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RELATIVE CHEBYSHEV CENTERS IN NORMED LINEAR SPACES. PART II.(U)
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Dan Amir and Zvi Ziegler

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RELATIVE CHEBYSHEV CENTERS IN NORMED LINEAR SPACES, PART II^{*}

Dan Amir⁽¹⁾ and Zvi Ziegler⁽²⁾

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ABSTRACT

Let E be a normed linear space, A a bounded set in E , and G in an arbitrary set in E . The relative Chebyshev center of A in G is the set of points in G best approximating A . We have obtained elsewhere general results characterizing the spaces in which the center reduces to a singleton in terms of structural properties related to uniform and strict convexity. In this paper an analysis of the Chebyshev norm case, which falls outside the scope of the previous analysis, is presented.

AMS (MOS) Subject Classification: 41A50

Key Words: Chebyshev Center, Best Simultaneous Approximation, Extended Unisolvence

Work Unit Number 3 (Numerical Analysis and Computer Science)

^{*} Work completed while the second author was visiting the Mathematics Research Center, University of Wisconsin-Madison.

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SIGNIFICANCE AND EXPLANATION

Suppose we have a series of experiments such that the results of each experiment are represented by a function. Let the set of such functions be denoted by A . It is then useful to find a function which would "best represent" (or "best approximate") the functions of the set A . This type of procedure is routinely followed, e.g., in the construction of weather maps. Several norms, i.e., measures of discrepancy, could be used, and the norm chosen depends on the problem. The set of "best" approximants is called the Chebyshev center of A . When the approximants are restricted to belong to a certain family, such as ordinary algebraic polynomials of restricted degree, trigonometric polynomials, exponential functions or a nonlinear family of rational functions, then it is called the relative Chebyshev center of A in the given family. It is especially important to know when the center consists of one function only. Questions of this type in general norms were studied by the authors in a previous publication. In the present report the frequently useful case of the Chebyshev (or uniform) norm, which falls outside of the scope of the general framework, is examined for quite general families. Necessary and sufficient conditions for the relative Chebyshev center to consist of one function are established.

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RELATIVE CHEBYSHEV CENTERS IN NORMED LINEAR SPACES, PART II^{*}

Dan Amir⁽¹⁾ and Zvi Ziegler⁽²⁾

INTRODUCTION

When E is a normed linear space and $A \subset E$ is a bounded set in E , then the Chebyshev center of A is the set in E of elements best approximating A .

When G is another set in E , we may consider the set of elements in G best approximating, from amongst all elements in G , the set A . This is called the relative Chebyshev center of A in G .

The first part of this work, to be published separately, develops the connection between structural properties of relative centers, convexity properties of the spaces, and the closeness of the resemblance of the space to a pre-Hilbert space. This extends the work of Garkavi [7], Day, James and Swaminathan [3] and of Rozema and Smith [17].

In the present paper we restrict our attention to the case where the space is $C[a,b]$ endowed with the uniform norm, i.e., we search for

$$\min(\max \|f - u\|)$$

$$u \in F \quad f \in S$$

where S is the set of functions to be approximated and F is the approximating family. This type of problem has been studied by several authors in recent years (see e.g., [4], [6], [9], [13]). Mixed norms have also been discussed.

For example, the problem of finding $\min(\|f_1 - u\|_\infty + \|f_2 - u\|_\infty)$,

involving the ℓ_1 and uniform norms, for two functions, has been investigated by Ling, McLaughlin and Smith [14]. Another, somewhat related problem, involves vectorial approximation (see e.g. [2], [8]).

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We focus our attention on the case where the approximating family is n -unisolvent. This is the natural framework for examining questions of uniqueness of best approximants. Despite the fact that the problem at hand can be reduced to the case where only the lower and upper envelopes are considered, there are inherent complications. A more manageable precursor of the type of difficulties encountered here occurs in the study of approximation of discontinuous functions (see e.g. [16]). We introduce the concept of an Extended n -unisolvent family, and then establish, through a somewhat delicate analysis of patterns of sign changes, the full characterization of the situations where the center consists of exactly one element.

1. General Characterization of the Center

This section is devoted to a brief discussion of general results concerning centers in $C[0,1]$. We recall the simple observation that in this particular norm a reduction to the case involving two functions, the upper and lower envelopes, is possible. We then present a proof of the characterization theorem for centers with respect to general n -unisolvent families, which employs ideas to be utilized in the proof of the main theorem in Section 2.

The following simple observation has been made by several authors (see e.g. [5]). When $A \subset C[0,1]$ is compact, then the functions $A_U(t) = \sup\{f(t); f \in A\}$ and $A_L(t) = \inf\{f(t); f \in A\}$ are continuous. Furthermore, when $g \in C[0,1]$ we have

$$\begin{aligned} r(g,A) &= \sup\{ \|f - g\| ; f \in A \} \\ &= \sup\{ |f(t) - g(t)| ; f \in A, t \in [0,1] \} \\ &= \sup\{ \max(A_U(t) - g(t), g(t) - A_L(t)); t \in [0,1] \} \\ &= \max(\|A_U - g\|, \|g - A_L\|) = r(g; A_U, A_L). \end{aligned}$$

Hence, the problem of relative centers of compact sets in $C[0,1]$ is reducible to a problem of relative centers for pairs of functions (f,g) , with $f \leq g$. The latter type was discussed in a general framework in Section 2 of [1]. In the subsequent analysis we restrict ourselves to unisolvent n -parameter approximating families, and for the corresponding problems will establish existence, characterization and uniqueness properties.

Let $F \subset C[0,1]$ be an n -parameter approximating family, and define the relative center of (f,g) with respect to F (in the Chebyshev sense), by

$$Z(F; f, g) = \{u^* \in F; r(u^*; f, g) = \min(r(u; f, g); u \in F)\} \quad (1.1)$$

Note that the existence of such u^* is assured by compactness; furthermore,

it is assured even for families which are dense compact on X (i.e., families G such that every bounded sequence of elements of G has a subsequence converging pointwise on a dense subset Y of X to an element g of G). This was proved by Dunham [6].

We now restrict ourselves further, to unisolvent families. We start by recalling some of the relevant definitions and properties. For details and a thorough discussion of the place such families occupy in Approximation Theory, see e.g. [15].

Definition 1.1: The n -parameter approximating family $F = \{F(\bar{a}; t); \bar{a} \in S \subset R^n\}$ of functions defined on $[0,1]$ is n -unisolvent if for any given set $\{t_i\}_{i=1}^n$ of distinct points in $[0,1]$ and any set $\{y_i\}_{i=1}^n$ of arbitrary numbers, there exists a unique $\bar{a}$ such that

$$F(\bar{a}; t_i) = y_i, \quad i=1, \dots, n. \quad (1.2)$$

Lemma 1.2 (see [15], p.72): The solution $F(\bar{a}; t)$ of (1.2) is a continuous function of the t_i 's and the y_i 's, i.e., given $\epsilon > 0$, $\bar{t}, \bar{y}$, there exists a $\delta > 0$ such that

$$\max(\|\bar{t} - \bar{t}'\|, \|\bar{y} - \bar{y}'\|) < \delta \Rightarrow \|F(\bar{a}; t) - F(\bar{a}'; t)\|_{\infty} < \epsilon \quad (1.3)$$

where $\bar{a}'$ is the solution of (1.2) for $\bar{t}', \bar{y}'$.

Applying the standard limit argument used for T-systems, we deduce

Corollary 1.3: If F is n -unisolvent and $\bar{a} \neq \bar{b}$, then $F(\bar{a}; t) - F(\bar{b}; t)$ has at most $n-1$ zeros in $[0,1]$. Here non-nodal zeros are counted twice (an interior point t_0 is a non-nodal zero of f if $f(t_0) = 0$ and f does not change sign at t_0).

We conclude that for a fixed $\bar{t}$, the mapping $\bar{y} \rightarrow F(\bar{a}; \cdot)$ is a homeomorphism of S onto F . Hence, each bounded compact set in $C[0,1]$ has a relative

Chebyshev center in F . We recall, furthermore, that analogues of the classical results for Chebyshev sets are valid for general n -unisolvent families, to-wit,

Lemma 1.4 ([15], p.93): Let F be n -unisolvent on $[0,1]$ and let $f \in C[0,1]$. Then f possesses a unique best Chebyshev approximation characterized by the existence of an $n+1$ -point alternance.

Coming back to the problem at hand, we introduce now some additional notations and definitions, tailored for our needs.

Definition 1.5: The set $(t_1, \dots, t_k)$ is called a k -point alternance for the approximation by u to f and g (abbreviated as the $(u;f,g)$ -approximation) if either

$$\left\{ \begin{array}{l} f(t_i) - u(t_i) = u(t_{i+1}) - g(t_{i+1}) = r(u;f,g), \quad i=1, \dots, k-1, \\ \text{or} \\ u(t_i) - g(t_i) = f(t_{i+1}) - u(t_{i+1}) = r(u,f,g), \quad i=1, \dots, k-1. \end{array} \right. \quad (1.4)$$

A point t_0 such that $f(t_0) - u(t_0) = r(u;f,g)$ is called a $(+)$ -point, while a point t_0 such that $u(t_0) - g(t_0) = r(u;f,g)$ is called a $(-)$ -point. Both kinds are called (e) -points. Following Dunham [5], we introduce the following definition:

Definition 1.6: The point t_0 is called a straddle point with respect to the $(u;f,g)$ -approximation if it is both a $(+)$ and a $(-)$ -point, i.e., if

$$f(t_0) - u(t_0) = u(t_0) - g(t_0) = r(u;f,g). \quad (1.5)$$

We are now ready to state the first theorem for relative centers of (f,g) with respect to unisolvent families. The theorem is due to Dunham [5]. We present here our own proof, which is different from Dunham's, since our methods

will be used subsequently to obtain further results.

Theorem 1.7 [5]: Let $f, g \in C[0,1]$, with $f \geq g$, and let F be an n -unisolvent family in $C[0,1]$. Then $u^* \in Z(F; f, g)$ if, and only if, either: a) $(u^*; f, g)$ has a straddle point, or b) $(u^*; f, g)$ has an $n+1$ point alternance. In the latter case, $Z(F; f, g) = \{u^*\}$.

Proof: 1) Sufficiency: If $(u^*; f, g)$ has the straddle point t_0 , then for each u

$$r(u; f, g) \geq \frac{1}{2}(f-g)(t_0) = r(u^*; f, g),$$

completing the proof in this case. Suppose next that $(u^*; f, g)$ has the $n+1$ -point alternance $t_0 \dots t_n$, and assume for definiteness that

$$f(t_i) - u^*(t_i) = u^*(t_{i+1}) - g(t_{i+1}) = r(u^*; f, g), \quad i=0, 1, \dots, n. \quad (1.6)$$

For each $u \in Z(F; f, g)$ we must have

$$\max\{f(t_i) - u(t_i), u(t_i) - g(t_i)\} \leq r(u^*; f, g), \quad i=0, 1, \dots, n.$$

Combining with (1.6), this yields

$$(-1)^{i+1}[u(t_i) - u^*(t_i)] \geq 0, \quad i=0, 1, \dots, n.$$

Thus, $u^* - u$ has at least n zeros (where multiplicities are counted as in Corollary 1.3). This is possible, in view of Corollary 1.3, only if $u^* = u$. Hence, u^* is the only element of the center.

Necessity: Assume that u^* has no straddle points and that it has only $k+1$ points of alternance, $0 \leq k < n$. Since $(u^*; f, g)$ has no straddle points, we have

$$2r(u^*; f, g) - \|f - g\| = 5\delta > 0. \quad (1.7)$$

With no loss of generality we may assume that the first (e) -point t_0 is a $(+)$ -point. We then sequentially define

$$\begin{aligned}
t_0 &= \min\{t; t \text{ is a } (+)\text{-point}\} \\
\underline{t}_0 &= \min\{t; f(t) - u^*(t) \geq r(u^*; f, g) - 2\delta\} \\
t_1 &= \min\{t; t \text{ is a } (-)\text{-point}\} \\
\bar{t}_0 &= \max\{t; t < t_1, f(t) - u^*(t) \geq r(u^*; f, g) - 2\delta\} \\
\underline{t}_1 &= \min\{t; t > \bar{t}_0, u^*(t) - g(t) \geq r(u^*; f, g) - 2\delta\} \\
t_2 &= \min\{t; t > t_1, t \text{ is a } (+)\text{-point}\}
\end{aligned}$$

etc. Let now $A_i = [\underline{t}_i, \bar{t}_i]$, $i=0, \dots, k$. There are $k+1$ such intervals since each interval contains precisely one of the alternance points. Observe that the A_i 's are disjoint closed intervals satisfying $A_0 < A_1 < \dots < A_k$.

Furthermore, all $(+)\text{-points}$ are in $\cup A_{2i}$, while all $(-)\text{-points}$ are in $\cup A_{2i+1}$. Note finally that

$$\begin{aligned}
\max(|f(t) - u^*(t)|, |u^*(t) - g(t)|) &< r(u^*; f, g) - 2\delta, \text{ for all} \\
t \in [0, 1] \setminus \bigcup_{i=0}^k A_i
\end{aligned} \tag{1.8}$$

Choose now a sequence of points $\tilde{t}_1 < \tilde{t}_2 < \dots < \tilde{t}_{n-1}$ satisfying the conditions

- a) $\tilde{t}_i \in (\underline{t}_{i-1}, \bar{t}_i)$, $1 \leq i \leq k$
- b) if $n \not\equiv k \pmod{2}$, then

$$\tilde{t}_i \in (\underline{t}_{k-1}, \bar{t}_k), \quad k+1 \leq i \leq n-1. \tag{1.9}$$

If $n \equiv k \pmod{2}$, then (1.9) is required to hold for $i \leq n-2$,

$$\text{and } \tilde{t}_{n-1} = 1.$$

We discuss first the case $n \not\equiv k$. We adjoin a point $\tilde{t}_0$ in A_0 , and construct a function $u \in F$ satisfying

$$\begin{cases} u(\tilde{t}_0) = u^*(\tilde{t}_0) + \eta, \\ u(\tilde{t}_i) = u^*(\tilde{t}_i), \quad i=1, \dots, n-1 \end{cases}$$

where η is chosen so small that $\|u-u^*\| < \delta$. This is possible in view of the continuity properties expressed in Lemma 1.2.

Note that $u-u^*$ cannot vanish at any point other than the $\tilde{t}_i$'s in view of the unisolvence. Hence, $u(t) > u^*(t)$ on A_0 , and in view of the way the $\tilde{t}_i$'s are placed, we have

$$(-1)^i [u(t) - u^*(t)] > 0, \text{ if } t \in A_i, \quad i=0,1,\dots,k.$$

Thus, we obtain

$$\begin{cases} 0 < r(u^*;f,g) - \delta < f(t) - u(t) < f(t) - u^*(t), & t \in A_{2i} \\ & i=0,1,\dots,k-1. \\ 0 < r(u^*;f,g) - \delta < u(t) - g(t) < u^*(t) - g(t), & t \in A_{2i+1} \end{cases}$$

On the complement of $\bigcup_{i=0}^k A_i$, we clearly have

$$\max(|f-u|, |u-g|) < r(u^*;f,g).$$

Combining these inequalities, we conclude that $r(u;f,g) < r(u^*;f,g)$, i.e., that u^* is not in the center.

The second case is similarly handled.

q.e.d.

Corollary 1.8: The point $\tilde{t}$ is a straddle point of some triplet $(u;f,g)$, if and only if

$$f(\tilde{t}) - g(\tilde{t}) = 2r(F;f,g) \quad (1.10)$$

Thus, if $\tilde{t}$ is a straddle point for one triplet, it is a straddle point for all triplets, and $u^*(\tilde{t}) = [f(\tilde{t}) + g(\tilde{t})]/2$ for all $u^* \in Z(F;f,g)$.

Proof: Suppose first that (1.10) is satisfied, and let $u^* \in Z(F;f,g)$. Then

$$\max[f(\tilde{t}) - u^*(\tilde{t}), u^*(\tilde{t}) - g(\tilde{t})] \leq r(u^*;f,g) = r(F;f,g)$$

Combining this with (1.10), it follows that

$$f(\tilde{t}) - u(\tilde{t}) = u(\tilde{t}) - g(\tilde{t}) = r(u^*;f,g) = r(F;f,g), \quad (1.11)$$

so that $\tilde{t}$ is a straddle point of $(u^*;f,g)$. Conversely, if $\tilde{t}$ is a straddle point of $(u;f,g)$, then by the previous theorem, $u^* \in Z(F;f,g)$, and using (1.11) we have (1.10).

The last observation in the corollary is a consequence of (1.11).

2. Uniqueness

We examine in this section the conditions under which the center will reduce to a singleton. It will be shown that the situation here is more interesting than the corresponding one in the approximation of one function, and an analogue does not exist. An intermediate situation, where some of the difficulties are beginning to show, occurs in the study of the approximation of discontinuous functions (see e.g. [16]).

The first result we have in this direction is a simple consequence of the definition of n -unisolvence and Corollary 1.8.

Lemma 2.1: Let F be an n -unisolvent family and let $f, g, f \succ g$ be two continuous functions. If there exist n straddle points (i.e., points satisfying (1.10)) then $Z(F; f, g)$ is a singleton.

The complete analysis of the conditions under which $Z(F; f, g)$ is a singleton requires more than standard perturbation methods, due to special phenomena which do not have a counterpart in the theory of approximation of one function. For example, it will follow from the subsequent discussion that if f, g are continuously differentiable and $F = [1, x]$, then the existence of one interior straddle point suffices to ensure that $Z(F; f, g)$ is a singleton.

We consider first the simplest case, where all the functions under consideration are n -times differentiable.

Definition 2.2: The n -parameter family F of n -times differentiable functions will be called an extended n -unisolvent family if for any prescribed set of "Hermite-data", i.e., data of the form

$$u^{(j)}(t_i) = \alpha_i^j, \quad i=1, \dots, m; \quad j=0, 1, \dots, k_i-1; \quad \sum_{i=1}^m k_i = n \quad (2.1)$$

there exists a unique $u \in F$ satisfying (2.1).

This generalizes, to unisolvent families, the concept of an Extended Tchebycheff system, which proved useful in the study of Tchebycheff systems (see [12]). Naturally, each Extended Tchebycheff system is an extended n -unisolvent family.

Remark: The natural analogue of Lemma 1.2 is valid for extended n -unisolvent families.

Let now $F \subset C^{(n)}[0,1]$, $f \geq g$, $f, g \in C^{(n)}[0,1]$; let t_0 be an interior straddle point, and let $u^* \in Z(F; f, g)$. Then

$$f'(t_0) = u^{*'}(t_0) = g'(t_0). \quad (2.2)$$

Indeed, assuming that $f'(t_0) > u^{*'}(t_0)$, we observe that in a small right neighborhood of t_0 , the inequality $f(t) - u^*(t) > f(t_0) - u^*(t_0)$ is valid in contradiction to the assumption that

$$f(t_0) - u^*(t_0) = r(u^*; f, g).$$

A similar analysis, involving the left neighborhood, obtains if $f'(t_0) < u^{*'}(t_0)$. Hence, $f'(t_0) = u^{*'}(t_0)$. The r.h.s. equality is similarly derived. Note that this type of result does not extend to second order derivatives, where only the weak inequalities $f''(t_0) \leq u^{*''}(t_0) \leq g''(t_0)$, have to hold. If $f''(t_0) = g''(t_0)$, then the chain collapses, and $u^{*''}(t_0)$ has to take the common value. A similar situation, where only weak inequalities are assured, exists at the end points, starting with the first derivative.

This discussion motivates the following definition.

Definition 2.3: Let F be an extended n -unisolvent family, and let $f \geq g$. The straddle point t_0 has the deficiency index k , $k \leq n$, with respect to $(F; f, g)$ if k Hermite-type conditions at t_0 are imposed on the elements of the center by the condition that t_0 is a straddle point.

The following observations can be easily deduced from the analysis preceding Def. 2.3.

Observation 1: The smoothness of f and g is not an integral part of the definition. If f and g are only continuous then a straddle point may have deficiency index 1. In the case of differentiable functions, the deficiency index of an interior point is at least 2. The subsequent analysis can be carried out, with technical modifications involving one-sided Dini derivatives, for non-smooth functions.

Observation 2: Let $f, g \in C^{(n)}(I)$. Let t be a straddle point in $\text{Int}(I)$, and let k be its deficiency index. If $k < n$, then it must be even.

Observation 3: Let $f, g \in C^{(n)}(I)$. The point t is a straddle point with deficiency k if and only if k is the largest integer such that

$$\begin{cases} f(t) = g(t) = \|f-g\| = 2r(F; f, g), \\ f^{(j)}(t) = g^{(j)}(t), \quad j=1, \dots, k-1. \end{cases} \quad (2.3)$$

We recall now some notation concerning sign changes of real valued sequences and functions (cf. [12], where the notation is extensively utilized.)

Notation: 1. Let $\bar{x} = (x_1, \dots, x_N)$ be a finite sequence of real numbers. Then $S^+(\bar{x})$ denotes the maximal number of sign changes of the sequence where the zeros (if they appear) are assigned arbitrary signs.

For example $S^+[(1, 0, 0, 1)] = 2$, $S^+[(1, 0, 0, 0)] = 3$.

2. Let α be a real valued function defined on a subset A of the real line. Then

$$S^+(\alpha) = \sup\{S^+[\alpha(t_1), \dots, \alpha(t_N)]\} \quad (2.4)$$

where the supremum is taken over all N and over all choices of ordered N -tuples from A .

Let $\tilde{u} \in Z(F; f, g)$ and let $\tilde{y}$ be a straddle point of deficiency k . We now proceed to define the concepts of a $\tilde{u}$ -deficiency and a $\tilde{u}$ -induced boundary straddle point.

We start by noting that

$$f^{(k)}(\tilde{y}) \leq \tilde{u}^{(k)}(\tilde{y}) \leq g^{(k)}(\tilde{y}) \quad (2.5)$$

Clearly, $f^{(k)}(\tilde{y}) < g^{(k)}(\tilde{y})$, since otherwise the deficiency is greater than k . Assume that $\tilde{u}^{(k)}(\tilde{y}) = g^{(k)}(\tilde{y})$. There are two possibilities: a) $\tilde{y}$ is not a cluster point of $(+)$ -points. b) $\tilde{y}$ is a cluster point of $(+)$ -points. We will show that in case b)

$$u^{(k)}(\tilde{y}) = \tilde{u}^{(k)}(\tilde{y}) = g^{(k)}(\tilde{y}) \quad (2.6)$$

for all $u \in Z(F; f, g)$. Indeed, since $\tilde{y}$ is a straddle point of deficiency k , we must have

$$u^{(k-1)}(\tilde{y}) = g^{(k-1)}(\tilde{y}) = f^{(k-1)}(\tilde{y})$$

for all $u \in Z(F; f, g)$. If $u^{(k)}(\tilde{y}) < \tilde{u}^{(k)}(\tilde{y})$, then, in a sufficiently small neighborhood of $\tilde{y}$, $u < \tilde{u}$. Let t^* be a $(+)$ -point of $\tilde{u}$, lying in this neighborhood. Then we have the following chain of inequalities

$$r(u; f, g) \geq \|f - u\| \geq (f - u)(t^*) > (f - \tilde{u})(t^*) = \|f - \tilde{u}\| = r(F; f, g)$$

contradicting the assumption that $\tilde{u} \in Z(F; f, g)$. Hence (2.6) must hold. The same argument can be extended to higher order derivatives, if $\tilde{u}$ has a higher degree of coincidence with g . We are thus led to the following definitions.

Definition 2.4: Let $\tilde{u} \in Z(F; f, g)$ and let $\tilde{y}$ be a straddle point of deficiency k . If $\tilde{u}^{(k)}(\tilde{y}) = g^{(k)}(\tilde{y})$ and $\tilde{y}$ is not a cluster point of $(+)$ -points, then $\tilde{y}$ is called a $(-)$ -boundary straddle point. If $\tilde{u}^{(k)}(\tilde{y}) = f^{(k)}(\tilde{y})$ and $\tilde{y}$ is not a

cluster point of (-)-points, then $\tilde{y}$ is called a (+)-boundary straddle point.

Definition 2.5: Let $\tilde{u} \in Z(F; f, g)$ and let $\tilde{y}$ be a straddle point of deficiency k . If $\tilde{y}$ is a cluster point of (+)-points and m is the largest integer $0 \leq m \leq n-k$ such that

$$\tilde{u}^{(j)}(\tilde{y}) = g^{(j)}(\tilde{y}), \quad j=0,1,\dots,k+m-1, \quad (2.7)$$

then m is the $\tilde{u}$ -induced g -associated deficiency of $\tilde{y}$. If $\tilde{y}$ is a cluster point of (-)-points and m is the largest integer $0 \leq m \leq n-k$ such that

$$\tilde{u}^{(j)}(\tilde{y}) = f^{(j)}(\tilde{y}), \quad j=0,1,\dots,k+m-1, \quad (2.7')$$

then m is the $\tilde{u}$ -induced f -associated deficiency of $\tilde{y}$. The number $m+k=h$ is called, in both cases, the total deficiency of $\tilde{y}$.

Remark: Note that the total deficiency of $\tilde{y}$ is independent of $\tilde{u}$, since by the previous analysis, if $\tilde{u}$ satisfies (2.7'), then

$$u^{(j)}(\tilde{y}) = g^{(j)}(\tilde{y}), \quad j=0,1,\dots,h-1 \quad (2.8)$$

for all $u \in Z(F; f, g)$, while if $\tilde{u}$ satisfies (2.7') then

$$u^{(j)}(\tilde{y}) = f^{(j)}(\tilde{y}), \quad j=0,1,\dots,h-1 \quad (2.8')$$

for all $u \in Z(F; f, g)$.

We return now to the characterization problem, and recall that unicity has been established for the case where there are n straddle points. Hence, we may assume in the subsequent discussion that the number of straddle points is smaller than n .

Let $u \in Z(F; f, g)$ and let E_u be the set of its (e)-points. Define a mapping x on E_u as follows:

$$x(t) = \begin{cases} t + H_t + l_t, & \text{if } t \text{ is not a straddle point} \\ \bigcup_{j=0}^{h_t-1} \{t + H_t + l_t + j\}, & \text{if } t \text{ is a non-boundary straddle point} \\ \bigcup_{j=0}^{h_t} \{t + H_t + l_t + j\}, & \text{if } t \text{ is a boundary straddle point} \end{cases} \quad (2.9)$$

Here h_t is the total deficiency of t , $H_t = \sum_{s < t} h_s$, and l_t is the number of boundary straddle points that are smaller than t .

Define a function α on $x(E_u)$ as follows:

$$\alpha(s) = \begin{cases} +1 & \text{if } x^{-1}\{s\} \text{ is a (+)-point} \\ -1 & \text{if } x^{-1}\{s\} \text{ is a (-)-point} \\ +1 & \text{if } s = t + H_t + l_t + h_t, \text{ where } t \neq 1 \text{ is a (+)-boundary straddle point,} \\ & \text{or if } s = 1 + H_1 + l_1 \text{ when } t = 1 \text{ is a (+)-boundary straddle point.} \\ -1 & \text{if } s = t + H_t + l_t + h_t, \text{ where } t \neq 1 \text{ is a (-)-boundary straddle point,} \\ & \text{or if } s = 1 + H_1 + l_1, \text{ when } t = 1 \text{ is a (-)-boundary straddle point.} \\ 0 & \text{if } x^{-1}\{s\} \text{ is a straddle point which is not a boundary straddle point.} \end{cases}$$

We are ready to fully characterize the case of uniqueness.

Theorem 2.6: Let $f, g \in C^{(n)}(I)$, $f \geq g$, and let F be an extended n -unisolvent family. Then the set $Z(F; f, g)$ is a singleton if and only if either:

$$a) \quad \sum_{i=1}^r h_i \geq n \quad (2.11)$$

where $h_1, \dots, h_r$ are the total deficiencies of the straddle points.

or,

$$b) \quad \text{There exists a function } u^* \in Z(F; f, g) \text{ such that } S^+(\alpha) \geq n \quad (2.12)$$

where α is the function corresponding to u^* , defined in (2.10).

Remark 1: Note that the theorem implies that if there are no straddle points, the function u^* is the only element of $Z(F;f,g)$ if and only if there exists an $(n+1)$ -alternance.

Remark 2: The proof carries over, mutatis mutandis, for the case where f, g are non-smooth. The technical modifications involve the use of one-sided Dini derivatives.

Proof: Sufficiency: Assume first that (2.11) holds. Then we have n Hermite type conditions that u^* must satisfy in order to be in $Z(F;f,g)$. Since F is an extended n -unisolvent system, we conclude that these conditions determine u^* uniquely.

Assume next that there exists a function $u^* \in Z(F;f,g)$ such that $S^+(\alpha) \geq n$. Let $x_1, \dots, x_{n+1}$ be a sequence of points of $x(E_{u^*})$ for which $S^+[(\alpha(x_1), \dots, \alpha(x_{n+1}))] = n$.

Let u be any other function in $Z(F;f,g)$, and consider the difference $v = u - u^*$. We will prove that $v \equiv 0$. Observe that although v is not a function of F , it has to vanish identically if it has n zeros (counting multiplicities). Indeed, if v has n zeros then u^* and u satisfy the same n Hermite data, and therefore must coincide since they belong to an extended n -unisolvent family.

Consider the ordered sequence $x_1, \dots, x_{n+1}$. If x_i is the image of a $(+)$ -point then $v[x_i^{-1}(x_i)] \geq 0$. Similarly, if x_i is the image of a $(-)$ -point, then $v[x_i^{-1}(x_i)] \leq 0$. Note that if $\tilde{t}$ is a nonboundary straddle point with total deficiency h then there are at most h points in the x_i -sequence whose pre-image is $\tilde{t}$, and that

$$v^{(j)}(\tilde{t}) = 0, \quad j=0,1,\dots,h-1. \quad (2.13)$$

If $\tilde{t}$ is a boundary straddle point, then it has a most $h+1$ image points in the x_i -sequence, and we have $v^{(h)}(\tilde{t}) \geq 0$ for a (+)-boundary point, $v^{(h)}(\tilde{t}) \leq 0$ for a (-)-boundary point.

We now construct the vector $(t_1, \dots, t_{n+1})$ as follows: $\{t_i\}_1^{n+1}$ is a weakly ordered sequence composed of pre-images of the x_i 's, according to the following rules: (1) pre-images of the (e)-points which are not straddle points are in $(t_1, \dots, t_{n+1})$. (2) Let $\tilde{t}$ be a straddle point of total deficiency h , which either is not a boundary straddle point, or is such that $v^{(h)}(\tilde{t}) = 0$. If, in the x_i -sequence, there are j points whose pre-image is $\tilde{t}$, then $\tilde{t}$ will appear in the t_i -sequence j times. (3) Let $\tilde{t}$ be a boundary straddle point of total deficiency h , such that $v^{(h)}(\tilde{t}) \neq 0$. If there are $j \leq h$ points whose pre-image is $\tilde{t}$, then $\tilde{t}$ will appear j times in the t_i -sequence.

If, however, there are $h+1$ points in the x_i -sequence whose pre-image is $\tilde{t}$, the point $\tilde{t}$ will appear only h times, and an additional point t' near $\tilde{t}$ will be chosen. If $\tilde{t} = 1$, then $t' < \tilde{t}$, whereas if $\tilde{t} \neq 1$, $\tilde{t} < t'$. We observe that if t' is sufficiently near $\tilde{t}$, the sign of $v(t')$ is positive if $\tilde{t}$ is a (+)-boundary point, and is negative if $\tilde{t}$ is a (-)-boundary point.

The conformity of signs between the $v(t_i)$'s and the $\alpha(x_i)$'s implies now that

$$s^+[v(t_1), \dots, v(t_{n+1})] = n \quad (2.14)$$

Let $v(t_p)$ be the first non-zero entry in this sequence. If such an entry does not exist, then $v(t)$ has more than n zeros (counting multiplicities), and the proof is complete. Thus, v has $p-1$ zeros, $\{t_i\}_1^{p-1}$, in $[t_1, t_p]$. Let next $v(t_q)$, $q \geq p+1$ be the last entry in the chain of non-zero entries following $v(t_p)$. By (2.14), the values $v(t_p), \dots, v(t_q)$ alternate in sign, so that continuity implies the existence of $q-p$ zeros in (t_p, t_q) . We have therefore $q-1$ zeros in $[t_1, t_q]$. If $q=n+1$, the proof is finished. If not, $v(t_{q+1})=0$, and we have to examine two possibilities:

(i) $v(t_i)=0$, $i \geq q+1$. In this case we are assured of $q-1+(n+1)-q = n$ zeros and the proof is finished.

(ii) There exists a first non-zero entry $v(t_r)$, $r > q+1$. It will suffice to show that in (t_q, t_r) there exist $r-q$ zeros, so that in $[t_1, t_r)$ there are $r-1$ zeros. The rest of the proof then follows by repeating (a finite number of times) the steps outlined above.

If $r-q$ is odd then the signs of $v(t_r)$ and $v(t_q)$ are different by (2.14). On the other hand, the number $r-q-1$ of zeros in (t_q, t_r) following from the definition of t_r is even. Thus, there has to be another point of sign change, or a higher multiplicity of one of the zeros. In either case, there will be $r-q$ zeros in (t_q, t_r) , concluding the proof.

Necessity: We assume that (2.11) does not hold, and that there exists a function $u_0 \in Z(F; f, g)$ such that $S^+(u) = p < n$. Note that, in view of Theorem 1.7, this implies the existence of straddle points, and $\|f-g\| = 2r(F; f, g)$. We now proceed to exhibit another function $u_1, u_1 \neq u_0$, in $Z(F; f, g)$. The method of proof bears some resemblance to that used in the proof of Theorem 1.7, with appropriate modifications necessitated by the existence of straddle points. We start with the case where no straddle point is a cluster point of (e)-points. Let $y_1, \dots, y_r, 1 \leq r$, be the straddle points, and let their deficiencies be $k_1, \dots, k_r$, with

$$k = \sum_{i=1}^r k_i < n \quad (2.15)$$

Since no y_i is a cluster point, it follows that the total deficiencies in this case are equal to the ordinary deficiencies. For each i , let $\epsilon(y_i)$ be chosen sufficiently small, so that

$$(y_i - \epsilon(y_i), y_i + \epsilon(y_i))$$

does not contain any (e)-points except y_i . Let

$$\varepsilon = \min_i \varepsilon(y_i), \text{ and } v_i = (y_i - \varepsilon, y_i + \varepsilon), i=1, \dots, n.$$

Let $I_0 = [0,1] \setminus \bigcup_{i=1}^r v_i$, and observe that, since I_0 is a closed set containing no straddle points, we have

$$2r(u_0; f, g) - \max(|f(t) - g(t)|; t \in I_0) = 5\delta > 0 \quad (2.16)$$

Let (y_i, y_{i+1}) , $1 \leq i \leq r-1$, be an interval between straddle points containing "signed" (e)-points. If $y_1 > 0$ or $y_r < 1$ a similar analysis can be carried out for $[0, y_1)$ or $(y_r, 1]$, respectively.

Assume, for concreteness, that (y_1, y_r) contains a (+)-point; then it is in I_0 , by the construction of the v_i 's, and we may assume that the leftmost (e)-point in $(y_1, y_r) \cap I_0$ is a (+)-point, which we denote by $t_{i,1}$. Note that

$$t_{i,1} = \min\{t; t \in (y_i, y_{i+1}), t \text{ is a (+)-point}\}.$$

Define $\underline{t}_{i,1} = \min\{t; t \in (y_i, y_{i+1}) \cap I_0, f(t) - u_0(t) \geq r(u_0; f, g) - 2\delta\}$.

Consider now two possibilities: 1) There exist no (-)-points in (y_i, y_{i+1}) .

Then define

$$\bar{t}_{i,1} = \max\{t; t \in (y_i, y_{i+1}) \cap I_0, f(t) - u_0(t) \geq r(u_0; f, g) - 2\delta\}$$

2) There exist (-)-points in (y_i, y_{i+1}) . Define

$$t'_i = \min\{t; t \in (y_i, y_{i+1}) \cap I_0, t \text{ is a (-)-point}\}$$

$$\bar{t}_{i,1} = \max\{t; t \in (y_i, y_{i+1}) \cap I_0, t < t'_i; f(t) - u_0(t) \geq r(u_0; f, g) - 2\delta\}$$

$$\underline{t}'_i = \min\{t; t \in (y_i, y_{i+1}) \cap I_0, t > \bar{t}_{i,1}; u_0(t) - g(t) \geq r(u_0; f, g) - 2\delta\}$$

Note that by (2.16), $\underline{t}'_i > \bar{t}_{i,1}$. We may now continue this process, depending on the existence of (+) points to the right of t'_i . If there are none, the process is ended by defining

$$\bar{t}_i' = \max\{t; t \in (y_i, y_{i+1}) \cap I_0, u_0(t) - g(t) \geq r(u_0; f, g) - 2\delta\}.$$

Otherwise, we define

$$t_{i,2} = \min\{t; t \in (y_i, y_{i+1}) \cap I_0, t_i' < t, t - a(+)\text{-point}\}$$

and continue along the same lines. Note that in view of the finiteness of $S^+(\alpha)$ (we have $S^+(\alpha) < n$, in fact), the process has a finite number of steps.

We apply this procedure for all intervals containing (e)-points.

We have thus constructed a set of intervals

$$\{A_j\}_{j=1}^S, \quad \bigcup_{i=1}^S A_i \subset I_0,$$

with the following properties:

- a) Each interval contains an (e)-point. All (e)-points are contained in the union of these intervals.
- b) If A_j contains a (+)-point, then

$$f(t) - u_0(t) \geq r(u_0; f, g) - 2\delta, \text{ for all } t \in A_j. \quad (2.17)$$

We call this A_j a (+)-interval. If A_j contains a (-)-point, then

$$u_0(t) - g(t) \geq r(u_0; f, g) - 2\delta, \text{ for all } t \in A_j. \quad (2.18)$$

This A_j will be called a (-)-interval.

- c) If $\{A_j, \dots, A_{j+l}\}$ are in the same interval (y_i, y_{i+1}) , then their signs alternate, and there exists an interval of positive length between adjacent A_i 's. Choose an ordered sequence in E_{u_0} consisting of one (e)-point from each A_i , and the straddle points. Apply the mapping $x(t)$ to the sequence and construct the vector $\{\alpha(s_i)\}_{i=1}^N$. Here $x(t)$ and $\alpha(s)$ are as defined in (2.9) and (2.10). Note that $S^+[(\alpha(s_1), \dots, \alpha(s_N))] = p < n$.

We will show that there exists a function $u_1, u_1 \neq u_0$, in $Z(F; f, g)$. We start by noting that u_1 has to satisfy the p conditions implied by the fact that $y_1, \dots, y_r$ are straddle points, with corresponding multiplicities $k_1, \dots, k_r$, viz.

$$u_1^{(j)}(y_i) = u_0^{(j)}(y_i), \quad i=1, \dots, r; \quad j=0, \dots, k_i-1 \quad (2.19)$$

Consider next a sequence of consecutive zeros in $\{\alpha(s_i)\}_{i=1}^N$. Suppose there are l zeros. These may correspond to the deficiency of one straddle point, or to the combined deficiencies of several consecutive straddle points, where no intervening $(+)$ or $(-)$ -points exist. There are two possibilities: 1) The l zeros are an initial or a final segment of the vector $\{\alpha(s_i)\}_{i=1}^N$. In this case we do not impose additional conditions on u at the corresponding straddle points. 2) On both sides of the segment of zeros, there exist non-zero terms. Let the adjacent sign from the left (right) be denoted by $(\text{sgn})_L$ [$(\text{sgn})_R$, respectively]. If $(-1)^l (\text{sgn})_L (\text{sgn})_R = 1$, then no additional requirements are imposed on u_1 at the corresponding straddle points. If, however, $(-1)^l (\text{sgn})_L (\text{sgn})_R = -1$, then we require

$$u_1^{(k_{i^*})}(y_{i^*}) = u_0^{(k_{i^*})}(y_{i^*}) \quad (2.20)$$

where y_{i^*} is the first straddle point corresponding to the block of l zeros.

Consider finally two adjacent non zero terms. If the signs are identical (this may happen only if at least one of the signs stems from a "signed"-boundary straddle point) then no additional requirements are imposed on u_1 . Suppose the terms are of opposite signs. This can happen when both correspond to (e) -points chosen from adjacent A_i 's, say A_m, A_{m+1} , or when at least one of the points is a "signed"-boundary straddle point. In the first case, we choose a point t^* in $(\max A_m, \min A_{m+1})$ and require

$$u_1(t^*) = u_0(t^*) \quad (2.21)$$

In the second case, assume that the first of the two terms corresponds to a straddle point $\tilde{y}$. We then require

$$u_1(\tilde{y} + \frac{3}{4}\epsilon) = u_0(\tilde{y} + \frac{3}{4}\epsilon). \quad (2.22)$$

Observe that the total number of zeros prescribed for $u_1 - u_0$ by the conditions of the form (2.19) - (2.22) is equal to $S^+(\alpha) = p$. Indeed, consider the case where $(-1)^{\ell}(\text{sgn})_L(\text{sgn})_R = -1$. The contribution of the sequence of ℓ zeros to $S^+(\alpha)$ is then $\ell+1$, and we have, in (2.20), adjoined one zero to the ℓ zeros prescribed by (2.19). The other cases are even simpler.

We now impose $n-p-1$ additional conditions of coincidence at 0,

$$u_1^{(\mu+j)}(0) = u_0^{(\mu+j)}(0), \quad j=0,1,\dots,n-p-2 \quad (2.23)$$

where μ is the smallest derivative at 0 not previously prescribed.

Finally, if there exist "signed" (e)-points or "signed" boundary straddle points, then we choose one such point $\tilde{t}$, and impose an n -th condition of the form

$$u_1^{(v)}(\tilde{t}) = u_0^{(v)}(\tilde{t}) + \eta \quad (2.24)$$

where v is the smallest derivative at $\tilde{t}$ not previously prescribed, and η is a small number whose sign agrees with the "sign" of the point. If there exist no "signed" points, we choose any straddle point $\tilde{t}$ and require (2.24) with $\eta > 0$.

Since F is an extended u -unisolvent family, there exists a (unique) u_1 satisfying all of the n above mentioned conditions. Furthermore, $u_1 \neq u$ by (2.24), so that $u_1 - u$ can have no additional zeros (counting multiplicities) besides the $n-1$ zero prescribed in the construction.

Hence, $u_1 - u$ changes sign exactly at the interior zeros of odd multiplicity. It follows that $u_1 > u_0$ on each (+)-interval, $u_1 < u_0$ on each (-)-interval.

Furthermore, if y is a (+)-boundary straddle point of deficiency k , then

$u_1 > u_0$ in the vicinity of y , so that in view of (2.19), we must have $u_1^{(k)}(y) > u_0^{(k)}(y)$.

The case of (-)-boundary straddle points is similarly handled. Finally, if η is chosen to be sufficiently small, then by the continuity property of elements of F (Lemma 1.2) we have $\|u_0 - u_1\| < \delta$, so that

$$\max\{|u_1(t) - f(t)|, |u_1(t) - g(t)|; t \in [0, 1] \setminus (\bigcup_{i=1}^r V_i) \setminus (\bigcup_{i=1}^r A_i)\} < r(u_0; f, g).$$

Collecting these results, we deduce that $u_1 \in Z(F; f, g)$, completing the proof in the case where no straddle point is a cluster point of (e)-points.

We consider now the general case, and describe the necessary adjustments in the proof. Let y be a straddle point which is a cluster point of (e)-points. As we have noted before, the finiteness of $S^+(\alpha)$ implies that if $\epsilon_1 > 0$ is sufficiently small, then in $(\tilde{y} - \epsilon_1, \tilde{y})$ all (e)-points are of one sign, and in $(\tilde{y}, \tilde{y} + \epsilon_1)$ all (e)-points are of one sign (not necessarily the same sign as before). Note in passing that if $\tilde{y}$ is a (-)-boundary point, then it can be a cluster point of (-)-points only, by the analysis preceding definition 2.4. The analogous result holds for (+)-boundary points.

Choose ϵ_1 as above, and let $\tilde{z}, \tilde{w}$ be the largest (e)-point in $(\tilde{y}, \tilde{y} + \epsilon_1)$ and the smallest (e)-point in $(\tilde{y} - \epsilon_1, \tilde{y})$, respectively. Let $\epsilon(\tilde{y}) = \min\{(\tilde{z} - \tilde{y}, \tilde{y} - \tilde{w})$, and let

$$\tilde{\epsilon} = \min\{\epsilon(\tilde{y}); \tilde{y} \text{ is a cluster point of (e)-points}\},$$

$$\epsilon^* = \min\{\epsilon(y_i); y_i \text{ is not a cluster point of (e)-points}\},$$

$$\epsilon = \min(\tilde{\epsilon}, \epsilon^*),$$

where $\epsilon(y_i)$ is as defined in the beginning of the proof of the necessity part. Define next v_i, I_0 as before and the rest of the proof can be carried out with no further modifications.

q.e.d.

We have shown that, in contrast to the situation where one function is approximated, the Chebyshev center of a set is not necessarily a singleton. We will now record some simple observations concerning the set of pairs for which $Z(F;f,g)$ is a singleton.

We consider the space of pairs of functions (f,g) , $f,g \in C[0,1]$, and define $\rho[(f,g),(f',g')] = \max[\|f-f'\|, \|g-g'\|]$.

Assertion 2.7: Let $f,g, f \geq g$ be a pair such that $Z(F;f,g)$ is not a singleton. Then, for each $\epsilon > 0$, there exists another pair (f',g') such that $\rho[(f,g),(f',g')] < \epsilon$, and $Z(F;f,g)$ is not a singleton.

Proof: Let $\bar{u} \in Z(F;f,g)$. There exist r straddle points, $y_1, \dots, y_r$, with total deficiencies $h_1, \dots, h_r$, $\ell = \sum_{i=1}^r h_i < n$. Perturb f slightly downward on one interval not containing straddle points, obtaining f' in this way. Then clearly $\bar{u} \in Z(F;f',g)$, the r straddle points remain the only straddle points, and no new (e) -points are created. Hence $Z(F;f',g)$ is not a singleton.

Remark: The same proof shows that if $Z(F;f,g)$ is a singleton, but there exist straddle points, then there exists a pair (f',g') near (f,g) for which $Z(F;f',g')$ is not a singleton.

However, the situation is different if $(F;f,g)$ has no straddle points. The following assertion can be easily established, using straightforward continuity arguments.

Assertion 2.8: Let $Z(F;f,g)$ be a singleton, and assume no straddle points exist. Then there exists a neighborhood V of (f,g) , such that for each pair (f',g') in V , the center $Z(F;f',g')$ is a singleton, and no straddle points exist.

Using the standard methods, we can deduce a local continuity property for the "best approximation" operator defined for such pairs, viz.

Assertion 2.9: Let (f,g) be a pair such that $(F;f,g)$ has no straddle points. Let T be defined on the set of such pairs by $T(f,g) = Z(F;f,g)$. Then T is continuous at (f,g) .

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