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BEHAVIOUR AT TIME  $T=0$  OF THE SOLUTIONS OF SEMI-LINEAR EVOLUTION--ETC(U)

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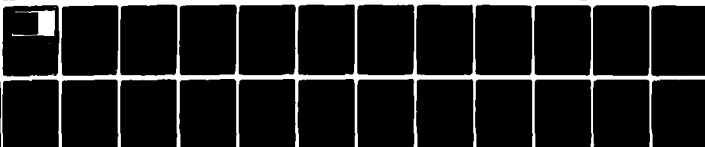
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BEHAVIOUR AT TIME  $t = 0$   
OF THE SOLUTIONS OF  
SEMI-LINEAR EVOLUTION EQUATIONS

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BEHAVIOUR AT TIME  $t = 0$  OF THE SOLUTIONS OF  
SEMI-LINEAR EVOLUTION EQUATIONS

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ABSTRACT

We investigate the regularity at time  $t = 0$  of the solutions of linear and semi-linear evolution equations (including the Stokes and Navier-Stokes equations). Necessary and sufficient conditions on the data, for an arbitrary order of regularity are given (the classical "compatibility conditions"). In the case of the Stokes and Navier-Stokes equations the compatibility conditions which we find are global conditions on the data. The presentation given here seems to improve and generalize the known results even in the simplest case of linear evolution equations.

AMS (MOS) Subject Classifications: 35K55, 35K45

Key Words: Semi-linear evolution equations, regularity, Stokes and Navier-Stokes equations, compatibility conditions.

Work Unit Number 1 - Applied Analysis

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### SIGNIFICANCE AND EXPLANATION

It is well-known that the heat equation

$$(*) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0, & x \in \Omega, t \in (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x), & x \in \Omega \end{cases},$$

possesses some regularizing effects: if  $u_0$  is a square integrable initial datum for (\*),  $u_0 \in L^2(\Omega)$ , then the function  $u(t) : \{x \in \Omega \rightarrow u(x,t) \in \mathbb{R}\}$ , is  $C^\infty$  in  $\Omega \times (0,T)$  (and even in  $\bar{\Omega} \times (0,T)$  if  $\partial\Omega$  is  $C^\infty$ ). However, even if the initial datum  $u_0 = u(0)$  is smooth (say  $u_0 \in C^\infty(\bar{\Omega})$ ), the solution  $u$  of (\*) may not be  $C^\infty$  or even  $C^1$  in  $\bar{\Omega} \times [0,T]$ . It is only for special initial data  $u_0$ , that  $u$  will belong to  $C^1$  or  $C^k$  of  $\bar{\Omega} \times [0,T]$ , those  $u_0$  which satisfy the so-called "compatibility conditions". The study of this question is of interest in the theory of evolution equations, for singular perturbations and for the numerical approximation of evolution equations.

For general linear equations and some semi-linear equations (including Stokes and Navier-Stokes equations), we describe in a simple way the conditions of regularity at  $t = 0$  of the solutions.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BEHAVIOUR AT TIME  $t = 0$  OF THE SOLUTIONS OF  
SEMI-LINEAR EVOLUTION EQUATIONS

R. Temam

We are considering, here, semi-linear evolution equations of the type

$$(0.1) \quad \frac{du}{dt}(t) + Au(t) = g(t, u), \quad 0 < t < T$$

$$(0.2) \quad u(0) = u_0$$

and we are interested to determine in what sense the initial condition (0.2) is achieved, i.e., to determine the best space in which

$$(0.3) \quad u(t) \rightarrow u_0, \quad \text{as } t \rightarrow 0.$$

We assume that  $A$  is a linear unbounded operator in a Hilbert space  $H$  with domain  $D(A)$ , and that  $g$  is a smooth function on  $[0, T] \times D(A)$ , dominated in some sense by  $A$ . The precise assumptions are described later on.

The realization of the convergence (0.3) in a space smaller than  $H$ , is a problem connected with the regularity near  $t = 0$  of the solutions of (0.1) (0.2) and is not in general expected even in the simple case of the linear heat equation. However, it is known for linear equations ( $g(t, u) \equiv g(t)$ ), that if  $u_0$  and  $g$  satisfy some compatibility conditions then  $u$  is more regular at time  $t = 0$  and the convergence (0.3) is achieved in a space better than  $H$ , cf. Friedman [5], Ladyzhenskaya-Solonnikov-Uralceva [8], Rauch-Massey [11], Smale [12].

Our task is to determine the compatibility conditions which guarantee the regularity near  $t = 0$  of the solutions of (0.1) (0.2). Actually we will show that under appropriate assumptions, some very simple and natural necessary conditions of regularity, are also sufficient. Some examples are given and in particular our results apply to the evolution Navier-Stokes equations for which they seem to be new. In the case of Navier-Stokes equations (or even of the linearized Stokes equations) the compatibility conditions on  $u_0$  are of a global

type and are totally unusual. Besides the intrinsic interest of this result we had some other motivations in doing this work, which will be developed elsewhere (in particular the extension of the results of [3] from the space periodic case to the case of the flow in a domain with boundary, and the numerical approximation of the evolution Navier-Stokes equations). We believe that the method is general and applies to many equations.

## 1. Preliminaries

### 1.1. Notations

Let  $H$  be a real Hilbert space, and  $A$  a linear self-adjoint operator from  $D(A)$  into  $H$ , where  $D(A)$  = the domain of  $A$  is a subspace of  $H$ . We assume that  $A$  is an isomorphism from  $D(A)$  (equipped with the graph norm) onto  $H$ , that  $A^{-1}$  is a linear compact operator in  $H$  and that

$$(1.1) \quad (Au, u) > 0, \quad \forall u \in D(A), \quad u \neq 0.$$

One can then define the powers  $A^\alpha$  of  $A$  for any  $\alpha \in \mathbb{R}$ , with domain  $D(A^\alpha)$ ;  $D(A^\alpha)$  is a Hilbert space for the norm  $|A^\alpha u|_H$  and  $A^\alpha$  is an isomorphism from  $D(A^\alpha)$  onto  $H$ . We set

$$(1.2) \quad V_\alpha = D(A^{\alpha/2}), \quad \forall \alpha \in \mathbb{R},$$

and in particular  $V = V_1$ ,  $H = V_0$ . For  $\alpha > 0$  the space  $V_{-\alpha}$  can be identified to the dual  $(V_\alpha)'$  of  $V_\alpha$ . We denote by  $|\cdot|_\alpha$  the norm in  $V_\alpha$ ;  $|\cdot|$  the norm in  $H$ .

It is well-known that for  $T > 0$ , and  $f$  and  $u_0$  given, satisfying

$$(1.3) \quad f \in L^2(0, T; V'), \quad u_0 \in H,$$

the initial value problem

$$(1.4) \quad \frac{du}{dt}(t) + Au(t) = f(t), \quad 0 < t < T$$

$$(1.5) \quad u(0) = u_0$$

possesses a unique solution  $u$  which satisfies

$$(1.6) \quad u \in L^2(0, T; V) \cap C([0, T]; H),$$

which implies

$$(1.7) \quad u(t) \rightarrow u_0 \text{ in } H \text{ as } t \rightarrow 0.$$

If furthermore

$$(1.8) \quad f \in L^2(0, T; H), \quad u_0 \in V,$$

then  $u$  satisfies

$$(1.9) \quad u \in L^2(0, T; D(A)) \cap C([0, T]; V) ,$$

and

$$(1.10) \quad u(t) \rightarrow u_0 \text{ in } V \text{ as } t \rightarrow 0 .$$

Now if  $u_0 \in V_m$ ,  $m \geq 3$ , it is not true, in general, that  $u(t) \rightarrow u_0$  in  $V_m$ , as  $t \rightarrow 0$ , even if  $f \equiv 0$ . The necessary and sufficient conditions that  $u_0$  and  $f$  must satisfy in order that  $u(t) \rightarrow u_0$  in a norm stronger than the norm of  $V$  are given in [8] [12] for special classes of operators  $A$ . We intend here to extend these results to general operators  $A$  and then to nonlinear equations of evolution.

## 1.2. Hypotheses

Besides the scale of spaces  $V_\alpha$ , we introduce a family of Hilbert spaces  $E_m$ ,  $m \in \mathbb{N}$ , with

$$(1.11) \quad E_{m+1} \subset E_m, \forall m \in \mathbb{N}, \text{ the injection being continuous, and } E_0 = H$$

$$(1.12) \quad V_m \text{ is a closed subspace of } E_m, \forall m \in \mathbb{N}, \text{ the norm induced by } E_m \text{ on } V_m \text{ being equivalent to } |\cdot|_m.$$

We are also given a linear operator  $A$  satisfying

$$(1.13) \quad A \text{ is continuous from } E_{m+2} \text{ into } E_m, \forall m \geq 0,$$

$$(1.14) \quad A \text{ is an isomorphism from } E_{m+2} \cap V \text{ onto } E_m, \forall m \geq 0,$$

$$(1.15) \quad Au = Au, \forall u \in V_m, m \geq 2. \quad (1)$$

We denote by  $G$  the inverse  $A^{-1}$  of  $A$  which is linear continuous from  $V_\alpha$  onto  $V_{\alpha+2}$ ,  $\forall \alpha \in \mathbb{R}$ . Because of (1.11) (1.14) (1.15),

$$(1.16) \quad GAu = u, \forall u \in E_{m+2} \cap V, m \geq 0 ,$$

while, in general,  $GAu \neq u$ , for an arbitrary  $u$  in  $E_{m+2}$ ,  $m \geq 0$ .

(1) In the applications the  $E_m$ 's are Sobolev type spaces,  $A$  is the differential operator,  $A$  the abstract operator associated to  $A$ ; cf the examples.



We then have the following scheme

$$(1.17) \quad \begin{cases} E_m \subset E_1 \subset E_0 \\ u \quad u \quad " \\ V_m \subset V \subset H \subset V_{-1} = V' \subset V_{-m} = V'_m . \end{cases}$$

#### Example 1

We show how the previous assumptions are satisfied in the easy case of the heat equation

$$(1.18) \quad \frac{\partial u}{\partial t} - \Delta u = f \quad \text{in } \Omega \times (0, T) ,$$

where  $\Omega$  is a bounded open set of  $\mathbb{R}^n$  with boundary  $\Gamma$ . New examples are described in Section 2.3.

We assume that  $\Gamma$  is a  $C^\infty$  manifold of dimension  $n - 1$  and we set  $H = L^2(\Omega)$ ,  $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ ,  $Au = -\Delta u$ ,  $\forall u \in D(A)$ , and  $E_m = H^m(\Omega)$ ,  $\forall m \geq 0$ ,  $Au = -\Delta u$ ,  $\forall u \in H^2(\Omega)$ . The operator  $A$  is an isomorphism from  $D(A)$  onto  $H$ , because of the regularity theory for elliptic boundary value problems (cf. Agmon-Douglis-Nirenberg [1]), and assumption (1.14) also follows from [1]. It is clear that  $V_1$  is a closed subspace of  $E_1$ ,  $V_2 = D(A)$  is a closed subspace of  $E_2$ : assumption (1.12) for  $m > 2$  follows then from this remark, the definition of  $A$  and (1.14). All the other assumptions are trivial or well-known. The spaces  $V_m$  and  $E_m$  are different for  $m \geq 2$ . For instance  $V_4 = D(A^2) = \{u \in E_4 = H^4(\Omega), u = \Delta u = 0 \text{ on } \Gamma\}$ .

#### 1.3. The successive derivatives of $u$ .

For any integer  $m$ , we denote by  $W_m$  the space

$$W_m = \{v \in C([0, T]; E_m), \frac{d^j v}{dt^j} \in C([0, T]; E_{m-2j}), j = 1, \dots, \ell\} ,$$

where  $\ell$  is the integer part of  $\frac{m}{2}$ .

The following remark will be useful.

#### Lemma 1.1.

We assume that (1.8) is satisfied and that, for some  $m \geq 2$ ,

$$(1.19) \quad f \in W_{m-2} ,$$

and the solution  $u$  of (1.4) (1.5) satisfies

$$(1.20) \quad u \in C([0, T]; E_m) .$$

Then

$$(1.21) \quad u \in W_m .$$

Proof

Due to (1.15), the equation (1.4) can be rewritten

$$(1.22) \quad \frac{du}{dt} + Au = f .$$

Because of (1.19) (1.20) (1.13),

$$\frac{du}{dt} = f - Au \in C([0, T]; E_{m-2}) .$$

Applying then the operator  $\frac{d}{dt} - A$  to each member of (1.22), we get<sup>(1)</sup>

$$\frac{d^2 u}{dt^2} - A^2 u = \left(\frac{d}{dt} - A\right)f .$$

Whence, with (1.19) (1.20) (1.13)

$$\frac{d^2 u}{dt^2} = A^2 u + \left(\frac{d}{dt} - A\right)f \in C([0, T]; E_{m-4}) .$$

More generally, with  $\delta = \frac{d}{dt}$ ,

$$(1.23) \quad -(-\delta)^{j+1} + A^{j+1} = \alpha_j(\delta, A)(\delta + A) = (\delta + A)\alpha_j(\delta, A)$$

where

$$(1.24) \quad \alpha_j(\delta, A) = \sum_{i=0}^j (-\delta)^i A^{j-i}$$

and therefore, for  $j = 0, \dots, l-1$ ,

$$(1.25) \quad (-1)^{j+1} \frac{d^{j+1} u}{dt^{j+1}} = A^{j+1} u = \alpha_j\left(\frac{d}{dt}, A\right)f ,$$

<sup>(1)</sup> The time derivatives are understood in the sense of vector valued distributions.

and we infer from (1.19) (1.20) (1.13) that the right-hand side of (1.25) belongs to  $C([0,T]; E_{m-2j-2})$  and the result follows. □

Remark 1.1

Under the assumptions of Lemma 1.1 one can compute all the derivatives

$$\frac{d^j u}{dt^j}(t), \quad j = 1, \dots, l, \quad t \in [0, T],$$

in term of  $u(t)$  and the data  $f$ . This can be obtained by differentiating  $l - 1$  times the equation (1.4) to obtain

$$(1.26) \quad \begin{aligned} \frac{d^2 u}{dt^2} + A \frac{du}{dt} &= \frac{df}{dt} \\ \frac{d^3 u}{dt^3} + A \frac{d^2 u}{dt^2} &= \frac{d^2 f}{dt^2}, \quad \text{etc.} \dots \end{aligned}$$

and then eliminating  $\frac{du}{dt}, \dots, \frac{d^{j-1} u}{dt^{j-1}}$ . More directly, with (1.25) we get

$$(1.27) \quad (-1)^{j+1} \frac{d^{j+1} u}{dt^{j+1}}(t) = A^{j+1} u(t) - [\varphi_j(f)](t), \quad j = 0, \dots, l-1, \quad t \in [0, T],$$

where we set

$$(1.28) \quad \varphi_j(f) = \alpha_j \left( \frac{d}{dt}, A \right) f = \sum_{i=0}^j \left( -\frac{d}{dt} \right)^i A^{j-i} f.$$

In particular at  $t = 0$

$$(1.29) \quad \frac{d^{j+1} u}{dt^{j+1}}(0) = (-1)^{j+1} A^{j+1} u_0 + (-1)^j \psi_j(f)$$

where

$$(1.30) \quad \psi_j(f) = [\varphi_j(f)](0).$$

## 2. Linear problems

### 2.1. The result in the linear case.

#### Theorem 2.1.

The assumptions are those of Section 1.1 and 1.2 and we assume that for some  $m \geq 2$

$$(2.1) \quad u_0 \in E_m \cap D(A), \quad f \in W_{m-2}$$

$$(2.2) \quad \frac{d^{\ell} f}{dt^{\ell}} \in \begin{cases} L^2(0, T; H) & \text{if } m = 2\ell + 1 \\ L^2(0, T; V') & \text{if } m = 2\ell \end{cases}.$$

Then a necessary and sufficient condition for the solution of (1.4) (1.5) to belong to  $W_m$  is that

$$(2.3) \quad \frac{d^j u}{dt^j}(0) \in V \quad \text{for} \quad \begin{cases} 1 \leq j \leq \ell & \text{if } m = 2\ell + 1 \\ 1 \leq j \leq \ell - 1 & \text{if } m = 2\ell \end{cases},$$

the expression of the derivatives  $\frac{d^j u}{dt^j}(0)$  in term of the data being given by formula (1.29) (1.30), i.e.

$$(2.4) \quad \frac{d^j u}{dt^j}(0) = (-A)^j u_0 + \sum_{i=0}^{j-1} \left( \frac{d}{dt} \right)^i (-A)^{j-i-1} f \Big|_{t=0}$$

( $j = 1, \dots, \ell$ )

□

Two proofs of the theorem are given in Section 2.2 and 2.3, the first one simpler, the second one giving more information.

#### 2.2. Proof of Theorem 2.1.

It is obvious that the conditions (2.3) are necessary, assuming that  $f$  and  $u_0$  satisfy (2.1). Indeed, if  $u \in C([0, T]; V)$  and  $\frac{du}{dt} \in C([0, T]; E_p)$ ,  $p \geq 1$ , then  $\frac{du}{dt} \in C([0, T]; E_1)$  and, since  $V$  is a closed subspace of  $E_1$ ,  $\frac{du}{dt}(t) \in V$  for each  $t$  and in particular for  $t = 0$ . The argument is the same for all the other derivatives of  $u$ , except for  $\frac{d^{\ell} u}{dt^{\ell}}$  when

$m = 2l$ , in which case we replace  $E_p$  by  $E_0$  and  $V$  by  $H$  and there is no condition on  $\frac{d^l u}{dt^l}(0)$  which belongs to  $H$ .

In order to prove that the conditions are sufficient, we differentiate equation (1.4)  $j$  times, and we consider the following initial value problem for  $u^{(j)} = \frac{d^j u}{dt^j}$ :

$$(2.5) \quad \frac{du^{(j)}}{dt} + Au^{(j)} = f^{(j)}$$

$$(2.6) \quad u^{(j)}(0) \text{ given by (2.4) .}$$

For  $j = 1, \dots, l$  (if  $m = 2l+1$ ), or  $j = 1, \dots, l-1$ , (if  $m = 2l$ ), we apply (1.9) and get

$$(2.7) \quad u^{(j)} \in L^2(0, T; D(A)) \cap C([0, T]; V) .$$

If  $m = 2l$ , we infer from (2.4) and (2.7) that  $u^{(l)}(0) \in H$  and, applying (1.6) we get

$$(2.8) \quad u^{(l)} \in L^2(0, T; V) \cap C([0, T]; H) .$$

Then by (2.1), the relations (2.5) and hypothesis (1.14) we find for  $m = 2l+1$ :

$$u^{(l)} \in C([0, T]; V), u^{(l-1)} \in C([0, T]; E_3), \dots, u \in C([0, T]; E_{2l+1})$$

and for  $m = 2l$

$$u^{(l)} \in C([0, T]; H), u^{(l-1)} \in C([0, T]; E_2), \dots, u \in C([0, T]; E_{2l}) .$$

Whence the result.

### 2.3. Another proof.

We give an alternate proof of the sufficiency, i.e., that (2.1) (2.2) (2.3) imply that the solution  $u$  of (1.4) (1.5) belongs to  $W_m$ . Actually, due to Lemma 1.1 we only have to prove that

$$(2.9) \quad u \in C([0, T]; E_m) .$$

Another simplification: we can assume that  $u_0 = 0$ . If we set  $\hat{u} = u - u_0$ ,  $\hat{f} = f - Au_0$ , then  $\hat{u}_0 = 0$  and  $\hat{f}$  satisfy (2.1) - (2.3) and  $u \in C([0, T]; E_m)$  if and only if  $\hat{u} \in C([0, T]; E_m)$ .

The expression  $w$

We set

$$(2.10) \quad w = \sum_{i=0}^{\ell-1} \left(-G \frac{d}{dt}\right)^i Gf.$$

Since  $f \in W_{m-2}$ , it is clear with (1.12) and (1.14) that

$$(2.11) \quad w \in C([0, T; E_m \cap V).$$

Then we infer from assumption (2.3) the following remark

Lemma 2.1.

$$(2.12) \quad w(0) \in V_m.$$

Proof

We are going to show that

$$(2.13) \quad w(0) = G^\ell \psi_{\ell-1}(f)$$

(cf the notations in Remark 1.1). Since, by assumption,  $\psi_{\ell-1}(f) \in V$  (if  $m = 2\ell+1$ ) or  $H$  (if  $m = 2\ell$ ), and  $G$  coincides with  $A^{-1}$  on  $V$  or  $H$ , we find that  $w(0) \in V_{2\ell+1}$  or  $V_{2\ell}$  and (2.12) follows.

In order to prove (2.13), we note that we have the recurrence formula

$$(2.14) \quad \psi_j(f) = \left(-\frac{d}{dt}\right)^j f(0) + A\psi_{j-1}(f), \quad \forall j \geq 0$$

(assuming  $\psi_{-1}(f) = 0$ ). Whence

$$(2.15) \quad G^\ell \psi_{\ell-1}(f) = \left(-\frac{d}{dt}G\right)^{\ell-1} Gf \Big|_{t=0} + G^\ell A\psi_{\ell-2}(f).$$

It follows from (1.14) (cf (1.28) (1.30)) that

$$(2.16) \quad \varphi_j(f) \in C([0, T; E_{m-2j-2}), \quad \psi_j(f) \in E_{m-2j-2}, \quad j = 0, \dots, \ell-2, \quad \ell = \left[\frac{m}{2}\right],$$

and because of assumptions (2.1) and (2.3)

$$\psi_{\ell-2}(f) \in V \cap E_p, p = 2 \text{ if } m = 2\ell, p = 3 \text{ if } m = 2\ell+1.$$

But  $GA$  is the identity on  $V \cap E_2$  or  $(V \cap E_3)$  and therefore

$$G^\ell A \psi_{\ell-2}(f) = G^{\ell-1} \psi_{\ell-2}(f).$$

By reiterating the argument until  $j = 0$ , we arrive at (2.13).

□

The function  $v = u - w$

The difference  $v = u - w$  is solution of an evolution equation which is easy to determine:

$$\begin{aligned} \left(\frac{d}{dt} + A\right)v &= f - \left(\frac{d}{dt} + A\right)w \\ &= f + \sum_{i=0}^{\ell-1} \left(-G \frac{d}{dt}\right)^{i+1} f - \sum_{i=0}^{\ell-1} A \left(-G \frac{d}{dt}\right)^i G f. \end{aligned}$$

Since, by (2.1),  $f(t) \in H \forall t \in [0, T]$  and  $AG$  is the identity on  $H$ , we get

$$\begin{aligned} \frac{dv}{dt} + Av &= f + \sum_{j=1}^{\ell} \left(-G \frac{d}{dt}\right)^j f - \sum_{j=0}^{\ell-1} \left(-G \frac{d}{dt}\right)^j f \\ &= \left(-G \frac{d}{dt}\right)^\ell f \end{aligned}$$

and  $v$  is solution to the initial value problem

$$(2.17) \quad \frac{dv}{dt} + Av = (-G)^\ell \frac{d^\ell f}{dt^\ell}$$

$$(2.18) \quad v(0) = -w(0).$$

By lemma 2.1,  $-w(0) \in V_m$  and, by assumption (2.2), the right-hand side of (2.17) belongs to  $L^2(0, T; V_{m-1})$ . If we apply the classical theorem on linear evolution equations as in (1.3) - (1.6), with  $V, H$  and  $V'$  replaced by  $V_{m+1}, V_m$  and  $V_{m-1}$ , we get that

(2.19) The problem (2.17) (2.18) possesses a unique solution

$$v \in L^2(0, T; V_{m+1}) \cap C([0, T]; V_m).$$

Since by (1.13)  $V_m$  is a closed subspace of  $E_m$ ,  $v \in C([0, T]; E_m)$  and then using (2.11) we conclude that  $u$  satisfies (2.9).

The proof of Theorem 2.1 is complete. □

### 2.3. Examples.

We study an equation associated to a linear operator of the fourth order and the linear Stokes equations. We assume that:

(2.20)  $\Omega$  is an open bounded set of  $\mathbb{R}^n$ , whose boundary  $\Gamma$  is a manifold of class  $C^{m+2}$  of dimension  $n - 1$ .

#### Example 2.1.

$$(2.21) \quad \begin{cases} \frac{\partial u}{\partial t} + \Delta^2 u = f & \text{in } \Omega \times (0, T) \\ u = \Delta u = 0 & \text{on } \Gamma \times (0, T) \\ u(0) = u_0 \end{cases}$$

We set  $H = L^2(\Omega)$ ,  $V = H^2(\Omega) \cap H_0^1(\Omega)$ ,  $D(A) = \{u \in H^4(\Omega), u = \Delta u = 0 \text{ on } \Gamma\}$ ,  $Au = \Delta^2 u$ ,  $E_m = H^{2m}(\Omega)$ ,  $Au = \Delta^2 u$ .

All the assumptions concerning the spaces and operators are clearly satisfied, the main one, (1.14), following from [1].

For  $m = 2$ , if  $u_0$  and  $f$  satisfy (2.1) (2.2), then (2.3) is automatically satisfied and Theorem 2.1 gives that  $u \in C([0, T]; D(A))$  which is well-known. For  $m = 4$ , assuming (2.1) (2.2), the necessary and sufficient condition for  $u$  to be in  $C([0, T]; H^8(\Omega))$  is that

$$(2.22) \quad -\Delta^2 u_0 + f(0) = 0 \text{ on } \Gamma.$$

The left-hand side of (2.22) must belong to  $H^4(\Omega) \cap H_0^1(\Omega)$  and, in particular, must not belong to  $D(A)$ .

#### Example 2.2: Stokes problem.

The unknown functions are  $u = (u_1, \dots, u_n)$  and  $p$  solutions to:

$$(2.23) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \text{grad } p = f & \text{in } \Omega \times (0, T) \\ \text{div } u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \Gamma \times (0, T) \\ u(x, 0) = u_0(x), \quad x \in \Omega \end{cases}$$



We set (cf [14])

$$H = \{u \in L^2(\Omega)^n, \operatorname{div} u = 0, u \cdot \vec{N} = 0 \text{ on } \Gamma\},$$

$\vec{N}$  the unit outward normal on  $\Gamma$ ,

$$V = \{u \in H_0^1(\Omega)^n, \operatorname{div} u = 0\}, D(A) = H^2(\Omega)^n \cap V.$$

We denote by  $P$  the orthogonal projector in  $L^2(\Omega)^n$  onto  $H$ , and for  $u \in D(A)$ , we set

$$(2.24) \quad Au = -\nu P \Delta u.$$

It follows from the Cattabriga-Solonnikov-Vorovich-Yudovich theorem [2-12-14], that  $A$  is an isomorphism from  $D(A)$  onto  $H$ . We set also  $E_m = H^m(\Omega)^n \cap H$  and again for  $u \in E_m$ ,  $m \geq 2$ ,  $Au = -\nu P \Delta u$ . The details of the proofs of the assumptions of Sections 1.1 and 1.2 can be found in [14]. In particular (1.13) follows (cf [14] p. 13) from the fact that

$$(2.25) \quad P \text{ is linear continuous from } H^m(\Omega)^n \text{ into } H^m(\Omega)^n \cap H, \forall m \geq 0.$$

We recall [14] that the orthogonal  $H^\perp$  of  $H$  in  $L^2(\Omega)^n$  is

$$(2.26) \quad H^\perp = \{u = \operatorname{grad} q, q \in H^1(\Omega)\},$$

and then by applying the operator  $P$  to both members of the first relation in (2.23), the pressure  $p$  disappears and we are left with the functional form of (2.23):

$$(2.27) \quad \begin{cases} \frac{du}{dt} + Au = f \\ u(0) = u_0 \end{cases}$$

(we assume as usual that  $Pf = f$ ).

Theorem 2.1 applies. For  $m = 2$ ,  $\ell = 1$ , we recover a classical result: if  $u_0 \in D(A)$ ,  $f \in L^2(0, T; H)$ ,  $\frac{df}{dt} \in L^2(0, T; V')$ , then (2.27) possesses a unique solution  $u$  which belongs in particular to  $C([0, T]; D(A))$ .

For  $m = 3$  or  $4$  we get the following:

Theorem 2.2.

We assume that  $u_0 \in H^3(\Omega)^n \cap V$ ,  $f \in C([0, T]; H^1(\Omega)^n \cap H)$ ,  $\frac{df}{dt} \in L^2(0, T; H)$ , then the solution  $u$  of (2.23) (or (2.27)) belongs to  $C([0, T]; H^3(\Omega)^n \cap V)$  if and only if

$$(2.28) \quad vP\Delta u_0 + f(0) = 0 \text{ on } \Gamma.$$

If  $u_0 \in H^4(\Omega)^n \cap V$ ,  $f \in C([0, T]; H^2(\Omega)^n \cap H)$ ,  $\frac{df}{dt} \in C([0, T]; H)$ ,  $\frac{d^2 f}{dt^2} \in L^2(0, T; V')$ , then (2.28) is necessary and sufficient for  $u$  to belong to  $C([0, T]; H^4(\Omega)^n \cap V)$ .

Proof

In both cases, applying Theorem 2.1, we just have to write that  $\frac{du}{dt}(0) \in V$ . Using (2.17), we find

$$(2.29) \quad \frac{du}{dt}(0) = vP\Delta u_0 + f(0)$$

which already belongs to  $H^1(\Omega)^n \cap H$  (if  $m = 3$ ) or  $H^2(\Omega)^n \cap H$  (if  $m = 4$ ). Then the expression (2.29) belongs to  $V$  if and only if its trace on  $\Gamma$  vanishes.

□

Remark 2.3

Let  $q$  be the solution to the Neumann problem

$$(2.30) \quad \begin{cases} \Delta q = 0 & \text{in } \Omega \\ \frac{\partial q}{\partial \vec{N}} = \Delta u_0 \cdot \vec{N} & \text{on } \Gamma \end{cases}.$$

Then cf [14],  $\Delta u_0 = \Delta u_0 - \nabla q$  and (2.28) amounts to:

$$(2.31) \quad v\Delta u_0 - v\Delta q + f(0) = 0 \text{ on } \Gamma.$$

This is equivalent to a condition given by J. G. Heywood [6].

2.4. Comments

i) Provided the spaces  $V_m$  are properly defined, the assumption that  $A$  is self-adjoint is not necessary for Theorem 2.1 (using the proof in Section 2.2).

ii) Under some appropriate assumptions, Theorem 2.1 and the proof in Section 2.2 extend to the case where the operator  $A$  depends on  $t$  (with  $D(A(t)) \equiv D(A(0))$  independent of  $t$ ).

The result is essentially the same, except that the expression (2.4) of the derivatives

$\frac{d^j u}{dt^j}(0)$  is not valid anymore.

iii) Under the assumptions of Theorem 2.1, but with (2.3) not satisfied, the proof in Section 2.3 allows us to split the difference  $v = u - w$  into the sum of two terms  $v_1, v_2$ , respectively solutions of

$$(2.32) \quad \begin{cases} \frac{dv_1}{dt} + Av_1 = (-G)^{\ell} \frac{d^{\ell} f}{dt^{\ell}} \\ v_1(0) = 0 \end{cases}$$

$$(2.33) \quad \begin{cases} \frac{dv_2}{dt} + Av_2 = 0 \\ v_2(0) = -w(0) \end{cases}$$

It is still true that  $w \in C([0, T]; E_m)$ ,  $v_1 \in C([0, T]; V_m)$  but  $v_2$  is not in  $C([0, T]; V_m)$  since  $w(0) \notin V_m$ . The expansion of  $v_2$  in the basis of eigenfunctions of  $A$  gives us, in this case, precise informations on the singularity of  $u$  at  $t = 0$ .

iv). The proof of Theorem 2.1 given in Section 2.3 generalizes that of Smale [12]. Example 2.2 and points i) and ii) above answer to questions in [12].

### 3. Nonlinear problems

#### 3.1. Hypotheses

In this section we consider nonlinear evolution equations of the type

$$(3.1) \quad \frac{du}{dt}(t) + Au(t) = g(t, u(t)), \quad 0 < t < T$$

$$(3.2) \quad u(0) = u_0,$$

all the assumptions of Sections 1.1 and 1.2 on  $A$  and  $H$  being maintained. Although more general equations can be handled, we will assume for simplicity that

$$(3.3) \quad g(t, u) = f(t) - B(u)$$

where  $B(u) = B(u, u)$  and  $B(u, v)$  is a bilinear continuous operator from  $V \times V$  into  $V'$  and furthermore<sup>(1)</sup>

$$(3.4) \quad (B(u, v), v) \geq 0, \quad u, v \in V$$

$$(3.5) \quad B(E_{m+1} \times E_{m+1}) \subset E_m, \quad \text{for } m \geq 1$$

$$(3.6) \quad (B(u, v), v) \text{ can be extended as a bilinear continuous form on } E_{s_1} \times E_{s_2+1} \times E_{s_3}, \text{ with}$$

$$s_i \geq 0 \text{ and } s_1 + s_2 + s_3 > \frac{3}{2} \text{ or}$$

$$s_i > 0 \text{ and } s_1 + s_2 + s_3 \geq \frac{3}{2}.$$

It is well-known that under these assumptions, for  $f$  and  $u_0$  given

$$(3.7) \quad f \in L^2(0, T; H), \quad u_0 \in V,$$

there exists  $T_*$ ,  $0 < T_* \leq T$  and a unique function  $u$

$$(3.8) \quad u \in L^2(0, T_*; D(A)) \cap C([0, T_*]; V)$$

which satisfies (3.1) (3.2). By changing the notation, we can assume that  $T_* = T$ .

<sup>(1)</sup> These assumptions are satisfied by the nonlinear Navier-Stokes operator, and related operators, cf. Section.

### 3.2. The result in the nonlinear case

We have the following analog of Theorem 2.1:

#### Theorem 3.1

We assume that the hypotheses of Section 1.1, 1.2 and (3.3) - (3.7) are satisfied and that for some  $m \geq 2$

$$(3.9) \quad u_0 \in E_m \cap D(A), \quad f \in W_{m-2}$$

$$(3.10) \quad \frac{d^l f}{dt^l} \in \begin{cases} L^2(0, T; H) & \text{if } m = 2l+1, \\ L^2(0, T; V') & \text{if } m = 2l. \end{cases}$$

Then a necessary and sufficient condition for the solution  $u$  of (3.1) (3.2) to belong to  $W_m$  is that<sup>(1)</sup>

$$(3.11) \quad \frac{d^j u}{dt^j}(0) \in V \quad \text{for} \quad \begin{cases} 1 \leq j \leq l & \text{if } m = 2l+1, \\ 1 \leq j \leq l-1 & \text{if } m = 2l. \end{cases}$$

#### Proof

i) We can prove as in Theorem 2.1 that the conditions (3.11) are necessary, provided we check that the derivatives

$$(3.12) \quad u^{(j)}(0) \in E_{m-2j}, \quad j = 1, \dots, l$$

when (3.9) is satisfied

By successive differentiations of (3.1) we find

$$(3.13) \quad \frac{du^{(j)}}{dt} + Au^{(j)} + \beta^{(j)} = f^{(j)}$$

where we set  $\varphi^{(j)} = \frac{d^j \varphi}{dt^j}$ ,  $\beta(t) = B(u(t))$  so that

$$(3.14) \quad \beta^{(j)} = \sum_{i=0}^j \binom{j}{i} B(u^{(j-i)}, u^{(i)})$$

---

<sup>(1)</sup> The expression of the derivatives  $\frac{d^j u}{dt^j}(0)$  in term of the data  $u_0, f$ , are given by the recurrent formula (3.15).

Now by (1.13) (3.5) and (3.9),  $\beta(0) = B(u(0)) \in E_{m-1}$  and

$$u'(0) = -Au(0) - B(u(0)) + f(0) \in E_{m-2}.$$

Similarly, if  $m \geq 4$ :

$$u''(0) = -Au'(0) - B(u'(0), u(0)) - B(u(0), u'(0)) + f'(0) \in E_{m-4}.$$

The proof continues by induction on  $j$  and using (1.13) and (3.5), shows that:

$$(3.15) \quad u^{(j)}(0) = -Au^{(j-1)}(0) - \sum_{i=0}^{j-1} \binom{j-1}{i} B(u^{(j-1-i)}(0), u^{(i)}(0)) + f^{(j-1)}(0) \in E_{m-2j},$$

for  $j = 1, \dots, \ell$ .

ii) We now prove the sufficiency of conditions (3.11), as in Section 2.2. For  $j = 1$ , the equation

$$(3.16) \quad \frac{du^{(1)}}{dt} + Au^{(1)} + B(u, u^{(1)}) + B(u^{(1)}, u) = f'$$

( $u^{(1)} = u'$ ) together with  $u^{(1)}(0)$  given by (3.15) and belonging to  $V$ , allow us to show that

$$(3.17) \quad u^{(1)} \in L^2(0, T; D(A)) \cap C([0, T]; V).$$

We continue by induction; once we establish that

$$(3.18) \quad u^{(j)} \in L^2(0, T; D(A)) \cap C([0, T]; V), \quad j = 0, \dots, \ell-1,$$

we consider the equation (3.13) (3.14) with  $j = \ell$  and  $u^{(\ell)}(0)$  given by (3.15) and belonging to  $V$  (if  $m = 2\ell+1$ ) or to  $H$  (if  $m = 2\ell$ ). This allows us to show that

$$(3.19) \quad u^{(\ell)} \in \begin{cases} L^2(0, T; D(A)) \cap C([0, T]; V) & \text{if } m = 2\ell+1, \\ L^2(0, T; V) \cap C([0, T]; H) & \text{if } m = 2\ell. \end{cases}$$

The next step consists in proving that

$$(3.20) \quad u^{(j)} \in C([0, T]; D(A)), \quad \text{for } j = 0, \dots, \ell-1.$$

For that purpose we write (3.13) (3.14) in the form

$$(3.21) \quad Au^{(j)} = -u^{(j+1)} + \sum_{i=0}^j \binom{j}{i} B(u^{(j-i)}, u^{(i)}) + f^{(j)}.$$

Because of (3.6)  $B$  is a bilinear continuous form on  $V \times V$  with value in  $V_{-1/2}$ . Therefore, by (3.18), for  $i = 0, \dots, j$ , and  $j = 0, \dots, l-1$ :

$$(3.22) \quad B(u^{(j-i)}, u^{(i)}) \in C([0, T]; V_{-1/2}).$$

Since  $A$  is an isomorphism from  $V_{3/2}$  onto  $V_{-1/2}$ , (3.21) implies then, that  $u^{(j)} \in C([0, T]; V_{3/2})$ . Using again (3.6) we get, for the same values of  $i$  and  $j$  as in (3.22):

$$B(u^{(j-i)}, u^{(i)}) \in C([0, T]; H),$$

and (3.20) follows.

Finally, we show that  $u \in W_m$ , i.e.

$$(3.23) \quad u^{(j)} \in C([0, T]; E_{m-2j}), \quad j = 1, \dots, l.$$

For  $j = l$ , this is included in (3.19). For  $j = l-1$ , using (3.5) and (3.20) we find that

$$\sum_{i=0}^j \binom{j}{i} B(u^{(j-i)}, u^{(i)}) \in C([0, T]; E_1),$$

and then (1.11) (3.21) and (3.19) show that  $u^{(l-1)} \in C([0, T]; E_{m-2l+2})$ . The proof continues then by induction for  $j = l-2, \dots, 0$ .

Theorem 3.1 is proved. □

### 3.3. Applications to Navier-Stokes equation

Instead of (2.23), we consider now the full Navier-Stokes equations

$$(3.24) \quad \begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \sum_{i=1}^n u_i \frac{\partial u}{\partial x_i} + \text{grad } p = f & \text{in } \Omega \times (0, T) \\ \text{div } u = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \Gamma \times (0, T) \\ u(x, 0) = u_0(x) & . \end{cases}$$

The functional setting is the same as in Section 2.3 and we write

$$(3.25) \quad B(u, v) = P \sum_{i=1}^n (u_i \frac{\partial v}{\partial x_i}), \quad B(u) = B(u, u)$$

so that (3.24) becomes

$$(3.26) \quad \begin{cases} \frac{du}{dt} + Au + B(u) = f \\ u(0) = u_0 \end{cases}$$

The properties (3.4) - (3.6) are well known, cf [14]. Instead of (3.4) we have

$$(B(u, v), v) = 0, \forall u, v \in V.$$

Result (3.6) is proved in [4], and (3.5) follows from the fact that  $H^m(\Omega)$  is a multiplicative algebra if  $m \geq 2$  and the dimension of space  $n$  is  $\leq 3$ :

$$(3.27) \quad \begin{cases} \text{If } u, v \in H^m(\Omega), m \geq 2, n \leq 3, \text{ then} \\ uv \in H^m(\Omega) \text{ and } |uv|_m \leq C_m(\Omega) |u|_m |v|_m. \end{cases}$$

Therefore if  $u, v \in H^{m+1}(\Omega)^n, m \geq 2$ , then  $u_i$  and  $\frac{\partial v_j}{\partial x_i} \in H^m(\Omega)$ ,  $u_i \frac{\partial v_j}{\partial x_i} \in H^m(\Omega)$  and since  $P$  maps  $H^m(\Omega)^n \cap H$  into itself,  $B(u, v)$  is in  $H^m(\Omega)^n \cap H$ . Property (3.7) (3.8) is well known too, cf [14], p. 316.

Theorem 3.1 gives for instance the following ( $m = 3$  or  $4$ ).

### Theorem 3.2

We assume that  $u_0 \in H^3(\Omega)^n \cap V, f \in C([0, T]; H^1(\Omega)^n \cap H), \frac{df}{dt} \in L^2(0, T; H), n = 2$  or  $3$ , then the solution  $u$  of (3.24) (or (3.26)) defined on some interval  $[0, T_*], 0 < T_* \leq T$  belongs to  $C([0, T_*]; H^3(\Omega)^n \cap V)$  if and only if

$$(3.28) \quad \nu P \Delta u_0 + P \left( \sum_{i=1}^n u_i \frac{\partial u_0}{\partial x_i} \right) + f(0) = 0 \text{ on } \Gamma.$$

If  $u_0 \in H^4(\Omega)^n \cap V, f \in C([0, T]; H^2(\Omega)^n \cap H), \frac{df}{dt} \in C([0, T]; H), \frac{d^2 f}{dt^2} \in L^2(0, T; V')$  then (3.28) is necessary and sufficient for  $u$  to belong to  $C([0, T_*]; H^4(\Omega)^n \cap V)$ .

Theorem 3.2 follows immediately from Theorem 3.1.



Remark 3.1

If  $u_0 \in C^\infty(\bar{\Omega})^n \cap V$ , if  $f \in C^\infty(\bar{\Omega} \times [0, T])^n \cap H$  and the conditions (3.11) are satisfied for all  $m$ , then  $u \in C^\infty(\bar{\Omega} \times [0, T])^n$  ( $T$  sufficiently small if  $n = 3$ ).<sup>(1)</sup>

Remark 3.2

If  $u_0 \in H$ ,  $f \in C^\infty(\bar{\Omega} \times [0, T])^n \cap H$ , the assumptions of Theorem 3.1 are not satisfied but we can prove, using Theorem 3.1 that  $u \in C^\infty(\bar{\Omega} \times (0, T])^n$  ( $T$  sufficiently small if  $n = 3$ ).

We proceed as follows:

- We have a weak solution  $u$  of (3.24) or (3.26) in  $L^\infty(0, T; H) \cap L^2(0, T; V)$  (cf [14]).

We choose  $t_0$  arbitrarily small such that  $u(t_0) \in V$ ;

- By (3.7) (3.8) we find that there exists  $T_*$ ,  $t_0 < T_* \leq T$ , such that the restriction of  $u$  to  $[t_0, T_*]$  satisfy (3.1). This restriction is still denoted  $u$ .

- We choose  $t_1 > t_0$ ,  $t_1 < T_*$ ,  $t_1$  arbitrarily close from  $t_0$ , such that  $u(t_1) \in D(A)$ . Then (3.26) shows as for (3.12) that  $u'(t_1) \in H$ . We conclude as in Theorem 3.1 that  $u' \in L^2(t_1, T_*; V) \cap C([t_1, T_*]; H)$ .

- We choose  $t_2 > t_1$ ,  $t_2 < T_*$ ,  $t_2 - t_1$  arbitrarily small, such that  $u'(t_2) \in V$ , and conclude that  $u' \in L^2(t_2, T_*; D(A)) \cap C([t_2, T_*]; V)$ . We then choose  $t_2 < t_3 < T_*$ ,  $t_3 - t_2$  arbitrarily small such that  $u'(t_3) \in D(A)$ , etc. ....

Finally we get that  $u \in C([t_\ell, T_*]; E_m)$ , for  $m = 2\ell$ ,  $t_\ell$  arbitrarily close from 0, and  $m$  arbitrarily large. The regularity follows. The  $C^\infty$  regularity in  $\Omega \times (0, T)$  was proved in O. A. Ladyzhenskaya [8].

Remark 3.3

Theorem 3.1 can be easily extended to more general semi-linear equations. We did not present a more general abstract theorem to avoid purely technical difficulties.

Let us consider for instance a nonlinear perturbation of Example 2.1 in Section 2.3.

$$(3.29) \quad \begin{cases} \frac{\partial u}{\partial t} + \Delta^2 u + u^{2p+1} = f & \text{in } \Omega \times (0, T) \\ u = \Delta u = 0 & \text{on } \Gamma \times (0, T) \\ u(0) = u_0 \end{cases}$$

<sup>(1)</sup> Remarks 3.7 and 3.8 p. 303 and 307 of [14] which overlook the compatibility conditions are not correct. To make the result correct one should replace  $\bar{Q}$  by  $Q$  on p. 303 1.4.

with  $2p + 2 < \frac{2n}{n-2}$  so that  $L^{2p+2}(\Omega) \subset H^2(\Omega)$ . Then a necessary and sufficient condition for  $u$  to be in  $C([0, T]; H^8(\Omega))$  is that

$$-\Delta^2 u_0 - u_0^{2p+1} + f(0) = 0 \quad \text{on } \Gamma$$

(assuming (3.9) (3.10) with  $m = 4$ ).

□

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| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br>We investigate the regularity at time $t = 0$ of the solutions of linear and semi-linear evolution equations (including the Stokes and Navier-Stokes equations). Necessary and sufficient conditions on the data, for an arbitrary order of regularity are given (the classical "compatibility conditions"). In the case of the Stokes and Navier-Stokes equations the compatibility conditions which we find are global conditions on the data. The presentation given here seems to improve and generalize the known results even in the simplest case of linear evolution equations. |                                     |                                                                                                            |