and take it as a positive rotation about $\underset{\sim}{t}$.

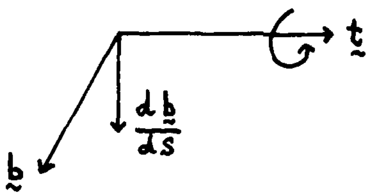


Figure 6.

Thus

$$\underset{\sim}{b} \times \frac{d\underset{\sim}{b}}{ds} = \tau\underset{\sim}{t} \tag{18a}$$

and

$$\frac{d\underset{\sim}{b}}{ds} = -\tau\underset{\sim}{p} \tag{18b}$$

§1.1. Serret-Frenet Equations.

A set of equations known as the Serret-Frenet equations, which are the intrinsic equations of a curve, are the following. Differentiating the equation

$$\underset{\sim}{p} = \underset{\sim}{b} \times \underset{\sim}{t}$$

with respect to s, we have

$$\frac{d\underset{\sim}{p}}{ds} = \tau\underset{\sim}{b} - k\underset{\sim}{t} \tag{19}$$

Equations (6), (18b) and (19) are the Serret-Frenet equations, and are collected below

$$\left.\begin{aligned} \frac{d\underset{\sim}{t}}{ds} &= k\underset{\sim}{p} \quad ; \quad k = \text{curvature} \\ \frac{d\underset{\sim}{b}}{ds} &= -\tau\underset{\sim}{p} \quad ; \quad \tau = \textit{torsion} \\ \frac{d\underset{\sim}{p}}{ds} &= \tau\underset{\sim}{b} - k\underset{\sim}{t} \end{aligned}\right\} \tag{20}$$

For a plane curve, $\tau = 0$, so that

$$\left.\begin{aligned} \underset{\sim}{b} &= \text{constant} \\ \frac{d\underset{\sim}{t}}{ds} &= k\underset{\sim}{p} \\ \frac{d\underset{\sim}{p}}{ds} &= -k\underset{\sim}{t} \end{aligned}\right\} \tag{21}$$

§2. Geometry of Two-Dimensional Surfaces Embedded in E^3.

In the theory of surfaces, embedded in E^3, we shall use u^1, u^2, or u^α as the coordinates in the surface and x^i ($i = 1,2,3$) as any general coordinate system in E^3. An element of directed segment $d\underset{\sim}{r}$ is then represented as

$$d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial x^i} dx^i = \underset{\sim}{a}_i dx^i \tag{22a}$$

If the same element belongs to a surface S, then

$$d\underset{\sim}{r} = \frac{\partial \underset{\sim}{r}}{\partial u^\alpha} du^\alpha = \underset{\sim}{r}_\alpha du^\alpha \tag{22b}$$

since in principle

$$\underset{\sim}{r} = \underset{\sim}{r}(x^i) = \underset{\sim}{r}(x^i(u^1,u^2))$$

for points belonging to the surface. Also since

$$x^i = x^i(u^1,u^2) \tag{23}$$

hence

$$dx^i = \frac{\partial x^i}{\partial u^\alpha} du^\alpha = x^i_\alpha du^\alpha \tag{24}$$

where we have used the notation

$$x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \tag{25}$$

Now the metric formed from (22) is

$$(ds)^2 = g_{ij}dx^i dx^j \tag{26}$$

so that on using (24), we get

$$(ds)^2 = g_{ij}x^i_\alpha x^i_\beta du^\alpha du^\beta$$

$$= a_{\alpha\beta}du^\alpha du^\beta \tag{27}$$

The three quantities

$$a_{\alpha\beta} = g_{ij}x^i_\alpha x^j_\beta \tag{28}$$

form the components of a symmetric tensor, called the fundamental metric tensor of a surface. In the old literature, the following non-tensorial notation is also used.

$$a_{11} = E \; , \; a_{12} = a_{21} = F \; , \; a_{22} = G \; .$$

Since in an Euclidean space we can always choose a rectangular Cartesian system, so that

$$g_{ij} = \delta_{ij}$$

and then

$$a_{\alpha\beta} = \delta_{ij}x^i_\alpha x^j_\beta \tag{29}$$

From here onward we shall return to the symbolism of Part I and use the notion g_{ij} for a_{ij} as there is no chance for confusion if the meaning is clear from the context. Thus, we shall take the metric of a surface with coefficients $g_{\alpha\beta}$ rather than $a_{\alpha\beta}$, as

$$(ds)^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{30}$$

Sometimes when there is no use for an index notation we have used the symbols

$$x^1 = u \quad \text{or} \quad \xi \ , \ u^2 = v \quad \text{or} \quad \eta$$

Therefore

$$(ds)^2 = g_{11}(du)^2 + 2g_{12}dudv + g_{22}(dv)^2 \tag{31}$$

The metric (30) for an element of length in the surface is called the "first fundamental form" for a surface. Some expansions for future reference are listed below. A variable subscript in the formulae given below stands for a partial derivative.

$$g_{11} = x_u^2 + y_u^2 + z_u^2 = \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_1 \tag{32a}$$

$$g_{12} = x_u x_v + y_u y_v + z_u z_v = \underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 \tag{32b}$$

$$= g_{21}$$

$$g_{22} = x_v^2 + y_v^2 + z_v^2 = \underset{\sim}{r}_2 \cdot \underset{\sim}{r}_2 \tag{32c}$$

$$g_{(\nu)} = g_{11}g_{22} - (g_{12})^2 \tag{33†}$$

---

† To distinguish the "g" of the general coordinates (cf. Part I) from the "g" formed by the coefficients of the first fundamental form of a surface, we denote the latter by $g_{(\nu)}$, where ν stands for a coordinate held fixed on the surface.

$$g_{\alpha\beta}g^{\alpha\delta} = \delta^{\alpha}_{\beta} \tag{34a}$$

$$g^{11} = g_{22}/g_{(\nu)} \; , \; g^{12} = g^{21} = -g_{12}/g_{(\nu)} \; , \; g^{22} = g_{11}/g_{(\nu)} \tag{34b}$$

where ν is a parameter which remains fixed on a surface.

Let θ be the angle between $\underset{\sim}{r}_1 = \frac{\partial \underset{\sim}{r}}{\partial u}$ and $\underset{\sim}{r}_2 = \frac{\partial \underset{\sim}{r}}{\partial v}$. Then

$$\cos\theta = (\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2)/\sqrt{g_{11}g_{22}} = g_{12}/\sqrt{g_{11}g_{22}} \tag{35}$$

and

$$|\underset{\sim}{r}_1 \times \underset{\sim}{r}_2|^2 = g_{11}g_{22}\sin^2\theta$$

$$= g_{11}g_{22}(1 - \cos^2\theta)$$

$$= g_{(\nu)} \tag{36}$$

From (35) we see that the surface coordinates are orthogonal if

$$g_{12} = 0 \; .$$

The base vectors in the surface defined in (22b), viz.,

$$\underset{\sim}{r}_\alpha = \frac{\partial \underset{\sim}{r}}{\partial u^\alpha}{}^{\dagger}$$

---

†We again emphasize the notation that a subscripted $\underset{\sim}{r}$ such as $\underset{\sim}{r}_1$ or $\underset{\sim}{r}_2$ stands for differentiation with respect to u^1 or u^2. Only when expressions have been opened in full, the notation $u^1 = u$, $u^2 = v$ has been used.

define the unit normal vector $\underset{\sim}{n}$ at each point of the surface through the equation

$$\underset{\sim}{n} = \frac{\underset{\sim}{r}_1 \times \underset{\sim}{r}_2}{|\underset{\sim}{r}_1 \times \underset{\sim}{r}_2|} = \frac{1}{\sqrt{g_{(\nu)}}}\left(\frac{\partial \underset{\sim}{r}}{\partial u} \times \frac{\partial \underset{\sim}{r}}{\partial v}\right) \tag{37}$$

The Cartesian components of $\underset{\sim}{n}$ will be denoted by X, Y, and Z, so that from eq. (37)

$$X = J_1/\sqrt{g_{(\nu)}}\ ,\ Y = J_2/\sqrt{g_{(\nu)}}\ ,\ Z = J_3/\sqrt{g_{(\nu)}} \tag{38}$$

where

$$\left.\begin{aligned} J_1 &= y_u z_v - y_v z_u \\ J_2 &= x_v z_u - x_u z_v \\ J_3 &= x_u y_v - x_v y_u \end{aligned}\right\} \tag{39}$$

§2.1. Normal Curvature of a Surface: Second Fundamental Form.

A plane containing $\underset{\sim}{t}$ and $\underset{\sim}{n}$ at a point P of the surface cuts the surface in different curves when rotated about $\underset{\sim}{n}$ as an axis. Each curve is known as a normal section of the surface at the point P. Since these curves belong both to the surface and also to the embedding space, a study of the curvature properties of these as space curves also reveals the curvature properties of the surfaces in which they lie.

We decompose the curvature vector $\hat{\underset{\sim}{k}}$ at P of C, defined in eq. (5), into a vector $\underset{\sim}{k}_n$ normal to the surface and a vector $\underset{\sim}{k}_g$ tangential to the surface as shown in Fig. 7.

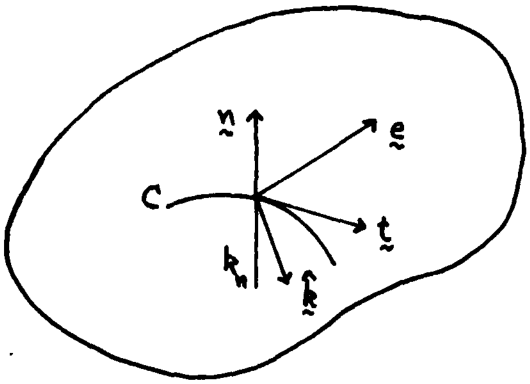


Figure 7.

Thus

$$\hat{\mathbf{k}} = \mathbf{k}_n + \mathbf{k}_g \tag{40}$$

The vector $\mathbf{k}_n$ is called the normal curvature vector at the point P. It is directed either toward or against the direction of $\mathbf{n}$, so that

$$\mathbf{k}_n = \mathbf{n}\, k_n \tag{41}$$

where k_n is the normal curvature of the normal section of the surface, and is an algebraic number.

To find the expression for k_n, we consider the equation

$$\mathbf{n} \cdot \mathbf{t} = 0$$

and differentiate it with respect to s

$$\frac{d\mathbf{n}}{ds} \cdot \mathbf{t} + \mathbf{n} \cdot (\mathbf{n}\, k_n + \mathbf{k}_g) = 0$$

or,

$$k_n = \frac{-d\mathbf{n} \cdot d\mathbf{r}}{(ds)^2} \tag{42}$$

Also, differentiating the equation

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_\beta = 0$$

with u^α to have

$$\underset{\sim}{n}_\alpha \cdot \underset{\sim}{r}_\beta + \underset{\sim}{n} \cdot \underset{\sim}{r}_{\alpha\beta} = 0 \tag{43a}$$

or,

$$\underset{\sim}{n}_\alpha \cdot \underset{\sim}{r}_\beta = -\underset{\sim}{n} \cdot \underset{\sim}{r}_{\alpha\beta} \tag{43b}$$

Further,

$$\begin{aligned} d\underset{\sim}{n} &= \underset{\sim}{n}_\alpha du^\alpha \\ d\underset{\sim}{r} &= \underset{\sim}{r}_\beta du^\beta \end{aligned} \tag{44}$$

Using (43b) and (44) in (42), we get

$$k_n = (\underset{\sim}{n} \cdot \underset{\sim}{r}_{\alpha\beta}) \frac{du^\alpha du^\beta}{(ds)^2} \tag{45}$$

We now introduce the quantities $b_{\alpha\beta}$ as

$$b_{\alpha\beta} = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\alpha\beta} \tag{46a}$$

$$= -\underset{\sim}{n}_\alpha \cdot \underset{\sim}{r}_\beta \tag{46b}$$

$$= \frac{1}{\sqrt{g_{(\nu)}}} \underset{\sim}{r}_1 \cdot (\underset{\sim}{r}_2 \times \underset{\sim}{r}_{\alpha\beta}) \tag{46c}$$

Thus (45) becomes

$$k_n = \frac{b_{\alpha\beta} du^\alpha du^\beta}{(ds)^2} \tag{47a}$$

$$= \frac{b_{\alpha\beta} du^\alpha du^\beta}{g_{\mu\nu} du^\mu du^\nu} \tag{47b}$$

The form

$$b_{\alpha\beta} du^\alpha du^\beta \tag{48}$$

is called the "second fundamental form" of the surface theory. The expanded forms of $b_{\alpha\beta}$ are

$$\begin{aligned} b_{11} &= Xx_{uu} + Yy_{uu} + Zz_{uu} \\ b_{12} &= Xx_{uv} + Yy_{uv} + Zz_{uv} = b_{21} \\ b_{22} &= Xx_{vv} + Yy_{vv} + Zz_{vv} \end{aligned} \tag{49}$$

Similar to $g_{(\nu)}$, we also define

$$b = b_{11} b_{22} - (b_{12})^2 \tag{50}$$

It is shown in standard texts on differential geometry, e.g., [6], [13], [17], etc., that points on a surface can be classified as follows.

$$\begin{aligned} b > 0 \quad &, \quad \text{elliptic point} \\ b = 0 \quad &, \quad \text{parabolic point} \\ b < 0 \quad &, \quad \text{hyperbolic point} \end{aligned} \tag{51}$$

We now return to a consideration of k_n. First notice that

$$\underset{\sim}{n} \cdot \hat{\underset{\sim}{k}} = \underset{\sim}{n} \cdot (\underset{\sim}{n}\, k_n + \underset{\sim}{k}_g)$$

$$= k_n \tag{52}$$

Since $\underset{\sim}{p}$ is the unit principal normal to the curve, hence

$$\hat{\underset{\sim}{k}} = k\underset{\sim}{p} \tag{53}$$

Using (53) in (52) and denoting

$$\underset{\sim}{n} \cdot \underset{\sim}{p} = \cos\gamma \tag{54}$$

we obtain

$$k_n = k\cos\gamma \tag{55}$$

Therefore if $\gamma = 0$, then $k = k_n$; if $\gamma = \frac{\pi}{2}$, then $k_n = 0$ and the curve is a plane curve; if $\gamma = \pi$, then $k = -k_n$. Let

$$k = \frac{1}{\rho}\ , \ k_n = \frac{1}{\rho_n}$$

then

$$\rho = \rho_n \cos\gamma \tag{56}$$

Equation (56) gives a theorem, called Meusnier's theorem: The center of curvature 0 of all curves on S at P having the same tangent $\underset{\sim}{t}$ lie on a circle of radius $\frac{1}{2}|\rho_n|$, Fig. 8.

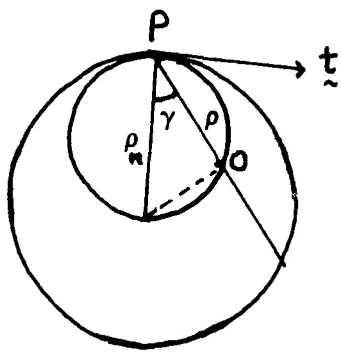


Figure 8.

§2.2. Principal Normal Curvatures.

From (47b), writing

$$\lambda = \frac{dv}{du}$$

we have

$$k_n = \frac{b_{11} + 2\lambda b_{12} + \lambda^2 b_{22}}{g_{11} + 2\lambda g_{12} + \lambda^2 g_{22}} \tag{57}$$

The coefficients $g_{\alpha\beta}$ and $b_{\alpha\beta}$ are constants at P, so that k_n is determined by the direction λ. Thus all curves through P having the same tangent and the same sense of $\underset{\sim}{n}$ have the same normal curvature k_n.

To find the extreme values of k_n, we differentiate k_n with respect to λ and set it equal to zero. Thus

$$\frac{dk_n}{d\lambda} = 0 \tag{58}$$

The roots of the above equation determine those directions for which the normal curvature k_n assumes extreme values. These directions are called the principal directions and the corresponding values of k_n are called the principal normal curvatures at P of the surface. To find these values, we first write (57) as

$$k_n = \frac{A + \lambda B}{C + \lambda D}$$

where

$$A = b_{11} + \lambda b_{12}$$

$$B = b_{12} + \lambda b_{22}$$

$$C = g_{11} + \lambda g_{12}$$

$$D = g_{12} + \lambda g_{22}$$

On using (58), we get

$$\frac{B}{D} = \frac{A}{C} = \frac{A + \lambda B}{C + \lambda D} = k_n$$

Elimination of λ between the two equations

$$B - D\, k_n = 0$$

$$A - C\, k_n = 0$$

gives

$$k_n^2 - \frac{1}{g_{(\nu)}}(g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11})k_n + \frac{b}{g_{(\nu)}} = 0 \tag{59a}$$

or

$$k_n^2 - b_{\alpha\beta}g^{\alpha\beta}k_n + \frac{b}{g_{(\nu)}} = 0 \tag{59b}$$

The roots of the above equation denoted as k_1 and k_2 are the principal normal curvatures. Obviously

$$k_1 + k_2 = b_{\alpha\beta}g^{\alpha\beta} \tag{60}$$

$$k_1 k_2 = \frac{b}{g_{(\nu)}} \tag{61}$$

Some definitions based on the above derivations are given below.

(i) Asymptotic directions:

Points on a surface where $k_n = 0$ give two directions through the equation

$$b_{22}\lambda^2 + 2b_{12}\lambda + b_{11} = 0$$

or

$$\frac{dv}{du} = \frac{-b_{12} \pm \sqrt{(b_{12})^2 - b_{11}b_{22}}}{b_{22}} \tag{62}$$

These directions are called the asymptotic directions. If a straight line can be drawn on a surface then it is obviously an asymptotic curve.

(ii) Lines of curvature:

The line of curvature is a curve whose direction is a principal direction at any of its points. That is, at every point of a line of curvature the normal curvature is either k_1 or k_2. Thus the lines of curvature are the solutions of the equation (58), viz.,

$$(g_{12}b_{22} - g_{22}b_{12})(\frac{dv}{du})^2 + (g_{11}b_{22} - g_{22}b_{11})\frac{dv}{du}$$

$$+ (g_{11}b_{12} - g_{12}b_{11}) = 0 \qquad (63)$$

Note that the equation (63) is equivalent to two equations of the first degree. Thus eq. (63) defines two families of curves on a surface. Moreover, the two curves are also orthogonal. (Refer to eqs. (123) for proof.)

(iii) Coordinate curves as lines of curvature:

If the curves u and v on a surface are lines of curvature, then from (63) we have for

$$u = \text{const.}$$

$$g_{12}b_{22} - g_{22}b_{12} = 0 \quad ,$$

and for

$$v = \text{const.}$$

$$g_{11}b_{12} - g_{12}b_{11} = 0$$

Thus in these coordinates

$$g_{12} = 0 \ , \ b_{12} = 0 \qquad (64)$$

Because of (64), (57) becomes

$$k_n = b_{11}(\frac{du}{ds})^2 + b_{22}(\frac{dv}{ds})^2$$

so that

$$\text{for}\quad v = \text{const.}\ ,\ (u\text{-curve})\ :\ k_1 = \frac{b_{11}}{g_{11}}$$

$$\text{for}\quad u = \text{const.}\ ,\ (v\text{-curve})\ :\ k_2 = \frac{b_{22}}{g_{22}} \tag{65}$$

(iv) Gaussian and mean curvatures:

The product of the principal normal curvatures as defined by (61) is the Gaussian curvature K.

$$K = k_1 k_2 = b/g_{(\nu)} \tag{66}$$

Similarly the mean curvature K_m is defined by (60) as

$$K_m = \frac{1}{2}(k_1 + k_2) \tag{67}$$

Surfaces for which $K_m = 0$ are called minimal surfaces.

The structure of the formula for the Gaussian curvature K given in (66) shows that it is an extrinsic property. In fact K is an intrinsic property of the surface, viz., it depends only on the coefficients $g_{\alpha\beta}$ of the first fundamental form and their derivatives. Refer to eq. (91) for this aspect of K.

Note that if $K > 0$ then both the principal normal curvatures have the same signs, while if $K < 0$ then they differ in sign. For example $K > 0$ for ellipsoids, elliptic paraboloids and spheres, etc., while $K < 0$ for hyperbolic paraboloids, hyperboloids, etc.

§2.3. Equations for the Derivatives of Surface Normal (Weingarten Equations).

Since

$$\underset{\sim}{n} \cdot \underset{\sim}{n} = 1$$

hence

$$\underset{\sim}{n} \cdot \underset{\sim}{n}_\alpha = 0 \ , \ \alpha = 1,2 \tag{68}$$

which shows that $\underset{\sim}{n}_\alpha$ lies in the tangent plane. Consequently $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ must be linear in $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$.

$$\underset{\sim}{n}_1 = P\underset{\sim}{r}_1 + Q\underset{\sim}{r}_2 \tag{69a}$$

$$\underset{\sim}{n}_2 = R\underset{\sim}{r}_1 + S\underset{\sim}{r}_2 \tag{69b}$$

To find P, ···, S, we first note that

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_1 = 0 \ , \ \underset{\sim}{n} \cdot \underset{\sim}{r}_2 = 0 \tag{70}$$

Differentiating the first equation in (70) by u^2 and the second by u^1, we get

$$\underset{\sim}{n}_1 \cdot \underset{\sim}{r}_2 = \underset{\sim}{n}_2 \cdot \underset{\sim}{r}_1 \tag{71}$$

Thus from (46b)

$$\left.\begin{aligned} -b_{11} &= \underset{\sim}{n}_1 \cdot \underset{\sim}{r}_1 \\ -b_{12} &= \underset{\sim}{n}_1 \cdot \underset{\sim}{r}_2 \\ &= \underset{\sim}{n}_2 \cdot \underset{\sim}{r}_1 \\ -b_{22} &= \underset{\sim}{n}_2 \cdot \underset{\sim}{r}_2 \end{aligned}\right\} \tag{72}$$

Using (72) in eqs. (69), we get

$$\left.\begin{aligned} Pg_{11} + Qg_{12} &= -b_{11} \\ Pg_{12} + Qg_{22} &= -b_{12} \\ Rg_{11} + Sg_{12} &= -b_{12} \\ Rg_{12} + Sg_{22} &= -b_{22} \end{aligned}\right\} \qquad (73)$$

Solving eqs. (73) we get P, Q, R, S, and hence

$$\underset{\sim}{n}_1 = \frac{1}{g_{(\nu)}}(b_{12}g_{12} - b_{11}g_{22})\underset{\sim}{r}_1 + \frac{1}{g_{(\nu)}}(b_{11}g_{12} - b_{12}g_{11})\underset{\sim}{r}_2 \qquad (74a)$$

$$\underset{\sim}{n}_2 = \frac{1}{g_{(\nu)}}(b_{22}g_{12} - b_{12}g_{22})\underset{\sim}{r}_1 + \frac{1}{g_{(\nu)}}(b_{12}g_{12} - b_{22}g_{11})\underset{\sim}{r}_2 \qquad (74b)$$

In suffix notation, eqs. (74) are written as

$$\underset{\sim}{n}_\alpha = -b_{\alpha\beta}g^{\beta\gamma}\underset{\sim}{r}_\gamma \qquad (74c)$$

Equations (74a,b) or (74c) are known as the Weingarten equations.

§2.4. Formulae of Gauss and the Surface Christoffel Symbols.

The vectors $\underset{\sim}{r}_1$, $\underset{\sim}{r}_2$ and $\underset{\sim}{n}$ form a system of independent vectors in a three-dimensional space. It should therefore be possible to express the vector $\underset{\sim}{r}_{\alpha\beta}$ in terms of these vectors. Thus we assume

$$\underset{\sim}{r}_{\alpha\beta} = T^\gamma_{\alpha\beta}\underset{\sim}{r}_\gamma + B_{\alpha\beta}\underset{\sim}{n} \qquad (75)$$

Since $\mathbf{n}$ is orthogonal to b $_1$ and $\mathbf{r}_2$, hence from (46a) we find that

$$B_{\alpha\beta} = b_{\alpha\beta} \quad .$$

Next taking the dot product of (75) with $\mathbf{r}_\delta$ we get

$$T^{\gamma}_{\alpha\beta} g_{\gamma\delta} = \mathbf{r}_{\alpha\beta} \cdot \mathbf{r}_\delta \tag{76}$$

We now write

$$\mathbf{r}_{\alpha\beta} \cdot \mathbf{r}_\delta = [\alpha\beta, \delta] \tag{77}$$

where as before (refer to eq. (129) of Part I), the quantities $[\alpha\beta,\delta]$ are called the Christoffel symbols of the first kind. If we now take the inner multiplication of both sides of eq. (76) with $g^{\sigma\delta}$, we get

$$T^{\sigma}_{\alpha\beta} = g^{\delta\sigma}[\alpha\beta, \delta] \tag{78}$$

The quantities $T^{\sigma}_{\alpha\beta}$ defined in (78) are the Christoffel symbols of the second kind. (Refer to eq. (133) of Part I.)

The equations

$$\mathbf{r}_{\alpha\beta} = T^{\gamma}_{\alpha\beta}\mathbf{r}_\gamma + b_{\alpha\beta}\mathbf{n} \tag{79}$$

are called the formulae of Gauss for the second derivatives $\mathbf{r}_{\alpha\beta}$.

The Christoffel symbols defined in (77) and (78) have got exactly the same structure as in the general case discussed in Part I, §4.

Note: The symbol T is the capital upsilon of the Greek alphabets.

However, because of the two-dimensional manifold under consideration, the Greek indices range only from 1 to 2. We shall keep the notation T in place of Γ for the Christoffel symbols of the second kind so as not to cause confusion in their use to be discussed in Part III.

§2.4.1. Christoffel Symbols.

For future references, we now list the expanded forms of the Christoffel symbols for a surface.

$$[\alpha\beta,\delta] = \frac{1}{2}\left(\frac{\partial g_{\beta\delta}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}}\right) \tag{80}$$

$$T^{\sigma}_{\alpha\beta} = g^{\delta\sigma}[\alpha\beta,\delta] \tag{81}$$

Writing $u^1 = u$, $u^2 = v$, we have

$$T^{1}_{11} = [g_{22}\frac{\partial g_{11}}{\partial u} + g_{12}(\frac{\partial g_{11}}{\partial v} - 2\frac{\partial g_{12}}{\partial u})]/2g_{(\nu)} \tag{82a}$$

$$T^{2}_{22} = [g_{11}\frac{\partial g_{22}}{\partial v} + g_{12}(\frac{\partial g_{22}}{\partial u} - 2\frac{\partial g_{12}}{\partial v})]/2g_{(\nu)} \tag{82b}$$

$$T^{1}_{22} = [g_{22}(2\frac{\partial g_{12}}{\partial v} - \frac{\partial g_{22}}{\partial u}) - g_{12}\frac{\partial g_{22}}{\partial v}]/2g_{(\nu)} \tag{82c}$$

$$T^{2}_{11} = [g_{11}(2\frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}) - g_{12}\frac{\partial g_{11}}{\partial u}]/2g_{(\nu)} \tag{82d}$$

$$T^{1}_{12} = T^{1}_{21} = (g_{22}\frac{\partial g_{11}}{\partial v} - g_{12}\frac{\partial g_{22}}{\partial u})/2g_{(\nu)} \tag{82e}$$

$$T^2_{12} = T^2_{21} = (g_{11} \frac{\partial g_{22}}{\partial u} - g_{12} \frac{\partial g_{11}}{\partial v})/2g_{(\nu)} \tag{82f}$$

$$\frac{1}{2g_{(\nu)}} \frac{\partial g_{(\nu)}}{\partial u} = T^1_{11} + T^2_{12} \tag{82g}$$

$$\frac{1}{2g_{(\nu)}} \frac{\partial g_{(\nu)}}{\partial v} = T^1_{12} + T^2_{22} \tag{82h}$$

Note that the $g_{\alpha\beta}$ in eqs. (82) are those which have been defined in (32) and (33).

From eq. (134) of Part I, we have the result

$$\underset{\sim}{r}_{ij} = \Gamma^{\ell}_{ij} \underset{\sim}{r}_{\ell} \tag{83}$$

where

$$\underset{\sim}{r}_{\ell} = \frac{\partial \underset{\sim}{r}}{\partial x^{\ell}} = \underset{\sim}{a}_{\ell}$$

$$\underset{\sim}{r}_{ij} = \frac{\partial^2 \underset{\sim}{r}}{\partial x^i \partial x^j} = \frac{\partial \underset{\sim}{a}_i}{\partial x^j}$$

It is worthwhile to compare (79) and (83) in the same coordinates in the sense that at the surface both should coincide. This idea will be explored fully in Part III.

§2.5. Intrinsic Nature of the Gaussian Curvature (Equations of Codazzi and Mainardi).

The position vector $\underset{\sim}{r}$ in an Euclidean space can always be represented in terms of the constant unit vecotrs. Thus it is clear that

$$\frac{\partial}{\partial u^{\gamma}}(\mathbf{r}_{\alpha\beta}) = \frac{\partial}{\partial u^{\beta}}(\mathbf{r}_{\alpha\gamma}) \tag{84}$$

for any choice of α, β and γ. We now use eq. (84) to obtain some important results of the surface theory.

On differentiating eq. (79) and using (74c), while properly taking care of the dummy indices, eq. (84) yields

$$\begin{aligned}
&[\{\frac{\partial T^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial T^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} + T^{\sigma}_{\alpha\beta}T^{\delta}_{\gamma\sigma} - T^{\lambda}_{\alpha\gamma}T^{\delta}_{\beta\lambda}\} \\
&\qquad -g^{\sigma\delta}(b_{\alpha\beta}b_{\gamma\sigma} - b_{\alpha\gamma}b_{\beta\sigma})]\mathbf{r}_{\delta} \\
&+ (\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} + T^{\delta}_{\alpha\beta}b_{\gamma\delta} - T^{\lambda}_{\alpha\gamma}b_{\beta\lambda})\mathbf{n} = 0
\end{aligned} \tag{85}$$

Since $\mathbf{r}_{\delta}$ ($\delta = 1,2$) and $\mathbf{n}$ are independent vectors, hence the coefficients of $\mathbf{r}_{\delta}$ and $\mathbf{n}$ must vanish separately.

The term in curly brackets in (85) is the two-dimensional version of the Riemann-Christoffel tensor defined in Part I, eq. (240). For the sake of clarity, we use R^{*} in place of R for the two-dimensional case. Thus (85) yields the equations

$$R^{*\delta}_{\cdot\alpha\gamma\beta} - g^{\sigma\delta}(b_{\alpha\beta}b_{\gamma\sigma} - b_{\alpha\gamma}b_{\beta\sigma}) = 0 \tag{86}$$

$$\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} + T^{\delta}_{\alpha\beta}b_{\gamma\delta} - T^{\lambda}_{\alpha\gamma}b_{\beta\lambda} = 0 \tag{87}$$

A two-dimensional Riemann curvature tensor (similar to eq. (244), Part I) is now introduced as

$$\overset{*}{R}_{\mu\alpha\gamma\beta} = g_{\mu\delta}\overset{*}{R}{}^{\delta}_{\cdot\alpha\gamma\beta} \tag{88}$$

Thus (86) becomes

$$\overset{*}{R}_{\mu\alpha\gamma\beta} = b_{\alpha\beta}b_{\gamma\mu} - b_{\alpha\gamma}b_{\beta\mu} \tag{89}$$

As discussed in Part I, §8.1, the covariant Riemann tensor has in all 16 components in a two-dimensional space out of which, apart from sign, only four are non-zero. Thus the four components are

$$\overset{*}{R}_{1212}\ ,\ \overset{*}{R}_{2121}\ ,\ \overset{*}{R}_{2112}\ ,\ \overset{*}{R}_{1221}$$

where

$$\overset{*}{R}_{1212} = \overset{*}{R}_{2121}$$

and

$$\overset{*}{R}_{2112} = \overset{*}{R}_{1221}$$

Therefore from (89)

$$\overset{*}{R}_{1212} = \overset{*}{R}_{2121} = b \tag{90a}$$

$$\overset{*}{R}_{2112} = \overset{*}{R}_{1221} = -b \tag{90b}$$

Using eq. (66), we get from (90a)

$$K = \overset{*}{R}_{1212}/g_{(\nu)} \tag{91}$$

Equation (91) shows that the Gaussian curvature is an intrinsic property of the surface, since $\overset{*}{R}_{1212}$ is formed purely of the coefficients $g_{\alpha\beta}$ and their derivatives. This is the "Theorema egregium" of Gauss.

Equations (87) are known as the Codazzi-Mainardi equations. The only two possible equations from (87) are

$$\frac{\partial b_{11}}{\partial v} - \frac{\partial b_{12}}{\partial u} - b_{11}T^1_{12} + (T^1_{11} - T^2_{12})b_{12} + T^2_{11}b_{22} = 0 \tag{92a}$$

$$\frac{\partial b_{22}}{\partial u} - \frac{\partial b_{12}}{\partial v} + b_{11}T^1_{22} + (T^2_{22} - T^1_{12})b_{12} - T^2_{12}b_{22} = 0 \tag{92b}$$

§2.5.1. A Particular Form of Codazzi Equations.

Consider the case when u and v are the lines of curvature, so that

$$g_{12} = 0 \ , \ b_{12} = 0 \ . \tag{93}$$

From (82):

$$\begin{aligned} T^1_{12} &= \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial v} \ , \quad T^2_{11} = -\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial v} \\ T^2_{12} &= \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial u} \ , \quad T^1_{22} = -\frac{1}{2g_{11}}\frac{\partial g_{22}}{\partial u} \end{aligned} \tag{94}$$

The set of eqs. (92) take the form (by the use of eq. (65)),

$$\frac{\partial b_{11}}{\partial v} = \frac{1}{2}(k_1 + k_2)\frac{\partial g_{11}}{\partial v} \tag{95a}$$

$$\frac{\partial b_{22}}{\partial u} = \frac{1}{2}(k_1 + k_2)\frac{\partial g_{22}}{\partial u} \tag{95b}$$

Opening the derivatives $\frac{\partial}{\partial v}(k_1 g_{11})$ and $\frac{\partial}{\partial u}(k_2 g_{22})$ and using (95), we get

$$\frac{\partial k_1}{\partial v} = \frac{1}{2g_{11}}(k_2 - k_1)\frac{\partial g_{11}}{\partial v} \tag{96a}$$

$$\frac{\partial k_2}{\partial u} = \frac{1}{2g_{22}}(k_1 - k_2)\frac{\partial g_{22}}{\partial u} \tag{96b}$$

§2.5.2. The Third Fundamental Form.

Let all the unit normal vectors $\underset{\sim}{n}$ to a surface have been translated parallel to themselves such that their initial points are tied at the origin of coordinates. The terminal points will then lie on the surface of a unit sphere, for ordinary surfaces. The first fundamental form for this sphere will then be

$$c_{\alpha\beta}du^{\alpha}du^{\beta}$$

where

$$c_{\alpha\beta} = \underset{\sim}{n}_{\alpha} \cdot \underset{\sim}{n}_{\beta} \tag{97}$$

As before, denoting the components of $\underset{\sim}{n}$ with respect to the Cartesian coordinates as X, Y, Z, we have the expansions

$$c_{11} = X_u^2 + Y_u^2 + Z_u^2$$

$$c_{12} = X_u X_v + Y_u Y_v + Z_u Z_v \tag{98}$$

$$c_{22} = X_v^2 + Y_v^2 + Z_v^2$$

Using the Weingarten equations (74) or (74c), we can also write (98) as

$$c_{11} = (g_{22}b_{11}^2 - 2g_{12}b_{12}b_{11} + g_{11}b_{12}^2)/g_{(\nu)}$$

$$c_{12} = [g_{22}b_{11}b_{12} - g_{12}(b_{11}b_{22} + b_{12}^2) + g_{11}b_{12}b_{22}]/g_{(\nu)} \tag{99}$$

$$c_{22} = (g_{22}b_{12}^2 - 2g_{12}b_{12}b_{22} + g_{11}b_{22}^2)/g_{(\nu)}$$

Also

$$c = c_{11}c_{22} - (c_{12})^2 = b^2/g_{(\nu)} \tag{100}$$

§2.6. The Geodesic Curvature.

In §2, eq. (40) we wrote the curvature vector $\hat{\underset{\sim}{k}}$ as the sum of the normal curvature $\underset{\sim}{k}_n$ and a tangential curvature $\underset{\sim}{k}_g$. The vector $\underset{\sim}{k}_g$ is called the geodesic curvature vector, and its magnitude as the geodesic curvature k_g.

Since the vector $\underset{\sim}{k}_g$ lies in the tangent plane to a surface, we define a unit vector $\underset{\sim}{e}$ as

$$\underset{\sim}{e} = \underset{\sim}{n} \times \underset{\sim}{t} \qquad \text{(Fig. 7)} \tag{101}$$

and write

$$\underset{\sim}{k}_g = \underset{\sim}{e}\, k_g \tag{102}$$

Now

$$k_g = \underset{\sim}{e} \cdot \hat{\underset{\sim}{k}}$$

$$= \underset{\sim}{e} \cdot \frac{d\underset{\sim}{t}}{ds} = (\underset{\sim}{n} \times \underset{\sim}{t}) \cdot \frac{d\underset{\sim}{t}}{ds}$$

$$= (\underset{\sim}{t} \times \frac{d\underset{\sim}{t}}{ds}) \cdot \underset{\sim}{n} \tag{103}$$

Further

$$\underset{\sim}{t} = \frac{d\underset{\sim}{r}}{ds} = r_\alpha \frac{du^\alpha}{ds} \tag{104a}$$

and

$$\frac{d\underset{\sim}{t}}{ds} = \frac{d^2\underset{\sim}{r}}{ds^2} = \underset{\sim}{r}_{\alpha\beta} \frac{du^\alpha}{ds} \frac{du^\beta}{ds} + \underset{\sim}{r}_\alpha \frac{d^2u^\alpha}{ds^2} \tag{104b}$$

Using (79) in (104b) and putting in (103), we get after some simplifications

$$k_g = \sqrt{g_{(\nu)}}\,[\Gamma^2_{11}(\frac{du}{ds})^3 - \Gamma^1_{22}(\frac{dv}{ds})^3 + (2\Gamma^2_{12} - \Gamma^1_{11})(\frac{du}{ds})^2 \frac{dv}{ds}$$

$$- (2\Gamma^1_{12} - \Gamma^2_{22})(\frac{dv}{ds})^2 \frac{du}{ds} + \frac{du}{ds}\frac{d^2v}{ds^2} - \frac{dv}{ds}\frac{d^2u}{ds^2}] \tag{105}$$

Thus, on the curves u = const., and v = const., we have

$$(k_g)_{u\,=\,\text{const.}} = -\sqrt{g_{(\nu)}}\;\Gamma^1_{22}/g_{22}^{3/2} \tag{106a}$$

$$(k_g)_{v\,=\,\text{const.}} = \sqrt{g_{(\nu)}}\;T^2_{11}/g^{3/2}_{11} \tag{106b}$$

If the coordinates are orthogonal, then

$$(k_g)_{u\,=\,\text{const.}} = \frac{1}{\sqrt{g_{11}}}\frac{\partial}{\partial u}(\ln\sqrt{g_{22}}) \tag{107a}$$

$$(k_g)_{v\,=\,\text{const.}} = \frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial v}(\ln\sqrt{g_{11}}) \tag{107b}$$

A curve C on a surface S is called a geodesic curve or simply geodesic if its geodesic curvature vanishes. Therefore for u = const. to be a geodesic

$$T^1_{22} = 0 \tag{108a}$$

similarly for v = const. to be a geodesic

$$T^2_{11} = 0 \tag{108b}$$

§2.6.1. Geodesics and Parallelism on a Surface.

Having defined the geodesics as curves on a surface whose geodesic curvature at each point is zero, we must now find the differential equations of the geodesics.

A vector in the tangent plane of a surface is known as a surface vector, or a vector in the surface. If u^1 and u^2 are the surface coordinates, then according to eq. (22b) the vectors $\underset{\sim}{r}_\alpha$ are the surface base vectors, and they are related with the space base vectors $\underset{\sim}{a}_i$ as

$$\underset{\sim}{r}_\alpha = \underset{\sim}{a}_i x^i_\alpha \;, \quad x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \quad .$$

Let $\underset{\sim}{A}$ be a surface vector field. Then

$$\underset{\sim}{A} = \underset{\sim}{r}_\alpha B^\alpha$$

Since $\underset{\sim}{A}$ can also be regarded as a vector field in E^3, hence

$$\underset{\sim}{A} = \underset{\sim}{a}_i A^i$$

Thus

$$A^i = B^\alpha x^i_\alpha \quad .$$

Let us consider a curve on the surface whose parametric equations are

$$u^\alpha = u^\alpha(t) \quad .$$

Then

$$\underset{\sim}{A} = \underset{\sim}{A}(u^\alpha(t)) \quad .$$

Consequently

$$\frac{d\underset{\sim}{A}}{dt} = \frac{dB^\alpha}{dt}\underset{\sim}{r}_\alpha + B^\alpha \underset{\sim}{r}_{\alpha\beta}\frac{du^\beta}{dt}$$

Using (79), we get

$$\frac{d\underset{\sim}{A}}{dt} = [B^\alpha{}_{,\beta}\underset{\sim}{r}_\alpha + B^\alpha b_{\alpha\beta}\underset{\sim}{n}]\frac{du^\beta}{dt} \tag{109a}$$

where $B^{\alpha}_{,\beta}$ is the covariant derivative[†] of B^{α} defined as

$$B^{\alpha}_{,\beta} = \frac{\partial B^{\alpha}}{\partial u^{\beta}} + B^{\gamma} T^{\alpha}_{\gamma\beta} \tag{109b}$$

The quantity

$$\frac{\delta B^{\alpha}}{\delta t} = B^{\alpha}_{,\beta} \frac{du^{\beta}}{dt} \tag{109c}$$

is called the intrinsic derivative of $\underset{\sim}{A}$ on the curve $u^{\alpha} = u^{\alpha}(t)$.

It is interesting to note that

$$\underset{\sim}{A} \cdot \frac{d\underset{\sim}{A}}{dt} = B_{\alpha} \frac{\delta B^{\alpha}}{\delta t}$$

where B_{α} are the covariant surface components of $\underset{\sim}{A}$. Consequently

$$\frac{d}{dt}(|\underset{\sim}{A}|^2) = \frac{\delta}{\delta t}(B_{\alpha} B^{\alpha}) \tag{109d}$$

In place of an arbitrary vector $\underset{\sim}{A}$, let us consider the tangent vector field $\underset{\sim}{t}(s)$ on a surface curve $u^{\alpha} = u^{\alpha}(s)$. Then

$$\underset{\sim}{t} = \underset{\sim}{r}_{\alpha} \frac{du^{\alpha}}{ds}$$

and

$$\frac{d\underset{\sim}{t}}{ds} = [t^{\alpha}_{,\beta} \underset{\sim}{r}_{\alpha} + t^{\alpha} b_{\alpha\beta} \underset{\sim}{n}] \frac{du^{\beta}}{ds} \tag{110}$$

[†]The definition of the covariant derivative in a surface is the same as in any other space. Refer to Part I, §4. The only care one should take is to replace Γ by T for the Christoffel symbols.

$$= \hat{\underset{\sim}{k}}$$

$$= \underset{\sim}{k}_n + \underset{\sim}{k}_g$$

Since on a geodesic curve $\underset{\sim}{k}_g = 0$, hence from (110) we find that for the geodesics

$$t^{\alpha}_{,\beta} \frac{du^{\beta}}{ds} = 0 \tag{111}$$

Thus the geodesics in a surface are the solution curves of the equations

$$\frac{d^2u^{\alpha}}{ds^2} + T^{\alpha}_{\beta\gamma} \frac{du^{\beta}}{ds} \frac{du^{\gamma}}{ds} = 0 \ , \ \alpha = 1,2 \ . \tag{112}$$

The definition of a parallel field of vectors in a space of any dimension was given in Part I, §7, eq. (220b). The same definition is applicable in the surface, viz., for a parallel field of vectors $\underset{\sim}{A}$, the intrinsic derivative is zero.

$$\frac{dB^{\alpha}}{dt} + B^{\gamma}T^{\alpha}_{\beta\gamma} \frac{du^{\beta}}{dt} = 0 \tag{113}$$

Equation (113) is also called the condition of parallel displacement in the sense of Levi-Civita. This means that in the covariant differentiation the Christoffel symbols $T^{\alpha}_{\beta\gamma}$ are used. It must be noted that a covariant differentiation can also be defined in which another three index symbol, say $G^{\alpha}_{\beta\gamma}$, is introduced, Weyl [33].

§2.7. Differential Parameters of Beltrami.

E. Beltrami in 1864 introduced four differential parameters which greatly simplify the representation of some formulae in the surface theory. If ϕ is a function of the surface coordinates u and v, then the differential parameters of the first and second orders are as follows.

(i) First order:

$$\Delta_1\phi = (g_{11}\phi_v^2 - 2g_{12}\phi_u\phi_v + g_{22}\phi_u^2)/g_{(\nu)} \tag{114a}$$

$$\Delta_1(\phi,\psi) = [g_{11}\phi_v\psi_v - g_{12}(\phi_u\psi_v + \phi_v\psi_u) + g_{22}\phi_u\psi_u]/g_{(\nu)} \tag{114b}$$

(ii) Second order:

$$\Delta_2\phi = [\frac{\partial}{\partial u}(\frac{g_{22}\phi_u - g_{12}\phi_v}{\sqrt{g_{(\nu)}}}) + \frac{\partial}{\partial v}(\frac{g_{11}\phi_v - g_{12}\phi_u}{\sqrt{g_{(\nu)}}})]/\sqrt{g_{(\nu)}} \tag{114c}$$

$$\begin{aligned}\Delta_{22}\phi = [(\phi_{uu} - T_{11}^1\phi_u - T_{11}^2\phi_v)(\phi_{vv} - T_{22}^1\phi_u - T_{22}^2\phi_v) \\ - (\phi_{uv} - T_{12}^1\phi_u - T_{12}^2\phi_v)^2]/g_{(\nu)}\end{aligned} \tag{114d}$$

The parameter $\Delta_1\phi$ is the surface gradient of ϕ, viz.,

$$\Delta_1\phi = g^{\alpha\beta}\,\frac{\partial\phi}{\partial u^\alpha}\,\frac{\partial\phi}{\partial u^\beta}$$

The parameter $\Delta_1(\phi,\psi)$ is related with the angle θ between two curves ϕ = const., ψ = const., as

$$\Delta_1(\phi,\psi) = -\sqrt{(\Delta_1\phi)(\Delta_1\psi)}\cos\theta \tag{115}$$

Thus the curves ϕ = const., and ψ = const., are orthogonal if

$$\Delta_1(\phi,\psi) = 0 \tag{116}$$

The condition equation (116) can be linked with a second degree equation in dv/du whose solution curves are orthogonal. We recover this theorem in a way different from that in Ref. [6], p. 80, as follows.

For ϕ = const., and ψ = const., we have the auxiliary equations

$$\phi_u du + \phi_v dv = 0$$

$$\psi_u du + \psi_v dv = 0$$

Thus

$$\frac{dv}{du} = \frac{-\phi_u}{\phi_v} = \frac{-\psi_u}{\psi_v} \tag{117}$$

so that

$$\phi_u\psi_v - \phi_v\psi_u = 0 \tag{118}$$

Using (118) in (116), we have

$$g_{11}\phi_v\psi_v - 2g_{12}\phi_v\psi_u + g_{22}\phi_u\psi_u = 0 \tag{119}$$

We now introduce the symbols

$$\phi_v\psi_v = T \ , \ \phi_v\psi_u = S \ , \ \phi_u\psi_u = R \tag{120}$$

so that the condition of orthogonality is

$$g_{11}T - 2g_{12}S + g_{22}R = 0 \tag{121}$$

Using (117) in (120), we also have

$$T = \phi_u \psi_u \left(\frac{du}{dv}\right)^2$$

$$S = -\phi_u \psi_u \frac{du}{dv} \qquad (122)$$

$$R = \phi_u \psi_u$$

From (122), we find that

$$R(du)^2 + 2S\, dudv + T(dv)^2$$

is identically zero. We therefore state the main result as follows.

"The ordinary differential equation

$$R(du)^2 + 2S\, dudv + T(dv)^2 = 0 \qquad (123a)$$

for arbitrary R, S and T yields orthogonal solution curves if and only if

$$g_{11}T - 2g_{12}S + g_{22}R = 0 \quad ." \qquad (123b)$$

The second order differential <u>parameter</u> given in (114c) also allows us to define a second order differential <u>operator</u>,

$$\Delta_2 = \frac{1}{\sqrt{g_{(\nu)}}}\left[\frac{\partial}{\partial u}\left\{\frac{1}{\sqrt{g_{(\nu)}}}\left(g_{22}\frac{\partial}{\partial u} - g_{12}\frac{\partial}{\partial v}\right)\right\}\right.$$

$$\left. + \frac{\partial}{\partial v}\left\{\frac{1}{\sqrt{g_{(\nu)}}}\left(g_{11}\frac{\partial}{\partial v} - g_{12}\frac{\partial}{\partial u}\right)\right\}\right] \qquad (124)$$

Thus

$$\Delta_2 u = \frac{1}{\sqrt{g_{(\nu)}}}\left[\frac{\partial}{\partial u}\left(\frac{g_{22}}{\sqrt{g_{(\nu)}}}\right) - \frac{\partial}{\partial v}\left(\frac{g_{12}}{\sqrt{g_{(\nu)}}}\right)\right] \tag{125}$$

$$\Delta_2 v = \frac{1}{\sqrt{g_{(\nu)}}}\left[\frac{\partial}{\partial v}\left(\frac{g_{11}}{\sqrt{g_{(\nu)}}}\right) - \frac{\partial}{\partial u}\left(\frac{g_{12}}{\sqrt{g_{(\nu)}}}\right)\right] \tag{126}$$

In Part I, eq. (190) we introduced a second order differential operator in a plane. For a two-dimensional surface it assumes the same form

$$D = g_{22}\partial_{uu} - 2g_{12}\partial_{uv} + g_{11}\partial_{vv} \tag{127}$$

Using (125)-(127) with

$$F = g_{(\nu)}\Delta_2 u \ , \ G = g_{(\nu)}\Delta_2 v \tag{128}$$

we can write (114c) as

$$\Delta_2\phi = (D\phi + F\phi_u + G\phi_v)/g_{(\nu)} \quad . \tag{129}$$

§2.7.1. First Differential Parameters.

Let x, y, z be the Cartesian coordinates. It is easy to show based on the expansions (32) and the components of the unit normal to a surface, eq. (38), that

$$\Delta_1 x = 1 - X^2$$

$$\Delta_1 y = 1 - Y^2$$

$$\Delta_1 z = 1 - Z^2$$

$$\Delta_1(x,y) = -XY$$

$$\Delta_1(y,z) = -YZ$$

$$\Delta_1(x,z) = -XZ \tag{130}$$

If u and v are the surface coordinates, then

$$\Delta_1 u = g^{11} = g_{22}/g_{(\nu)}$$

$$\Delta_1 v = g^{22} = g_{11}/g_{(\nu)} \tag{131}$$

$$\Delta_1(u,v) = g^{12} = -g_{12}/g_{(\nu)}$$

§3. Mapping of Surfaces.

Let there be two surfaces S and $\overline{S}$ in which the parametric coordinates are denoted as (ξ,η) and $(\overline{\xi},\overline{\eta})$ respectively. The mapping of a portion of S onto a portion of $\overline{S}$ is called a one-to-one mapping when it is possible to establish the functional relations

$$\overline{\xi} = f_1(\xi,\eta) \ , \ \overline{\eta} = f_2(\xi,\eta) \tag{132}$$

where f_1 and f_2 are differentiable functions of the desired orders, and the Jacobian of the transformation is not zero, viz.,

$$\overline{\xi}_\xi \overline{\eta}_\eta - \overline{\xi}_\eta \overline{\eta}_\xi \neq 0 \ .$$

In a three-dimensional space E^3, the two points on S and $\bar{S}$, which transform into one another under the mapping (132), are respectively

$$\underset{\sim}{r} = \underset{\sim}{\phi}(\xi,\eta) \tag{133a}$$

$$\bar{\underset{\sim}{r}} = \underset{\sim}{\psi}(\bar{\xi},\bar{\eta}) \quad . \tag{133b}$$

Using (132) in (133b), we get

$$\begin{aligned}\bar{\underset{\sim}{r}} &= \underset{\sim}{\psi}(f_1(\xi,\eta),\ f_2(\xi,\eta)) \\ &= \underset{\sim}{\chi}(\xi,\eta) \quad .\end{aligned}$$

Thus the two points which are the images of one another are representable through the same parametric coordinates as

$$\left.\begin{aligned}\underset{\sim}{r} &= \underset{\sim}{\phi}(\xi,\eta) \\ \bar{\underset{\sim}{r}} &= \underset{\sim}{\chi}(\xi,\eta)\end{aligned}\right. \tag{134}$$

Equation (134) expresses the meaning of the sentence, "<u>the coordinate systems on S and $\bar{S}$ are the same</u>."

Below we discuss various mappings from one surface to another. Some definitions have been taken directly from Ref. [18].

(i) Isometric mapping:

A mapping of a portion S of a surface onto a portion $\bar{S}$ of a surface is isometric if and only if at corresponding points of S and $\bar{S}$, when referred to the same coordinate systems on S and $\bar{S}$, the values of $g_{\alpha\beta}$ on S and $\bar{S}$ are the same.

Thus for isometric mapping

$$\bar{g}_{\alpha\beta}(\xi,\eta) = g_{\alpha\beta}(\xi,\eta) \quad . \tag{135}$$

(ii) Equiareal mapping:

A mapping of a portion S of a surface onto a portion $\bar{S}$ of a surface is equiareal if and only if at corresponding points of S and $\bar{S}$, when referred to the same coordinate systems on S and $\bar{S}$, the values of $g_{(\nu)}$ and $\bar{g}_{(\nu)}$ of the first fundamental form are equal.

Thus for equiareal mapping

$$\bar{g}_{(\nu)}(\xi,\eta) = g_{(\nu)}(\xi,\eta) \tag{136}$$

(iii) Geodesic mapping:

A mapping of a portion S of a surface onto a portion $\bar{S}$ of a surface is geodesic if and only if at corresponding points, when referred to the same coordinate systems on S and $\bar{S}$, the following relation holds.

$$T^{\gamma}_{\alpha\beta} = \bar{T}^{\gamma}_{\alpha\beta} + \delta^{\gamma}_{\alpha}\frac{\partial f}{\partial \bar{u}^{\beta}} + \delta^{\gamma}_{\beta}\frac{\partial f}{\partial \bar{u}^{\alpha}} \tag{137}$$

where

$$f = \frac{1}{6}\,\ell n(g/\bar{g}) \tag{138}$$

and

$$\bar{u}^1 = \bar{\xi} = \xi$$

$$\bar{u}^2 = \bar{\eta} = \eta \quad .$$

The proof of (137) follows directly from (105) equated to zero. Since the proof of all the statements in (i), (ii), and (iii) above are already available in Ref. [18], the reader is referred to that work.

(iv) Conformal mapping:

A mapping of a portion of a surface S onto a portion of a surface $\overline{S}$ is conformal if and only if, when referred to the same coordinate systems on S and $\overline{S}$, the coefficients $g_{\alpha\beta}$ and $\overline{g}_{\alpha\beta}$ are proportional at each point, viz.,

$$\overline{g}_{\alpha\beta} = \lambda(\xi,\eta)g_{\alpha\beta} \quad . \tag{139}$$

As the name "conformal" suggests, the angle between the two intersecting arcs in S is preserved in mapping to $\overline{S}$.

(v) Conformal mapping of surfaces in a plane:

A theorem on conformal mapping of surfaces in a plane states that: "Every portion of a surface S, which is at least three times continuously differentiable, can be conformally mapped into a plane."

In a plane it is always possible to introduce Cartesian coordinates. If we denote these coordinates by u and v, then we will first show that from a general coordinates ξ,η in a surface in which the metric is given as

$$(ds)^2 = g_{11}(d\xi)^2 + 2g_{12}d\xi d\eta + g_{22}(d\eta)^2 \tag{140}$$

we can devise a transformation such that the same ds is given by

$$(ds)^2 = \lambda(u,v)[(du)^2 + (dv)^2] \tag{141}$$

The coordinates u and v are called the <u>isothermic coordinates</u>.

First note that (140) can be factored as

$$(ds)^2 = [\sqrt{g_{11}}\, d\xi + \frac{1}{\sqrt{g_{11}}}(g_{12} + i\sqrt{g_{(\nu)}})d\eta][\sqrt{g_{11}}\, d\xi + \frac{1}{\sqrt{g_{11}}}(g_{12} - i\sqrt{g_{(\nu)}}d\eta)]$$

where $i = \sqrt{-1}$. For each term in the brackets there exists an integrating factor. Let $f_1(\xi,\eta)$ and $f_2(\xi,\eta)$ be real functions, then we can form perfect differentials

$$d\xi' = (f_1 + if_2)[\sqrt{g_{11}}\, d\xi + \frac{1}{\sqrt{g_{11}}}(g_{12} + i\sqrt{g_{(\nu)}})d\eta]$$

$$d\eta' = (f_1 - if_2)[\sqrt{g_{11}}\, d\xi + \frac{1}{\sqrt{g_{11}}}(g_{12} - i\sqrt{g_{(\nu)}})d\eta]$$

Thus

$$(ds)^2 = \frac{d\xi' d\eta'}{f_1^{\,2} + f_2^{\,2}} \tag{142}$$

The curves ξ' = const., and η' = const., are called isotropic curves.

Since ξ' and η' are complex conjugates, hence

$$\xi' = u + iv \ , \ \eta' = u - iv \tag{143}$$

Using (143) in (142), we get

$$(ds)^2 = \lambda(u,v)[(du)^2 + (dv)^2]$$

where

$$\lambda = (f_1^{\,2} + f_2^{\,2})^{-1} \ .$$

We thus find that a coordinate transformation from (ξ,η) to (u,v) exists in which

$$g_{22} = g_{11} \; , \; g_{12} = 0 \qquad (144)$$

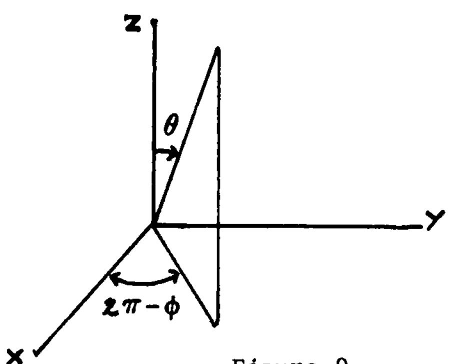
The above analysis proves the theorem of conformal mapping of portions of S into a plane, and also introduces the concept of the <u>isothermic coordinates</u>. In essence, the isothermic coordinates in a surface are those coordinates which are orthogonal and in which $g_{22} = g_{11} = \lambda$, so that the metric in the surface is given by (141).

§3.1. Isothermic and Equiareal Coordinates on a Sphere.

We take the parametric equation of a sphere of unit radius as, (refer to Fig. 9 and eq. (153)),

$$x = \sin\theta\cos\phi \; , \; y = -\sin\theta\sin\phi \; , \; z = \cos\theta$$

where ϕ and θ are measured clockwise from the x- and z-axes respectively, and $\theta = 0$, $\theta = \pi$ represent the north and south poles respectively.



$$0 \le \theta < \pi$$

$$0 \le \phi < 2\pi$$

Figure 9.

Thus $g_{11} = \sin^2\theta$, $g_{22} = 1$, $g_{(\nu)} = \sin^2\theta$, so that the metric on the surface is

$$(ds)^2 = \sin^2\theta(d\phi)^2 + (d\theta)^2 \tag{144}$$

(i) Isothermic coordinates on a sphere:

We follow the technique shown previously from eqs. (141)-(143).

First

$$d\xi' = (f_1 + if_2)\sin\theta \cdot d(\phi + i \ \ln \tan \frac{\theta}{2})$$

$$d\eta' = (f_1 - if_2)\sin\theta \cdot d(\phi - i \ \ln \tan \frac{\theta}{2})$$

Second

$$f_1 = \frac{1}{\sin\theta} , \ f_2 = 0$$

then

$$\xi' = \phi + i \ \ln \tan \frac{\theta}{2}$$

$$= u + iv$$

$$\eta' = \phi - i \ \ln \tan \frac{\theta}{2}$$

$$= u - iv$$

Equating the real and imaginary parts, we get the mapping

$$\phi = u \ , \ \theta = 2\tan^{-1}(e^v) \tag{145a}$$

or

$$u = \phi \ , \ v = \ln \tan \frac{\theta}{2} \tag{145b}$$

The equations (145) define the isothermic coordinates on a unit sphere. It is an easy matter to verify that using (145), the Cartesian coordinates are

$$x = \frac{2e^{v}\cos u}{1 + e^{2v}}, \quad y = \frac{-2e^{v}\sin u}{1 + e^{2v}}, \quad z = \frac{1 - e^{2v}}{1 + e^{2v}} \tag{146}$$

in which the metric coefficients are

$$g_{22} = g_{11} = \frac{4e^{2v}}{(1 + e^{2v})^2} \tag{147}$$

and the metric has the required form,

$$(ds) = \frac{4e^{2v}}{(1 + e^{2v})^2}[(du)^2 + (dv)^2] \tag{148}$$

(ii) Equiareal (mapping) coordinates:

The mapping

$$u = \phi, \; v = 2\sin\frac{\theta}{2} \tag{149}$$

yields the metric

$$(ds)^2 = \frac{(dv)^2}{1 - \frac{v^2}{4}} + v^2(1 - \frac{v^2}{4})(du)^2 \tag{150}$$

The mapping (149) is equiareal, for if we take an auxiliary Cartesian plane $x^* y^*$ in which the polar coordinates are v and u so that

$$x^* = v \cos u$$

$$y^* = -v \sin u$$

then the metric in this plane will be given by

$$(ds^*)^2 = (dv)^2 + v^2(du)^2 \tag{151}$$

Thus the value of $g_{(\nu)}$ in both spaces are the same, which is the condition of equiareal mapping.

The transformation (149) establishes a one-to-one correspondence between the points of a unit sphere and the points of a plane. As θ varies from 0 to π, v varies from 0 to 2. The north pole is the center of the concentric circles. The limiting circle on the outside is the south pole. (Fig. 10)

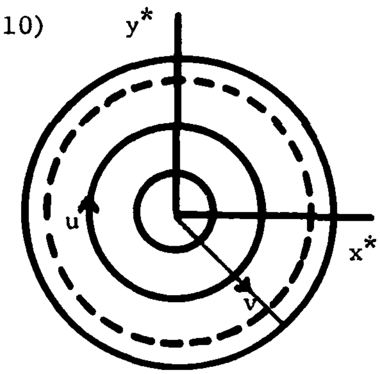


Figure 10.

§4. Some Standard Parametric Representations.

For reference purposes, we list some parametric representations for known surfaces. In the following we have used $u^\alpha (\alpha = 1,2)$ to represent the surface coordinates. (Taken from Ref. [18]).

(i) Sphere of radius a:

$$\underset{\sim}{r}(u^\alpha) = (a \cos u^2 \cos u^1,\ a \cos u^2 \sin u^1,\ a \sin u^2) \tag{152}$$

$$0 \leq u^1 < 2\pi, \; -\frac{\pi}{2} \leq u^2 \leq \frac{\pi}{2}$$

or

$$\underset{\sim}{r}(u^\alpha) = (a \sin u^2 \cos u^1, \; a \sin u^2 \sin u^1, \; a \cos u^2) \tag{153}$$

$$0 \leq u^1 < 2\pi, \; 0 \leq u^2 < \pi$$

(ii) Cone of revolution:

$$\underset{\sim}{r}(u^\alpha) = (u^1 \cos u^2, \; u^1 \sin u^2, \; a\, u^1) \tag{154}$$

$$0 \leq u^2 < 2\pi$$

(iii) Ellipsoid:

$$\underset{\sim}{r}(u^\alpha) = (a \cos u^2 \cos u^1, \; b \cos u^2 \sin u^1, \; c \sin u^2) \tag{155}$$

(iv) Elliptic paraboloid:

$$\underset{\sim}{r}(u^\alpha) = (a\, u^1 \cos u^2, \; b\, u^1 \sin u^2, \; (u^1)^2) \tag{156}$$

(v) Hyperbolic paraboloid:

$$\underset{\sim}{r}(u^\alpha) = (a\, u^1 \cosh u^2, \; b\, u^1 \sinh u^2, \; (u^1)^2) \tag{157}$$

(vi) Hyperboloid of two sheets:

$$\underset{\sim}{r}(u^\alpha) = (a \sinh u^1 \cos u^2, \; b \sinh u^1 \sin u^2, \; c \cosh u^1) \tag{158}$$

Part III

Basic Differential Models for Coordinate Generation

§1. Problem Formulation

The problem of generating spatial coordinates, either by analytic or numerical methods, is a problem of much interest in practically all branches of engineering mechanics and physics. A look at the older literature shows that most problems in fluid mechanics, electrostatics, potential theory, space mechanics, even relativity, etc., which have been classified as solutions of permanent value, are for discs, flat plates, circles, spheres, spheoriods, cones, ellipsoids, and paraboloids, etc. The main reason for interest in these shapes is because of the availability of exact analytic coordinates which are body conforming, so that the physical conditions at their respective surfaces can be exactly imposed. In some cases the governing equations in these coordinates are much simpler than in any other coordinates.

The coordinates for the above mentioned shapes and a score of others are obtained by the use of elementary geometrical and algebraic methods, which are introduced at a very early stage of one's mathematical and engineering education. Later, at a slightly higher level, in courses on differential geometry, these coordinates are repeatedly used in exercises to investigate the geometric properties of surfaces and of the curves which are formed in them. These geometric properties are obtained by using the differential relations which have been developed in Part II of this report. A question which naturally arises at this stage is this: Is it possible to develop a set of consistent differential relations and equations from the available body of differential-geometric results so as to generate coordinates for arbitrary shaped given bodies? In fact this question has been addressed by various researchers after Gauss, not from the point of view of arbitrary shaped bodies, but for specific characteristics of a body. As an example, the most widely studied problem has been of generating a surface, and so its coordinates, when the mean curvature is zero everywhere in the surface. Such surfaces are called the "minimal" surfaces. Weingarten surfaces provide another example. Eisenhart in 1923 published a book [34] on conjugate and other forms of coordinate net in surfaces.

The material of this chapter should not be taken as a review of the existing methods of coordinate generation but rather as an attempt to bring in the ideas of tensors and differential geometry in formulating problems of coordinate generation. (A comprehensive review of the existing methods in coordinate generation is to be published shortly [35].) The following two basic criteria have been used in the selection of material for this chapter.

(i) Derive only those differential relations and equations which have a direct bearing on the geometry of the generated surfaces, and which are of a nature of lasting interest for future research.

(ii) Methods to obtain solutions of the developed equations, if possible.

Two methods, which satisfy the above criteria are discussed below. However, it is important first to list a few expansions for the ensuing material from Parts I and II.

§1.1. Collection of Some Useful Expansions and Notation.

In what follows, the general curvilinear coordinates are again denoted as x^i. However, when an expression has been expanded out in full and there is no use for an index notation then we shall use the symbols ξ, η, ζ, where

$$x^1 = \xi \; , \; x^2 = \eta \; , \; x^3 = \zeta \tag{1}$$

Rectangular Cartesian coordinates are the components of the position vector $\underset{\sim}{r}$, i.e.,

$$\underset{\sim}{r} = (x,y,z) \tag{2}$$

From Part I, eq. (39), the covariant base vectors in space are

$$\underset{\sim}{a}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i}$$

Thus

$$\underset{\sim}{a}_1 = \underset{\sim}{r}_\xi \; , \; \underset{\sim}{a}_2 = \underset{\sim}{r}_\eta \; , \; \underset{\sim}{a}_3 = \underset{\sim}{r}_\zeta \tag{3}$$

where a variable subscript will always denote a partial derivative.

The metric tensor g_{ij} in three dimensions has six distinct components. The determinant g is then

$$g = g_{11}g_{22}g_{33} + 2g_{12}g_{13}g_{23} - (g_{23})^2 g_{11} - (g_{13})^2 g_{22} - (g_{12})^2 g_{33} \tag{4}$$

Writing

$$G_1 = g_{22}g_{33} - (g_{23})^2 \qquad (5a)$$

$$G_2 = g_{11}g_{33} - (g_{13})^2 \qquad (5b)$$

$$G_3 = g_{11}g_{22} - (g_{12})^2 \qquad (5c)$$

$$G_4 = g_{13}g_{23} - g_{12}g_{33} \qquad (5d)$$

$$G_5 = g_{12}g_{23} - g_{13}g_{22} \qquad (5e)$$

$$G_6 = g_{12}g_{13} - g_{11}g_{23} \qquad (5f)$$

we have

$$g^{11} = G_1/g \ , \ g^{22} = G_2/g \ , \ g^{33} = G_3/g \qquad (6a)$$

$$g^{12} = G_4/g \ , \ g^{13} = G_5/g \ , \ g^{23} = G_6/g \qquad (6b)$$

The space Christoffel symbols [ij,k] and Γ^i_{jk} defined in Part I, eqs. (130) and (133) have been expanded for a three-dimensional space and listed in Appendix 1. The surface Christoffel symbols of the second kind $\Gamma^\alpha_{\beta\gamma}$ for various coordinates held fixed are listed in Appendix 2. (Refer also to Part II, eqs. (82).)

As stated earlier, the x^i or ξ, η, ζ are the coordinates in a three-dimensional space. In place of using different symbols for a two-dimensional surface imbedded in the three-dimensional space, we have used ξ, η as coordinates on a surface on which ζ is held constant. This and two other possibilities are listed below.

(i) Coordinates (x^1,x^2) or (ξ,η) on a surface on which $x^3 = \zeta = \text{const.}$

(ii) " (x^3,x^1) or (ζ,ξ) " " " " " $x^2 = \eta = \text{const.}$

(iii) " (x^2,x^3) or (η,ζ) " " " " " $x^1 = \xi = \text{const.}$

(7)

Note that the right-handed convention is implicit in the ordering of the coordinates.

The index symbol ν is used in parentheses to denote which index or coordinate has been held fixed, with the exception of G_ν † (without parentheses) which stands for the value of G at ν = const. as defined in eqs. (5).

For variations from 1 to 2, or 3 to 1, or 2 to 3, we use Greek indices. Thus, according to Part II, eq. (37), the unit surface normal on a surface ν = const. will be

$$\underset{\sim}{n}^{(\nu)} = (\underset{\sim}{r}_\alpha \times \underset{\sim}{r}_\beta)/|\underset{\sim}{r}_\alpha \times \underset{\sim}{r}_\beta| \tag{8}$$

where

$$\begin{aligned} &\nu = 1 : \alpha = 2\ ,\ \beta = 3 &&(\text{surface } x^1 = \xi = \text{const.})\\ &\nu = 2 : \alpha = 3\ ,\ \beta = 1 &&(\text{surface } x^2 = \eta = \text{const.})\\ &\nu = 3 : \alpha = 1\ ,\ \beta = 2 &&(\text{surface } x^3 = \zeta = \text{const.})\end{aligned} \tag{9}$$

The rectangular components of $\underset{\sim}{n}^{(\nu)}$ are

---

† In Part II, the G_ν appearing here was denoted as $g_{(\nu)}$.

$$\underset{\sim}{n}^{(\nu)} = (X^{(\nu)}, Y^{(\nu)}, Z^{(\nu)}) \quad . \tag{10}$$

The coefficients of the second fundamental form for a surface, $b_{\alpha\beta}$, have been defined in Part II, eq. (46a). We now adopt the following notation in place of $b_{\alpha\beta}$.

$$S^{(\nu)} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\alpha\alpha} \qquad \text{(no sum on } \alpha\text{)}$$

$$T^{(\nu)} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\alpha\beta} \tag{11}$$

$$U^{(\nu)} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\beta\beta} \qquad \text{(no sum on } \beta\text{)}$$

where (ν,α,β) are in the permutational sequences of (1,2,3) as shown in (9).

The Gauss equations, defined in Part II, eq. (79) are now written as

$$\underset{\sim}{r}_{\alpha\alpha} = T^{\gamma}_{\alpha\alpha}\underset{\sim}{r}_{\gamma} + S^{(\nu)}\underset{\sim}{n}^{(\nu)}$$

$$\underset{\sim}{r}_{\alpha\beta} = T^{\gamma}_{\alpha\beta}\underset{\sim}{r}_{\gamma} + T^{(\nu)}\underset{\sim}{n}^{(\nu)} \tag{12}$$

$$\underset{\sim}{r}_{\beta\beta} = T^{\gamma}_{\beta\beta}\underset{\sim}{r}_{\gamma} + U^{(\nu)}\underset{\sim}{n}^{(\nu)}$$

where the summation is to be performed only on γ, and (ν,α,β) are in the permutational sequences of (1,2,3) as shown in (9).

The sum of principal curvatures of the surface ν = const., (defined in Part II, eq. (60)) is now written as

$$k_1^{(\nu)} + k_2^{(\nu)} = (g_{\alpha\alpha}U^{(\nu)} - 2g_{\alpha\beta}T^{(\nu)} + g_{\beta\beta}S^{(\nu)})/G_\nu \tag{13}$$

where, in writing eq. (13) for a particular value of ν, use must be made of (9).

The two second order differential operators introduced in Part II, eqs. (124) and (127), are now written as

$$\Delta_2^{(\nu)} \equiv \frac{1}{\sqrt{G_\nu}}[\partial_\alpha\{\frac{1}{\sqrt{G_\nu}}(g_{\beta\beta}\partial_\alpha - g_{\alpha\beta}\partial_\beta)\}$$

$$+ \partial_\beta\{\frac{1}{\sqrt{G_\nu}}(g_{\alpha\alpha}\partial_\beta - g_{\alpha\beta}\partial_\alpha)\}] \tag{14}$$

$$D^{(\nu)} \equiv g_{\beta\beta}\partial_{\alpha\alpha} - 2g_{\alpha\beta}\partial_{\alpha\beta} + g_{\alpha\alpha}\partial_{\beta\beta} \tag{15}$$

§2. Differential Equations for Coordinate Generation Based on the Riemann Tensor.

Earlier in Part I, §8, we discussed the curvature of a general space in terms of the Riemann tensor. It was shown in Part I, eq. (251) that the six distinct components of the Riemann tensor $R_{ijk\ell}$ for a three-dimension space are

$$R_{1212}\ ,\ R_{1313}\ ,\ R_{2323}\ ,\ R_{1213}\ ,\ R_{1232}\ ,\ R_{1323}\ .$$

If the space is Euclidean, i.e., E^3, then the above components are identically zero no matter which coordinate system is introduced in this space. Thus

$$R_{1212} = 0\ ,\ R_{1313} = 0\ ,\ R_{2323} = 0 \tag{16a}$$

$$R_{1213} = 0 \ , \ R_{1232} = 0 \ , \ R_{1323} = 0 \tag{16b}$$

Equations (16) are those consistent set of partial differential equations which must always be satisfied by the metric coefficients g_{ij}. It should be noted that there are six distinct coefficients to be obtained from the six equations (16), so that we have a closed system of equations. In contrast, a two dimensional space has only one curvature equation and three metric coefficients, the four-dimensional space in general relativity has twenty curvature equations for the ten metric coefficients. In these cases the system is either under determined or over determined, respectively.

Using eq. (247) of Part I, we now write the six second order partial differential equations as dictated by (16).

$$R_{1212} = \frac{\partial^2 g_{11}}{\partial \eta^2} - 2 \frac{\partial^2 g_{12}}{\partial \xi \partial \eta} + \frac{\partial^2 g_{22}}{\partial \xi^2}$$

$$+ 2g^{ts}([22,s][11,t] - [12,s][12,t]) = 0 \tag{17}$$

$$R_{1313} = \frac{\partial^2 g_{11}}{\partial \zeta^2} - 2 \frac{\partial^2 g_{13}}{\partial \xi \partial \zeta} + \frac{\partial^2 g_{33}}{\partial \xi^2}$$

$$+ 2g^{ts}([33,s][11,t] - [13,s][13,t]) = 0 \tag{18}$$

$$R_{2323} = \frac{\partial^2 g_{22}}{\partial \zeta^2} - 2 \frac{\partial^2 g_{23}}{\partial \eta \partial \zeta} + \frac{\partial^2 g_{33}}{\partial \eta^2}$$

$$+ 2g^{ts}([33,s][22,t] - [23,s][23,t]) = 0 \tag{19}$$

$$R_{1213} = \frac{\partial^2 g_{11}}{\partial\eta\partial\zeta} - \frac{\partial^2 g_{12}}{\partial\xi\partial\zeta} - \frac{\partial^2 g_{13}}{\partial\xi\partial\eta} + \frac{\partial^2 g_{23}}{\partial\xi^2}$$

$$+ 2g^{ts}([23,s][11,t] - [12,s][13,t]) = 0 \qquad (20)$$

$$R_{1232} = \frac{\partial^2 g_{22}}{\partial\xi\partial\zeta} - \frac{\partial^2 g_{12}}{\partial\eta\partial\zeta} - \frac{\partial^2 g_{23}}{\partial\xi\partial\eta} + \frac{\partial^2 g_{13}}{\partial\eta^2}$$

$$+ 2g^{ts}([22,s][13,t] - [23,s][12,t]) = 0 \qquad (21)$$

$$R_{1323} = \frac{\partial^2 g_{33}}{\partial\xi\partial\eta} - \frac{\partial^2 g_{13}}{\partial\eta\partial\zeta} - \frac{\partial^2 g_{23}}{\partial\xi\partial\zeta} + \frac{\partial^2 g_{12}}{\partial\zeta^2}$$

$$+ 2g^{ts}([33,s][12,t] - [23,s][13,t]) = 0 \qquad (22)$$

For a triply orthogonal system of coordinates

$$g_{12} = g_{13} = g_{23} = 0 \qquad (23a)$$

$$[12,3] = [13,2] = [23,1] = 0 \qquad (23b)$$

$$\Gamma^3_{12} = \Gamma^2_{13} = \Gamma^1_{23} = 0 \qquad (23c)$$

$$g = g_{11}g_{22}g_{33} \qquad (23d)$$

Under the constraint of orthogonality, eqs. (17)-(22) reduce somewhat. Using eqs. (23) and then multiplying the first equation by $1/\sqrt{g_{11}g_{22}}$, second by $1/\sqrt{g_{11}g_{33}}$, and the third by $1/\sqrt{g_{22}g_{33}}$, we can put the equations in the following form.

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial\eta}\right) + \frac{1}{2g_{33}\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial\zeta}\frac{\partial g_{22}}{\partial\zeta} = 0 \qquad (24)$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g_{11}g_{33}}}\frac{\partial g_{33}}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{\sqrt{g_{11}g_{33}}}\frac{\partial g_{11}}{\partial\zeta}\right) + \frac{1}{2g_{22}\sqrt{g_{11}g_{33}}}\frac{\partial g_{11}}{\partial\eta}\frac{\partial g_{33}}{\partial\eta} = 0 \qquad (25)$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{g_{22}g_{33}}}\frac{\partial g_{33}}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{\sqrt{g_{22}g_{33}}}\frac{\partial g_{22}}{\partial\zeta}\right) + \frac{1}{2g_{11}\sqrt{g_{22}g_{33}}}\frac{\partial g_{22}}{\partial\xi}\frac{\partial g_{33}}{\partial\xi} = 0 \qquad (26)$$

$$\frac{\partial^2 g_{11}}{\partial\eta\partial\zeta} = \frac{1}{2}\frac{\partial g_{11}}{\partial\eta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\zeta} + \frac{1}{g_{22}}\frac{\partial g_{22}}{\partial\zeta}\right) + \frac{1}{2g_{33}}\frac{\partial g_{11}}{\partial\zeta}\frac{\partial g_{33}}{\partial\eta} \qquad (27)$$

$$\frac{\partial^2 g_{22}}{\partial\xi\partial\zeta} = \frac{1}{2}\frac{\partial g_{22}}{\partial\zeta}\left(\frac{1}{g_{22}}\frac{\partial g_{22}}{\partial\xi} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial\xi}\right) + \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial\zeta}\frac{\partial g_{22}}{\partial\xi} \qquad (28)$$

$$\frac{\partial^2 g_{33}}{\partial\xi\partial\eta} = \frac{1}{2}\frac{\partial g_{33}}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial\eta}\right) + \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial\xi}\frac{\partial g_{33}}{\partial\eta} \qquad (29)$$

Equations (24)-(29) are the celebrated Lamé's equations, which he obtained in 1859 by following a different approach.

§2.1. Laplacians of ξ, η, and ζ and Their Inversions.

In Part I, eq. (214), we obtained the Laplacian $\nabla^2\phi$ of a scalar ϕ, where

$$\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz} \quad .$$

From the equation for $\nabla^2\phi$ we obtained the Laplacian of any curvilinear coordinate x^m, the coordinates being assumed to be functions of the Cartesian coordinates x, y, z. Thus

$$\nabla^2 x^m = -g^{ij}\Gamma^m_{ij} \qquad (30a)$$

$$= -\frac{1}{g}(G_1\Gamma^m_{11} + G_2\Gamma^m_{22} + G_3\Gamma^m_{33} + 2G_4\Gamma^m_{12}$$

$$+ 2G_5\Gamma^m_{13} + 2G_6\Gamma^m_{23}) \tag{30b}$$

where $x^1 = \xi$, $x^2 = \eta$, $x^3 = \zeta$, and G_ν have been defined in (5).

The inversion of these equations can be written down by using eq. (215) of Part I as

$$g^{ij} \frac{\partial^2 x}{\partial x^i \partial x^j} = - \frac{\partial x}{\partial x^m} \nabla^2 x^m \tag{31a}$$

$$g^{ij} \frac{\partial^2 y}{\partial x^i \partial x^j} = - \frac{\partial y}{\partial x^m} \nabla^2 x^m \tag{31b}$$

$$g^{ij} \frac{\partial^2 z}{\partial x^i \partial x^j} = - \frac{\partial z}{\partial x^m} \nabla^2 x^m \tag{31c}$$

Introducing the operator

$$L = G_1\partial_{\xi\xi} + G_2\partial_{\eta\eta} + G_3\partial_{\zeta\zeta} + 2G_4\partial_{\xi\eta} + 2G_5\partial_{\xi\zeta} + 2G_6\partial_{\eta\zeta} \tag{32}$$

we can write eqs. (31) as

$$Lx = -g(x_\xi\nabla^2\xi + x_\eta\nabla^2\eta + x_\zeta\nabla^2\zeta) \tag{33a}$$

$$Ly = -g(y_\xi\nabla^2\xi + y_\eta\nabla^2\eta + y_\zeta\nabla^2\zeta) \tag{33b}$$

$$Lz = -g(z_\xi\nabla^2\xi + z_\eta\nabla^2\eta + z_\zeta\nabla^2\zeta) \tag{33c}$$

The operator L reduces to the operator D (defined in Part I, eq. (190))

for the two-dimensional case. The corresponding equations are then eqs. (216) of Part I for the surface ζ = constant.

§2.1.1. Laplacians in Orthogonal Coordinates.

In the case of orthogonal coordinates the equation (30b) can be simplified to have the following forms for ξ, η, ζ.

$$\nabla^2\xi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi}\left(\sqrt{\frac{g_{22}g_{33}}{g_{11}}}\right) \tag{34a}$$

$$\nabla^2\eta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\eta}\left(\sqrt{\frac{g_{11}g_{33}}{g_{22}}}\right) \tag{34b}$$

$$\nabla^2\zeta = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\zeta}\left(\sqrt{\frac{g_{11}g_{22}}{g_{33}}}\right) \tag{34c}$$

§2.2. Riemann Curvature Tensor for Specific Surfaces.

It is worthwhile for us at this stage to list the Riemann curvature tensor for specific surfaces, ξ = const., η = const., and ζ = const.

We refer to Eqs. (89) and (91) of Part II where the single Riemann tensor for a surface $\overset{*}{R}_{\mu\alpha\gamma\beta}$ and the Gaussian curvature K were defined. In expanded form the expression for $\overset{*}{R}_{\mu\alpha\gamma\beta}$ is

$$\overset{*}{R}_{\mu\alpha\gamma\beta} = \frac{\partial}{\partial u^\gamma}[\alpha\beta,\mu] - \frac{\partial}{\partial u^\beta}[\alpha\gamma,\mu] + T^\sigma_{\alpha\gamma}[\mu\beta,\sigma]$$

$$- T^\sigma_{\alpha\beta}[\mu\gamma,\sigma] \tag{35}$$

Thus, for ζ = const.

$$\overset{*}{R}_{1212} = \frac{\partial}{\partial\xi}[22,1] - \frac{\partial}{\partial\eta}[21,1] + T^1_{21}[12,1] + T^2_{21}[12,2]$$

$$- T^1_{22}[11,1] - T^2_{22}[11,2] \quad . \tag{36}$$

for η = const.

$$\overset{*}{R}_{1313} = \frac{\partial}{\partial\xi}[33,1] - \frac{\partial}{\partial\zeta}[31,1] + T^3_{31}[13,3] + T^1_{31}[13,1]$$

$$- T^3_{33}[11,3] - T^1_{33}[11,1] \quad . \tag{37}$$

for ξ = const.

$$\overset{*}{R}_{2323} = \frac{\partial}{\partial\eta}[33,2] - \frac{\partial}{\partial\zeta}[32,2] + T^2_{32}[23,2] + T^3_{32}[23,3]$$

$$- T^2_{33}[22,2] - T^3_{33}[22,3] \tag{38}$$

For the expressions of T's refer to Appendix 2.

Each one of (36)-(38) can be reduced to different forms. For example, eq. (36) can also be expressed as follows.

$$\overset{*}{R}_{1212} = \sqrt{G_3}\left[\frac{\partial}{\partial\eta}\left(\frac{\sqrt{G_3}}{g_{11}} T^2_{11}\right) - \frac{\partial}{\partial\xi}\left(\frac{\sqrt{G_3}}{g_{11}} T^2_{12}\right)\right] \tag{39a}$$

$$= \sqrt{G_3}\left[\frac{\partial}{\partial\xi}\left(\frac{\sqrt{G_3}}{g_{22}} T^1_{22}\right) - \frac{\partial}{\partial\eta}\left(\frac{\sqrt{G_3}}{g_{22}} T^1_{12}\right)\right] \tag{39b}$$

where G_3 is defined in (5c). The forms (39a,b) are due to J. Liouville.

The Gaussian curvature for each surface is

$$K^{(\zeta)} = k_1{}^{(\zeta)} k_2{}^{(\zeta)} = R^*_{1212}/G_3 \tag{40a}$$

$$K^{(\eta)} = k_1{}^{(\eta)} k_2{}^{(\eta)} = R^*_{1313}/G_2 \tag{40b}$$

$$K^{(\xi)} = k_1{}^{(\xi)} k_2{}^{(\xi)} = R^*_{2323}/G_1 \tag{40c}$$

It must be noted that in any one of the formulae, from (36)-(40), all quantities have to be evaluated for the coordinate held fixed. There is no difficulty in this process, since any of these quantities have no derivatives with respect to the variable held fixed.

As obtained in Part II, eq. (90a), we can also write the equations in (40) as

$$K^{(\zeta)} = [S^{(\zeta)} U^{(\zeta)} - (T^{(\zeta)})^2]/G_3 \tag{41a}$$

$$K^{(\eta)} = [S^{(\eta)} U^{(\eta)} - (T^{(\eta)})^2]/G_2 \tag{41b}$$

$$K^{(\xi)} = [S^{(\xi)} U^{(\xi)} - (T^{(\xi)})^2]/G_1 \tag{41c}$$

In the representation (41), the quantities S, T, U can also be determined through other quantities which are dependent on the derivatives with respect to ζ. These representations thus establish a connection of the given surface with the neighboring surfaces. This idea has later been used (see §3, eqs. (74)) to develop a method of coordinate generation from the data of the given surfaces.

§2.2.1. Coordinates in a Plane.

If the surface ζ = const. (say), on which ξ and η are the parametric

coordinates, reduces to a plane, then for this surface $K^{(\zeta)} = 0$.†
Consequently, we have a single equation (selecting either (39a) or (38b)),

$$\frac{\partial}{\partial\eta}\left(\frac{\sqrt{G_3}}{g_{11}} T^2_{11}\right) - \frac{\partial}{\partial\xi}\left(\frac{\sqrt{G_3}}{g_{11}} T^2_{12}\right) = 0 \tag{42}$$

In contrast to the six equations (24)-(29) for a three-dimensional space, we have only a simple equation for two-dimensional space. All the three coefficients $g_{\alpha\beta}$ cannot be determined from this single equation and additional relations, either algebraic or differential, have to be imposed to solve eq. (42). We shall return to these problems in §2.4.

§2.3. Determination of the Cartesian Coordinates.

The solutions of eqs. (17)-(22) under the prescribed boundary conditions should provide all the metric coefficients as a field distribution, so that by differentiation one can calculate also all the Christoffel symbols Γ^i_{jk}. Now in any physical problem, e.g., the Navier-Stokes equations, only the metric coefficients g_{ij} and the Christoffel symbols Γ^i_{jk} appear in the transformed equations, so that the solutions of the equations (17)-(22) provide all the essential coefficients to solve the physical problem. Nevertheless, one sometimes also needs the values of the Cartesian coordinates x, y, z as functions of ξ, η, ζ. Our purpose is now to describe a technique for the determination of x, y, z based on the availability of the metric coefficients.

We define the unit base vectors

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†Refer also to Appendix 3 for the Beltrami equations in a plane.

$$\underset{\sim}{\lambda}_i = \underset{\sim}{a}_i / |\underset{\sim}{a}_i| = \underset{\sim}{a}_i / \sqrt{g_{ii}} \tag{43}$$

where there is no summation on i. The components of $\underset{\sim}{\lambda}_i$ along the rectangular Cartesian axes are denoted as u_i, v_i, w_i respectively, i.e.,

$$\underset{\sim}{\lambda}_i = (u_i, v_i, w_i) \tag{44}$$

In total there will be nine values of u_i, v_i, w_i.

Now

$$d\underset{\sim}{r} = \underset{\sim}{\lambda}_1 \sqrt{g_{11}}\, d\xi + \underset{\sim}{\lambda}_2 \sqrt{g_{22}}\, d\eta + \underset{\sim}{\lambda}_3 \sqrt{g_{33}}\, d\zeta$$

so that the values of x, y, and z are given by the line integrals

$$x = \int u_1 \sqrt{g_{11}}\, d\xi + u_2 \sqrt{g_{22}}\, d\eta + u_3 \sqrt{g_{33}}\, d\zeta \tag{45a}$$

$$y = \int v_1 \sqrt{g_{11}}\, d\xi + v_2 \sqrt{g_{22}}\, d\eta + v_3 \sqrt{g_{33}}\, d\zeta \tag{45b}$$

$$z = \int w_1 \sqrt{g_{11}}\, d\xi + w_2 \sqrt{g_{22}}\, d\eta + w_3 \sqrt{g_{33}}\, d\zeta \tag{45c}$$

The determination of u_i, v_i, w_i $(i = 1,3)$ which is needed in eqs. (45) poses another problem. Their derivatives can, however, be expressed in terms of u_i, v_i, w_i by substituting (43) in eq. (134) of Part I as

$$\frac{\partial \underset{\sim}{\lambda}_i}{\partial x^j} = \underset{\sim}{\lambda}_1 \sqrt{\frac{g_{11}}{g_{ii}}}\, \Gamma^1_{ij} + \underset{\sim}{\lambda}_2 \sqrt{\frac{g_{22}}{g_{ii}}}\, \Gamma^2_{ij} + \underset{\sim}{\lambda}_3 \sqrt{\frac{g_{33}}{g_{ii}}}\, \Gamma^3_{ij}$$

$$- \frac{\underset{\sim}{\lambda}_i}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j} \tag{46}$$

where there is <u>no</u> summation on i.

On changing i and j from 1 to 3, we find twenty seven values of the derivatives from (46). Thus, they form a system of twenty seven first order partial differential equations to be solved under a prescribed Cauchy data.

§2.4. Coordinate Generation Capabilities of the Developed Equations.

The derivation of eqs. (17)-(22) has demonstrated quite clearly that these equations are neither arbitrary nor randomly selected to generate some coordinate system. They are actually the very basic equations which every coordinate system in E^3 <u>must</u> satisfy. Any six symmetric functions g_{ij} of a coordinate system ξ, η, ζ which satisfy eqs. (17)-(22) are qualified to be called the metric coefficients of the introduced coordinate system in E^3.

Despite the versatility and power of these equations, the solution of these equations is difficult to obtain. The set of equations (17)-(22) form a highly nonlinear system of coupled partial differential equations. Even if they have been solved, the determination of x, y, z requires a solution of twenty seven first order partial differential equations as shown in §2.3. Nevertheless, these equations must form a basis for future developments. An indepth study into the nature of these equations, e.g., the compatability conditions and the type of data to be prescribed as the boundary conditions, etc., has to be investigated. In the following sub-sections we consider two particular cases of these equations.

§2.4.1. Two-Dimensional Orthogonal Coordinates in a Plane.

For the case of orthogonal coordinates in a plane, the basic equation to start with is eq. (42). When the constraint of orthogonality, viz.,

$$g_{12} = 0$$

is imposed, we get the equation

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial\eta}\right) = 0 \tag{47}$$

In Ref. [36] a method has been developed to compute orthogonal curvilinear coordinates about arbitrarily given inner and outer boundaries. Equation (47) is first simplified for the case of isothermic coordinates (refer to §3 of Part II, particularly eq. (144) and the definition that follows), in which

$$g_{22} = g_{11} \tag{48}$$

and eq. (47) takes the much simpler form

$$\frac{\partial^2 P}{\partial\xi^2} + \frac{\partial^2 P}{\partial\eta} = 0 \tag{49}$$

where

$$P = \ln g_{11} \quad .$$

From eqs. (34a,b) we also have the additional conditions, that for isothermic coordinates

$$\nabla^2\xi = 0 \ , \ \nabla^2\eta = 0 \tag{50}$$

where now $\nabla^2 = \partial_{xx} + \partial_{yy}$ and $g_{33} = 1$.

Equation (49) can be exactly solved in a Fourier series form by prescribing the values of g_{11} or P at the inner boundary, denoted at $\eta = \eta_\beta$, and the outer boundary denoted as $\eta = \eta_\infty$, as shown in Ref. [36]. The equations (50) are then used to pick out those ξ distributions which establish an orthogonal correspondence between the points of the inner and outer boundaries.

§2.4.2. Three-Dimensional Orthogonal Coordinates.

The governing equations for the three-dimensional orthogonal coordinates are the Lamé's equations and have been stated earlier in eqs. (24)-(29). These equations are as complicated as their non-orthogonal counterparts. In this section we shall study two particular forms of these equations which are amenable to analysis and computation.

In this connection we need the following definitions.

(i) Gaussian curvature in orthogonal coordinates:

For a surface ζ = const. in which ξ and η are the orthogonal coordinates, and ζ is the coordinate normal to the surface, we have (cf. Appendix 2)

$$\Gamma^2_{11} = -\frac{1}{2g_{22}}\frac{\partial g_{11}}{\partial\eta} \ , \ \Gamma^2_{12} = \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial\xi} \tag{51}$$

where $G_3 = g_{11}g_{22}$. Thus from (39a), we have

$$R^*_{1212} = -\frac{1}{2}\sqrt{g_{11}g_{22}}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial\eta}\right)\right] \tag{52}$$

Using eq. (24) in (52), we obtain

$$K^{(\zeta)} = R^{*}_{1212}/G_3$$

$$= \frac{1}{4g_{11}g_{22}g_{33}} \frac{\partial g_{11}}{\partial \zeta} \frac{\partial g_{22}}{\partial \zeta} \quad \text{at } \zeta = \text{const.} \tag{53}$$

(ii) Surfaces of constant Gaussian curvature:

Surfaces for which the Gaussian curvature has a constant value at every point in the surface are known as surfaces of constant curvature. The Gaussian curvature can be either zero, positive, or a negative constant throughout the surface. A surface ζ = const. for which $K^{(\zeta)} = 0$ is a developable surface. A developable surface can be mapped isometrically onto a plane. Recall from Part II, eq. (135) that the isometric correspondence between two surfaces, when the coordinates on the two surfaces are the same, is such that the length element $\bar{ds}$ between two corresponding points remains the same.

The simplest example of a surface for which $K^{(\zeta)} > 0$ is a sphere of radius R for which $K^{(\zeta)} = 1/R^2$. If a sphere or a spherical cap can be deformed in any other shape whatsoever without stretching, then its Gaussian curvature will not be altered. In the case of $K^{(\zeta)} > 0$, every surface of constant curvature can be mapped isometrically on a sphere of radius $(K^{(\zeta)})^{-1/2}$. All these results are explained in books on differential geometry, e.g., [17].

We now consider the following two cases of orthogonal coordinates.

Case I:

Since the coordinates in the surface are orthogonal, the length element for ζ = const. is

$$(ds^{(\zeta)})^2 = g_{11}(d\xi)^2 + g_{22}(d\eta)^2 \qquad (54)$$

We now select ξ = const. as any arbitrary curve. Through every point of this curve a geodesic can be drawn. We call these curves as η = const. Obviously ξ, η are orthogonal. From Part II, eq. (107b) we have the result that for η = const. to be a geodesic we must have

$$\frac{\partial g_{11}}{\partial \eta} = 0 \quad .$$

Thus the metric which we select for Case I is

$$g_{11}(\xi,\zeta) \ , \ g_{22}(\xi,\eta,\zeta) \ , \ g_{33} = 1 \qquad (55)$$

where ζ remains fixed on each selected surface. Since g_{11} is not a function of η, we can define an arc legnth as a perfect differential du for each ζ = const. as

$$du = \sqrt{g_{11}}\, d\xi$$

Thus

$$(ds^{(\zeta)})^2 = (du)^2 + g_{22}(d\eta)^2 \qquad (56)$$

In the literature, the coordinates u, η are known as geodesic polar coordinates, since a point O on the surface can be chosen where

$$u = 0 \ , \ g_{22}(0,\eta) = 0$$

so that $ds^{(\zeta)} = 0$.

We now substitute (55) in (52), and have

$$R^*_{1212} = -\frac{1}{2} g_{11}\sqrt{g_{22}} \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial u}\right) \tag{57a}$$

or,

$$K^{(\zeta)} = -\frac{1}{2\sqrt{g_{22}}} \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{g_{22}}} \frac{\partial g_{22}}{\partial u}\right) \tag{57b}$$

Thus

$$\frac{\partial^2}{\partial u^2}(\sqrt{g_{22}}) + K^{(\zeta)}\sqrt{g_{22}} = 0 \tag{58}$$

Solving eq. (58)

$$\sqrt{g_{22}} = \frac{1}{\sqrt{K^{(\zeta)}}} \sin(u\sqrt{K^{(\zeta)}}) \quad , \tag{59}$$

the parameter u being the arc length along the geodesic coordinates η = const.

A study of eq. (53) shows that for $K^{(\zeta)}$ to remain constant, the forms of g_{11} and g_{22} should be

$$g_{11} = \phi(\zeta)\ f(\xi)$$

$$g_{22} = \psi(\zeta)\ F(\xi,\eta)$$

so that

$$K^{(\zeta)} = \left(\frac{\phi'\psi'}{4\phi\psi}\right)_{\zeta\,=\,\text{const.}}$$

The form of the function $F(\xi,\eta)$ is fixed by the solution (59).

The preceding method can be made a basis of numerical coordinate generation for those surfaces which can be isometrically mapped on spheres of varying radii.

Case II:

In this case we select the metric such that in the surface ζ = const., the coordinates are isothermic. Thus we take

$$g_{22} = g_{11} \text{ , } g_{33} = 1 \tag{60}$$

Under the constraint (60), eqs. (24)-(29) simplify to the following equations:

$$\frac{\partial}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta}\right) + \frac{1}{2g_{11}}\left(\frac{\partial g_{11}}{\partial\zeta}\right)^2 = 0 \tag{61}$$

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial g_{11}}{\partial\zeta}\right) = 0 \tag{62}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\zeta}\right) = 0 \tag{63}$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\zeta}\right) = 0 \tag{64}$$

while eq. (53) becomes

$$K^{(\zeta)} = \frac{1}{4(g_{11})^2}\left(\frac{\partial g_{11}}{\partial \zeta}\right)^2 \tag{65}$$

A study of eqs. (62)-(64) suggests that the only form g_{11} can have is

$$g_{11} = (A + B\zeta)^2 f(\xi,\eta) \tag{66}$$

where A and B are arbitrary constants, and $f > 0$.

Substituting (66) in (61), we obtain the equation for f as

$$\frac{\partial}{\partial \xi}\left(\frac{1}{f}\frac{\partial f}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{f}\frac{\partial f}{\partial \eta}\right) + 2B^2 f = 0 \tag{67}$$

Writing

$$Q = \ell n f$$

we get

$$\frac{\partial^2 Q}{\partial \xi^2} + \frac{\partial^2 Q}{\partial \eta^2} + 2B^2 e^Q = 0 \tag{68}$$

which is an equation similar to eq. (49) except for the last term.

We now substitute (66) in (65), and have

$$K^{(\zeta)} = B^2/(A + B\zeta)^2 \tag{69}$$

Thus, for each ζ = const. the surfaces generated will be of constant Gaussian curvature. Numerical techniques can be used to solve eq. (67).

In the context of isothermic coordinates in the surface and $g_{33} = 1$, we have the following additional equations from eqs. (34).

$$\nabla^2 \xi = 0 \tag{70a}$$

$$\nabla^2 \eta = 0 \tag{70b}$$

$$\nabla^2 \zeta = \frac{1}{g_{11}} \frac{\partial g_{11}}{\partial \zeta} \tag{70c}$$

With eqs. (70) available, it is possible to develop a complete algorithm for numerical coordinate generation.

§3. Differential Equations for Coordinate Generation Based on the Formulae of Gauss.

In this section we shall discuss another method of coordinate generation suitable for three-dimensional situations and which has the added property that the method reduces to the method of Ref. [29] for two-dimensional plane regions. Some details are available in a previous publication, [37].

Before developing the proposed method it is important to have the following formulae.

From Part I, eq. (134), we have

$$\underset{\sim}{r}_{\xi\xi} = \Gamma^1_{11}\underset{\sim}{r}_\xi + \Gamma^2_{11}\underset{\sim}{r}_\eta + \Gamma^3_{11}\underset{\sim}{r}_\zeta \tag{71a}$$

$$\underset{\sim}{r}_{\xi\eta} = \Gamma^1_{12}\underset{\sim}{r}_\xi + \Gamma^2_{12}\underset{\sim}{r}_\eta + \Gamma^3_{12}\underset{\sim}{r}_\zeta \tag{71b}$$

$$\underset{\sim}{r}_{\eta\eta} = \Gamma^1_{22}\underset{\sim}{r}_\xi + \Gamma^2_{22}\underset{\sim}{r}_\eta + \Gamma^3_{22}\underset{\sim}{r}_\zeta \tag{71c}$$

where the 3-space Christoffel symbols are given in Appendix 1.

We now consider a surface designated as ζ = const. on which ξ and η are the parametric coordinate. Then from eq. (12), we have

$$\underset{\sim}{r}_{\xi\xi} = T^1_{11}\underset{\sim}{r}_\xi + T^2_{11}\underset{\sim}{r}_\eta + S^{(\zeta)}\underset{\sim}{n}^{(\zeta)} \tag{72a}$$

$$\underset{\sim}{r}_{\xi\eta} = T^1_{12}\underset{\sim}{r}_\xi + T^2_{12}\underset{\sim}{r}_\eta + T^{(\zeta)}\underset{\sim}{n}^{(\zeta)} \tag{72b}$$

$$\underset{\sim}{r}_{\eta\eta} = T^1_{22}\underset{\sim}{r}_\xi + T^2_{22}\underset{\sim}{r}_\eta + U^{(\zeta)}\underset{\sim}{n}^{(\zeta)} \tag{72c}$$

where the 2-space Christoffel symbols for ζ = const. are given in Appendix 2.

Taking now the dot product of every term with $\underset{\sim}{n}^{(\zeta)}$ in both eqs. (71) and (72), we obtain

$$S^{(\zeta)} = (\underset{\sim}{n}^{(\zeta)} \cdot \underset{\sim}{r}_\zeta)\Gamma^3_{11} \tag{73a}$$

$$T^{(\zeta)} = (\underset{\sim}{n}^{(\zeta)} \cdot \underset{\sim}{r}_\alpha)\Gamma^3_{12} \tag{73b}$$

$$U^{(\zeta)} = (\underset{\sim}{n}^{(\zeta)} \cdot \underset{\sim}{r}_\zeta)\Gamma^3_{22} \tag{73c}$$

All ζ-derivatives in eqs. (73) are assumed to be evaluated at ζ = const.

The above procedure can be repeated for constant ξ and η surfaces. However, in what follows we shall be obtaining formulae only for ζ = const. surfaces, and for brevity of notation drop the superscript (ζ) from the formulae. Thus

$$S = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\xi} = (\underset{\sim}{n} \cdot \underset{\sim}{r}_\zeta)\Gamma^3_{11} \tag{74a}$$

$$T = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\xi\eta} = (\underset{\sim}{n} \cdot \underset{\sim}{r}_{\zeta})\Gamma^3_{12} \tag{74b}$$

$$U = \underset{\sim}{n} \cdot \underset{\sim}{r}_{\eta\eta} = (\underset{\sim}{n} \cdot \underset{\sim}{r}_{\zeta})\Gamma^3_{22} \tag{74c}$$

where

$$\underset{\sim}{n} \cdot \underset{\sim}{r}_{\zeta} = Xx_{\zeta} + Yy_{\zeta} + Zz_{\zeta} \quad . \tag{74d}$$

§3.1. Formulation of the Problem.

We multiply eqs. (72a)-(72c) respectively by g_{22}, $-2g_{12}$, g_{11}, adding and using eqs. (13)-(15) to have

$$D\underset{\sim}{r} + G_3(\underset{\sim}{r}_{\xi}\Delta_2\xi + \underset{\sim}{r}_{\eta}\Delta_2\eta) = G_3\underset{\sim}{n}(k_1 + k_2) \tag{75}$$

where

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta}$$

$$\left.\begin{aligned} \Delta_2\xi &= \frac{1}{G_3}(2g_{12}T^1_{12} - g_{22}T^1_{11} - g_{11}T^1_{22}) \\ \Delta_2\eta &= \frac{1}{G_3}(2g_{12}T^2_{12} - g_{22}T^2_{11} - g_{11}T^2_{22}) \end{aligned}\right\} \tag{76}$$

$$G_3 = g_{11}g_{22} - (g_{12})^2$$

To obtain an expression for $k_1 + k_2$, we take the scalar product of (75) by $\underset{\sim}{n}$ and use eqs. (74)

$$G_3(k_1 + k_2) = (\underset{\sim}{n} \cdot \underset{\sim}{r}_{\zeta})(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \tag{77}$$

We now propose to put constraints on the coordinates ξ and η such that

$$\Delta_2\xi = 0 \tag{78}$$

$$\Delta_2\eta = 0 \tag{79}$$

With these constraints, the differential equations for the determination of the Cartesian coordinates are given by

$$D\underset{\sim}{r} = G_3\underset{\sim}{n}(k_1 + k_2) \tag{80}$$

In expanded form, eqs. (80) are

$$g_{22}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{11}x_{\eta\eta} = X \cdot R \tag{81}$$

$$g_{22}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{11}y_{\eta\eta} = Y \cdot R \tag{82}$$

$$g_{22}z_{\xi\xi} - 2g_{12}z_{\xi\eta} + g_{11}z_{\eta\eta} = Z \cdot R \tag{83}$$

where

$$R = (Xx_\zeta + Yy_\zeta + Zz_\zeta)(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \tag{84}$$

and

$$\begin{aligned} X &= (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3} \\ Y &= (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3} \\ Z &= (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3} \end{aligned} \tag{85}$$

The proposed constraining equations form the core of the method. Firstly it must be noted that Δ_2 is neither a Laplace operator in the Cartesian plane (x,y), nor in the Cartesian space (x,y,z), though it reduces to a two-dimensional Laplace operator when the surface reduces to a plane surface, viz., no dependence on z. Secondly, the eqs. (78) and (79) express an attempt in providing a set of basic constraints on the distribution of $g_{\alpha\beta}$ in a surface, which is perfectly legitimate.

Another important observation in favor of these equations is the following. Using the expressions for the Christoffel symbols appearing in eqs. (76), we can also write

$$\Delta_2\xi = \frac{1}{\sqrt{G_3}}\left[\frac{\partial}{\partial\xi}\left(\frac{g_{22}}{\sqrt{G_3}}\right) - \frac{\partial}{\partial\eta}\left(\frac{g_{12}}{\sqrt{G_3}}\right)\right] \tag{86}$$

$$\Delta_2\eta = \frac{1}{\sqrt{G_3}}\left[\frac{\partial}{\partial\eta}\left(\frac{g_{11}}{\sqrt{G_3}}\right) - \frac{\partial}{\partial\xi}\left(\frac{g_{12}}{\sqrt{G_3}}\right)\right] \tag{87}$$

In the case of isothermic coordinates, viz., when $g_{22} = g_{11}$, eqs. (86) and (87) are identically satisfied. There is a parallel situation in the case of conformal coordinates in a plane where Laplace equations are satisfied identically.

§3.1.1. Particular Case of Eqs. (81)-(83). (Minimal Surfaces).

For surfaces in which isothermic coordinates have been introduced and at each of its points the mean curvature is zero, we have from eqs. (81)-(83),

$$x_{\xi\xi} + x_{\eta\eta} = 0 \tag{88a}$$

$$y_{\xi\xi} + y_{\eta\eta} = 0 \tag{88b}$$

$$z_{\xi\xi} + z_{\eta\eta} = 0 \tag{88c}$$

Such surfaces are called the minimal surfaces.

As an example, a minimal surface of revolution can be obtained by first assuming

$$x = f(\eta)\cos\xi \ , \ y = f(\eta)\sin\xi \ , \ z = g(\eta) \quad . \tag{89}$$

From (89), we obtain

$$g_{11} = f^2 \ , \ g_{12} = 0 \ , \ g_{22} = f'^2 + g'^2$$

prime denoting differentiation with respect to η.

The isothermic condition gives

$$f'^2 + g'^2 = f^2 \tag{90}$$

while eqs. (88) give

$$f'' - f = 0 \tag{91a}$$

$$g'' = 0 \tag{91b}$$

A solution satisfying (90) and (91) is

$$\begin{aligned} f(\eta) &= A \cosh \eta \\ g(\eta) &= B + A\eta \end{aligned} \tag{92}$$

where A and B are arbitrary constants. Thus when (92) is substituted for f and g in (89) we obtain a minimal surface of revolution.

§3.2. Coordinate Generation Between Two Prescribed Surfaces.

We now consider the problem of coordinate generation between two surfaces denoted as $\eta = \eta_B$ and $\eta = \eta_\infty$ in Fig. 11, where ξ and ζ are the parametric coordinates in these surfaces.

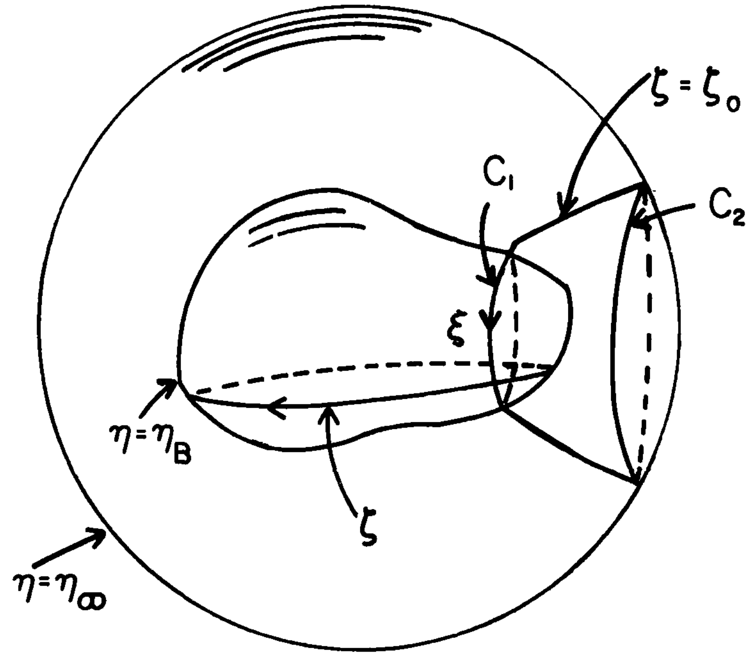
Equations (81)-(83) form a quasilinear system of partial differential equations in which the components of the vector $\underset{\sim}{r}_\zeta$ are assumed to be prescribed or available through some interpolation/extrapolation numerical scheme. Since the values of x, y, z are known on the basic inner and outer boundaries (cf. Fig. 11), the values of $(\underset{\sim}{r}_\zeta)_{\eta = \eta_B}$ and $(\underset{\sim}{r}_\zeta)_{\eta = \eta_\infty}$ are known. Thus a suitable way of prescribing $\underset{\sim}{r}_\zeta$ in space can be

$$\underset{\sim}{r}_\zeta = f_1(\eta)(\underset{\sim}{r}_\zeta)_{\eta = \eta_B} + f_2(\eta)(\underset{\sim}{r}_\zeta)_{\eta = \eta_\infty} \tag{93}$$

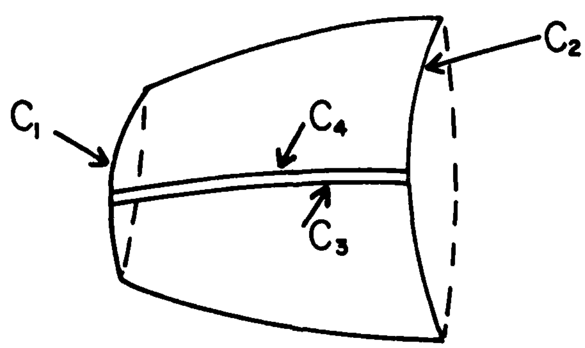
where $f_1(\eta)$ and $f_2(\eta)$ are suitable weights having the properties

$$\begin{aligned} f_1(\eta_B) &= 1 \ , \ f_2(\eta_B) = 0 \\ f_1(\eta_\infty) &= 0 \ , \ f_2(\eta_\infty) = 1 \end{aligned} \tag{94}$$

Referring to Fig. 11, we now solve eqs. (81)-(83) for each ζ = const., on a rectangular plane by prescribing the values of x, y, and z on the lower side (C_1) and upper side (C_2) which represents the curves on B and ∞ respectively. The side (C_3) and (C_4) are the cut lines on which periodic boundary conditions are to be imposed. The preceding analysis thus completes the formulation of the problem.



(a)



(b)

Figure 11: (a) Topology of the given surfaces. Inner $\eta = \eta_B$, outer $\eta = \eta_\infty$, current variables ξ, ζ. (b) Surface to be generated for each ζ = const., current variables ξ, η.

§3.3. Coordinate Redistribution.

For the purpose of generating coordinates between the space of the inner and outer boundary, which can be distributed in a desired manner, we consider a coordinate transformation from ξ to χ and η to σ. Let

$$\xi = \xi(\chi) + \xi_0$$
$$\eta = \eta(\sigma) + \eta_B \tag{95}$$

then

$$\xi = \xi_0 \quad \text{at} \quad \chi = \chi_0 \;,\; \xi(\chi_0) = 0$$
$$\eta = \eta_B \quad \text{at} \quad \sigma = \sigma_B \;,\; \eta(\sigma_B) = 0 \tag{96}$$

Writing

$$\lambda(\chi) = \frac{d\xi}{d\chi} \;,\; \theta(\sigma) = \frac{d\eta}{d\sigma} \tag{97a}$$

and denoting the transformed metric tensor as $\bar{g}_{ij}$, we have

$$g_{11} = \bar{g}_{11}/\lambda^2 \;,\; \bar{g}_{11} = x_\chi^2 + y_\chi^2 + z_\chi^2 \tag{97b}$$

$$g_{12} = \bar{g}_{12}/\theta\lambda \;,\; \bar{g}_{12} = x_\chi x_\sigma + y_\chi y_\sigma + z_\chi z_\sigma \tag{97c}$$

$$g_{22} = \bar{g}_{22}/\theta^2 \;,\; \bar{g}_{22} = x_\sigma^2 + y_\sigma^2 + z_\sigma^2 \tag{97d}$$

$$G_3 = \bar{G}_3/\theta^2\lambda^2 \;,\; \bar{G}_3 = \bar{g}_{11}\bar{g}_{22} - (\bar{g}_{12})^2 \tag{97e}$$

$$X = \bar{X} \;,\; Y = \bar{Y} \;,\; Z = \bar{Z} \tag{97f}$$

$$k_1 + k_2 = \bar{k}_1 + \bar{k}_2 \tag{97g}$$

$$R = \bar{R}/\theta^2\lambda^2 \tag{97h}$$

Further noting that

$$\underset{\sim}{r}_{\xi\xi} = (\underset{\sim}{r}_{\chi\chi} - \frac{\underset{\sim}{r}_{\chi}\lambda_{\chi}}{\lambda})/\lambda^2 \tag{98a}$$

$$\underset{\sim}{r}_{\xi\eta} = \underset{\sim}{r}_{\chi\sigma}/\theta\lambda \tag{98b}$$

$$\underset{\sim}{r}_{\eta\eta} = (\underset{\sim}{r}_{\sigma\sigma} - \frac{\underset{\sim}{r}_{\sigma}\theta_{\sigma}}{\theta})\theta^2 \tag{98c}$$

Substituting eqs. (97) and (98) in eqs. (81)-(83) we now have the following set of equations

$$\bar{g}_{22}x_{\chi\chi} - 2\bar{g}_{12}x_{\chi\sigma} + \bar{g}_{11}x_{\sigma\sigma} = Px_{\chi} + Qx_{\sigma} + \bar{X}\cdot\bar{R} \tag{99}$$

$$\bar{g}_{22}y_{\chi\chi} - 2\bar{g}_{12}y_{\chi\sigma} + \bar{g}_{11}y_{\sigma\sigma} = Py_{\chi} + Qy_{\sigma} + \bar{Y}\cdot\bar{R} \tag{100}$$

$$\bar{g}_{22}z_{\chi\chi} - 2\bar{g}_{12}z_{\chi\sigma} + \bar{g}_{11}z_{\sigma\sigma} = Pz_{\chi} + Qz_{\sigma} + \bar{Z}\cdot\bar{R} \tag{101}$$

where

$$P = \frac{\bar{g}_{22}}{\lambda}\lambda_{\chi}$$

$$Q = \frac{\bar{g}_{11}}{\theta}\theta_{\sigma} \tag{102}$$

Thus, by choosing λ and θ arbitrarily we can redistribute the coordinates in the desired manner. An example of this choice is given in the next section.

§3.4. An Analytical Example of Coordinate Generation.

In this section we shall consider the problem of coordinate generation between a prolate ellipsoid (considered as an inner body) and a sphere (considered as an outer boundary), with coordinate contraction near the inner surface. This problem yields an exact solution of the equations (99)-(101).

Let $\eta = \eta_B$ and $\eta = \eta_\infty$ be the inner prolate ellipsoid and the outer sphere respectively. The coordinates which vary on these two surfaces are ξ and ζ. We now establish a net of lines made of ξ = const. and ζ = const. on both surfaces. A curve C_1 on the inner surface designated as $\zeta = \zeta_0$ is

$$\begin{aligned} x &= \cosh \eta_B \cos \zeta_0 \\ y &= \sinh \eta_B \sin \zeta_0 \cos \xi \\ z &= \sinh \eta_B \sin \zeta_0 \sin \xi \end{aligned} \tag{103}$$

Similarly, the curve C_2 corresponding to $\zeta = \zeta_0$ on the outer surface is

$$x = e^{\eta_\infty} \cos \zeta_0$$

$$y = e^{\eta_\infty} \sin \zeta_0 \sin \xi$$

$$z = e^{\eta_\infty} \sin \zeta_0 \sin \xi \tag{104}$$

Based on the forms of the functions x, y, z in (103) and (104), we assume the following forms of x, y, z for the surface $\zeta = \zeta_0$:

$$\begin{aligned} x &= f(\sigma)\cos \zeta_0 \\ y &= \phi(\sigma)\sin \zeta_0 \cos \xi \\ z &= \phi(\sigma)\sin \zeta_0 \sin \xi \end{aligned} \tag{105}$$

The boundary conditions for f and ϕ are

$$\begin{aligned} f(\sigma_B) &= \cosh \eta_B \\ f(\sigma_\infty) &= e^{\eta_\infty} \\ \phi(\sigma_\infty) &= \sinh \eta_B \\ \Phi(\sigma_\infty) &= e^{\eta_\infty} \end{aligned} \tag{106}$$

Calculating the various derivatives, metric coefficients, and all other data needed in eqs. (99)-(101), we get on substitution an equation which has $\sin^2\zeta_0$ and $\cos^2\zeta_0$. Equating to zero the coefficients of $\sin^2\zeta_0$ and $\cos^2\zeta_0$, we obtain

$$\frac{f''}{f'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \tag{107}$$

$$\frac{\phi''}{\phi'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \tag{108}$$

where a prime denotes differentiation with respect to σ. Equations (107) and (108) can be directly integrated. The solution under the boundary conditions (105) and (106) is

$$f(\sigma) = A\exp(B\eta(\sigma)) + C \tag{109}$$

$$\phi(\sigma) = D\exp(B\eta(\sigma)) \tag{110}$$

where

$$A = (\exp(\eta_\infty) - \cosh\eta_B)\sinh\eta_B/(\exp(\eta_\infty) - \sinh\eta_B) \tag{111a}$$

$$B = \ell n[(\exp(\eta_\infty) - \sinh\eta_B)^{1/(\eta_\infty - \eta_B)}] \tag{111b}$$

$$C = (\cosh\eta_B - \sinh\eta_B)\exp(\eta_\infty)/(\exp(\eta_\infty) - \sinh\eta_B) \tag{111c}$$

$$D = \sinh\eta_B \tag{111d}$$

As an application we take the functions $\xi(\chi)$ and $\eta(\sigma)$ from Ref. [38],

$$\xi(\chi) = a\chi$$

$$\eta(\sigma) = b(\sigma - \sigma_B)K^\sigma$$

where a and b are constants. Since at η_∞,

$$\eta(\sigma_\infty) = \eta_\infty - \eta_B$$

hence

$$\eta(\sigma) = \frac{(\eta_\infty - \eta_B)(\sigma - \sigma_B)}{\sigma_\infty - \sigma_B} K^{(\sigma - \sigma_B)} \tag{112}$$

where $K > 1$ is an arbitrarily chosen constant. A value of $K \approx 1.1$ gives sufficient contraction of coordinates near the inner surface.

For the chosen problem, since the dependence on ζ is quite simple, we find that the coordinates between a prolate ellipsoid and a sphere with contraction are given by

$$\begin{aligned} x &= [C + A \exp(B\eta(\sigma))]\cos\zeta \\ y &= D \exp(B\eta(\sigma))\sin\zeta \cos\xi \\ z &= D \exp(B\eta(\sigma))\sin\zeta \sin\xi \end{aligned} \tag{113}$$

where A, B, C and D are given in eq. (111).

A computer program based on eqs. (99)-(101) has been developed by Ziebarth* by using the method of finite difference approximation. The differenced equations are solved by using the point-successive over relaxation method. Complete duplication of the exact solution obtained above has been achieved.

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*John Ziebarth, private communication.

Appendix 1

Christoffel Symbols in Three-Dimensional Coordinates

First kind:

$$[ij,k] = [ji,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

$$[11,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial \xi}$$

$$[12,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial \eta}$$

$$[13,1] = \frac{1}{2}\frac{\partial g_{11}}{\partial \zeta}$$

$$[22,1] = \frac{1}{2}\left(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}\right)$$

$$[23,1] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{13}}{\partial \eta} - \frac{\partial g_{23}}{\partial \xi}\right)$$

$$[33,1] = \frac{1}{2}\left(2\frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}\right)$$

$$[11,2] = \frac{1}{2}\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right)$$

$$[12,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial \xi}$$

$$[13,2] = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{13}}{\partial \eta}\right)$$

$$[22,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial \eta}$$

$$[23,2] = \frac{1}{2}\frac{\partial g_{22}}{\partial \zeta}$$

$$[33,2] = \frac{1}{2}(2\frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta})$$

$$[11,3] = \frac{1}{2}(2\frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta})$$

$$[12,3] = \frac{1}{2}(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta})$$

$$[13,3] = \frac{1}{2}\frac{\partial g_{33}}{\partial \xi}$$

$$[22,3] = \frac{1}{2}(2\frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta})$$

$$[23,3] = \frac{1}{2}\frac{\partial g_{33}}{\partial \eta}$$

$$[33,3] = \frac{1}{2}\frac{\partial g_{33}}{\partial \zeta}$$

For triply orthogonal systems: $[23,1] = [13,2] = [12,3] = 0$.

Second kind:

$$\Gamma^{i}_{jk} = \Gamma^{i}_{kj} = g^{i\ell}[jk,\ell]$$

Therefore, using eqs. (5) and (6) of Part III, we have

$$\Gamma^1_{jk} = \frac{1}{g}\{G_1[jk,1] + G_4[jk,2] + G_5[jk,3]\}$$

$$\Gamma^2_{jk} = \frac{1}{g}\{G_4[jk,1] + G_2[jk,2] + G_6[jk,3]\}$$

$$\Gamma^3_{jk} = \frac{1}{g}\{G_5[jk,1] + G_6[jk,2] + G_3[jk,3]\}$$

$$\Gamma^1_{11} = \frac{1}{2g}\{G_1 \frac{\partial g_{11}}{\partial \xi} + G_4(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) + G_5(2 \frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta})\}$$

$$\Gamma^1_{12} = \frac{1}{2g}\{G_1 \frac{\partial g_{11}}{\partial \eta} + G_4 \frac{\partial g_{22}}{\partial \xi} + G_5(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta})\}$$

$$\Gamma^1_{13} = \frac{1}{2g}\{G_1 \frac{\partial g_{11}}{\partial \zeta} + G_4(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{13}}{\partial \eta}) + G_5 \frac{\partial g_{33}}{\partial \xi}\}$$

$$\Gamma^1_{22} = \frac{1}{2g}\{G_1(2 \frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_4 \frac{\partial g_{22}}{\partial \eta} + G_5(2 \frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta})\}$$

$$\Gamma^1_{23} = \frac{1}{2g}\{G_1(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{13}}{\partial \eta} - \frac{\partial g_{23}}{\partial \xi}) + G_4 \frac{\partial g_{22}}{\partial \zeta} + G_5 \frac{\partial g_{33}}{\partial \eta}\}$$

$$\Gamma^1_{33} = \frac{1}{2g}\{G_1(2 \frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}) + G_4(2 \frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta}) + G_5 \frac{\partial g_{33}}{\partial \zeta}\}$$

$$\Gamma^2_{11} = \frac{1}{2g}\{G_4 \frac{\partial g_{11}}{\partial \xi} + G_2(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) + G_6(2 \frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta})\}$$

$$\Gamma^2_{12} = \frac{1}{2g}\{G_4 \frac{\partial g_{11}}{\partial \eta} + G_2 \frac{\partial g_{22}}{\partial \xi} + G_6(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta})\}$$

$$\Gamma^2_{13} = \frac{1}{2g}\{G_4 \frac{\partial g_{11}}{\partial \zeta} + G_2(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{13}}{\partial \eta}) + G_6 \frac{\partial g_{33}}{\partial \xi}\}$$

$$\Gamma^2_{22} = \frac{1}{2g}\{G_4(2\,\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_2\,\frac{\partial g_{22}}{\partial \eta} + G_6(2\,\frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta})\}$$

$$\Gamma^2_{23} = \frac{1}{2g}\{G_4(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{13}}{\partial \eta} - \frac{\partial g_{23}}{\partial \xi}) + G_2\,\frac{\partial g_{22}}{\partial \zeta} + G_6\,\frac{\partial g_{33}}{\partial \eta}$$

$$\Gamma^2_{33} = \frac{1}{2g}\{G_4(2\,\frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}) + G_2(2\,\frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta}) + G_6\,\frac{\partial g_{33}}{\partial \zeta}\}$$

$$\Gamma^3_{11} = \frac{1}{2g}\{G_5\,\frac{\partial g_{11}}{\partial \xi} + G_6(2\,\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) + G_3(2\,\frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta})\}$$

$$\Gamma^3_{12} = \frac{1}{2g}\{G_5\,\frac{\partial g_{11}}{\partial \eta} + G_6\,\frac{\partial g_{22}}{\partial \xi} + G_3(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta})\}$$

$$\Gamma^3_{13} = \frac{1}{2g}\{G_5\,\frac{\partial g_{11}}{\partial \zeta} + G_6(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{13}}{\partial \eta}) + G_3\,\frac{\partial g_{33}}{\partial \xi}\}$$

$$\Gamma^3_{22} = \frac{1}{2g}\{G_5(2\,\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_6\,\frac{\partial g_{22}}{\partial \eta} + G_3(2\,\frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta})\}$$

$$\Gamma^3_{23} = \frac{1}{2g}\{G_5(\frac{\partial g_{12}}{\partial \zeta} + \frac{\partial g_{13}}{\partial \eta} - \frac{\partial g_{23}}{\partial \xi}) + G_6\,\frac{\partial g_{22}}{\partial \zeta} + G_3\,\frac{\partial g_{33}}{\partial \eta}\}$$

$$\Gamma^3_{33} = \frac{1}{2g}\{G_5(2\,\frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}) + G_6(2\,\frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta}) + G_3\,\frac{\partial g_{33}}{\partial \zeta}\}$$

For triply orthogonal systems: $\Gamma^1_{23} = \Gamma^2_{13} = \Gamma^3_{12} = 0.$

Appendix 2

Christoffel Symbols Based on Surface Coefficients

(i) Surface ζ = constant:

$$T^1_{11} = \frac{1}{2G_3}[g_{22} \frac{\partial g_{11}}{\partial \xi} + g_{12}(\frac{\partial g_{11}}{\partial \eta} - 2 \frac{\partial g_{12}}{\partial \xi})]$$

$$T^2_{22} = \frac{1}{2G_3}[g_{11} \frac{\partial g_{22}}{\partial \eta} + g_{12}(\frac{\partial g_{22}}{\partial \xi} - 2 \frac{\partial g_{12}}{\partial \eta})]$$

$$T^1_{22} = \frac{1}{2G_3}[g_{22}(2 \frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) - g_{12} \frac{\partial g_{22}}{\partial \eta}]$$

$$T^2_{11} = \frac{1}{2G_3}[g_{11}(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) - g_{12} \frac{\partial g_{11}}{\partial \xi}]$$

$$T^1_{12} = T^1_{21} = \frac{1}{2G_3}(g_{22} \frac{\partial g_{11}}{\partial \eta} - g_{12} \frac{\partial g_{22}}{\partial \xi})$$

$$T^2_{12} = T^2_{21} = \frac{1}{2G_3}(g_{11} \frac{\partial g_{22}}{\partial \xi} - g_{12} \frac{\partial g_{11}}{\partial \eta})$$

(ii) Surface η = constant:

$$T^1_{11} = \frac{1}{2G_2}[g_{33} \frac{\partial g_{11}}{\partial \xi} + g_{13}(\frac{\partial g_{11}}{\partial \zeta} - 2 \frac{\partial g_{13}}{\partial \xi})]$$

$$T^3_{33} = \frac{1}{2G_2}[g_{11} \frac{\partial g_{33}}{\partial \zeta} + g_{13}(\frac{\partial g_{33}}{\partial \xi} - 2 \frac{\partial g_{13}}{\partial \zeta})]$$

$$T^1_{33} = \frac{1}{2G_2}[g_{33}(2 \frac{\partial g_{13}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \xi}) - g_{13} \frac{\partial g_{33}}{\partial \zeta}]$$

$$T^3_{11} = \frac{1}{2G_2}[g_{11}(2\frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta}) - g_{13}\frac{\partial g_{11}}{\partial \xi}]$$

$$T^1_{13} = T^1_{31} = \frac{1}{2G_2}(g_{33}\frac{\partial g_{11}}{\partial \zeta} - g_{13}\frac{\partial g_{33}}{\partial \xi})$$

$$T^3_{13} = T^3_{31} = \frac{1}{2G_2}(g_{11}\frac{\partial g_{33}}{\partial \xi} - g_{13}\frac{\partial g_{11}}{\partial \zeta})$$

(iii) Surface ξ = constant:

$$T^2_{22} = \frac{1}{2G_1}[g_{33}\frac{\partial g_{22}}{\partial \eta} + g_{23}(\frac{\partial g_{22}}{\partial \zeta} - 2\frac{\partial g_{23}}{\partial \eta})]$$

$$T^3_{33} = \frac{1}{2G_1}[g_{22}\frac{\partial g_{33}}{\partial \zeta} + g_{23}(\frac{\partial g_{33}}{\partial \eta} - 2\frac{\partial g_{23}}{\partial \zeta})]$$

$$T^2_{33} = \frac{1}{2G_1}[g_{33}(2\frac{\partial g_{23}}{\partial \zeta} - \frac{\partial g_{33}}{\partial \eta}) - g_{23}\frac{\partial g_{33}}{\partial \zeta}]$$

$$T^3_{22} = \frac{1}{2G_1}[g_{22}(2\frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta}) - g_{23}\frac{\partial g_{22}}{\partial \eta}]$$

$$T^2_{23} = T^2_{32} = \frac{1}{2G_1}(g_{33}\frac{\partial g_{22}}{\partial \zeta} - g_{23}\frac{\partial g_{33}}{\partial \eta})$$

$$T^3_{23} = T^3_{32} = \frac{1}{2G_1}(g_{22}\frac{\partial g_{33}}{\partial \eta} - g_{23}\frac{\partial g_{22}}{\partial \zeta})$$

In the preceding formulae, the coefficients G_1, G_2 and G_3 are those which have been defined in eqs. (5) and (6) of Part III.

Appendix 3

The Beltrami Equations

For a study of the curvilinear coordinates in plane two-dimensional regions the technique of quasiconformal mapping is frequently used. Quasiconformal mappings are more general and flexible than the usual conformal mappings. For details on the mathematical aspects of the quasiconformal mappings, refer to [39] and [40].

A quasiconformal mapping of a region D onto a region D^* is given by a one-to-one continuous mapping whose inverse is also continuous (homeomorphism). The mapping function $w = f(z,\bar{z})$ for this case is taken as a solution of the complex equation

$$f_{\bar{z}} - H(z,\bar{z})f_z = 0 \tag{1}$$

where

$$z = x + iy \ , \ \bar{z} = x - iy \ , \ i = \sqrt{-1} \ .$$

The complex equation (1) is called the Beltrami equation, which is equal to the two real equations

$$-\eta_x = \beta\xi_x + \gamma\xi_y \tag{2}$$

$$\eta_y = \alpha\xi_x + \beta\xi_y \tag{3}$$

where

$$f(z,\bar{z}) = \xi(x,y) + i\eta(x,y) \tag{4a}$$

$$H(z,\bar{z}) = \mu(x,y) + i\nu(x,y) \tag{4b}$$

$$\alpha = [(1 - \mu)^2 + \nu^2]/\Delta \tag{4c}$$

$$\beta = -2\nu/\Delta \tag{4d}$$

$$\gamma = [(1 + \mu)^2 + \nu^2]/\Delta \tag{4e}$$

$$\Delta = 1 - (\mu^2 + \nu^2) \tag{4f}$$

Note that

$$\alpha\gamma - \beta^2 = 1 \tag{6}$$

$$\alpha + \gamma = 2(2 - \Delta)/\Delta \tag{7}$$

A quasiconformal mapping becomes conformal when H = 0, or, equivalently $\alpha = \gamma = 1$, $\beta = 0$. In this case eqs. (2) and (3) reduce to the Cauchy-Riemann equations

$$\xi_x = \eta_y \ , \ \xi_y = -\eta_x \tag{8}$$

and f(z) is then a holomorphic or analytic function in D.

Now, from eq. (1)

$$|H|^2 = \mu^2 + \nu^2 = |f_{\bar{z}}/f_z|^2$$

so that on using eqs. (175)-(177) of Part I, we obtain

$$\Delta = 4\sqrt{g}/[2\sqrt{g} + (g_{11} + g_{22})] \tag{9}$$

Substituting (9) in (7), we get

$$\alpha + \gamma = (g_{11} + g_{22})/\sqrt{g} \tag{10}$$

Equations (2) and (3) can also be written by using the inversion relations given in Part I, eq. (177), as

$$x_\xi = \alpha y_\eta - \beta x_\eta \tag{11}$$

$$y_\xi = \beta y_\eta - \gamma x_\eta \tag{12}$$

Solving eqs. (11) and (12) for x_η and y_η, we have

$$x_\eta = \beta x_\xi - \alpha y_\xi \tag{13}$$

$$y_\eta = \gamma x_\xi - \beta y_\xi \tag{14}$$

The Beltrami equations (13), (14) form a system of first order partial differential equations for numerical coordinate generation. The coefficients α, β and γ are related, as can be seen by solving eqs. (6) and (10),

$$\alpha,\gamma = [g_{11} + g_{22} \mp \{(g_{11} + g_{22})^2 - 4(1 + \beta^2)g\}^{1/2}]/2\sqrt{g} \tag{15}$$

The choice of β can be based on the minimization of a certain functional to ensure uniqueness. This algorithm has been followed in Ref. [41].

If orthogonal coordinates are desired, then using eqs. (13) and (14) in the orthogonality condition

$$g_{12} = x_\xi x_\eta + y_\xi y_\eta = 0$$

we obtain β through the algebraic equation

$$\beta = \frac{(\gamma - \alpha)x_\xi y_\xi}{{y_\xi}^2 - {x_\xi}^2} \tag{16}$$

An iterative numerical scheme can now be used to solve the coupled system of equations (13)-(16).

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