AD-A096 655 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
ANALYSIS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS.(U)
NOV 80 P A MARKOWICH DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2138 NL

1 OF 1
AD A
096655

END
DATE
FILMED
4-81
DTIC

LEVEL II

(2)

AD A096655

MRC Technical Summary Report #2138

ANALYSIS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

Peter A. Markowich

(14) MRC-TSR-2138

(12) 55

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

(11) November 1980

Received August 26, 1980

(15) DAAG29-80-C-0041, NSF-MCS79-27062

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

221200

81 3 19 066

UNIVERSITY OF WISCONSIN - MADISON

MATHEMATICS RESEARCH CENTER

ANALYSIS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

Peter A. Markowich

Technical Summary Report #2138
November 1980

ABSTRACT

In this paper boundary value problems on infinite intervals are treated. There is a theory for problems of this kind which requires the fundamental matrix of the system of differential equations to have certain decay properties near infinity. The aim of this paper is to establish a theory which holds under weaker and more realistic assumptions. The analysis for linear problems is done by determining the fundamental matrix of the system of differential equations asymptotically. For inhomogeneous problems a suitable particular solution having a 'nice' asymptotic behaviour is chosen and so global existence and uniqueness theorems are established in the linear case. The asymptotic behaviour of this solution follows immediately. Nonlinear problems are treated by using perturbation techniques meaning linearization near infinity and by using the methods for the linear case. Moreover, some practical problems from fluid dynamics and thermodynamics are dealt with and they illustrate the power of the asymptotic methods used.

AMS(MOS) Subject Classification: 34B15, 34C05, 34C11, 34D05, 34B05.

Key Words: Nonlinear boundary value problems, singular points, boundedness, asymptotical properties, asymptotic expansions.

Work Unit No. 1 - Applied Analysis.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062.

ρ-1

SIGNIFICANCE AND EXPLANATION

Boundary value problems on infinite intervals occur in many areas of physics, for example in fluid mechanics when similarity solutions of the Navier-Stokes equations describing the flow over an infinite medium are sought. Problems which are dealt with in this paper have the following form. We look for a solution of a system of differential equations over an infinite interval which is continuous at infinity and which fulfills certain boundary conditions at a finite point and at infinity. The following questions arise immediately: which condition on the differential equation and on the boundary conditions assure the existence and the uniqueness of a solution and how does this solution depend on the data. There is a well known theory for problems of this kind, but it can only be applied to problems where the fundamental matrix of the system of differential equations has certain convergence properties near infinity. However this assumption is not fulfilled for many practical problems. This paper answers the above named question under very weak assumptions on the problems. Moreover some fluid dynamical problems illustrating the power of the theory developed are dealt with.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ANALYSIS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS

PETER A. MARKOWICH

1. Introduction.

This paper is concerned with the analysis of boundary value problems on infinite intervals posed in the following way

$$y' = t^{\alpha} f(t,y), \quad 1 \leq t < \infty, \quad \alpha \in \mathbb{N}_0 \tag{1.1}$$

$$y \in C([1,\infty]): \iff y \in C([1,\infty)) \text{ and } \lim_{t\to\infty} y(t) = y(\infty) \text{ exists} \tag{1.2}$$

$$b(y(1),y(\infty)) = 0 \tag{1.3}$$

where y is an n-vector, f and b are nonlinear mappings.

Equation (1.1) has a singularity of the second kind at $t = \infty$ of rank $\alpha + 1$, because we assume that f is continuous in $(\infty,y(\infty))$. The goal is to establish existence and uniqueness theorems for very general f's and b's , to gain information on the behaviour of y for large t and - in the linear inhomogenous case - to investigate the dependence of y on the boundary data and the inhomogenity.

Problems of this kind frequently occur in fluid dynamics when stationary similarity solutions of the Navier-Stokes equations for certain flow-constellations are sought (see for example McLeod (1969); Markowich (1980); Lentini and Keller (1980), Cohen, Fokas and Lagerström (1978)).

For application in other areas of physics see for example Lentini (1978).

Much analytical work has been done on singular boundary value problems of the second kind. F. de Hoog and Richard Weiss (1980a,b) investigated the case where $\frac{\partial f}{\partial y}(\infty,y(\infty))$ has

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This material is based upon work supported by the National Science Foundation under Grant No. MCS-7927062.

no eigenvalue on the imaginary axis by linearizing f around $y(\infty)$ and evaluating at $t = \infty$ so getting the constant coefficient problem

$$z' = t^{\alpha} f_y(\infty, y(\infty)) z \tag{1.5}$$

and by employing perturbation techniques which are based on estimates derived for a certain particular solution of linear inhomogenous problems. They could establish uniqueness and existence theorems. M. Lentini and H. B. Keller (1980) extended this approach neglecting the assumption on the eigenvalues but they required that the projection of $f_y(t,y(\infty))$ onto the direct sum of invariant subspaces of $f_y(\infty,y(\infty))$ which correspond to an imaginary eigenvalue, converge, at least as $t^{-(\alpha+1)r-\varepsilon}$, where r is the largest dimension of these subspaces and $\varepsilon > 0$. It turns out that this assumption is crucial and the perturbation approach breaks down if $f_y(\infty,y(\infty))$ has eigenvalues with real part zero and if the convergence requirement is neglected. However many practical problems do not fulfill this convergence-requirement in the presence of imaginary eigenvalues (see for example Cohen, Fokas and Lagerstrom (1978) and M. Lentini and V. Pereyra (1977) and therefore a more general approach is necessary.

In this paper asymptotic series are used in order to determine asymptotically fundamental solution matrices of linear systems of the form:

$$y' = t^{\alpha} A(t) y \qquad t > \delta . \tag{1.6}$$

The basic assumption is that A is analytical in $[\delta,\infty]$ for some $\delta > 1$, so that

$$A(t) = \sum_{i=0}^{\infty} A_i \, t^{-i} \quad \text{where} \quad A_i = \frac{1}{i!} \lim_{x \to 0+} \frac{d^i}{dx^i} A(\tfrac{1}{x}) . \tag{1.7}$$

Then a fundamental matrix $\Phi(t)$ of (1.6) can be calculated as an asmptotic (formal) log-exponential series from the coefficients A_i by a recursive algorithm (see Coddington and Levinson (1955) and Wasow (1965)).

Assumption (1.7) can be weakened so that only a finite but large enough number of these derivatives exist. Equation (1.1) is treated by linearization around $y(\infty)$ obtaining the variable coefficient problem

$$z' = t^{\alpha} f_y(t, y(\infty)) z \tag{1.8}$$

and again by employing similar perturbation techniques. The advantages of the formal-series approach is twofold. Firstly no restrictions (except (1.7)) have to be made concerning the convergence behaviour of $f_y(t,y(\infty))$, secondly the asymptotic behaviour of the (basic) solutions is obtained directly. The asymptotic behaviour is crucial for the determination of appropriate numerical procedures for problems (1.1), (1.2), (1.3). (see Lentini and Keller (1980), Markowich (1980) and de Hoog and Weiss (1980b)).

This paper is organized in the following way, in chapter 1 some remarks are made on linear inhomogenous constant coefficient problems (see Lentini and Keller (1980)), in Chapter 2 the case where $A(\infty)$ has distinct eigenvalues is treated, in Chapter 4 we admit a general Jordan canonical form of $A(\infty)$ and in Chapter 5 we get to nonlinear problems of the form (1.1), (1.2), (1.3). Chapter 6 is concerned with practical examples which illustrate the power of the used asymptotic methods.

2. Linear problems with constant coefficients.

We consider problems of the form

$$y' - t^{\alpha}Ay = t^{\alpha}f(t) \qquad 1 \leq t < \infty , \quad \alpha \in R , \alpha > -1 \tag{2.1}$$

$$y \in C([1,\infty]) \tag{2.2}$$

$$B_1y(1) + B_\infty y(\infty) = \hat{\beta} \tag{2.3}$$

where the $n \times n$-Matrix $A \neq 0$.

At first we transform A to its Jordan canonical form J

$$A = E\,J\,E^{-1} \tag{2.4}$$

and substitute

$$u = E^{-1}y \ . \tag{2.4a}$$

So we get the new problem

$$u' - t^{\alpha} J(t)u = t^{\alpha}E^{-1}f(t) \tag{2.5}$$

$$u \in C([1,\infty]) \tag{2.6}$$

$$B_1E\;u(1) + B_\infty E\;u(\infty) = \hat{\beta} \ . \tag{2.7}$$

Without loss of generality we can assume that J has the block diagonal form

$$J = \mathrm{diag}(J^{+}, J^{0}, J^{-}) \tag{2.8}$$

where the real part of the eigenvalues of J^{+} are positive. The real parts of the eigenvalues of J^0 are equal to zero and the real parts of the eigenvalues of J^- are negative. This structure can always be obtained by reordering the columns of E . Let the dimensions of these three matrices be r_+, r_0 resp. r_-.

The diagonal projections D_+, D_0, D_- are obtained by taking the main-diagonal of J and by replacing every eigenvalue with positive, zero resp. negative real part by 1 and all others by zero so that

$$I = D_+ + D_0 + D_- \tag{2.9}$$

holds.

Furthermore let $\bar{D}_0$ be the projection onto the direct sum of eigenspaces of J associated with zero eigenvalues, which is obtained by replacing by zero every (diagonal) element of D_0 which is not associated with the first column of a Jordan block of J belonging to a zero eigenvalue.

Let the number of nonzero columns of $\bar{D}_0$ which equals the geometrical multiplicity of the eigenvalue zero, be $\bar{r}_0$. The general solution of the homogenous problem (2.5), (2.6) is:

$$u_h(t) = \phi(t)(\bar{D}_0 + D_-)\xi = \exp\left(J\,\frac{t^{\alpha+1}}{\alpha+1}\right)(\bar{D}_0 + D_-)\xi\ ,\quad \xi \in C^n. \tag{2.10}$$

In order to solve the inhomogenous problem (2.5) we look for a particular solution $u_p \in C([1,\infty])$.

F. de Hoog and R. Weiss (1980a,b) and M. Lentini and H. B. Keller (1980) suggested the following choice:

$$\begin{aligned} u_p(t) = (Hf)(t) &= \phi(t)\int_\infty^t D_+\phi^{-1}(s)E^{-1}f(s)s\,ds\ + \\ &+ \phi(t)\int_\infty^t D_0\phi^{-1}(s)E^{-1}f(s)s^\alpha ds\ + \\ &+ \phi(t)\int_\delta^t D_-\phi^{-1}(s)E^{-1}f(s)s^\alpha ds \end{aligned} \tag{2.11}$$

with $\delta \in [1,\infty)$.

We denote the three terms on the right hand side of (2.11) by $u_{p_+} = H_+f$, $u_{p_0} = H_0f$ resp. $u_{p_-} = H_-f$. F. de Hoog and Richard Weiss (1980a) showed that u_{p_+} and u_{p_-} are in $C([1,\infty])$ if $D_-E^{-1}f$ and $D_+E^{-1}f$ are in $C([1,\infty])$ and that $J(D_+ + D_-)u_p(\infty) = -(D_+ + D_-)E^{-1}f(\infty)$ holds.

M. Lentini and H. B. Keller (1980) showed that

$$\|u_{p_0}(t)\| = O(t^{-\varepsilon}) \quad \text{if} \quad \|D_0E^{-1}f\| = O\left(t^{-(\alpha+1)r-\varepsilon}\right)$$

where $\varepsilon > 0$ and r is the maximal dimension of the invariant subspaces of J associated with eigenvalues on the imaginary axis. Therefore the operator H operates on the space of all functions f, which fulfill:

$$f \in C([1,\infty]) \quad \text{and} \quad D_0E^{-1}f(t) = F_0(t)t^{-(\alpha+1)r-\varepsilon} \tag{2.12}$$

with $F_0 \in C_b([1,\infty))$, where $C_b([1,\infty))$ is the space of functions which are continuous on $[1,\infty)$ and bounded as $t \to \infty$.

Inserting the general solution of (2.5), (2.6) into (2.7) we get

$$(2.13)\qquad \begin{aligned}&\left[(B_1E + B_\infty E)\bar{D}_0 + B_1E\exp\left(\frac{J}{\alpha+1}\right)D_-\right]\xi = \\ &= \hat{\beta} - [B_1Eu_p(1) + B_\infty Eu_p(\infty)].\end{aligned}$$

Therefore we conclude

<u>Thereom 2.1.</u> The problem (2.1), (2.2), (2.3) has a unique solution y for all f which fulfill (2.12) and $\hat{\beta} \in \mathbb{R}^{\bar{r}_0 + r_-}$, if and only if

$$(2.14)\qquad \operatorname{rank}\left[(B_1E + B_\infty E)\bar{D}_0 + B_1\exp\left(\frac{J}{\alpha+1}\right)D_-\right] = \bar{r}_0 + r_-$$

where B_1 and B_∞ are $(\bar{r}_0 + r_-) \times n$ matrices.

In this case y depends continuously (in the norm $\|y\|_{[1,\infty]} := \max_{t \in [1,\infty]} \|y(t)\|$) on the data $\hat{\beta}$, $(D_+ + D_-)E^{-1}f$ and F_0.

The 1st statement follows directly from estimates given in the papers cited above. We see that (2.2) is an additional boundary condition at $t = \infty$ of the rank $r_+ + (r_0 - \bar{r}_0)$.

Now we investigate the decay properties of u_p in dependence of the decay properties of f:

<u>Theroem 2.2.</u> If f fulfills (2.12) then the following estimates hold for $t \geq \delta$:

$$(2.15)\qquad \|(H_+f)(t)\| \leq \text{const.}\ \|D_+E^{-1}f\|_{[t,\infty]}$$

$$(2.16)\qquad \|(H_0f)(t)\| \leq \text{const.}\ t^{-\varepsilon} \max_{s \geq t} \|s^{(\alpha+1)r+\varepsilon} D_0E^{-1}f(s)\|$$

with $\varepsilon > 0$.

If $\gamma \geq 0$ then

$$(2.17)\qquad \|(H_-f)(t)\| \leq \text{const.}\ t^{-\gamma} \max_{\delta \leq s \leq t} \|s^{\gamma}D_-E^{-1}f(s)\|$$

all constants are independent of f and δ.

The first two estimates have been proved by M. Lentini and H. B. Keller (1980).

The third estimate follows from

$$\|u_{p_-}(t)\| \leq c.\max_{i=1(1)k}\left(e^{-\frac{\lambda}{\alpha+1}t^{\alpha+1}}\int_\delta^t e^{\frac{\lambda}{\alpha+1}s^{\alpha+1}}(t^{\alpha+1}-s^{\alpha+1})^{i-1}s^{\alpha-\gamma}ds\right)\max_{\delta\leq s\leq t}\|s^\gamma D_-E^{-1}(s)\|$$

where $-\lambda$ is the largest real part of eigenvalues of J with $\lambda > 0$ and k is the maximal dimension of the associated Jordan-blocks.

By applying Taylor's formula to

$$g_i(t) = \frac{\int_\delta^t e^{\frac{\lambda}{\alpha+1}s^{\alpha+1}}(t^{\alpha+1}-s^{\alpha+1})^{i-1}s^{\alpha-\gamma}ds}{e^{\frac{\lambda}{\alpha+1}t^{\alpha+1}}t^{-\gamma}}$$

it is easy to conclude that $\lim_{t\to\infty} g_i(t) = \text{const.}$ In particular Theorem 2.2 tells us that inhomogenities which converge to zero as a power function produce a particular solution which converges as a power function whose exponent is increased by $(\alpha + 1)r$.

Now we want to investigate exponentially decreasing inhomogenities.

If $\int_\delta^\infty D_-\phi^{-1}(s)E^{-1}f(s)s^\alpha ds$ exists we can substitute H_- by $\tilde{H}_-$ which is defined by

(2.18) $$(\tilde{H}_-f)(t) = \phi(t)\int_\infty^t D_-\phi^{-1}(s)E^{-1}f(s)s^\alpha ds \ .$$

Now we prove:

<u>Theorem 2.3.</u> Let J^- consist of Jordan blocks belonging to the same eigenvalue $-\lambda$ and let k be the dimension of the largest of these blocks.
Furthermore let $D_-E^{-1}f(t) = t^\beta\exp\left(-\frac{\omega}{\alpha+1}t^{\alpha+1}\right)F_-(t)$ with $F_- \in C_b([1,\infty))$, $\beta \in R$ and $\omega > 0$. Then for $t > \delta$:

(2.19) $$\|(H_-f)(t)\| \leq \text{const. } t^\beta\exp\left(-\frac{\omega}{\alpha+1}t^{\alpha+1}\right)\|F_-\|_{[\delta,t]}$$

if $\operatorname{Re}\lambda - \omega > 0$.

(2.20) $$\|(H_-f)(t)\| \leq \text{const. } \exp\left(-\frac{\omega}{\alpha+1}t^{\alpha+1}\right)t^{(\alpha+1)k+\beta}\ell nt\cdot\|F_-\|_{[\delta,t]}$$

if $\operatorname{Re}\lambda - \omega = 0$ and $\beta \geq -k(\alpha+1)$ The factor ℓnt only appears if $\beta = -k(\alpha+1)$.

(2.21) $$\|(\tilde{H}_-f)(t)\| \leq \text{const. } \exp\left(-\frac{\omega}{\alpha+1}t^{\alpha+1}\right)\cdot t^{k(\alpha+1)+\beta}\|F_-\|_{[t,\infty]}$$

if $\operatorname{Re}\lambda - \omega = 0$ and $\beta < -k(\alpha + 1)$.

$$(2.22) \qquad \|(\tilde{H}_- f)(t)\| \leq \text{const.} \exp\left(-\frac{\omega}{\alpha+1} t^{\alpha+1}\right) t^{\beta} \|F_-\|_{[t,\infty]}$$

if $\mathrm{Re}\lambda - \omega < 0$.

All constants are independent of f and δ .

The proofs work analogously to Theorem 2.2. Theorem 2.3 implys that exponentially decaying inhomogenities produce inhomogenities which converge with the same exponential factor, only the algebraic factor may change and a logarithmic factor may appear. If $\tilde{H}_-$ exists then it cuts off the terms of the particular solution which are already included in $\phi(t)D_-\xi$.

Assume now that J^- consists of more than one Jordan block with different eigenvalues and that $D_-E^{-1}f(t)$ has the form as in Theorem 3. Then H_- and $\tilde{H}_-$ may be used in order to gain a particular solution which decays as fast as possible according to the different cases of Theorem 2.3. Doing this D_- has to be split up into the projections onto the direct sums of the invariant subspaces associated with different eigenvalues with negative real part and H_- resp $\tilde{H}_-$ have to be applied to the resulting subsystems. We will call the resulting operator $\bar{H}$. Its composition depends on the decay properties of f and on J^- .

3. Linear Variable Coefficient Problems - Distinct Eigenvalues.

Now we analyze

(3.1) $$y' - t^{\alpha}A(t)y = t^{\alpha}f(t), \quad \alpha \in N_0$$

(3.2) $$y \in C([1,\infty])$$

(3.3) $$B_1 y(1) + B_{\infty} y(\infty) = \hat{\beta}$$

The $n \times n$ matrix $A(t)$ fulfills:

(3.4) $$A \in C([1,\infty]), \quad A(\infty) \neq 0$$

(3.5) $$A \text{ is analytic on } [\delta,\infty] \text{ for some } \delta > 1$$

so that

(3.6a) $$A(t) = \sum_{i=0}^{\infty} A_i t^{-i} \quad \text{for } t \text{ sufficiently large where}$$

(3.6b) $$A_i = \frac{1}{i!} \lim_{x \to 0+} \frac{d^i}{dx^i} A\left(\frac{1}{x}\right).$$

The basic assumption of this chapter is that the eigenvalues of $A_0 = A(\infty)$ are distinct.

Let J_0 be the Jordan canonical form of A_0 obtained by the transformation

(3.7) $$A_0 = E\, J_0\, E^{-1}$$

and let the J_i's be defined by

(3.8) $$A_i = E\, J_i\, E^{-1}.$$

The matrices J_i *are the coefficients* of the series

(3.9) $$J(t) = E^{-1}A(t)E = \sum_{i=0}^{\infty} J_i\, t^{-i} \quad \text{for } t \to \infty.$$

We set

(3.10) $$J_0 = \operatorname{diag}(\lambda_1, \cdots, \lambda_n), \quad \lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j$$

and assume that:

(3.11) $$\left.\begin{array}{lll} \operatorname{Re}\lambda_i > 0 & \text{for} & 1 \leq i \leq r_+ \\ \operatorname{Re}\lambda_{r_+ + j} = 0 & \text{for} & 1 \leq j \leq r_0 \\ \operatorname{Re}\lambda_{r_+ + r_0 + k} < 0 & \text{for} & 1 \leq k \leq r_- \end{array}\right\} \; r_+ + r_0 + r_- = n.$$

Again we substitute

(3.12) $$u = E^{-1}y$$

and get the problem

$$(3.13) \qquad u' = t^{\alpha} J(t)u = t^{\alpha} E^{-1} f(t), \quad 1 \leq t < \infty$$

$$(3.14) \qquad u \in C([1,\infty]).$$

For the following we need the definition of an *asymptotic series*. *A function* $P(t)$ *is* said to be represented asymptotically by a formal series:

$$(3.15) \qquad P(t) \sim \sum_{i=0}^{\infty} P_i t^{-i}, \quad t \to \infty$$

if

$$(3.16) \qquad t^m \Big[P(t) - \sum_{i=0}^{m} P_i t^{-i}\Big] \to 0 \quad \text{for} \quad t \to \infty \quad \text{and} \quad m > 0 .$$

Therefore

$$(3.17) \qquad P(t) = \sum_{i=0}^{m} P_i t^{-i} + O(t^{-m-1}) \quad \text{for} \quad t \to \infty \quad \text{and} \quad m > 0$$

holds. To get more information on asymptotic series see Wasow (1965).

Coddington and Levinson (1955) and Wasow (1965) prove an asymptotic representation of the fundamental matrix of the homogenous system (3.13) which we state in

<u>Theorem 3.1.</u> Under the given assumptions on $J(t)$ the homogenous system (3.13) possesses a fundamental solution of the form:

$$\phi(t) = P(t)\, t^D e^{Q(t)} .$$

Here $D = diag(d_1, \cdots, d_n)$, $Q(t) = diag(q_1(t), \cdots, q_n(t))$ *where the* $q_i(t)$ *are polynomials* in t of degree $\alpha + 1$, so that

$$Q(t) = J_0 \frac{t^{\alpha+1}}{\alpha+1} + Q_1 \frac{t^{\alpha}}{\alpha} + \cdots + Q_{\alpha} t$$

with diagonal matrices Q_i. $P(t)$ admits the asymptotic expansion:

$$P(t) \sim \sum_{i=0}^{\infty} P_i t^{-i} \qquad \text{for} \qquad t \to \infty$$

and $P_0 = P(\infty) = I$.

The unknown coefficients P_i, Q_i and D can be calculated by algebraic operations from the J_i's. We get an algorithm for the calculation of these coefficients by inserting the *asymptotic series for* ϕ *and* J into the differential equation, by formal differentiation of ϕ and by comparing the coefficients of the corresponding powers of t (see Wasow (1965)).

For an arbitrary matrix B let $diag(B)$ denote the matrix which has only zero off-diagonal entries and the same main diagonal as B .

In the case $\alpha = 0$ we find $D = \mathrm{diag}(J_1)$. For $\alpha > 0$ we pursue in the following way:

1) Set $Q_1 := \mathrm{diag}(J_1)$ and let $\tilde{P}_1$ be the solution matrix of the equation:

$$\tilde{P}_1 J_0 - J_0 \tilde{P}_1 = J_1 - Q_1$$

with $\mathrm{diag}(\tilde{P}_1) = 0$. This equation is uniquely soluble because the diagonal elements of J_0 are distinct.

2) $\tilde{P}_k$ and Q_k are determined recursively. Set

$$Q_k = \mathrm{diag}\left(\sum_{\ell=1}^{k-1} (J_\ell \tilde{P}_{k-\ell} - \tilde{P}_{k-\ell} Q_\ell) + J_k\right) \qquad \text{for } 2 \leq k \leq \alpha$$

and let $\tilde{P}_k$ with $\mathrm{diag}(\tilde{P}_k) = 0$ be the solution matrix of:

$$\tilde{P}_k J_0 - J_0 \tilde{P}_k = \sum_{\ell=1}^{k-1} (J_\ell \tilde{P}_{k-\ell} - \tilde{P}_{k-\ell} Q_\ell) + (J_k - Q_k) .$$

3) Set $D := \mathrm{diag}\left(\sum_{\ell=1}^{\alpha} (J_\ell \tilde{P}_{\alpha+1-\ell} - \tilde{P}_{\alpha+1-\ell} Q_\ell) + J_{\alpha+1}\right)$.

The matrices $J_0, \cdots, J_{\alpha+1}$ determine $Q_0, \cdots, Q_\alpha$ and D. In the presence of eigenvalues of $A(\infty)$ with a real part zero it is therefore not sufficient to know $A(\infty) = A_0$ in order to determine whether the fundamental solutions which correspond to these eigenvalues are in $C([1,\infty])$ or not. More (at most $\alpha+2$) coefficients of the series expansion of $A(t)$ have to be known. Moreover an eigenvalue with real part zero can produce a basic solution which is exponentially increasing or decreasing, algebraically increasing or decreasing, constant or undampenedly oscillating as t approaches infinity. So the i-th basic solution is in $C([1,\infty])$ if and only if:

$$\lim_{t \to \infty} e^{q_i(t)} t^{d_i} \quad \text{exists.} \tag{3.18}$$

Lentini and Keller (1980) treated the case $\alpha = 0$ and $D_0 J_1 = 0$ where D_0 is again the diagonal projection onto the direct sum of invariant subspaces associated with eigenvalues of J_0 with real part zero (D_-, D_+ are also defined as in Chapter 2). Under this assumptions a zero eigenvalue of J_0 generates a solution which has a power-series expansion in t^{-1} without an exponential and algebraic factor. Therefore it is sufficient to investigate the system

(3.19) $$v' - t^{\alpha} J_0 v = 0$$

and to apply a perturbation approach.

Let $\tilde{r}_0$ be the number of solutions generated by imaginary eigenvalues of J_0 fulfilling the condition (3.18) and let $\tilde{D}_0$ be the projection onto the direct sum of invariant subspaces associated with these eigenvalues. Then the general solution of the homogenous problem (3.13), (3.14) is

(3.20) $$u_h = \phi(t)(\tilde{D}_0 + D_-)\xi \quad , \qquad \xi \in C^n .$$

Now we have to determine a particular solution $u_p \in C([1,\infty))$.

In order to do this we denote by D_{0i} for $1 \leq i \leq r_0$ the diagonal projection onto the eigenspace associated with the i-th imaginary eigenvalue of J_0 . D_{0i} has only one nonzero entry in the $(r_+ + i)$-th column of the main diagonal.

Now we impose conditions on f:

(a) $f \in C([1,\infty))$,

(b) $D_{0i} P^{-1}(t)E^{-1}f(t) = t^{-k_i} F_{0i}(t) \in C([1,\infty])$

(3.21) if $|\mathrm{Re}\, q_{r_+ + i}(t)| = O(t^{\alpha+1-k_i})$ and $\mathrm{Re}\, q_{r_+ + 1} \not\equiv 0$

for $i = 1(1)r_0$, $1 \leq k_i \leq \alpha$

(c) $D_{0i}P^{-1}(t)E^{-1}f(t) = t^{-\alpha-1-\varepsilon_i} F_{0i}(t)$,

$F_{0i} \in C_b([1,\infty)]$, $\varepsilon_i > 0$ if $\mathrm{Re}\, q_{r_+ + i} \equiv 0$.

Therefore we require at most that

$$\|f(t)\| = O(t^{-\alpha-1-\varepsilon}) \qquad \varepsilon > 0 . \tag{3.22}$$

Now we define:

$$\text{(a)}\quad u_p = u_{p_+} + u_{p_0} + u_{p_-} , \quad \text{(b)}\quad u_{p_0} = \sum_{j=1}^{r_0} u_{p_{0i}} \tag{3.23}$$

$$\begin{aligned} &\text{(a)}\quad u_p = Hf,\ \ u_{p_+} = H_+f,\ \ u_{p_0} = H_0 f,\ \ u_{p_-} = H_-f, \\ &\text{(b)}\quad u_{p_{0i}} = H_{0i}f \end{aligned} \tag{3.24}$$

and

$$u_{p_+}(t) = \phi(t) \int_\infty^t D_+\phi^{-1}(s)s^{\alpha}E^{-1}f(s)ds \tag{3.25}$$

$$u_{p_-}(t) = \phi(t) \int_\delta^t D_-\phi^{-1}(s)s^{\alpha}E^{-1}f(s)ds \tag{3.26}$$

$$u_{p_{0i}}(t) = \begin{cases} \text{(a)}\quad \phi(t) \int_\infty^t D_{0i}\phi^{-1}(s)s^{\alpha}E^{-1}f(s)ds , & \text{if (A) holds} \\ \text{(b)}\quad \phi(t) \int_\delta^t D_{0i}\phi^{-1}(s)s^{\alpha}E^{-1}f(s)ds , & \text{if (B) holds} \end{cases} \tag{3.27}$$

where:

(A) $\quad \operatorname{Re} q_{r_++i}(t) \to \infty \quad \text{or} \quad (\operatorname{Re} q_{r_++i} \equiv 0 \ \text{ and } \ \varepsilon_i > -\operatorname{Re} d_{r_++i})$

(B) $\quad \operatorname{Re} q_{r_++i}(t) \to -\infty \quad \text{or} \quad (\operatorname{Re} q_{r_++i} \equiv 0 \ \text{ and } \ \varepsilon_i < -\operatorname{Re} d_{r_++i})$.

Then $u_p(\infty)$ exists and fulfills the linear equation:

$$J_0(D_+ + D_-)u_p(\infty) = -(D_+ + D_-)E^{-1}f(\infty)$$

We prove that u_{p_+} is continuous in $t = \infty$, the proofs for u_{p_0} and u_{p_-} are similar.

Using Theorem 3.1 and (3.28) we get

$$u_{p_+}(t) = P(t)t^D e^{Q(t)} \int_\infty^t D_+e^{-Q(s)}s^{-D+\alpha I}P^{-1}(s)E^{-1}f(s)ds .$$

Looking at the i-th component of $P^{-1}(t)u_{p_+}(t)$ we find,

$$(P^{-1}(t)u_{p_+}(t))_i = \int_\infty^t e^{q_i(t) - q_i(s)} \left(\frac{t}{s}\right)^{d_i} s^{\alpha}(P^{-1}(s)E^{-1}f(s))_i ds$$

Applying del'Hospital's limit theorem to

$$g_i(t) = \frac{\int_\infty^t e^{-q_i(s)} s^{\alpha-d_i} (P^{-1}(s)E^{-1}f(s))_i ds}{e^{-q_i(t)} t^{-d_i}}$$

we get because of the convergence of the integral in the numerator

$$(u_{p_+}(\infty))_i = \lim_{t\to\infty} g_i(t) = \lim_{t\to\infty} \frac{(P^{-1}(t)E^{-1}f(t))_i}{-q_i'(t)t^{-\alpha} - d_i t^{-\alpha-1}} = \frac{(E^{-1}f(\infty))_i}{-\lambda_i} .$$

We have used that $P(t) = I + O(t^{-1})$ and $P^{-1}(t) = I + O(t^{-1})$ are continuous in $t = \infty$. As in paragraph 2 we substitute $u_h + u_p$ into the boundary condition:

$$(B_1E\phi(1)(\tilde{D}_0 + D_-) + B_\infty E\tilde{D}_{00})\xi = \hat{\beta} - B_1Eu_p(1) - B_\infty Eu_p(\infty) .$$

There $\tilde{D}_{00}$ is the projection onto the direct sum of invariant subspaces associated with those imaginary eigenvalues of J_0, which produce basic solutions which do not vanish at infinity.

Theorem 3.2. The problem (3.1), (3.2), (3.3) under the assumptions (3.4), (3.5), (3.10) has a unique solution y for all f fulfilling (3.21) and $\hat{\beta} \in R^{\tilde{r}_0+r_-}$ if and only if

$$\operatorname{rank}[B_1E\phi(1)(\tilde{D}_0 + D_-) + B_\infty E\tilde{D}_{00}] = \tilde{r}_0 + r_-$$

where B_1 and B_∞ are $(\tilde{r}_0 + r_-) \times n$ matrices. This solution depends continuously on $\hat{\beta}$, $(D_+ + D_-)E^{-1}P^{-1}f$ and F_0 where F_0 fulfills $D_{0i}F_0 = F_{0i}$.

For the proof of the continuity statement we need

Theorem 3.3. If f fulfills (3.21) then the following estimates hold for $t \geq \delta$:

(3.28) $\quad \|(H_+f)(t)\| \leq \text{const.} \|D_+P^{-1}E^{-1}f\|_{[t,\infty]}$.

If $\gamma > 0$ then:

(3.29) $\quad \|(H_-f)(t)\| \leq \text{const.}\ t^{-\gamma} \max_{\delta\leq s\leq t} \|s^\gamma D_-P^{-1}E^{-1}f(s)\|$.

For $i = 1(1)r_0$:

If $\operatorname{Re} q_{r_+ + i} \to +\infty$ then

(3.30) $\quad \|(H_{0i}f)(t)\| \leq \text{const.} \|F_i\|_{[t,\infty]}$.

If $\operatorname{Re} q_{r_+ + i} = 0$ and $\varepsilon_i > -\operatorname{Re} d_{r_+ + i}$ then

(3.31) $$\|(H_{0i}f)(t)\| \le \text{const.}\ t^{-\varepsilon_i}\|F_i\|_{[t,\infty]} .$$

If $\operatorname{Re} q_{r_+ + i} \to -\infty$ and $\gamma > 0$ then

(3.32) $$\|(H_{0i}f)(t)\| \le \text{const.}\ t^{-\gamma} \max_{\delta \le s \le t} \|s^{\gamma}F_i(s)\| .$$

If $\operatorname{Re} q_{r_+ + i} \equiv 0$ and $\varepsilon_i < -\operatorname{Re} d_{r_+ + i}$ then

(3.33) $$\|(H_{0i}f)(t)\| \le \text{const.}\ t^{-\varepsilon_i}\|F_i\|_{[\delta,t]}$$

If $\operatorname{Re} q_{r_+ + i} \equiv 0$ and $\varepsilon_i = -\operatorname{Re} d_{r_+ + i}$ then

(3.34) $$\|(H_{0i}f)(t)\| \le \text{const.}\ t^{-\varepsilon_i}\ \ell nt\|F_i\|_{[\delta,t]} .$$

All constants are independent of δ and f .

We prove estimate (3.30) and the others are proven quite analogously.

From (3.12) and (3.27) we derive

$$\|H_{0i}f)(t)\| \le \text{const.}\frac{\int_t^{\infty} \exp(-\operatorname{Re} \sigma_{r_+ + i}(s))s^{-\operatorname{Re} d_{r_+ + i} + \alpha - k_i}\, ds}{\exp(-\operatorname{Re} q_{r_+ + i}(t))t^{-\operatorname{Re} d_{r_+ + i}}}\|F_i\|_{[t,\infty]}$$

Applying del'Hospital's rule and using $|\operatorname{Re} q_{r_+ + i}(t)| = O\left(t^{\alpha+1-k_i}\right)$ the estimate (3.30) follows. Herewith Theorem (3.2) is proven completely. An inhomogenity f fulfilling (3.22) produces a particular solution which decays at least as fast as $t^{-\varepsilon}\ \ell nt$.

Now we again investigate exponentially decaying inhomogenities.

We define $\tilde{H}_-$ as in Chapter 2 , $\tilde{H}_{0i}$ as (3.27a), $\hat{H}_{0i}$ as (3.27b) no matter whether (A) or (B) holds. Of course the definitions make sense only for inhomogenities which make the appearing integrals exist.

We assume that

(3.35) $$f(t) = e^{p(t)}t^{\beta}F(t), \quad F \in C_b([1,\infty)), \quad \beta \in R$$

where $p(t)$ is a real polynomial of maximal degree $\alpha + 1$ and $p(t) \to -\infty$ for $t \to \infty$. We split up the operators H_-, $\tilde{H}_-$ into H_{-j}, $\tilde{H}_{-j}$ by substituting the projection D_- by D_{-j} (for $j = 1(1)r_-$) which is the projection onto the invariant subspace generated by the j-th eigenvalue of J_0 with real part smaller than zero.

Theorem 3.4. Let (3.35) hold. Then the following estimates can be derived for $t > \delta$:

$$\|(H_{-j} f)(t)\| \le \text{const. } e^{p(t)} t^{\beta+\alpha+1-\ell_i} \|F\|_{[\delta,t]}$$

if $p(t) - \mathrm{Re}(q_{r_+ + r_0 + j}(t)) \to +\infty$ and ℓ_i is the degree of the polynomial $p(t) - \mathrm{Re}(q_{r_+ + r_0 + j})$

(3.37) $$\|(H_{-j}f)(t)\| \le \text{const. } e^{p(t)} t^{\alpha+\beta+1} \|F\|_{[\delta,t]}$$

if $p(t) = \mathrm{Re}(q_{r_+ + r_0 + j}(t))$ and $\beta > \mathrm{Re}(d_{r_+ + r_0 + j}) - \alpha - 1$.

(3.38) $$\|(H_{-j}f)(t)\| \le \text{const. } e^{p(t)} t^{\alpha+\beta+1} \ln t \, \|F\|_{[\delta,t]}$$

if $p(t) = \mathrm{Re}(q_{r_+ + r_0 + j}(t))$ and $\beta = \mathrm{Re}(d_{r_+ + r_0 + j}) - \alpha - 1$

(3.39) $$\|(\tilde{H}_{-j}f)(t)\| \le \text{const. } e^{p(t)} t^{\alpha+\beta+1} \|F\|_{[t,\infty]}$$

if $p(t) = \mathrm{Re}(q_{r_+ + r_0 + j}(t))$ and $\beta < \mathrm{Re}(d_{r_+ + r_0 + j}) - \alpha - 1$.

(3.40) $$\|(\tilde{H}_{-j}f)(t)\| \le \text{const. } e^{p(t)} t^{\beta+\alpha+1-\ell_i} \|F\|_{[t,\infty]}$$

if $p(t) - \mathrm{Re}(q_{r_+ + r_0 + j}(t)) \to -\infty$.

The estimates (3.37), (3.38), (3.39) are also valid if H_{-j} resp. $\tilde{H}_{-j}$ and the index $r_+ + r_0 + j$ are substituted by $\hat{H}_{-j}$ resp $\tilde{H}_{0i}$ and the index $r_+ + i$ for $i = 1(1)r_0$. The proof of Theorem 3.4 is quite similar to that of Theorem 3.3.

As in the constant coefficient case exponentially decaying inhomogenities produce particular solutions which converge with the same exponential factor, only the algebraic factor may change and $\ln t$ as a factor may appear. If the inhomogenity f contains $(\ln t)^s$ with $s \in \mathbb{N}$ as a factor then all derived estimates are valid if their right hand sides are multiplied by $(\ln t)^s$. This follows by direct estimation of the appearing integrals and by using the Theorems 3.3 and 3.4. The operator which produces the best order of convergence according to Theorems 3.3 and 3.4 and which is composed of the operators H_+, $\tilde{H}_{0i}$, $\hat{H}_{0i}$, H_{-i}, $\tilde{H}_{-i}$ is again denoted by $\bar{H}$. The composition of $\bar{H}$ depends on the decay properties of f and on $J_0, J_1, \cdots, J_{\alpha+1}$.

We want to weaken the condition (3.6) which is a very strong assumption on $A(t)$. The basic idea for this is that the determination of the matrices $Q_0, \cdots, Q_\alpha$, D only requires the knowledge of $A_0, \cdots, A_{\alpha+1}$ while the matrices $A_{\alpha+2}, \cdots$ do not influence the exponential and algebraic factors of the fundamental matrix, they only influence the power series factor $P(t)$.

We assume that

$$A\left(\frac{1}{\cdot}\right) \in C^{\alpha+2}\left([0, \tfrac{1}{\delta}]\right), \qquad \delta > 1 \tag{3.41}$$

$$A \in C([1,\infty])$$

and therefore we can write

$$A(t) = A_0 + t^{-1}A_1 + \cdots + t^{-\alpha-1}A_{\alpha+1} + \tilde{A}(t), \tilde{A}(t) = a(t)t^{-\alpha-2-\beta} \tag{3.42}$$

where $a \in C_b([1,\infty))$ and $\beta > 0$.

The equation

$$y' = t^\alpha A(t)y + f(t), \quad t > 1, \quad y \in C([1,\infty])$$

where f fulfills (3.22) can be rewritten as

$$y' = t^\alpha\left(\sum_{i=0}^{\alpha+1} A_i t^{-i}\right)y + (\tilde{A}(t)y + f(t)), \quad y \in C([1,\infty]). \tag{3.43}$$

The homogenous problem (3.43) with $f \equiv 0$ has the general solution

$$y_h = E\,\phi(\tilde{D}_0 + D_-)\xi + E\,\bar{H}_1\tilde{A}y_h, \quad \xi \in C^n \tag{3.44}$$

where E transforms A_0 into its Jordan form and $E\phi$ is the fundamental matrix of the unperturbed problem

$$\tilde{y}_h' = t^\alpha\left(\sum_{i=0}^{\alpha+1} A_i t^{-i}\right)\tilde{y}_h .$$

Now let

$$\|\phi(t)(\tilde{D}_0 + D_-)\| = p(t)t^d e^{q(t)} = p(t)\sigma_h(t), \quad p \in C([1,\infty]) \tag{3.45}$$

$\bar{H}_1$ is composed so that inhomogenities which decay as $t^{-\alpha-2-\beta}\sigma_h(t)$ produce fast decaying particular solutions regarding Theorems 3.3 and 3.4.

From (3.44) we get the equation:

$$(I - E\bar{H}_1\tilde{A})y_h = E\phi(\tilde{D}_0 + D_-)\xi \tag{3.46}$$

for which we take as basic Banach-space:

$$(A_{\sigma_h,\delta} = \{u|u(t) = U(t)\sigma_h(t),\ U \in C_b([\delta,\infty))\}, \|u\|_{\sigma_h,\delta} = \|U\|_{[\delta,\infty]}). \tag{3.47}$$

If $\sigma_h(t) \equiv 1$ then we set $A_{\sigma_h,\delta} = C([\delta,\infty])$.

We get the estimate

$$\|\bar{H}_1\tilde{A}\|_{\sigma_h,\delta} = \max_{\|y_h\|_{\sigma_h,\delta}\le 1} \|\bar{H}_1\tilde{A}y_h\|_{\sigma_h,\delta} \le \text{const. } \delta^{-1-\beta}\ell n\delta < \frac{1}{2\|E\|} \tag{3.48}$$

if δ is sufficiently large. Therefore $(I - E\bar{H}_1\tilde{A})^{-1}$ exists as an operator on $A_{\sigma_h,\delta}$ and for $\xi \in C^n$:

$$y_h = (I - E\bar{H}_1\tilde{A})^{-1}E\phi(\tilde{D}_0 + D_-)\xi = \psi_-^0(\tilde{D}_0 + D_-)\xi \in A_{\sigma_h,\delta}\ . \tag{3.49}$$

As particular solution y_p of (3.43) we set,

$$y_p = E\bar{H}_3\tilde{A}y_p + E\bar{H}_2f\ . \tag{3.50}$$

The inhomogenity f fulfills

$$\|f(t)\| = O\left(t^{\bar{d}}e^{\bar{q}(t)}\right) \text{ and } \sigma_p(t) = t^{\alpha+1+\bar{d}}\ell nt\ e^{\bar{q}(t)} \to 0 \tag{3.51}$$

and $\bar{H}_2$ is composed so that $(\bar{H}_2f)(t)$ decays as fast as possible with regard to Theorems 3.3 and 3.4. Then

$$\|(\bar{H}_2f)(t)\| = O(\sigma_p(t)) \tag{3.52}$$

$\bar{H}_3$ is composed to make particular solutions belonging to inhomogenities which decay as $t^{-\alpha-2-\beta}\sigma_p(t)$ decrease as fast as possible. As basic space we now take $A_{\sigma_p,\delta}$ and conclude the invertibility of $(I - E\bar{H}_3\tilde{A})$ on $A_{\sigma_p,\delta}$ with δ sufficiently large and get

$$y_p = \psi(f) = (I - EH_3\tilde{A})^{-1}E\bar{H}_2f \in A_{\sigma_p,\delta} \tag{3.53}$$

Obviously

$$y(t) = y_h(t) + y_p(t) \in C([\delta,\infty]) \tag{3.54}$$

holds. By substituting H, which is defined by (3.24), for $\bar{H}_1$ in (3.44) it is easily shown that the solution manifold y_h (with the parameters $(\tilde{D}_0 + D_-)\xi$) is unique in $C([\delta,\infty])$, because $A_{\sigma_h,\delta} \in C([\delta,\infty])$ and because the solution space is $\tilde{r}_0 + r_-$ dimensional. Therefore y defined in (3.54) is unique in $C([\delta,\infty])$ (as manifold).

In order to get the solution in $C([1,\infty])$ we solve the 'regular' problem:

(3.55) $$y' = t^{\alpha}A(t)y + t^{\alpha}f(t) \qquad 1 \le t \le \delta$$

(3.56) $$y(\delta) = y_h(\delta) + y_p(\delta) .$$

From(3.44) and (3.49) we get, using the expansion

$$(I - G)^{-1} = \sum_{i=0}^{\infty} G^{i} \qquad \text{for } \|G\| < 1$$

the following estimates which hold for $t \ge \delta$:

(3.57) $$\|\psi_-^0(t) - E\phi(t)(\tilde{D}_0 + D_-)\| \le \text{const. } t^{-1-\beta}\, \ell nt \cdot \sigma_h(t)$$

and

(3.58) $$\|(\psi(f))(t) - E(\bar{H}_2 f)(t)\| \le \text{const. } t^{-1-\beta}\, \ell nt \cdot \sigma_p(t) .$$

Theorem 3.2 remains valid if $E\phi(1)(\tilde{D}_0 + D_-)$ is substituted by $\psi_-^0(1)$ where $\psi_-^0(t)$ has been continued to $[1,\infty]$.

4. Linear Variable Coefficient Problems - General case.

In this paragraph problems of the form (3.1), (3.2), (3.3) fulfilling (3.4), (3.5), (3.6) but not (3.10) are investigated. So we assume a general Jordan-structure of J_0 in the block diagonal form

$$J_0 = \mathrm{diag}(J_0^+, J_0^0, J_0^-),\ \dim(J_0^+) = r_+,\ \dim(J_0^0) = r_0,\ \dim(J_0^-) = r_- \tag{4.1}$$

where J_0^+, J_0^0 resp. J_0^- contain the Jordan blocks which are associated with eigenvalues with real part greater, equal resp. smaller than zero. As in the case of distinct eigenvalues of J_0 we determine an asymptotic expression for the fundamental matrix. This expression is given by Wasow (1965):

Theorem 4.1. Under the given assumption on $J(t)$ there is a fundamental matrix which has the form

$$\phi(t) = P(t)t^D e^{Q(t)}$$

where $Q(t)$ is a diagonal matrix:

$$Q(t) = \mathrm{diag}(J_0)\frac{t^{\alpha+1}}{\alpha+1} + Q_1\frac{t^{\alpha+1-\frac{1}{p}}}{p(\alpha+1)-1} + Q_2\frac{t^{\alpha+1-\frac{2}{p}}}{p(\alpha+1)-2} + \cdots$$

$$\cdots + Q_{p(\alpha+1)-2}\frac{t^{\frac{2}{p}}}{2} + Q_{p(\alpha+1)-1}\,t^{\frac{1}{p}}$$

with some $p \in \mathbb{N}$, , D is a constant matrix in Jordan canonical form and

$$P(t) = P_1(t)P_2(t)$$

where

$$P_1(t) \sim I + \sum_{i=1}^{\infty} P_{1i}t^{-i}, \quad T \to \infty$$

and

$$P_2(t) \sim \sum_{i=0}^{\infty} P_{2i}t^{-\frac{i}{p}}, \quad t \to \infty$$

The (diagonal) elements of $Q(t)$ which correspond to a particular Jordan block of D are equal. Therefore t^D and $e^{Q(t)}$ commute. Moreover the block structure of D is a subdivision of that blocking of J_0 which is obtained by gathering all Jordan blocks of J_0 belonging to the same eigenvalue. Also $P_2(t)$ has a block structure which is identical to the above mentioned blocking of J_0.

Contrary to the case of distinct eigenvalues $A(t)$ is now a polynomial in $t^{\frac{1}{p}}$, where p is some positive integer and $P(t)$ has an expansion in descending powers of $t^{\frac{1}{p}}$, D is no longer strictly diagonal, it is in Jordan canonical form and maybe the most important difference is, that $P(\infty) = P_{20}$ does no longer have to be regular which implies that $P(t)^{-1}$ may grow undoundedly as $t \to \infty$.

However, the proof of this asymptotic expansion for $\phi(t)$ given by Wasow (1965) is constructive and therefore contains an algorithm for the calculation of P, D and Q. We assume that J_0 has the different eigenvalues $\lambda_1, \cdots, \lambda_k$ and the block diagonal form

(4.2) $$J_0 = \mathrm{diag}(J_0^1, \cdots, J_0^k),\ \dim(J_0^i) = r_i$$

J_0^i has the only eigenvalue λ_i. Then the following algorithm results:

1) Substitute $u = T_1(t)u_1$ and determine

$$T_1(t) \sim \sum_{j=1}^{\infty} T_1^j t^{-j}$$

so that the resulting system

$$u_1' = t^{\alpha} B_1(t) u_1 \ ,\quad B_1(t) \sim J_0 + \sum_{j=1}^{\infty} B_1^j t^{-j}$$

split up into two separate ones. The first of them, which is r_1-dimensional, has the form

$$u_{(1)}' = t^{\alpha} J_{(1)}(t) u_{(1)} \quad \text{with} \quad J_{(1)}(t) \sim J_0^1 + \sum_{j=1}^{\infty} J_{(1)}^j t^{-j}$$

where $u_{(1)}$ is the vector consisting of the first r_1 components of u_1. The algorithm for the recursive determination of T_1^j and B_1^j is given by Wasow (1965), paragraph 11.

The remaining $(n-r_1)$-dimensional system is treated in the same way. A r_2-dimensional system with J_0^2 as leading matrix is split off. Finally we get k separate systems

(4.3) $$u_{(i)}' = t^{\alpha} J_{(i)}(t) u_{(i)} \quad \text{with} \quad J_{(i)}(t) \sim J_0^i + \sum_{j=1}^{\infty} J_{(i)}^j t^{-j}.$$

The transformation

(4.4) $$u = P_1(t) \begin{bmatrix} u_{(1)} \\ \vdots \\ u_{(k)} \end{bmatrix}$$

has the following form

$$(4.5)\qquad P_1(t) = T_1(t) \prod_{i=1}^{k-1} \left[\begin{array}{c:c} I_{r_1+\cdots+r_i} & 0 \\ \hdashline 0 & T_{i+1}(t) \end{array} \right], \quad P_1(\infty) = I$$

Because of (4.3) k systems, where the leading matrix of each of them has only one eigenvalue, have to be treated now.

2) We substitute

$$u_{(i)} = v_{(i)} \exp\left(\lambda_i \frac{t^{\alpha+1}}{\alpha+1}\right)$$

and get

$$(4.7)\qquad v'_{(i)} = t^{\alpha}(J_{(i)}(t) - \lambda_i I_{r_i})v_{(i)} \qquad \text{for } i = 1(1)k .$$

The leading matrices of the systems (4.7) are now $J_0^i - \lambda_i I_{r_i}$ having the only eigenvalue 0 .

3) We apply so called shearing transformations

$$(4.8)\qquad v_{(i)} = S_{(i)}(t)w_{(i)}$$

where

$$(4.9)\qquad S_{(i)}(t) = \mathrm{diag}\left(1, t^{-g_i}, t^{-2g_i}, \ldots, t^{-(r_i-1)g_i}\right) ,$$

with g_i rational and $g_i > 0$ to the systems (4.7). The g_i should be chosen so that the leading matrices of the resulting systems, which have $w_{(i)}$ as dependent variables, have more than one different eigenvalue or if not possible for a certain i , so that the rank of this new i-th system is smaller than $\alpha + 1$ or that this systems splits up into separate subsystems.

Wasow (1965) showed that it is always possible to achieve one of these simplifications. In order to get systems where only integral powers of the independent variable occur we substitute

$$(4.10)\qquad x_i = p_i^{1/(g_i-\alpha-1)} \cdot t^{1/p_i} \qquad \text{for } i = 1(1)k$$

where p_i is the smallest integer so that $g_i p_i$ is a natural number. Then we get systems of the form

$$(4.11)\qquad w'_{(i)}(x_i) = x_i^{h_i} C_{(i)}(x_i)w_{(i)}(x_i)$$

where

$$(4.12) \qquad \text{(a)} \quad h_i = (\alpha + 1 - g_i)p_i - 1, \quad \text{(b)} \quad C_{(i)}(x_i) \sim \sum_{j=0}^{\infty} C^j_{(i)} x_i^{-j}$$

4a) If $C^0_{(i)}$ has only 0 as eigenvalue then we have reduced the rank of the system or it splits up into separate subsystems of lower order. Applying more shearing transformations to (4.11) we end up with a system whose leading matrix has either more than one different eigenvalue or has rank equal to 0. In the second case we have a system with a singularity of the first kind for which the fundamental matrix $\phi_{(i)}(t)$ has the form

$$(4.13) \qquad \phi_{(i)}(t) = P_{(i)}(t)t^{D_{(i)}}, \quad P_{(i)}(t) = \sum_{j=0}^{\infty} P^j_{(i)} t^{-j}$$

and $D_{(i)}$ is a constant matrix in Jordan canonical form (see Wasow (1965)).

4b) Now we assume that $C^0_{(i)}$ has at least two different eigenvalues. Then we transform $C^0_{(i)}$ to its Jordan canonical form $\tilde{C}^0_{(i)}$

$$(4.14) \qquad C^0_{(i)} = E_{(i)}\tilde{C}^0_{(i)} \bar{E}^{-1}_{(i)}$$

and substitute

$$(4.15) \qquad w_{(i)}(x_i) = E_{(i)} \, z_{(i)}(x_i)$$

getting a system whose leading matrix is $\tilde{C}^0_{(i)}$, that means in Jordan form

$$(4.16) \qquad z'_{(i)}(x_i) = x_i^{h_i} \tilde{C}_{(i)}(x_i) z_{(i)}(x_i) \; .$$

5) We apply the transformation explained in 1) to the system (4.16) in order to get separate subsystems of lower order whose leading matrices have the only eigenvalue $\mu_{(i)_j}$. By the means of 2) we normalize these systems so that their leading matrices have only the eigenvalue 0. These transformations split off the factors

$$(4.17) \qquad \exp\Big(\mu_{(i)_j} \frac{x_i^{h_i+1}}{h_i+1}\Big) \qquad \text{for } \; i = 1(1)k \; .$$

Resubstituting in (4.10) and using (4.12)(a) we notice that the argument in (4.17) is of the order $t^{\alpha+1-g_i}$ that means of order lower than $t^{\alpha+1}$ which is the order of the argument of the first exponential factor because if $g_i = 0$ the system remained unchanged.

Applying another set of shearing transformations 3) we arrive at 4a) or 4b).

6) So a finite chain of all the in 1) -5) described transformations result in a set of one dimensional systems and systems with a singularity of the first kind. Setting $p = \bar{p}^{-m}$ where $\bar{p}$ in the smallest common multiple of all the p's used in the sequence of shearing transformations, and m is the number of these transformations which split the system into a set of systems described above, we get the formula for the fundamental matrix $\phi(t)$ given in Theorem 4.1 by taking into account (4.13). Moreover we get:

$$P_2(t) = \Big(\prod_{i=1}^{m} S_i(t)E_iP_{2i}(t)\Big)P_4(t), \quad P_{2i}(t) = I + \sum_{j=1}^{\infty} P_{2i}^{j}t^{-\frac{j}{p}} \tag{4.18}$$

and the $S_i(t)$ are composed of submatrices $S_{(ij)}(t)$ defined in (4.9). The E_i's are regular and $P_4(t)$ is derived by solving the systems with singularities of the first kind using (4.13).

We define:

$$P_3(t) = \prod_{i=1}^{m} S_i(t)E_iP_{2i}(t). \tag{4.19}$$

An estimate of $P_3^{-1}(t)$ can be obtained in the following way.

Let D_i be the projection onto the direct sum of eigenspaces associated with the eigenvalue λ_i of J_0. Then

$$\|P_3^{-1}(t)D_i\| \leq \text{const. } t^{\left[(r_i-1)\left(g_{i1}+\frac{g_{i2}}{p_{i1}}+\frac{g_{i3}}{p_{i1}p_{i2}}+\cdots+\frac{g_{im}}{p_{i1}\cdots p_{im-1}}\right)\right]} \tag{4.20}$$

holds. The sum in the exponent of t is derived by estimating $S_i^{-1}(t)$ and by taking into account the block structures of the E_i^{-1} and $P_{2i}^{-1}(t)$. p_{ij} and g_{ij} are as in (4.10) and represent that sequence of shearings starting off from the i-th r_i-dimensional subsystem and giving the largest exponent in (4.20).

For this sequence we can calculate the ranks of the corresponding sequence of subsystems as in (4.12)

$$
\begin{aligned}
&h_{i0} + 1 = \alpha + 1 \\
&h_{i1} + 1 = p_{i1}(h_{i0} + 1 - g_{i1}) \\
(4.21)\qquad &h_{i2} + 1 = p_{i2}(h_{i1} + 1 - g_{i2}) \\
&\vdots \\
&h_{i,m} + 1 = p_{im}(h_{i,m-1} + 1 - g_{im}) .
\end{aligned}
$$

By assumption $h_{i,m} + 1 > 0$ holds and we get recursively:

$$
(4.22)\qquad \alpha + 1 > g_{i1} + \frac{g_{i2}}{p_{i1}} + \cdots + \frac{g_{im}}{p_{i1},\cdots,p_{i,m-1}}
$$

and therefore the estimate

$$
\|P_3^{-1}(t)D_i\| \leq \text{const. } t^{(r_i-1)(\alpha+1)}
$$

holds.

The basic solutions φ_i with $\phi(t) = (\varphi_1(t),\cdots,\varphi_1(t))$ fulfill

$$
(4.24)\qquad \|\varphi_i(t)\| \leq p_i(t)e^{q_i(t)}\,t^{d_i}(\ell nt)^{j_i},\ p_i \in C([1,\infty]). \ .
$$

Eigenvalues of J_0 with positive real part produce exponentially increasing, eigenvalues with negative real part produce exponentially decreasing basic solutions. Imaginary eigenvalues of J_0 can produce exponentially and algebraically increasing and decreasing constant and oscillating and logarithmically increasing solutions. The asymptotic behaviour of a particular basic solution φ_i can be determined by the knowledge of $Q(t)$, D and $P_0,\cdots,P_{m_i}$ where m_i is sufficiently large. Therefore the solution of the homogenous problem (3.16), (3.17) (under the assumptions of Chapter 4) is:

$$
(4.25)\qquad u_h(t) = \phi(t)(\tilde{D}_0 + D_-)\xi\ , \qquad \xi \in C^n
$$

where $\tilde{D}_0$ and D_- are defined as in Chapter 3.

Now we construct a particular solution u_p of the problem

$$u_p' = t^{\alpha}J(t)u_p + t^{\alpha}E^{-1}f(t) \qquad 1 < t < \infty,\ f \in C([1,\infty]) \tag{4.26}$$

$$u_p \in C([1,\infty]) \ . \tag{4.27}$$

We substitute

$$u_p(t) = P_1(t)v_p(t) \quad \text{and} \quad v_p(t) = \begin{bmatrix} v_{p_+}(t) \\ v_{p_0}(t) \\ v_{p_-}(t) \end{bmatrix} \tag{4.28}$$

and define

$$u_{p_+}(t) = P_1(t)\begin{bmatrix} v_{p_+}(t) \\ 0 \\ 0 \end{bmatrix}, \quad u_{p_0}(t) = P_1(t)\begin{bmatrix} 0 \\ v_{p_0}(t) \\ 0 \end{bmatrix} \tag{4.29}$$

$$u_{p_-}(t) = P_1(t)\begin{bmatrix} 0 \\ 0 \\ v_{p_-}(t) \end{bmatrix}$$

so that the definitions (3.23a) and (3.24a) hold. We get the separated problems

$$\begin{bmatrix} v_{p_+} \\ v_{p_0} \\ v_{p_-} \end{bmatrix}' = t^{\alpha}(\operatorname{diag}(J_0^+ + J^+(t), J_0^0 + J^0(t), J_0^- + J^-(t))\begin{bmatrix} v_{p_+} \\ v_{p_0} \\ v_{p_-} \end{bmatrix} +$$

$$+ t^{\alpha}P_1^{-1}(t)E^{-1}f(t)$$

where $J^+(t)$, $J^0(t)$, $J^-(t)$ have an asymptotic power series expansion in t^{-1} without a constant term. We define the block components v_{p_+}, v_{p_-} as in de Hoog and Weiss (1980a,b) as solutions of the operator equations:

$$v_{p_+} = \hat{H}_+ J^+ v_{p_+} + \hat{H}_+(P_1^{-1}E^{-1}f)_+ \tag{4.30)(a}$$

$$v_{p_-} = \hat{H}_- J^- v_{p_-} + \hat{H}_-(P_1^{-1}E^{-1}f)_- \tag{4.30)(b}$$

where $(P_1^{-1}E^{-1}f)_+$ resp. $(P_1^{-1}E^{-1}f)_-$ are the first r_+ resp. the last r_- components of $P_1E^{-1}f$ and $\hat{H}_+$, $\hat{H}_-$ are the operators defined similar to (2.11).

$$(4.30)(c) \qquad (\hat{H}_+g)(t) = \int_\infty^t \exp\left(\frac{J_0^+}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1})\right)s^\alpha g(s)ds$$

$$(4.30)(d) \qquad (\hat{H}_-g)(t) = \int_\delta^t \exp\left(\frac{J_0^-}{\alpha+1}(t^{\alpha+1} - s^{\alpha+1})\right)s^a g(s)ds$$

for $\varepsilon\ C([1,\infty])$.

From (4.30) we derive

$$(4.31)(a) \qquad v_{p_+} = (I - \hat{H}_+J^+)^{-1}\hat{H}_+(P_1^{-1}E^{-1}f)_+ \ \varepsilon\ C([\delta,\infty])$$

$$(4.31)(b) \qquad v_{p_-} = (I - \hat{H}_-J^-)^{-1}\hat{H}_-(P_1^{-1}E^{-1}f)_- \ \varepsilon\ C([\delta,\infty])$$

with δ sufficiently large. The proof of the invertibility of $(I-H_{+,-}J^{+,-})$ is given in de Hoog and Weiss (1980a,b). Again we get:

$$(4.32) \qquad (a)\ \ v_{p_+}(\infty) = (J_0^+)^{-1}(E^{-1}f(\infty))_+, \quad (b)\ \ v_{p_-}(\infty) = (J_0^-)^{-1}(E^{-1}f(\infty))_-$$

because $J^+(\infty) = 0$ and $J^-(\infty) = 0$.

Now we assume that for some $\varepsilon > 0$:

$$(4.33) \qquad D_0P_1^{-1}(t)E^{-1}f(t) = F_0(t)t^{-\bar{r}(\alpha+1)-\varepsilon}, \quad F_0\ \varepsilon\ C_b([1,\infty))$$

holds, where $\bar{r}$ is the largest algebraic multiplicity of the eigenvalues of J_0 with real part zero.

So $\bar{r}$ is defined differently to r in Chapter 2. However, more sophisticated assumptions on f could be used (similar to (3.21)) but we will use (4.33) for simplicity.

The system

$$(4.34) \qquad v'_{p_0} = t^\alpha(J_0^0 + J^0(t))v_{p_0}$$

is composed of separate systems, each of them associated with one imaginary eigenvalue of J_0 and $\bar{r}$ is the maximal dimension of these subsystems.

We take one of these (inhomogeneous) subsystems

$$(4.35) \qquad v'_{p_{0(i)}} = t^\alpha J_{(i)}(t)v_{p_{0(i)}} + t^\alpha(P_1^{-1}(t)E^{-1}f(t))_{0(i)}$$

where $\hat{f}_{(i)}(t) := (P^{-1}(t)E^{-1}f(t))_{0(i)}$ consists of the corresponding components of the inhomogenity $P^{-1}(t)E^{-1}f(t)$. The leading matrix J_0^i of $J_{(i)}(t)$ has the only eigenvalue λ_i with $\mathrm{Re}\lambda_i = 0$.

Now we apply the transformations

$$(4.36) \qquad v_{p_{0(i)}} = \exp\left(\frac{\lambda_i t^{\alpha+1}}{\alpha+1}\right) \cdot S_{(i)}(t) w_{p_{0(i)}} \quad , \quad x_i = c(i) t^{\frac{1}{p_i}}$$

$$(4.37) \qquad w_{p_{0(i)}}(x_i) = E_{(i)} z_{p_{0(i)}}(x_i)$$

as defined in (4.6), (4.9), (4.10) and (4.15) and get

$$(4.38) \qquad z'_{p_{0(i)}}(x_i) = x_i^{h_i} \tilde{C}_{(i)}(x_i) z_{p_{0(i)}}(x_i) + x_i^{h_i} \hat{g}_{(i)}(x_i)$$

where

$$(4.39) \quad \hat{g}_{(i)}(x_i) = \left(\frac{x_i}{c(i)}\right)^{p_i g_i} E_{(i)}^{-1} S_{(i)}^{-1}\left(\left(\frac{x_i}{c(i)}\right)^{p_i}\right) \exp\left(-\frac{\lambda_i}{\alpha+1}\left(\frac{x_i}{c(i)}\right)^{p_i(\alpha+1)}\right) \cdot \hat{f}_{(i)}\left(\left(\frac{x_i}{c(i)}\right)^{p_i}\right).$$

From (4.23), (4.33) we derive:

$$(4.40) \qquad \|\hat{g}_{(i)}(x_i)\| \le \text{const.}\ x_i^{-p_i\varepsilon - \bar{r}p_i(\alpha+1-g_i)} .$$

If the leading matrix of the system (4.38) has at least one eigenvalue different from zero then the separating transformation

$$z_{p_{0(i)}} = P_{2(i)}(x_i) z^1_{p_{0(i)}} \quad , \quad P_{2(i)}(\infty) = I$$

can be applied. All resulting sybsystems, whose leading matrices have eigenvalues with real part different from zero can now be solved by the means of (4.30), (4.31) because the new inhomogenity has the form

$$(4.41) \qquad \hat{g}^1_{(i)}(x_i) = P^{-1}_{2(i)}(x_i) \hat{g}_{(i)}(x_i) \in C([1,\infty]) .$$

For all other subsystems this sequence of substitutions is repeated as long as we arrive at systems whose leading matrices have eigenvalues with real part different from zero or one-dimensional systems or we arrive at systems with a singularity of rank zero. In the second and third cases only eigenvalues with real part zero have been split off therefore the inhomogenities do not contain exponentially increasing or decreasing factors (see (4.39)) and therefore every inhomogenity g_0 which occurs in this sequence of transformations fulfills after resubstitution to t as independent variable:

(4.42) $$\|g_0(t)\| \le \text{const.}\ \|P_3^{-1}(t)D_i\|t^{\alpha+1}\|D_0P_1^{-1}(t)E^{-1}f(t)\| \le \text{const.}\ t^{-\varepsilon}$$

because (4.22) holds.

A particular solution for one-dimensional systems can be found according to Chapter 3, so we just have to treat systems with a singularity of rank zero.

We want to solve

(4.43) $$z_p' = \left(\frac{1}{x}B + \frac{1}{x}\tilde{B}(x)\right)z_p + \frac{1}{x}g_0(x), \quad \tilde{B}(x) = \bar{B}(x)\frac{1}{x}$$

(4.44) $$z_p(x) = Z_p(x)x^{-\varepsilon}(\ln x)^j, \quad Z_p \in C_b([1,\infty)), \quad j \in N_0$$

where B is a constant matrix in Jordan form, $\bar{B} \in C([1,\infty])$ and g_0 fulfills (4.42).

Let

(4.45) $$B = \operatorname{diag}(B_1,\cdots,B_q), \quad B_i = \begin{bmatrix} b_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & b_i \end{bmatrix}, \; i = 1(1)q$$

and let D^i be the projection onto the invariant subspace associated with b_i.

Then we define

(4.46) $$z_p = G\tilde{B}z_p + Gg_0, \quad G = G_1 + \cdots + G_q$$

and

(4.47)(a), (4.47)(b) $$(G_ig_0)(x) = \begin{cases} x^B \int_\delta^x D^i s^{-B-I} g_0(s)ds, & b_i < -\varepsilon \\ x^B \int_x^\infty D^i s^{-B-I} g_0(s)ds, & b_i > -\varepsilon . \end{cases}$$

It is easily checked that for $x > \delta$

(4.48) $$\|(Gg_0)(x)\| \le \text{const.}\ (\ln x)^j \max\left(\max_{s\in[x,\infty]}\|g_0(s)\|,\ x^{-\varepsilon}\cdot\max_{s\in[\delta,x]}\|s^{\varepsilon}g_0(s)\|\right)$$

holds where $j = \max_i(\dim(B_i))$. Therefore we get from (4.46)

(4.49) $$z_p = (I - G\tilde{B})^{-1}Gg_0$$

z_p fulfills (4.44) because of a contraction argument similar to that applied in (3.47), (3.48), (3.49). If $g_0(x) = O(x^{-\varepsilon}(\ln x)^i)$ then the right hand side of (4.48) has an additional factor $(\ln x)^i$.

After having performed all necessary resubstitutions we get a solution $v_{P_0}(t)$ fulfilling $v_{P_0}(\infty) = 0$. Defining $\tilde{r}_0$, $\tilde{D}_{00}$ in the same way as we did in Chapter 3 we get

<u>Theorem 4.2:</u> The problem (3.1), (3.2), (3.3) under the assumption (3.4), (3.5), (4.1) has a unique solution y for all f fulfilling (3.21)(a), (4.33) and for all $\hat{\beta} \in R^{\tilde{r}_0+r_-}$ if and only if

$$\operatorname{rank}[B_1 E\phi(1)(\tilde{D}_0 + D_-) + B_\infty E\tilde{D}_{00}] = \tilde{r}_0 + r_-$$

where B_1 and B_∞ are $(\tilde{r}_0 + r_-) \times n$ matrices. This solution y depends continuously on $\hat{\beta}$, $(D_+ + D_-)E^{-1}P_1^{-1}f$ and F_0 which is defined in (4.33).

The proof is complete if the continuity statement is proven.

<u>Theorem 4.3.</u> If $f \in C([1,\infty])$ and (4.33) holds then the following estimates hold for $t > \delta$:

(4.50) $$\|(H_+f)(t)\| < \text{const.}\ \|D_+P_1^{-1}E^{-1}f\|_{[t,\infty]}$$

(4.51) $$\|(H_-f)(t)\| < \text{const.}\ t^{-\gamma} \cdot \max_{s \in [\delta,t]} \|s^\gamma D_-P_1^{-1}(s)E^{-1}f(s)\| \text{ for } \gamma > 0.$$

(4.52) $$\|(H_0f)(t)\| < \text{const.}\ (\ln t)^{j_0} \cdot \max(t^{-\varepsilon} \max_{s\in[\delta,t]} \|s^{\varepsilon+(\alpha+1)\bar{r}} D_0P_1^{-1}(s)E^{-1}f(s)\|, \max_{s > t} \|s^{(\alpha+1)\bar{r}} D_0P_1^{-1}(s)E^{-1}f(s)\|)$$

where $j_0 = \max_{i_k} (\dim(D_{i_k}))$ and D_{i_k} are the Jordan blocks of the matrix D in Theorem 4.1 for which the corresponding polynomials $q_{i_k}(t)$ fulfill $\operatorname{Re} q_{i_k}(t) \equiv 0$. All constants are independent of δ and f.

Proof. From (4.31) we conclude

(4.53)(a) $$v_{P_+}(t) = \sum_{i=0}^{\infty} ((\hat{H}_+J^+)^i \hat{H}_+(P_1^{-1}E^{-1}f)_+)(t)$$

(4.53)(b) $$v_{P_-}(t) = \sum_{i=0}^{\infty} ((\hat{H}_-J^-)^i \hat{H}_-(P_1^{-1}E^{-1}f)_-)(t)$$

for δ sufficiently large.

$\hat{H}_+$ resp $\hat{H}_-$ fulfill the estimates (2.15) resp. (2.17) and therefore we get:

(4.54) (a) $$\|v_{p_+}(t)\| \leq \text{const.} \max_{s\in[t,\infty]} \|D_+P_1^{-1}(s)E^{-1}f(s)\| \sum_{i=0}^{\infty} (\tfrac{1}{2})^i$$

(4.54) (b) $$\|v_{p_-}(t)\| \leq \text{const. } t^{-\gamma} \max_{s\in[\delta,t]} \|s^{\gamma}D_-P_1^{-1}(s)E^{-1}f(s)\| \cdot \sum_{i=0}^{\infty} (\tfrac{1}{2})^i$$

if δ is so large that $\|J^+\|_{[\delta,\infty]} < 1/2$ and $\|J^-\|_{[\delta,\infty]} < 1/2$. The estimates (4.50), (4.51) follow by using (4.29) and (4.52) follows from the derivation of v_{p_0} and from (4.48), (4.49).

Theorem 4.4 implies that an inhomogenity f fulfilling

(4.55) $$f(t) = t^{-(\alpha+1)\bar{r}-\varepsilon}(\ln t)^{\ell}F(t), \quad F \in C_b([1,\infty))$$

produces a particular solution Hf for which the estimate

(4.56) $$\|(Hf)(t)\| \leq \text{const. } t^{-\varepsilon}(\ln t)^{j_0+\ell} \|F\|_{[\delta,\infty]}$$

holds.

Now we assume inhomogenities of the form

(4.57) $$f(t) = e^{p(t)}t^{\beta}(\ln t)^{\ell}F(t), \quad F \in C_b([1,\infty))$$

and $e^{p(t)}t^{\beta}F(t)$ fulfills (4.33) and $p(t)$ is a polynomial in $t^{\frac{1}{p}}$ with leading term $p_0 \frac{t^{\alpha+1}}{\alpha+1}$. It is possible to construct a particular solution $\bar{H}f = H_+f + \bar{H}_0f + \bar{H}_-f$ fulfilling

(4.58) (a) $$\|(\bar{H}_0f(t)\| \leq \text{const. } e^{p(t)}t^{\beta+\bar{r}(\alpha+1)}(\ln t)^{j_0+\ell} \|F\|_{[\delta,\infty]}$$

(4.58) (b) $$\|(\bar{H}_-f)(t)\| \leq \text{const. } e^{p(t)}t^{\beta+\bar{k}(\alpha+1)}(\ln t)^{j_-+\ell} \|F\|_{[\delta,\infty]}$$

where $\bar{k}$ is the maximal algebraic multiplicity of the eigenvalues of J_0 with negative real part and j_- is the maximal dimension of Jordan blocks in DD_-. To outline the proof we assume the system

$$(4.59)\qquad v' = t^{\alpha}(J_- + \tilde{J}_-(t)t^{-1})v + t^{\alpha}g(t),\ \tilde{J}_-(\infty) = 0$$

where $\tilde{J}_-$ has an asymptotic series expansion and J_- has only one eigenvalue λ with $\mathrm{Re}\lambda < 0$ and $g(t)$ fulfills (4.57). If $-\mathrm{Re}\lambda + \mathrm{Re}p_0 > 0$ we determine a particular solution v_{p_-} by applying (4.30)(b) . If $-\mathrm{Re}\lambda + \mathrm{Re}p_0 < 0$ we use the operator $\tilde{\tilde{H}}_-$ instead of H_- in (4.30) where $\tilde{\tilde{H}}_-$ is defined by (4.30)(a) with $\delta = \infty$. In both cases the estimate $\|v_{p_-}(t)\| \le \text{const. } e^{P(t)}t^{\beta}(\ell nt)^{\ell}\|F_-\|_{(\delta,\infty]}$ follows by a contraction argument and by (2.19), (2.22). However, if $\mathrm{Re}\lambda = \mathrm{Re}p_0$ we perform the substitution of the form (4.36), (4.37), (4.40) as long as we arrive at systems whose eigenvalues have real parts different from the real parts of the corresponding coefficients of $p(t)$. Then we apply H_- resp. $\tilde{H}_-$. If $\mathrm{Re}p(t) \equiv \mathrm{Re}q_j(t)$ where q_j is a polynomial in the exponential factor of the fundamental matrix of the homogenous system (4.59) this sequence of transformation either leads to one dimensional systems or to a system with a singularity of the first kind (rank zero). The first has been treated in Chapter 3 and the second is solved by (4.46), (4.47). Resubstitution and application to all separate subsystems corresponding to eigenvalues with negative real part yields the estimate (4.58)(b). The estimate (4.58)(a) is gained in a similar way.

Now we assume that the matrix $A(t)$ of the system (3.1) fulfills

$$(4.60)\qquad A\left(\tfrac{1}{\cdot}\right) \in C^{(\alpha+1)\bar{\ell}+1}\left(\left[0,\tfrac{1}{\delta}\right]\right),\qquad \delta > 1$$
$$A \in C([1,\infty])$$

instead of (3.5), where $\bar{\ell} = \max(\bar{r},\bar{k})$ is defined for the Jordan form of $A(\infty)$. Therefore,

$$(4.61)\qquad A(t) = A_0 + t^{-1}A_1 + \cdots + t^{-(\alpha+1)\bar{\ell}}A_{(\alpha+1)\bar{\ell}} + \tilde{A}(t),\quad \tilde{A}(t) = a(t)t^{-(\alpha+1)\bar{\ell}-1-\beta}$$

where $a \in C_b([1,\infty))$, $\beta > 0$.

The system (3.1) can now be written as:

$$(4.62)\qquad y' = t^{\alpha}(A_0 + \cdots + t^{-(\alpha+1)\bar{\ell}}A_{(\alpha+1)\bar{\ell}})y + t^{\alpha}(\tilde{A}(t)y(t) + f(t)).$$

By regarding (4.62) as a perturbation of

$$(4.63)\qquad \tilde{y} = t^{\alpha}\Big(\sum_{i=0}^{(\alpha+1)\bar{\ell}} t^{-i}A_i\Big)\tilde{y} + t^{\alpha}f$$

and proceeding similarly to Chapter 3 we find that

$$(4.64)\qquad \|\psi_-^0(t) - E\phi(t)(\tilde{D}_0 + D_-)\| \le \text{const. } t^{-1-\beta}(\ell nt)^{2\max(j_0,j_-)}\sigma_h(t)$$

where $E\phi(t)$ is the fundamental matrix of (4.63) $\sigma_h(t)$ is defined as in (3.45) and $\psi_-^0 \in C([\delta,\infty])$ is the matrix fulfilling $y_h(t) = \psi_-^0(t)(\tilde{D}_0 + D_-)\xi$, where y_h is the general $C([\delta,\infty])$- solution of (4.62) with $f \equiv 0$. Similarly a particular solution $\psi(f)$ of the inhomogenous problem (4.62) is obtained fulfilling

$$(4.65) \qquad \|(\psi(f))(t) - E(\bar{H}f)(t)\| \le \text{const.}\ t^{-1-\beta}(\ell n t)^{2\max(j_0,j_-)}\sigma_p(t)$$

where $E\bar{H}f$ is the particular solution of (4.3) which decays as fast as possible according to Theorem 4.3 and (4.58). σ_p is defined as:

$$(4.66) \qquad \|(\bar{H}f)(t)\| = O(\sigma_p(t)).$$

Theorem 3.2 remains valid if $E\phi(1)(\tilde{D}_0 + D_-)$ is substituted by $\psi_-^0(1)$ where ψ_-^0 has been continued to $[1,\infty]$.

If only an existence theorem for $\psi_-^0(t)$ is desired without taking care of the decay properties then $\tilde{\ell} = \bar{r}$ and $\beta > -1$ in (4.61) can be chosen. In this case the right hand sides of (4.64), (4.65) do not contain eventually occurring exponentially decaying factors in $\sigma_h(t)$ resp. $\sigma_p(t)$.

Moreover it is important to consider problems where the matrix $\tilde{A}(t)$ defined in (4.61) decays exponentially:

$$(4.67) \qquad \tilde{A}(t) = a(t)t^{\gamma}e^{q(t)}\ ,\ a \in C_b([1,\infty)),\quad q(t) \to -\infty$$

Using the same methods as in the case of algebraic decay we get:

$$(4.68) \qquad \|\psi_-^0(t) - E\phi(t)(\tilde{D}_0 + D_-)\| \le \text{const.}\ t^{\gamma+(\alpha+1)\tilde{\ell}}(\ell n t)^{2\max(j_0,j_-)}\sigma_h(t)e^{q(t)}$$

$$(4.69) \qquad \|(\psi_p(f))(t) - E(\bar{H}f)(t)\| \le \text{const.}\ t^{\gamma+(\bar{\alpha}+1)\ell}(\ell n t)^{2\max(j_0,j_-)}\sigma_p(t)e^{q(t)}\ .$$

Only the construction of the 'particular' solutions has to be changed in order to get these estimates.

Finally the author conjectures that it is possible to change $\bar{r}$ to r defined in Chapter 2 so that all statements made should hold with r instead of $\bar{r}$. Maybe someone else is successful in improving the derived estimates.

5. Nonlinear Problems.

We consider problems of the form

$$(5.1)\qquad y' = t^{\alpha} f(t,y), \quad 1 \leq t < \infty, \quad \alpha \in N_0$$

$$(5.2)\qquad y \in C([1,\infty])$$

$$(5.3)\qquad b(y(1), y(\infty)) = 0 .$$

From (5.2) and (5.1) we conclude that

$$(5.4)\qquad f(\infty, y(\infty)) = 0 .$$

Now we define for $a \in R^n$, $x, \bar{t} \in R$:

$$(5.5)\qquad S_x(a) = \{y \in R^n \mid \|y - a\| < x\}$$

$$(5.6)\qquad C_x(\bar{t},a) = \{(\bar{t},y) \in R^{n+1} \mid t > \bar{t},\ y \in S_x(a)\}$$

and assume that

$$(5.7)\qquad f, f_y \in C_{lip}(C_x(1,y(\infty)))$$

for some $x > 0$ sufficiently large.

(5.4) is a (nonlinear) system of equations from which $y(\infty)$ can be calculated as a solution manifold $y(\infty) = y_\infty(\mu)$, $y \in S \subset R^{n_1}$ $n_1 \leq n$, if the Problems (5.1), (5.2), (5.3) admits a solution. *The dimension of this manifold* $-n_1-$ *can be calculated a priori* if we require for a solution point y_∞ that

$$(5.8)\qquad \mathrm{rank}(f_y(\infty, y_\infty)) = n - n_1$$

$$(5.9)\qquad \mathrm{rank}(f_y(\infty, y)) \leq n - n_1 \quad \text{for} \quad y \in \overset{\circ}{S}_x(y_\infty), \quad x > 0 .$$

Then there is a $x_1 > 0$ and a n_1-dimensional manifold $y_\infty(\mu)$ which fulfills the equation

$$(5.10)\qquad f(\infty, y_\infty(\mu)) \equiv 0 \quad \text{for} \quad \mu \in S \subset R^{n_1}$$

$$(5.11)\qquad y_\infty(\mu) \in \overset{\circ}{S}_{x_1}(y_\infty) .$$

From (5.8) we conclude that n_1 equals the geometrical multiplicity of the eigenvalue zero of the matrix $f_y(\infty, y_\infty)$.

However as practical examples point out, the assumption that $f_y(\infty,y)$ does not decrease its rank in y_∞ is too strong and therefore we regard n_1 as a priori unknown but obviously the solution of the equation (5.1) determines n_1 for a practical problem.

Now we define

$$A(t,\mu) = f_y(t,y_\infty(\mu)) \tag{5.12}$$

and require that A fulfills (3.4), (3.5) so that

$$A(t,\mu) = \sum_{i=0}^{\infty} A_i(\mu)t^{-i} \quad \text{for} \quad t \geq \delta . \tag{5.13}$$

We transform $A_0(\mu)$ *to its Jordan canonical form* $J_0(\mu)$

$$A_0(\mu) = E(\mu)J_0(\mu)E^{-1}(\mu) \tag{5.14}$$

and introduce z as new dependent variable

$$E(\mu)z = y - y_\infty(\mu) \tag{5.15}$$

getting

$$z' = t^\alpha J(t,\mu)z + t^\alpha g(z,t,\mu) \tag{5.16}$$

$$z(\infty) = 0 . \tag{5.17}$$

Here

$$J(t,\mu) = E^{-1}(\mu)A(t,\mu)E(\mu) \tag{5.18}$$

and

$$g(z,t,\mu) = E^{-1}(\mu)f(t,E(\mu)z + y_\infty(\mu)) - J(t,\mu)z \tag{5.19}$$

hold. The perturbation g fulfills the estimates:

$$\|g(z,t,\mu)\| \leq C_1(\mu)(\|f(t,y_\infty(\mu))\| + \|z\|^2) \tag{5.20}$$

$$\|g(z_1,t,\mu) - g(z_2,t,\mu)\| \leq C_2(\mu)(\|z_1\| + \|z_2\|)\|z_1 - z_2\| \tag{5.21}$$

where $C_i(\mu)$ depend also on the Lipschitz-constant of f_y on $C_x(1,y_\infty(\mu))$.

We restrict μ to subsets $\tilde{S} \subset S$ which are defined as follows.

1) The projections onto the direct sums of invariant subspaces of $J_0(\mu)$, which belong to eigenvalues with positive, zero and negative real part are constant for $\mu \in \tilde{S}$. Moreover the projections onto the invariant subspaces of $J_0(\mu)$ are constant for $\mu \in \tilde{S}$, therefore $r_+, r_0, r_-, \bar{r}$ are defined for $J_0(\mu)$ as in the last chapters and are independent of $\mu \in \tilde{S}$.

2) $y_\infty(\mu)$, $E(\mu)$, $E^{-1}(\mu)$ are continuous for $\mu \in \tilde{S}$.

Let $\phi(t,\mu)$ be the fundamental matrix of the (homogenous) Problem (5.16).

3) The same columns $\varphi_i(t,\mu)$ of the matrix $\phi(t,\mu)$ fulfill

$$\|\varphi_i(t,\mu)\| \leq C_i(\mu)t^{-(\alpha+1)\bar{r}-\varepsilon_1(\mu)}(\ell nt)^{j_i}, \quad \varepsilon_1(\mu) > 0$$

for all $\mu \in \tilde{S}$. Therefore there is a projection matrix $\hat{D}_0$ independent of μ in $\tilde{S}$ so that

$$\|\phi(t,\mu)(\hat{D}_0 + D_-)\| \leq C(\mu)t^{-(\alpha+1)\bar{r}-\varepsilon_1(\mu)}(\ell nt)^{j_0} \quad \text{for } \mu \in \tilde{S} \tag{5.23}$$

holds. $\hat{r}_0$ be the number of 1's in the main diagonal of $\hat{D}_0$.

We require f to fulfill

$$\|f(t,y_\infty(\mu))\| \leq C(\mu)t^{-2(\alpha+1)\bar{r}-\varepsilon_2(\mu)}, \quad \varepsilon_2(\mu) > 0 \quad \text{for } \mu \in \tilde{S}. \tag{5.24}$$

For $\mu \in \tilde{S}$, $\xi \in C^n$ fixed we set

$$(\psi(z,\mu))(t) = \phi(t,\mu)(\hat{D}_0 + D_-)\xi + (Hg(z,\cdot,\mu))(t) \tag{5.25}$$

where H is as in (4.56) (with $E = I$). Regarding $\psi(\cdot,\mu)$ as an operator on the Banach space:

$$\begin{gathered}(A_{\varepsilon,\delta} = \{z|z(t) = Z(t)t^{-(\alpha+1)\bar{r}-\varepsilon}(\ell nt)^{j_0},\ Z \in C_b([\delta,\infty))\},\ \|z\|_\varepsilon = \|Z\|_{[\delta,\infty]}), \\ \delta > 1,\ 0 < \varepsilon = \min(\varepsilon_1,\varepsilon_2).\end{gathered} \tag{5.26}$$

Every fixed point z of $\psi(\cdot,\mu)$ establishes a solution for all $\xi \in C^n$. At first ψ maps $A_{\varepsilon,\delta}$ on $A_{\varepsilon,\delta}$ for δ sufficiently large because of (5.20), (5.23), (5.24), (4.55), (4.56).

Now we take a sufficiently large sphere $S_\varepsilon(\rho)$ in $A_{\varepsilon,\delta}$ with center $\phi(\cdot,\mu)(\hat{D}_0 + D_-)\xi$ and radius ρ and prove the contraction property of ψ on $S_\varepsilon(\delta)$

$$\begin{aligned}\|\psi(z_1,\mu) - \psi(z_2,\mu)\|_\varepsilon &= \|H(g(z_1,\cdot,\mu) - g(z_2,\cdot,\mu))\|_\varepsilon \leq \\ &\leq \text{const}(\mu)\ \rho\cdot\delta^{-\varepsilon}(\ell n\delta)^{2j_0}\|z_1 - z_2\|_\varepsilon\end{aligned} \tag{5.27}$$

for $z_1,z_2 \in S_\varepsilon(\rho)$ because of (5.21), (4.55) and (4.56).

Moreover if $z \in S_\varepsilon(\rho)$ then

$$\begin{aligned}\|\psi(z,\mu) - \phi(\cdot,\mu)(\hat{D}_0 + D_-)\xi\|_\varepsilon &= \|Hg(z,\cdot,\mu)\|_\varepsilon \\ &\leq (\text{const}(\mu)+\rho)^2\cdot\ \delta^{-\varepsilon}(\ell n\delta)^{2j_0}\end{aligned} \tag{5.28}$$

Therefore $\psi(z,\mu) \in S_\varepsilon(\rho)$ if δ is sufficiently large and from (5.27), (5.28) we conclude that $\psi(\cdot,\mu)$ has a unique fixed point $z \in S_\varepsilon(\rho)$ $A_{\varepsilon,\delta}$ for δ sufficiently large. The construction of H implies that $(\hat{D}_0 + D_-)P(\delta,\mu)^{-1}z(\delta) = \delta^{D(\mu)}e^{Q(\delta,\mu)}(\hat{D}_0 + D_-)\xi$. Therefore, for fixed $\mu \in \tilde{S}$, we have constructed a $\hat{r}_0 + r_-$ -dimensional solution manifold in $A_{\varepsilon,\delta}$, for δ sufficiently large but fixed whenever $((\hat{D}_0 + D_-)\xi$ varies in a compact set $K \subset C^{\hat{r}_0 + r_-}$. In order to get more information on the asymptotic behavior of the solution we now treat the important case:

(5.29) $$f(t,y_\infty(\mu)) \equiv 0 \quad \text{for} \quad t \geq \delta\,, \quad \mu \in \tilde{S}$$

and

(5.30) $$\|\phi(t,\mu)(\hat{D}_0 + D_-)\| = p(t,\mu)e^{q(t,\mu)}t^{\beta(\mu)}(\ell nt)^{j_0}, \qquad p \in C([1,\infty])$$

where

(5.31) $$q(t,\mu) \to -\infty \quad \text{for} \quad t \to \infty \quad \text{and} \quad \mu \in \tilde{S}$$

holds.

We define:

(5.32) $$\sigma(t,\mu) = e^{q(t,\mu)}t^{\beta(\mu)}(\ell nt)^{j_0}$$

and set

(5.33) $$(\bar{\psi}(z,\mu))(t) = \phi(t,\mu)(\hat{D}_0 + D_-)\zeta + (\bar{H}g(z,\cdot,\mu))(t), \quad \zeta \in C^n\,.$$

$\bar{H}$ is constructed according to Chapter 4 (with $E = I$) so that inhomogenities f which decay as $\sigma^2(t,\mu)$ produce a particular solution $\bar{H}f$ which decays as $t^{(\alpha+1)\bar{\ell}}\sigma^2(t,\mu)(\ell nt)^{\max(j_0,j_-)}$. We regard $\bar{\psi}$ as an operator on the Banach space

(5.34) $$(A_{\sigma,\delta} = \{u \mid u = \sigma(t,\mu)U,\ U \in C_b([\delta,\infty))\},\ \|u\|_\sigma = \|U\|_{[\delta,\infty]})\,.$$

The contraction mapping theorem, employed as in the case of algbraic decay, assures the existence of a unique fixed point z in $A_{\sigma,\delta}$.

From (5.33) we conclude

(5.35) $$\|z(t) - \phi(t,\mu)(\hat{D}_0 + D_-)\zeta\| \leq C(\mu)t^{(\alpha+1)\bar{\ell}}\sigma^2(t,\mu)(\ell nt)^j$$

where $j = \max(j_0,j_-)$.

It is easy to check that $\psi(\cdot,\mu)$ is also a contraction in a sphere around $\phi(\cdot,\mu)(\hat{D}_0 + D_-)\xi$ in $A_{\sigma,\delta}$. The uniqueness of the fixed point assures that

$$(5.36) \qquad \bar{H}g(z,\cdot,\mu) \in A_{\sigma,\delta}$$

for every fixed point z of $\psi(\cdot,\mu)$ in $A_{\varepsilon,\delta}$. Therefore

$$(5.37) \qquad (\bar{\psi}(z,\mu))(t) = \phi(t,\mu)(\hat{D}_0 + D_-)\xi + ((H - \bar{H})g(z,\cdot,\mu))(t) + (\bar{H}g(z,\cdot,\mu))(t)$$

holds. Because Hg and $\bar{H}g$ are particular solutions for fixed $z \in A_{\sigma,\delta}$ we get

$$(5.38) \qquad ((H - \bar{H})g(z,\cdot,\mu)(t) = \phi(t,\mu)(\hat{D}_0 + D_-)\gamma(z), \quad \gamma(z) \in C^n.$$

Choosing

$$(5.39) \qquad \zeta = \xi + \gamma(z)$$

assures that every fixed point of $\psi(z,\mu)$ is also a fixed point of $\bar{\psi}(z,\mu)$ (with ζ different to ξ) and vice versa.

In general our perturbation approach does not give us all $C([1,\infty])$ solutions of the Problem (5.1), (5.2), (5.3), it only gives us all solutions, which decay at least as fast as $t^{-(\alpha+1)\bar{r}-\varepsilon}$ where $\varepsilon > 0$. This is illustrated by the example:

$$(5.40) \qquad y' = y^2, \quad 1 < t < \infty$$

$$(5.41) \qquad y \in C([1,\infty])$$

which has the solution manifold $y = -\frac{1}{t+c}$, $c > -1$ and $y \equiv 0$. Our approach gives

$$(5.42) \qquad y_\infty = 0, \quad \frac{\partial f}{\partial y}(y_\infty) = 0, \quad \bar{r} = 1$$

and therefore the fixed point equation

$$(5.43) \qquad y = Hy^2$$

results. In $A_{\varepsilon,\delta}$ the only solution of (5.43) is $y \equiv 0$, which is the only solution of (5.40) decaying faster than t^{-1}.

If $f_y(t,y_\infty(\mu))$ is not analytical at $t = \infty$ but if it fulfills a relation of the form (4.61) then $\psi_-^0(t,\mu)(D_- + \hat{D}_0)$ resp. $\psi(g(z_1,\cdot,\mu))$ defined in Chapter 3.4 have to be used instead of $E(\mu)\phi(t,\mu)(\hat{D}_0 + D_-)$ resp. $E(\mu)\bar{H}g(z,\cdot,\mu)$. The results do not change.

The following theorem follows immediately:

<u>Theorem 5.1.</u> Let f fulfill a uniform Lipschitz condition in y on $[1,\infty]$. Then the problem (5.1), (5.2), (5.3) where (5.13) holds asymptotically has a solution $y \equiv y(\cdot,(\hat{D}_0 + D_-)\xi,\mu) \equiv E(\mu)z(\cdot,(\hat{D}_0 + D_-)\xi,\mu) + y_\infty(\mu)$ for every root $((\hat{D}_0 + D_-)\xi,\mu)$ of the equation

$$b(E(\mu)z(1,(\hat{D}_0 + D_-)\xi,\mu) + y_\infty(\mu),y_\infty(\mu)) = 0$$

where $(\xi,\mu) \in C^n \times \tilde{S}$ and $b:R^n \to R^{\hat{r}_0+r_-+n_1}$. Here $z(t,(\hat{D}_0 + D_-)\xi,\mu)$ denotes the continuated fixed points of $\psi(\cdot,\mu)$ with $\xi \in C^n$. On the other hand, if the boundary value problem (5.1), (5.2), (5.3) has a solution y, so that $y - y(\infty) \in A_{\varepsilon,1}$ for some $\varepsilon > 0$, then there is a $\mu \in R^{n_1}$ and a $(\hat{D}_0 + D_-)\xi$, $\xi \in C^n$ so that $z :=$ $E^{-1}(\mu)(y - y_\infty)$ has the asymptotic expansion (5.35).

6. Case Studies.

In this chapter problems from fluid dynamics resp. thermo dynamics described by boundary value problems on infinite intervals are analyzed. These problems are represnted by nonautonomous system of nonlinear differential equations. An autonomous problem, namely von Karman's swirling flow problem has been investigated by Markowich (1980b). The following equation represents a model for viscous flow past a solid at low Reynolds number:

(6.1) $$u'' + \frac{k}{x}u' + \alpha uu' + \beta(u')^2 = 0 , \quad \delta \leq x < \infty , \quad \alpha > 0$$

(6.2) $$u(\delta) = 0 , \quad \delta > 0 , \quad u(\infty) = U > 0 .$$

The parameters and variables are described by Cohen, Fokers and Lagerstrom (1978). The transformation

(6.3) $$y_1 = u , \quad y_2 = u' , \quad y = (y_1,y_2)^T$$

takes (6.1), (6.2) into

(6.4) $$y' = \begin{bmatrix} y_2 \\ -\frac{k}{x} y_2 - \alpha y_1 y_2 - \beta y_2^2 \end{bmatrix} = f(x,y)$$

(6.5) $$\text{(a)} \quad [1,0]y(\delta) = 0 , \qquad \text{(b)} \quad [1,0]y(\infty) = U$$

(6.6) $$y \in C([\delta,\infty]) .$$

We get $y_\infty = y_\infty(U) = \binom{U}{0}$ and calculate

(6.7) $$f_y(x,y_\infty(U)) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -\alpha U \end{bmatrix}}_{A_0(U)} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -k \end{bmatrix}}_{A_1}$$

$A_0(U)$ has the distinct eigenvalues 0 and $-\alpha U$ and we get

(6.8) $$J_0(U) = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha U \end{bmatrix} , \quad E(U) = \begin{bmatrix} 1 & -\frac{1}{\alpha U} \\ 0 & 1 \end{bmatrix}.$$

The transformation (5.15) with $\mu = U$ results in the system

$$(6.9)\qquad z' = \left(\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\alpha U \end{bmatrix}}_{J_0(U)} + \frac{1}{x}\underbrace{\begin{bmatrix} 0 & -\frac{k}{\alpha U} \\ 0 & -k \end{bmatrix}}_{J_1(U)}\right) z + g(z,U)$$

$$(6.10)\qquad z(\infty) = 0 .$$

The homogenous problem $z_h' = (J_0(U) + \frac{1}{x} J_1(U))z_h$ has a fundamental matrix $\phi(x,U)$ of the form

$$(6.11)\qquad \phi(x,U) = P(x,U) \begin{bmatrix} 1 & 0 \\ 0 & e^{-\alpha Ux} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x^{-k} \end{bmatrix}$$

where

$$(6.12)\qquad P(x,U) = I + O(x^{-1})$$

holds. This follows from Theorem 3.1 and from the algorithm for the calculation of the coefficients. Moreover

$$(6.13)\qquad \hat{D}_0 = 0 , \quad D_- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

holds and the fixed point equation

$$(6.14)\qquad z(t) = P(t,U) \begin{bmatrix} 0 \\ e^{-\alpha Ux} x^{-k} \xi \end{bmatrix} + (\bar{H}g(z,U))(t), \quad \xi \in R$$

results because we are only interested in real solutions.

$g(z,U)$ fulfills

$$(6.15)\qquad \|g(z,U)\| \le const(U)\|z\|^2$$

because $f(x,y_\infty(U)) = 0$ holds.

Chapter 5 assures the existence of solutions $z(\cdot,\xi,U) \in A_{\varepsilon,\bar{\delta}} =$ $= \{u | u(x) = x^{-1-\varepsilon} \ell n x U(x),\ U \in C_b([\bar{\delta},\infty))\}$ and from (5.35) we conclude:

$$(6.16)\qquad \left\| z(x,\xi,U) - P(x,U) \begin{bmatrix} 0 \\ e^{-\alpha Ux} x^{-k} \xi \end{bmatrix} \right\| \le C(U) x^{1-2k} e^{-2\alpha Ux} (\ell n x)^2 .$$

Using Theorem 3.4 this estimate can be improved, so that its right hand side is of the order $x^{-2k} e^{-2\alpha Ux}$. Resubstituting we get

(6.17) $$u(x,\xi,U) = U - \frac{1}{\alpha U}(1 + O(x^{-1}))e^{-\alpha Ux}x^{-k}\xi + O(e^{-2\alpha Ux}x^{-2k}), \quad x \to \infty .$$

Since $U \in R^+$ is given (6.5)(a) has to be used in order to determine $\xi \in R$ and the problem is well posed concerning the number of conditions as $x = \delta$ and $x = \infty$.

The second problem is a similarity equation for a combined forced and free convection flow over a horizontal plate (see Schneider (1979)):

(6.18) $$\left.\begin{aligned} &(a) \quad 2f''' + ff'' + kxg = 0 \\ &(b) \quad 2g' + fg = 0 \end{aligned}\right\} \quad 0 < x < \infty$$

(6.19) $$(a) \quad f(0) = f'(0) = 0 , \quad g(0) = 1 , \qquad (b) \quad f'(\infty) = 1 , \ g(\infty) = 0$$

The variables and the parameter k are explained in Schneider (1979). For simplicity we have set the Prandtl number to 1 . Because of (6.19)(b) we substitute

(6.20) $$f(x) = x + h(x)$$

and get the new problems:

(6.21) $$\left.\begin{aligned} &(a) \quad 2h''' + (x + h)h'' + kxg = 0 \\ &(b) \quad 2g' + (x + h)g = 0 \end{aligned}\right\} \quad 0 < x < \infty$$

(6.22) $$(a) \quad h(0) = 0 , \quad h'(0) = -1 , \quad g(0) = 1 , \qquad (b) \quad h'(\infty) = 0 , \quad g(\infty) = 0 .$$

Substituting

(6.23) $$y_1 = h , \quad y_2 = h' , \quad y_3 = h'' , \quad y_4 = q , \quad y = (y_1,y_2,y_3,y_4)^T$$

we get the system

$$(6.24) \qquad y' = x \begin{bmatrix} \dfrac{y_2}{x} \\[2ex] \dfrac{y_3}{x} \\[2ex] -\tfrac{1}{2}\left(1 + \dfrac{y_1}{x}\right)y_3 - \dfrac{k}{2}\, y_4 \\[2ex] -\tfrac{1}{2}\left(1 + \dfrac{y_1}{x}\right)y_4 \end{bmatrix} = xf(x,y) \quad , \quad 0 < x < \infty$$

$$(6.25)(a) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} y(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} , \quad (b) \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} y(\infty) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We only admit solutions fulfilling $h(\infty) \in R$ therefore we require that

$$(6.26) \qquad y \in C([0,\infty])$$

holds. From (6.23), (6.24) we conclude

$$(6.27) \qquad y_\infty = y_\infty(h_\infty) = (h_\infty,0,0,0) \ , \quad h_\infty = h(\infty) \in R \ .$$

We calculate:

$$(6.28) \qquad f_y(x,y_\infty(h_\infty)) = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\tfrac{1}{2} & -\dfrac{k}{2} \\ 0 & 0 & 0 & -\tfrac{1}{2} \end{bmatrix}}_{A_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\dfrac{h_\infty}{2} & 0 \\ 0 & 0 & 0 & -\dfrac{h_\infty}{2} \end{bmatrix}}_{A_1(h_\infty)}$$

and

$$(6.29) \qquad E \equiv E(h_\infty) = \mathrm{diag}(1,1,1,-\frac{2}{k}) \ .$$

The substitution $E(h_\infty)z = y - y_\infty(h_\infty)$ gives the system

$$(6.30)\qquad z' = x \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 1 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}}_{J_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{h_\infty}{2} & 0 \\ 0 & 0 & 0 & -\frac{h_\infty}{2} \end{bmatrix}}_{J_1(h_\infty) = A_1(h_\infty)} z + xg(z,h_\infty)$$

J_0 has the eigenvalues 0 with algebraic and geometric multiplicity 2 and $-1/2$ with algebraic multiplicity 2 and geometric multiplicity 1. Because $f(x,y_\infty(h_\infty)) = 0$ holds we get

$$(6.31)(a)\qquad \|g(z,h_\infty)\| \le C(h_\infty)\|z\|^2.$$

We have to set up the fundamental matrix $\phi(x,h_\infty)$ of the system

$$(6.31)(b)\qquad z_h' = x(J_0 + \frac{1}{x} J_1(h_\infty))z_h\ ,\quad z_h = (z_h^1, z_h^2, z_h^3, z_h^4).$$

Because of the simple structure of $J_0 + \frac{1}{x} J_1$ we do not have to apply the algorithm of Chapter 4 , we can proceed in the following way.

The last equation of (6.31) is

$$(6.32)\qquad z_h^{4\,'} = x(-1/2 - \frac{h_\infty}{2}\frac{1}{x})z_h^4\ .$$

It can be integrated at once giving

$$(6.33)\qquad z_h^4 = e^{-1/4 x^2 - \frac{h_\infty}{2} x} c\ ,\quad c \in R\ .$$

Setting $c = 1$ we have to find a particular solution of

$$(6.34)\qquad z_h^{3\,'} = x(-1/2 - \frac{h_\infty}{2}\frac{1}{x})z_h^3 + xz_h^4\ .$$

We take

$$(6.35)\qquad z_h^3 = e^{-1/4 x^2 - \frac{h_\infty}{2}\frac{1}{x}}\ \frac{x^2}{2}(1 - \frac{1}{x^2})\ .$$

Integrating

$$(6.36)\qquad z_h^{2\,'} = z_h^3$$

we find

$$z_h^2 = e^{-\frac{1}{4}x^2 - \frac{h_\infty}{2}x} \times O(1) \ . \tag{6.37}$$

Analogously we integrate

$$z_h^{1'} = z_h^2 \tag{6.38}$$

and get

$$z_h^1 = e^{-\frac{1}{4}x^2 - \frac{h_\infty}{2}x} O(1) \ . \tag{6.39}$$

The fourth column of $\phi(x,h_\infty)$ can be chosen as $(z_h^1, z_h^2, z_h^3, z_h^4)^T$.

In order to get the third column we set $C = 0$ in (6.33) and proceed as we did.

Finally we get the following fundamental matrix

$$\phi(x,h_\infty) = \begin{bmatrix} 1 & x & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(x^{-2}) & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(1) \\ 0 & 1 & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} O(x^{-1}) & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} \times O(1) \\ 0 & 0 & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} x^2(1 - \frac{1}{x^2}) \\ 0 & 0 & 0 & e^{-\frac{x^2}{4} - \frac{h_\infty}{2}x} \end{bmatrix} \tag{6.40}$$

which we can write as in Theorem 4.1

$$\phi(x,h_\infty) = P(x,h_\infty)x^D e^{Q(x,h_\infty)} \tag{6.41}$$

where

$$P(x,h_\infty) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{x} O(1) \tag{6.42}$$

$$D = \mathrm{diag}(1,x,1,x^2) \tag{6.43}$$

$$Q(x,h_\infty) = \mathrm{diag}\left(1,1,e^{-\frac{x^2}{4} - \frac{h_\infty}{2} x}, e^{-\frac{x^2}{4} - \frac{h_\infty}{2} x}\right) \tag{6.44}$$

holds.

Similarily to the first example we conclude that there are solutions $z(x,\xi_1,\xi_2,h_\infty)$ in $A_{\varepsilon,\delta}$, which is now the space of all functions in $C([\delta,\infty])$ which decay at least as $x^{-4-\varepsilon}\ell nx$, $\varepsilon > 0$ because $\tilde{r} = 2$ and $\alpha = 1$ holds. Moreover

$$z(x,\xi_1,\xi_2\, h_\infty) = P(x,h_\infty) \begin{bmatrix} 0 \\ 0 \\ xe^{-\frac{x^2}{4} - \frac{h_\infty}{2} x} \xi_1 \\ x^2 e^{-\frac{x^2}{4} - \frac{h_\infty}{2} x} \xi_2 \end{bmatrix} + O(x^4 e^{-\frac{x^2}{2} - h_\infty x}) \tag{6.45}$$

with h_∞, ξ_1, ξ_2 R results because the exponential factor $e^{-\frac{x^2}{2} - h_\infty x}$ does not appear in the fundamental matrix.

So we get the asymptotic expansions

$$f(x) = x + h_\infty + O(1)e^{-\frac{x^2}{4} - \frac{h_\infty}{2} x} + O(x^4 e^{-\frac{x^2}{2} - h_\infty x}) \tag{6.46}$$

$$g(x) = -\frac{2}{x}\, e^{-\frac{x^2}{4} - \frac{h_\infty}{2} x} \xi_2 + O(x^4 e^{-\frac{x^2}{2} - h_\infty x}) \tag{6.47}$$

where the $O(1)$ in (6.46) depends linearly on ξ_1 and ξ_2.

The constants $(h_\infty, \xi_1, \xi_2) \in R^3$ have to be determined from the three initial conditions (6.25)(a).

The third problem to be analyzed is the well known Falkner-Skan equation (see for example Lentini and Pereyra (1977)):

(6.48) $\quad f''' + ff'' + (1 - f'^2) = 0$

(6.49) $\quad f'(\infty) = 1.$

We do not pose any initial condition because we look for a solution manifold.

Because of (6.48) we substitute

(6.50) $\quad f(x) = x + g(x)\ ,\quad y_1 = g\ ,\quad y_2 = g'\ ,\quad y_3 = g''$

(6.51) $\quad y = (y_1, y_2, y_3)^T$

and get the system

(6.52) $$y' = x \begin{bmatrix} \dfrac{y_2}{x} \\ \dfrac{y_3}{x} \\ -y_3 - \dfrac{y_1 y_3}{x} + \dfrac{2y_2}{x} + \dfrac{y_2^2}{x} \end{bmatrix} = x\, f(x,y),\quad x \geq \delta\ .$$

Moreover we require

(6.53) $$y \in C([\delta,\infty])$$

so that

(6.54) $$y(\infty) = (g_\infty, 0, 0)^T\ ,\qquad g_\infty := g(\infty)$$

holds. We calculate

(6.55) $$f_y(x, y_\infty(g_\infty)) = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{J_0} + \frac{1}{x} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -g_\infty \end{bmatrix}}_{J_1(g_\infty)}\ .$$

The linearized system is therefore

(6.56) $$z_h' = x(J_0 + \frac{1}{x} J_1(g_\infty)) z_h$$

Because J_0 has the eigenvalue 0 with algebraic multiplicity 2 we apply the theory developed in Chapter 4.

At first we split up the system (6.56) by the transformation

$$(6.57)\qquad z_h = P(x,g_\infty)u \quad u = (u_1,u_2,u_3)^T,\ P(x,g_\infty) \sim I + \sum_{i=1}^{\infty} P_i(g_\infty)x^{-i}$$

to get subsystems whose leading matrices have the only eigenvalue 0 resp. -1. (6.57) gives a system of the form:

$$(6.58)\qquad u' = x\,B(x,g_\infty)u\ ,\quad B(x,g_\infty) \sim J_0 + \sum_{i=1}^{\infty} B_i(g_\infty)x^{-i}\ .$$

From Wasow (1965) we conclude that

$$(6.59)\qquad P_i = \begin{bmatrix} 0 & 0 & p_{i1} \\ 0 & 0 & p_{i2} \\ p_{i3} & p_{i3} & 0 \end{bmatrix}\ ,\quad B_i = \begin{bmatrix} b_{i1} & b_{i2} & 0 \\ b_{i3} & b_{i4} & 0 \\ 0 & 0 & b_{i5} \end{bmatrix}$$

holds and that the recursion

$$(6.60)\qquad J_0\,P_i - P_iJ_0 = \sum_{s=0}^{i-1} (P_s\,B_{i-s} - J_{i-s}\,P_s) - (i-2)P_{i-2}\ ,\qquad i > 0$$

with the last term absent for $i < 2$ and $J_k = 0$ for $k > 1$ holds.

From the investigation of the perturbed system we know that only the coefficients B_0, B_1, B_2, B_3 and B_4 influence the asymptotic behavior of the fundamental matrix because $(\alpha+1)\tilde{r} = 4$ for our example. (6.59) and (6.60) give

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -g_\infty \end{bmatrix}\ ,\quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\ ,$$

(6.61)

$$B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2g_\infty & 0 \\ 0 & 0 & -2g_\infty \end{bmatrix}\ ,\quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2+2g_\infty^2 & 0 \\ 0 & 0 & b_{45} \end{bmatrix}\ .$$

We do not have to know b_{45} explicitely because it does not influence the behavior of the solution of

$$(6.62)\qquad u_3' = x\left(-1 - \frac{g_\infty}{x} - \frac{2}{x^2} + \frac{2g_\infty}{x^3} + \cdots\right)u_3\ .$$

From Chapter 3 we get

$$(6.63)\quad u_3 = p_3(x,g_\infty)e^{-\frac{x^2}{2} - g_\infty x}x^{-2}\ ,\ p_3(x,g_\infty) \sim 1 + p_{31}(g_\infty)x^{-1} + p_{32}(g_\infty)x^{-2} + \cdots\ .$$

Moreover we get:

$$(6.64)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{x}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{x^2}\begin{bmatrix} 0 & 0 \\ 0 & -2g_\infty \end{bmatrix} + \frac{1}{x^3}\begin{bmatrix} 0 & 0 \\ 0 & -2+2g_\infty^2 \end{bmatrix} + \cdots\right)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

because the leading term which comes from B_0 vanishes. Therefore the coefficient of $\frac{1}{x^3}$ does not influence the behavior of u_1, u_2. It is sufficient to solve

$$(6.65)\qquad \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}' = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{x}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} + \frac{1}{x^2}\begin{bmatrix} 0 & 0 \\ 0 & -2g_\infty \end{bmatrix}\right)\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}.$$

We get directly by integration

$$(6.66)\qquad \tilde{u}_2 = x^2\exp\left(\frac{2g_\infty}{x}\right) \quad \text{and} \quad \tilde{u}_1 = x^3 O(1)\ .$$

Finally we conclude

$$(6.67)\quad f = x + g_\infty + p(x,g_\infty)e^{-\frac{x^2}{2} - g_\infty x}\xi + O\left(e^{-x^2-2g_\infty x}\right),\ p(x,g_\infty) \sim 1 + \sum_{i=1}^{\infty} p_i(g_\infty)x^{-i}$$

with $\xi \in R$ because we look for real solutions and because there is no column of the fundamental matrix of (6.56) which has the factor $e^{-x^2-2g_\infty x}$.

References

E. Coddington and N. Levinson (1955). Theory of Ordinary Differential Equations. McGraw Hill, New York.

D. S. Cohen, A. Fokas and P. A. Lagerstrom (1978). Proof of some asymptotic results for a model equation for low Reynolds number flow. SIAM J. Appl. Math. Vol. 35, No. 1. pp. 187-207

F. R. de Hoog and Richard Weiss (1980a). On the boundary value problem for systems of ordinary differential equations with a singularity of the second kind. SIAM Journal of Math. Anal. Vol. 11, No. 1, pp. 46-61.

F. R. de Hoog and R. Weiss (1980b). An approximation method for boundary value problems on infinite intervals. Computing 24, pp. 227-239.

M. Lentini and V. Pereyra (1977). An adaptive finite difference solver for nonlinear two point boundary problems with mild boundary layers. SIAM J. Numer. Math., Vol. 14, No. 1, pp. 91-112.

M. Lentini (1978). Boundary value problems over semi-finite intervals. Ph.D. Thesis, California Institute of Technology, Pasadena, CA 91125.

M. Lentini and H. B. Keller (1980). The von Karman swirling flows. SIAM J.Appl.Math., Vol. 38, No 1., pp. 52-64.

P. A. Markowich (1980a). Randwertprobleme auf unendlichen Intervallen. Dissertation, Technische Unitersitat Wien.

P. A. Markowich (1980b). Asymptotic analysis of von Karman flows, submitted.

J. B. McLeod (1969). Von Karman's Swirling Flow Problem. Arch. Rat. Mech. and Anal., Vol. 1. 33, pp. 91-102.

W. Schneider (1979). A similarity solution for combined forced and free convection flow over a horizontal plate. Int. J. Heat Mass Transfer, Vol 22, pp. 1401-1406.

W. Wasow (1965). Asymptotic Expansion for Ordinary Differential Equations. Series in Pure and Applied Mathematics, Vol XIV, Wiley & Sons, New York.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2138	2. GOVT ACCESSION NO. AD-A096655	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ANALYSIS OF BOUNDARY VALUE PROBLEMS ON INFINITE INTERVALS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Peter A. Markowich		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 MCS-7927062
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS SEE ITEM 18 below		12. REPORT DATE November 1980 13. NUMBER OF PAGES 50
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D. C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Nonlinear boundary value problems, singular points, boundedness, asymptotical properties, asymptotic expansions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper boundary value problems on infinite intervals are treated. There is a theory for problems of this kind which requires the fundamental matrix of the system of differential equations to have certain decay properties near infinity. The aim of this paper is to establish a theory which holds under weaker and more realistic assumptions. The analysis for linear problems is done by determining the fundamental matrix of the system of differential equations asymptotically. For inhomogeneous problems a suitable particular solution having a 'nice' asymptotic behaviour is chosen and so global existence and uniqueness theorems are

continued

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. (Abstract continued).

established in the linear case. The asymptotic behaviour of this solution follows immediately. Nonlinear problems are treated by using perturbation techniques meaning linearization near infinity and by using the methods for the linear case. Moreover, some practical problems from fluid dynamics and thermodynamics are dealt with and they illustrate the power of the asymptotic methods used.

DATE
ILMED
-8