AD-A096 661 WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER F/G 12/1
AN APPLICATION OF THE GENERALIZED MORSE INDEX TO TRAVELLING WAV--ETC(U)
NOV 80 C CONLEY, R GARDNER DAAG29-80-C-0041
UNCLASSIFIED MRC-TSR-2144 NL

1 OF 1
AD A
096661

END
DATE
FILMED
4-81
DTIC

MRC Technical Summary Report #2144

AN APPLICATION OF THE GENERALIZED MORSE INDEX TO TRAVELLING WAVE SOLUTIONS OF A COMPETITIVE REACTION-DIFFUSION MODEL

C. Conley and R. Gardner

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

November 1980

(Received September 9, 1980)

Sponsored by

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

AN APPLICATION OF THE GENERALIZED MORSE INDEX TO TRAVELLING WAVE SOLUTIONS OF A COMPETITIVE REACTION-DIFFUSION MODEL

C. Conley and R. Gardner

Technical Summary Report #2144

November 1980

ABSTRACT

The existence of travelling wave solutions of a diffusion reaction-system is studied via the generalized Morse index of isolated invariant sets. This index theory is analogous to degree theory, and the method of proof follows lines familiar from the latter theory. The equations in question are "deformed" to a "standard" system where the index can be easily computed, and the existence theorem follows from the "non-triviality" of the index.

The index theory has been described in other papers; here the main job is to construct "isolating neighborhoods" which are analogous (in the degree theory) to open sets with no critical points on the boundary. Some novel means of locating such neighborhoods are described.

The main theorem concerns a case where the reaction system is a competitive system: that is, the growth rate of one population decreases as the other population increases. In a plausible class of such models, each system admits three stable equilibria: the two at which one population is at a maximal stable level and the other is eliminated, and one at which both populations are at the zero level. In general there are (two) travelling waves connecting one or the other of the first two equilibria to the third. The theorem gives a criterion, in terms of the relative velocity of these two waves, in order that there be a third wave running between the first two equilibria.

AMS (MOS) Subject Classification: 34C05, 35B99, 92A17
Key Words: Competitive Systems; Travelling Waves; Diffusion-Reaction; Generalized Morse Index
Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the National Science Foundation under Grant No. MCS800-1816.

SIGNIFICANCE AND EXPLANATION

Two populations are in competition if an increase in one causes a decrease in the growth rate of the other. In the absence of any other effects, the interaction of the two populations can be modeled by a system of ordinary differential equations describing the change in population levels. Typically this system admits three attracting rest points: two at which one population has reached its maximum (stable) level and the other has been eliminated, and the third where both populations are at zero level.

Now, allowing for spatial effects, one can consider a (symmetric) situation where one population is at its maximal level in one part of the space (with the other eliminated) and the reverse situation holds in the other part. If the populations are able to move in space, it can be expected that one or the other of them might "win out" over the whole space. This report concerns mathematical problems related to the above situation where the spatial movement is modeled by diffusion.

Among a class of mathematical models of a plausible type, two possibilities are indicated. In the first, one of the two populations will eventually dominate and the other will be eliminated. This corresponds to the existence of a "travelling wave" solution (of the diffusion-reaction system) which connects the first two of the above mentioned rest points. In the second situation this solution does not exist; instead there exists a "composition" of two travelling waves, each of which connects one of the above two rest points to the third at which both populations are zero. Evidently, in this case neither population is strong enough to survive the conflict. (However, this is not proved in this report.)

The main result gives a simple way to distinguish these two possibilities on the basis of the "relative strengths" of the attractors.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

AN APPLICATION OF THE GENERALIZED MORSE INDEX TO TRAVELLING WAVE SOLUTIONS OF A COMPETITIVE REACTION-DIFFUSION MODEL

C. Conley and R. Gardner

§1. Introduction

A. The Equations

This report concerns the ordinary differential equations for travelling wave solutions of a diffusion-reaction system. The latter can be interpreted as a model for interspecific competition in which the two species diffuse over a "one-dimensional" spatial domain.

Without diffusion, the model takes the form

$$
\begin{aligned}
u_1' &= u_1 F_1(u_1, u_2) \\
u_2' &= u_2 F_2(u_1, u_2)
\end{aligned}
\tag{1}
$$

where u_1 and u_2 are the total populations of the two competing species and F_1 and F_2 are the exponential growth rates which depend on the total population. These equations will be called the "reaction" equations.

The competitive aspect of the model is contained in the hypothesis that $\partial F_1/\partial u_2$ and $\partial F_2/\partial u_1$ are both assumed to be negative; thus the growth rate of each species decreases with an increase in the competing population.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and the National Science Foundation under Grant No. MCS800-1816.

B. Example

The main emphasis is on the situation where the zero sets of F_1 and F_2 are as pictured in Figure 1. Then the F's are negative for large values of u_1 and u_2 (due to limited resources) and near $u_1 = u_2 = 0$ (due to the too low population level).

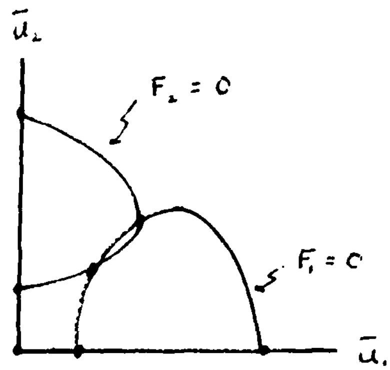

Figure 1.

These reaction equations have seven critical points. Three of these, $(0,0)$; $(0,\bar{u}_2)$ and $(\bar{u}_1,0)$, are attractors and the point interior to the quadrant and closest to the origin is a repeller. So long as the points are assumed to be non-degenerate, this forces the remaining three points to be saddles.

C. The Diffusion Terms and Travelling Wave Solutions

In the presence of diffusion, the model becomes

$$\partial u_1/\partial t = \delta_1 \partial^2 u_1/\partial x^2 + u_1 F_1(u_1, u_2)$$

(2)

$$\partial u_2/\partial t = \delta_2 \partial^2 u_2/\partial x^2 + u_2 F_2(u_1, u_2)$$

where δ_1 and δ_2 are positive constants measuring the diffusion rates of the two species (these are not assumed to be equal).

The aim is to show the existence of travelling wave solutions for the diffusion-reaction equations. These are solutions of the form $u(x-\theta t)$.

In particular they depend only on the one variable $\xi = x - \theta t$, so satisfy ordinary differential equations of the form:

$$(3)\qquad \begin{aligned} -\theta u_1' &= \delta_1 u_1'' + u_1 F_1(u_1, u_2) \\ -\theta u_2' &= \delta_2 u_2'' + u_2 F_2(u_1, u_2) . \end{aligned}$$

These are most conveniently written as a first order system:

$$(4)\qquad \begin{aligned} u_1' &= v_1 \\ u_2' &= v_2 \\ v_1' &= -\delta_1^{-1}[\theta v_1 + u_1 F_1(u_1, u_2)] \\ v_2' &= -\delta_2^{-1}[\theta v_2 + u_2 F_2(u_1, u_2)] . \end{aligned}$$

Travelling wave solutions are so called because they retain their shape as time progresses, but translate along the x-axis at velocity θ (since the "center" $\xi = 0$ is at $x = \theta t$).

The waves of interest are those which, for fixed t, have limiting values as x tends to plus or minus infinity. This means that as solutions of the ordinary differential equations, (4), they are orbits which run from one critical point to another. The critical points of most interest in this regard are the (three) attractors of the reaction system.

D. Strength of Attractors

The wave can be interpreted as measuring the relative "strength" of the attractors as follows: The system is in one stable state at $-\infty$ and another at $+\infty$; then the shape of the wave reflects the way the transition from one state to the other will most naturally take place. The wave velocity now determines which state will "take over": that is, if $\theta > 0$ then the state at minus infinity is "dominant" in the sense that the solution converges pointwise in x to this state as t tends to infinity. If $\theta < 0$, the other state is strongest. Of course the relative strength defined in this way generally depends on the way the diffusion takes place.

E. Relation to Clines

The existence of travelling wave solutions depends on the fact that the reaction terms in the equation are independent of x. However, the idea of the relative strength of attractors appears naturally even in the x-dependent case.

Suppose that the F's depend on x, but are asymptotically space-independent as x tends to infinity in either time direction. Also suppose the general appearance of the zero curves of F (for each fixed value of x) are as in Figure 1 for all x.

Then it is possible that one attractor dominates at minus infinity and the other at plus infinity, the dominance being measured as above by the travelling wave solutions of the asymptotically space independent systems. In this case the travelling wave is replaced by a static solution which limits to the relevant dominant attractor in either x-direction. Such solutions are called clines (in population genetics). This topic will be pursued further someplace else.

F. Method of Proof

The relevance of the remarks about clines in this report on travelling waves comes from the method of proof. Namely, the existence of either type of solution can be traced to the non-triviality of a topological (homotopy) invariant called the connection index. The construction of this invariant is described in detail in §2 (the relevant proofs are in [1], [2] and [3]). The main job of the paper is to compute this invariant for the example in Fig. 1.

The computation is carried out much as analogous computations in degree theory. The equations are deformed or "continued" (preserving the invariant) to a system for which the invariant is easy to compute. This simpler system, described in §3, is itself the product of a "hyperbolic point" and a lower dimensional (planar) system. A product theorem for the invariant thus reduces the computation to that for a planar system.

The planar system is described in §2 (with some detail deferred to the appendix); it is also used there to illustrate the abstract definition of the connection index.

In §4 "isolating neighborhoods" (analogous to open sets with no fixed points on the boundary) are constructed; these are needed to carry out the continuation described in §5. This construction illustrates some novel ways of locating isolating neighborhoods in higher dimensional systems of this type. The lemmas proved are somewhat more general than necessary, but allow for applications to examples other than that of Figure 1.

G. An Alternate Proof and Stability

In a forthcoming paper by the second author ([4]) a proof of existence of the travelling wave will be obtained from a topological degree argument.

It seems clear that the success of one approach implies that of the other in some general setting; however, a general theorem has not been formulated. A similar situation occurs in the study of shock waves (compare [5] and [6]).

Also in the above mentioned paper, [4], the stability of the waves as solutions of the partial differential equation will be treated.

H. <u>The Main Theorem</u>.

In the example of Figure 1, there are three attractors: namely $(\bar{u}_2, 0)$, $(0, 0)$ and $(0, \bar{u}_1)$. It is already well known (and the same invariant has been used in [2] to see) that there are travelling waves running from the first to the second and from the second to the third (these are, respectively, solutions of the two dimensional systems obtained on setting u_1 or u_2 equal to zero).

The main theorem concerns waves from $(\bar{u}_2, 0)$ to $(0, \bar{u}_1)$:

<u>Theorem</u>: With reference to A., B. and the above remarks, let θ' and θ'' be the wave velocities of the waves from $(\bar{u}_2, 0)$ to $(0, 0)$ and from $(0, 0)$ to $(0, \bar{u}_1)$ respectively. Then if $\theta' > \theta''$, there is also a wave from $(\bar{u}_2, 0)$ to $(0, \bar{u}_1)$.

The proof of this theorem is concluded in § 5.

§ 2. The Standard Problem.

As described in the introduction, the connection problem treated here can be continued to a "standard" problem which is described below. This model problem will also be used to illustrate the abstract definitions required to arrive at the connection index used in the existence proof.

The equations are

$$(5)\qquad \begin{aligned} w' &= z \\ z' &= -\theta z - w(1-w^2) \ . \end{aligned}$$

The phase portrait in the (w, z)-plane with $\theta = 0$ is sketched in Figure 2.

There is a solution in the upper half plane connecting the left hand, hyperbolic rest point, $p' = (-1, 0)$, to the right hand one, $p'' = (0, 1)$. A homotopy invariant related to the existence of this connection is constructed as follows.

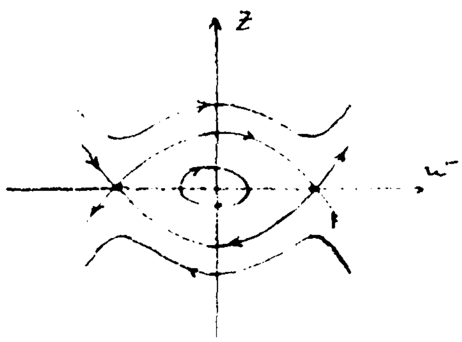

Figure 2

A. Definition 1.

Suppose a differential equation (or local flow) on a space X is given. If $N \subset X$ is compact, let $I(N)$ denote the set of points on solutions which stay in N for all time. (Then $I(N)$ is a compact invariant set).

If $S = I(N)$ is interior to N, it will be called an isolated invariant set and N will be called an isolating neighborhood for S.

Definition 2.

Now suppose there is given a family of differential equations on $\mathbf{R}^n$ parametrized (continuously) by a parameter θ in a closed interval $[\theta_0, \theta_1]$.

Let $X = \mathbf{R}^n \times [\theta_0, \theta_1]$. Then, in view of the continuity, the equations determine a (local) flow on X.

Let S', S'' and S be isolated invariant sets for this local flow and let $S'(\theta)$ etc. be the set of points in S' with parameter value θ.

Then the triple (S', S'', S) will be called a (codimension 1) connection triple if

a) $S' \cup S'' \subset S$

b) $S' \cap S'' = \varphi$

c) For $\theta = \theta_0$ or θ_1, $S(\theta) = S'(\theta) \cup S''(\theta)$.

B. Example.

With regard to Figure 2, let D' and D'' be disjoint compact neighborhoods of p' and p'' respectively, neither of which contains $(0,0)$. Let D be a compact neighborhood of the connecting solution disjoint from the non-positive z-axis. Then D' and D'' are isolating neighborhoods for all values of θ with corresponding isolated invariant sets p' and p''.

Also, D is an isolating neighborhood for all θ: if $\theta \neq 0$, $I(D) = \{p', p''\}$ and if $\theta = 0$, $I(D)$ is the closure of the connecting solution.

It follows that for any compact interval $[\theta_0, \theta_1]$, $N' \equiv D' \times [\theta_0, \theta_1]$, $N'' \equiv D'' \times [\theta_0, \theta_1]$ and $N \equiv D \times [\theta_0, \theta_1]$ are all isolating neighborhoods.

Also $S' = I(N')$ and $S'' = I(N'')$ are disjoint and contained in $S = I(N)$, so if condition c) of Definition 2 is satisfied then (S', S'', S) is a connection triple. This is the case if neither of θ_0 nor θ_1 is 0 ; however, if θ_0 (say) is 0 then $S(\theta_0) \neq S'(\theta_0) \cup S''(\theta_0)$ and condition c) is violated.

Observe that if θ_0 and θ_1 have the same sign then $S = S' \cup S''$ and there is no connection (for any θ in $[\theta_0, \theta_1]$) from S' to S''. However, if θ_0 and θ_1 have different signs then $S \neq S' \cup S''$ and the difference $S \setminus S' \cup S''$ is the connecting solution. The aim is to define a homotopy invariant for a connection triple which will distinguish these two cases. Before defining this invariant, "continuation" (or deformation) of isolated invariant sets and triples will be defined.

C. Definition 1.

Suppose now that there is given a family of local flows on a space X parametrized (continuously) by a parameter $\lambda \in [0,1]$. Let $Y = X \times [0,1]$ and consider the *family as a local flow on* Y. Given a compact set $N \subset Y$ and $\lambda \in [0,1]$, let $N(\lambda)$ be the set of points in N with parameter value λ. (*Observe that if* N *is an isolating neighborhood*, then for $\lambda \in [0,1]$, $N(\lambda)$ is also).

Let S_0 and S_1 be isolated invariant sets for the flows on X corresponding to $\lambda = 0$ and $\lambda = 1$ respectively.

Then S_0 and S_1 are related by continuation if there is an isolating neighborhood, N, for the flow on Y such that $S_0 = I(N(0))$ and $S_1 = I(N_1)$. (One could say that as λ varies from 0 to 1, S_0 is "deformed" to S_1. However, it is really the equations that are deformed— the sets may change radically).

Definition 2

In a like way one can define continuation of connection triples: assume that in the above, $X = \mathbb{R}^n \times [\theta_0, \theta_1]$ and that for each λ, the flow on X is actually a family of flows parametrized by θ. Suppose also that S, S', S'' are isolated invariant sets of the flow on Y and that for each λ, $(S'(\lambda), S''(\lambda), S(\lambda))$ is a connection triple. Then the triples at $\lambda = 0$ and $\lambda = 1$ are related by continuation.

The "continuation" relation is obviously an equivalence relation; the aim now is to define an invariant or "index" which is the same for all isolated invariant sets (or connection triples) that are in the same equivalence class.

D. Definition.

Let N be an isolating neighborhood for a flow on a space X and let N_1 and $N_2 \subset N_1$ be compact subsets of N. Suppose $N_1 \setminus N_2$ is a neighborhood of $I(N)$ and that N_1 and N_2 are positively invariant relative to N. (I.e. whenever the initial point of an orbit segment lies in N_1, respectively N_2, and the segment lies in N, then the segment lies in N_1, respectively N_2). Also assume that points of N_1 which are carried out of N hit N_2 before leaving N.

Then (N_1, N_2) is called an index pair for $S = I(N)$.

Theorem 1. For any isolated invariant set S, index pairs exist. Moreover the homotopy type, $[N_1, N_2]$, of the pointed space N_1/N_2 depends only on $S = I(N)$ and is the same for all isolated invariant sets which are related by continuation to S. It will be called the (homotopy) index of S and denoted $h(S)$.

Remark: If S_0 and S_1 are related by continuation (as in Definition 1 of C.) and N is the neighborhood used to define the continuation then $h(S_0) = h(I(N))$.

The definition of the connection index is played back to that above using the following lemma.

Lemma: Let (S', S'', S) be a connection triple for a family of differential equations on $\mathbf{R}^n$ parametrized by θ in the interval $[\theta_0, \theta_1]$. Assume the equations are defined for $\theta \in (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$ for some $\varepsilon > 0$ (this is no real restriction—they can be extended to such an interval). Let U' and U'' be open neighborhoods in $\mathbf{R}^n \times (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$ of $S'(\theta_0) \cup S'(\theta_1)$ and $S''(\theta_0) \cup S''(\theta_1)$ (respectively); choose these to have disjoint closures. Let φ be a continuous real valued function on $\mathbf{R}^n$ which is positive on U' and negative on U''. Append to the given family of equations the equation $\theta' = \mu\varphi(x)(\theta_0 + \theta_1)/2)$ where μ is a (small) positive parameter.

Then there is a $\mu_0 > 0$ such that if $\mu \in (0, \mu_0)$ then N is an isolating neighborhood for the appended equation. Let h_μ be the index of $S(N)$, $\mu \in (0, \mu_0)$. Then h_μ is independent of μ, and in fact depends only on the triple (S', S'', S).

Proof. This lemma is proved in [3] in a rather more general form (see also [2]).

Definition: With the above notation, the "index" of the triple (S', S'', S) is $h(S', S'', S) = h_\mu$.

<u>Theorem</u>: The index of a connection triple is constant on equivalence classes under the continuation relation.

<u>Proof</u>. See [3].

E. <u>Example</u>. (See Appendix A)

The indices of the hyperbolic rest points in (5) (Example B.) are both pointed circles, denoted here by Σ^1.

The index of the disjoint union of isolated invariant sets is the "sum" of their indices; namely, the pointed space obtained on identifying their distinguished points. Thus the index of the union of the two hyperbolic points is $\Sigma^1 \vee \Sigma^1$ where $\vee$ means sum.

By the continuation theorem, the index of the closure of the connecting solution is also $\Sigma^1 \vee \Sigma^1$. (This set is the continuation of the set $\{p', p''\}$ with $\theta \in [\theta_0, 0]$ the continuation parameter).

The index for the connection triple in B. is shown in Appendix A to be $\Sigma^2 \vee \Sigma^1$ if θ_0 and θ_1 have the same sign (i.e. when $\theta = 0$ and the connecting solution are absent) and $\overline{0}$ (the homotopy type of the one point space) when $\theta_0 \theta_1 < 0$.

<u>Remark</u>: In addition to the sum of indices, a product is also defined by $[N_1'/N_2'] \wedge [N_1'', N_2'']$ $[N_1' \times N_2' / \{N_1' \times N_2'' \cup N_2'' \times N_1'\}]$. Given a product of flows, the product of isolated invariant sets is isolated and its index is the product of the indices of the sets. This product is used in the statement of the "existence" theorem following.

F. The theorem needed for the existence proof is:

Theorem: Suppose (S', S'', S) is a connection triple with index $\bar{h}$, and let h' and h'' denote the indices of S' and S'' respectively. Then if $\bar{h} \neq (\Sigma^1 \wedge h') \vee h''$, $S \neq S' \cup S''$.

Example: (See the appendix). In the example (E.) if $\theta_0 \theta_1 > 0$ then $\bar{h} = \Sigma^2 \vee \Sigma^1$, $h' = h'' = \Sigma^1$, and $\Sigma^1 \wedge \Sigma^1 = \Sigma^2$. Therefore $\bar{h} = (\Sigma^1 \wedge h') \vee h''$ so the theorem gives no information. If $\theta_0 \theta_1 < 0$ then $\bar{h} = \bar{0} \neq \Sigma^2 \vee \Sigma^1$ so $S \neq S' \cup S''$. The difference between S and $S' \cup S''$ is, of course, the connecting solution in this case.

G. In §4 and §5 it will be seen that the problem of §1 can be resolved by continuing an appropriate connection triple to a product of that in B. with a hyperbolic rest point. The indices of the triple in the product will then be $\bar{h} = \Sigma^1 \wedge \bar{0} = \bar{0}$, $h' = \Sigma^1 \wedge \Sigma^1 = \Sigma^2$ and $h'' = \Sigma^2$. Then $\bar{0} \neq (\Sigma^1 \wedge \Sigma^2) \vee \Sigma^2$ $(= \Sigma^3 \vee \Sigma^2)$ will imply the existence of the connecting solution.

The continuation relies only on identifying isolating neighborhoods which is fairly easy. Before doing this, however, the "target" equation (i.e. the product) will be described in §3.

§ 3. A Four Dimensional Example

A. Consider the equations:

$$u_1' = v_1$$

$$u_2' = v_2$$

(6)

$$v_1' = -\theta v_1 - (4u_1 - 6u_1u_2 - 4u_1^3)$$

$$v_2' = -\theta v_2 - (4u_2 - 6u_1u_2 - 4u_2^3) .$$

The zero sets of F_1 and F_2 are pictured in Figure 3.

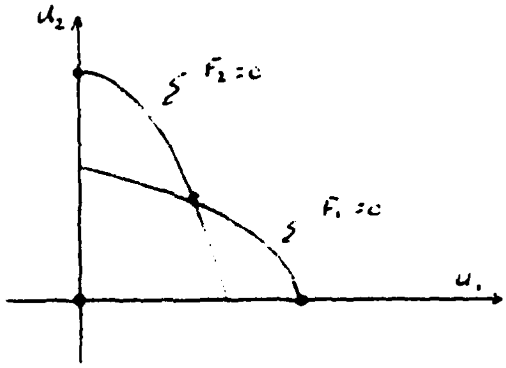

Figure 3.

Let $w_1 = u_1 + u_2$ and $w_2 = u_1 - u_2$. Then

$4u_1u_2 = w_1^2 - w_2^2$, $2(u_1^2 + u_2^2) = w_1^2 + w_2^2$,

$4(u_1^3 + u_2^3) = w_1 [2(w_1^2 + w_2^2) - (w_1^2 - w_2^2)] = w_1 [w_1^2 + 3w_2^2]$ and

$4(u_1^3 - u_2^3) = w_2 [2(w_1^2 + w_2^2) + (w_1^2 - w_2^2)] = w_2 [3w_1^2 + w_2^2]$.

The transformed equations are therefore:

$$
(7) \qquad
\begin{aligned}
w_1' &= z_1 \\
w_2' &= z_2 \\
z_1' &= -\theta z_1 - 4w_1 + 3(w_1^2 - w_2^2) + w_1(w_1^2 + 3w_2^2) \\
z_2' &= -\theta z_2 - 4w_2 + w_2(3w_1^2 + w_2^2) \quad .
\end{aligned}
$$

B. An Invariant Manifold Containing the Standard Problem.

When $w_1 = 1$ and $z_1 = 0$, $w_1' = z_1' = 0$. Therefore the set $\{w_1 = 1;\ z_1 = 0\}$ is an invariant manifold. On this manifold, the equations for w_2 and z_2 are:

$$
(8) \qquad
\begin{aligned}
w_2' &= z_2 \\
z_2' &= -\theta z_2 - w_2(1 - w_2^2) \ ;
\end{aligned}
$$

namely the equations of §2 .

The displacement equations describing the behavior of w_1 and z_1 near the invariant manifold are given by:

$$
(9) \qquad
\begin{aligned}
\xi' &= \zeta \\
\zeta' &= -\theta\zeta + (5 + 3w_2^2)\xi + 0(\xi^2)
\end{aligned}
$$

(wherein $w_1 = 1 + \xi$ and $z_1 = \zeta$).

Suppose now that N is any compact set in (w_2, z_2, θ)-space . Let K be a bound for $6w_2 z_2$ on N. Let $D_\varepsilon = \{(w_1, z_1) \mid |w_1 - 1| \le \varepsilon/K$ and $|z_1| \le \varepsilon$. The idea is to show that if ε is small enough, then $I(D_\varepsilon \times N) = \{(1,0)\} \times I(N)$.

To see this it is sufficient to see that for small ε, no solution in $I(D_\varepsilon \times N)$ can pass through a point in $\partial D_\varepsilon \times N$.

Suppose, then, that $|\xi| = \varepsilon / K$. If $\xi^1 = \zeta$ is not zero the solution can't be in $I(D_\varepsilon \times N)$. If $\zeta = 0$, then $\xi'' = -\theta\zeta + (5 + 3w_2^2)\xi + 0(\xi^2)$. Thus ξ'' has the same sign as ξ , so again the solution is not in $I(D_\varepsilon \times N)$ if ε is small enough.

Now suppose $|\zeta| = \varepsilon$. If $\zeta^1 = \xi \neq 0$ then the solution isn't in $I(D_\varepsilon \times N)$. If $\xi = 0$ then $\zeta'' = \zeta$ so ζ'' has the same sign as ζ and again the solution must leave $D_\varepsilon \times N$.

Observe that the same argument as that above works if the terms in (9) involving θ , w_2^2 and ξ^2 are multiplied by a parameter λ varying between 0 and 1 .

Now let N be an isolating for the (w_2, z_2, θ) -equation. Then for small ε , $D_\varepsilon \times N$ is an isolating neighborhood for (6) or equivalently, (8) with (9) . Namely, $I(D_\varepsilon \times N) = I(\{0\} \times N) \subset \text{int.} D_\varepsilon \times N$. Modifying the equations with a parameter λ as described above, one continues the invariant set to (the "same") one for the product of the equation (8) with the hyperbolic point:

$$\text{(10)} \qquad \begin{aligned} \xi^1 &= \zeta \\ \zeta^1 &= 5\xi \qquad\qquad (\lambda = 0) \end{aligned}$$

Thus the connection triple in the invariant manifold $w_1 = 1$, $z_1 = 0$ with $\theta \in [\theta_0, \theta_1]$ continues to the product of that in 2.B. with a hyperbolic point. As described in 2.G., when $\theta_0\theta_1 < 0$, the indices for triple are $\overline{h} = \overline{0}$, $h' = \Sigma^2$ and $h'' = \Sigma^2$.

In the next section, isolating neighborhoods will be identified which allow one to continue the general problem to the one treated here.

§ 4. The Main Problem

The equations of § 1 are written here in the more convenient form:

$$
(11) \qquad
\begin{aligned}
u_1' &= v_1 \\
u_2' &= v_2 \\
v_1' &= -\theta_1 v_1 - u_1 f_1(u_1, u_2) \\
v_2' &= -\theta_2 v_2 - u_2 f_2(u_1, u_2) \;;
\end{aligned}
$$

where $\theta_i = \delta_i^{-1}\theta$ and $f_i = \delta_i^{-1}F_i$ $(i = 1, 2)$.

Observe that θ_1 and θ_2 have the same sign (or are zero together); this is necessary for the results on isolating neighborhoods that follow.

Also note that δ_1 and δ_2 do not appear explicitly in the following: the aim is to continue a connection triple of the equations (11) above to that for equations (6) of § 3, and in so doing it does not matter that the relation between the δ_i and θ_i is lost.

To carry out the continuation, it is sufficient to find appropriate isolating neighborhoods, or in other words, compact sets N such that $I(N) \cap \partial N = \phi$. This will be done by "successive approximations."

The following hypotheses hold throughout:

Hypotheses. There is a positive constant K such that if u lies in the quadrant $u_i \geq 0$ $(i = 1, 2)$, then:

1) $u_1 \geq K$ implies $f_1(u_1, u_2) < 0$, and $u_2 > K$ implies $f_2(u_1, u_2) < 0$.

2) There are only finitely many points u for which $f_1 = f_2 = 0$.

3) $\partial f_1 / \partial u_2$ and $\partial f_2 / \partial u_1$ are both negative.

A. Definition.

Let $\overline{S}$ be the set of points on bounded solutions of (11) whose u-coordinates always satisfy $u_i \geq 0$ $(i = 1, 2)$.

Lemma: There is a constant $L > 0$ such that for any values of θ_1 and θ_2, $\overline{S} \subset \{(u, v) | \; 0 \leq u_i < K$ and $|v_i| < L \quad i = 1, 2\}$.

Proof. Choose any solution in $\overline{S}$ and let (u, v) be a point in its closure where u_1 achieves its maximum value. Then $u_1' = v_1$ must be zero so $u_1'' = -u_1 f_1(u_1, u_2)$. But also, u_1'' must be non-positive so f_1 must be non-negative. This implies $u_1 < K$, and it follows that $u_1 | \overline{S} < K$. A similar argument proves $u_2 | \overline{S} < K$.

Now let M be a bound for $|u_i f_i(u_1, u_2)|$ on the square $0 \leq u_i \leq K$ $(i = 1, 2)$. Suppose $\theta_1 \geq 1$. Consider a solution on which $|v_1|$ exceeds M at some time. Then at that time v_1 and v_1' have the same sign and this condition will be maintained either in forward time $(v_1 > 0)$ or backward time $(v_1 < 0)$ so that the solution will grow exponentially and not be in $\overline{S}$. Similar arguments show that if $|\theta_i| \geq 1$ then, on $\overline{S}$, $|v_i| \leq M$.

Suppose $|\theta_1| \leq 1$. Then for large enough v_1, $|v_1'| < 2|v_1|$. Therefore, there is a constant $L' > K$ such that if $|v_1|$ is larger than L'

at time zero, it is larger than K for time 1. It follows that u_1 at zero differs from u_1 at 1 by more than K and the solution cannot be in $\bar{S}$. Thus on $\bar{S}$, $|v_1|$, and in a similar way $|v_2|$, is strictly bounded by $L = \max(M+1, L')$ independently of θ_1 and θ_2.

B. Definition.

Let C denote the set of critical points in $\bar{S}$. Let U_1^+ denote the points on solutions in $\bar{S}$ with $u_2 = v_2 = 0$ and $v_1 \geq 0$ and U_2^- those with $u_1 = v_1 = 0$ and $v_2 \leq 0$.

Lemma: Let $N_1 \equiv \{(u,v) |\ 0 \leq u_1, u_2 \leq K \text{ and } 0 \leq v_1, \ -v_2 \leq L\}$.

Then $I(N_1)$ consists of critical points and orbits connecting them. Furthermore, if $p' \in \partial N_1$ then either

a) p' is on an unbounded solution, or

b) The solution through p' leaves N_1 immediately in one or the other time direction, or

c) The projection in u-space of the solution through p' is contained in one of the coordinate axis, or

d) $p' \in C$.

In particular, $I(N_1) \cap \partial N_1 = U_1^+ \cup U_1^- \cup C$.

Proof. Since $v_1 \geq 0$ and $v_2 \leq 0$ in N_1, the function $u_1 - u_2$ is strictly increasing on non-constant solutions in N_1. (If $u_1 - u_2$ were constant on some time interval, both v_1 (≥ 0) and v_2 (≤ 0) would be zero on the interval and the solution would be a critical point). By general theorems, it follows that $I(N_1)$ consists of critical points and solutions connecting them (cf. [1]).

Suppose $p' \in \partial N_1$. Then either some u_i is K or 0, or one of v_1 or $-v_2$ is L or 0.

1. If u_1 or u_2 equals K, or v_1 or $-v_2$ equals L, the solution through p' is unbounded (by A.).

2. Suppose $u_1 = 0$. If $u_1' = v_1$ is not zero the solution leaves N_1 immediately in one or the other time direction.

If $v_1 = 0$ the projection of the solution in u-space is in the u_2 axis (since the set $u_1 = v_1 = 0$ is obviously an invariant set).

The case $u_2 = 0$ is treated similarly.

3. Suppose $v_1 = 0$ and $u_1 \neq 0$. If $v_1' = -u_1 f_1$ is not zero the solution leaves N_1 immediately in one or the other time direction. If $v_1' = 0$ then $v_1'' = -u_1(\partial f_1/\partial u_2)v_2$. If this is not zero then $v_1'' < 0$ ($u_1 > 0$, $\partial f_1/\partial u_2 < 0$ and $v_2 < 0$) so v_1 has achieved a strict maximum and the solution leaves N_1 immediately in both time directions.

4. Continuing from 3., suppose $v_1'' = 0$; then $v_2 = 0$. If $u_2 = 0$ the projection is in the u_1 axis. Assume $u_2 \neq 0$. If $v_2' \neq 0$ the solution leaves N_1 immediately (as in 3.). Otherwise $v_1 = v_2 = v_1' = v_2' = 0$ and the point is in C.

This concludes the proof of Lemma B.

C. Increasing N_1 Near Attractors of the Reaction Equations.

Let $p = (\bar{u}_1, \bar{u}_2, 0, 0)$ be a critical point of (8) and let M_p be the Jacobean matrix $\partial(u_1 f_1, u_2 f_2)/\partial(u_1, u_2)$ at p. Let C_I be the set of critical points p such that the diagonal entries of M_p are negative and $\det M_p > 0$. (These points are easily seen to be attractors for the reaction equations (1) of the introduction, but that fact will not be used explicitly).

Given positive numbers Δ_1 and Δ_2 and $\mu \in [0,1]$, define
$N(\mu, p) \equiv \{(u,v) \,\big|\, |u_1 - \bar{u}_1| \le \Delta_1;\ |u_2 - \bar{u}_2| \le \Delta_2 \text{ and } |v_i| \le L,\ i = 1,2\}$.

Lemma: Positive numbers Δ_1 and Δ_2 can be chosen so that for all $\mu \in (0,1]$, $N(\mu, p)$ is an isolating neighborhood for p.

Having chosen numbers (Δ_1, Δ_2) as above for each point of C_I so that the sets $N(1,p)$ $(p \in C_I)$ are disjoint, let $N_2(\mu)$ be the union of N_1 with all the $N(\mu, p)$ for $p \in C_I$. Let $N_2 \equiv N_2(1)$.

Then for $\mu \in (0,1]$, $I(N_2) = I(N_1)$. In particular, the points of C_I are not in $I(N_2) \cap \partial N_2$.

Proof.

1. Since the off-diagonal terms of M_p are known to be non-positive, the hypotheses imply that the gradient vectors of $u_1 f_1$ and $u_2 f_2$ at $(\bar{u}_1, \bar{u}_2)$ both point into the closed third quadrant and that the orientation determined by the ordering $(\nabla(u_1 f_1), \nabla u_2 f_2)$ agrees with the standard orientation of the (u_1, u_2) plane.

2. In view of 1., there is a line with negative slope through $(\bar{u}_1, \bar{u}_2)$ such that at points on the line near $(\bar{u}_1, \bar{u}_2)$, both of $(u_1 - \bar{u}_1) u_1 f_1(u_1, u_2)$

and $(u_2 - \bar{u}_2) u_2 f_2(u_1, u_2)$ are non-positive and if either one is zero then $u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$. (See Figure 4).

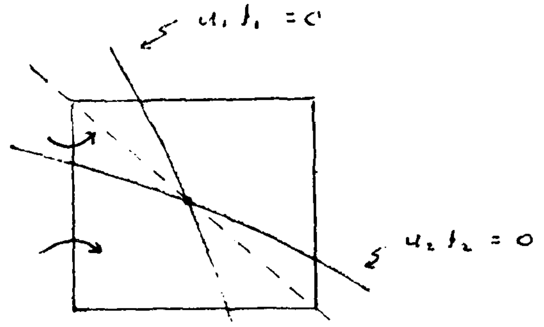

Figure 4

Let this line be given by $\Delta_1(u_1 - \bar{u}_1) + \Delta_2(u_2 - \bar{u}_2) = 0$ where Δ_1 and Δ_2 are positive and chosen small enough that the above inequalities are satisfied on the segment of the line determined by $|u_i - \bar{u}_i| \le \Delta_i$ $(i = 1, 2)$ (Fig. 4).

3. Now choose $\mu \in (0, 1]$. The u-coordinates of points in $N(\mu, p)$ then range over the rectangle $|u_1 - \bar{u}_1| \le \mu\Delta_1$; $|u_2 - \bar{u}_2| \le \mu\Delta_2$. Observe that $u_1 f_1$ is positive on the left hand side of the rectangle and negative on the right, while $u_2 f_2$ is negative on the top and positive on the bottom.

4. To see that $N(\mu, p)$ is an isolating neighborhood, first recall that no orbits on which v_1 or v_2 equals L are bounded. Therefore any point in $I(N(\mu, p)) \cap \partial N(\mu, p)$ must have u-coordinates in the boundary of the rectangle. Also the orbit through the point must be tangent to the boundary (cusp points counting as tangencies).

Suppose, for example, that the point is in the left hand boundary so that $u_1 - \bar{u}_1 = -\mu\Delta_1$ and $v_1 = 0$. Then $v_1' = -u_1 f_1$ and is negative. It follows that $u_1 - \bar{u}_1$ has a strict maximum at the point and from there that the solution leaves $N(\mu)$ in both time directions so in particular the point can't be in $I(N(\mu, p)) \cap \partial N(\mu, p)$.

The other three cases are treated in a similar way and it is found that if $|u_i - \bar{u}_i| = \mu\Delta_i$ for $i = 1$ or 2, then the orbit through u leaves $N(\mu, p)$ in both time directions.

5. Now, since $N(\mu, p)$ is an isolating neighborhood for all $\mu \in (0,1]$, the corresponding isolated invariant set must be in the intersection of the $N(\mu, p)$. The points in this intersection have the form $(\bar{u}_1, \bar{u}_2, v_1, v_2)$ so the only orbit in the intersection is the critical point.

6. To prove the last paragraph of the lemma, suppose $N_2(\mu)$ is defined as it is there and choose μ so that the $N(\mu, p)$ $(p \in C_I)$ are disjoint. It will be shown that $I(N_2(\mu)) \cap \partial N_2(\mu) \subset I(N_1)$ for all μ.

Assuming this for now, one sees that $I(N_1) = I(N_2(\mu))$ as follows. First let $N_2(0) = \bigcap \{N_2(\mu) | \mu \in (0,1]\}$. Then points in $N_2(0) \setminus N_1$ have the form $(\bar{u}_1, \bar{u}_2, v_1, v_2)$ where $(\bar{u}_1, \bar{u}_2, 0, 0) = p \in C_I$ and v violates the condition $0 \le v_1$, $-v_2 \le L$. These points are obviously not in $I(N_2(0))$ (since they leave $N_2(0)$ immediately in both time direction) and so $I(N_2(0)) = I(N_1)$.

Now if $I(N_2(\bar{\mu})) \neq I(N_1)$ for some $\bar{\mu}$, then for some μ, $I(N_2(\mu)) \cap \partial N_2(\mu)$ contains a point not in $I(N_1)$. But this violates $I(N_2(\mu)) \cap \partial N_2(\mu) \subset I(N_1)$.

7. To show that for all $\mu \in (0,1]$, $I(N_2(\mu)) \cap \partial N_2(\mu) \subset I(N_1)$, first observe that $\partial N_2(\mu)$ consist of three types of points: Namely,

a) points not in N_1 but in the boundary of some $N(\mu, p)$ with $p \in C_I$;

b) points in ∂N_1 but not in any $N(\mu, p)$; and

c) points in $\partial N_1 \cap \partial N(\mu, p)$ for some $p \in C_I$.

8. Suppose $p' \in \partial N(\mu, p) \setminus N_1$ for some $p \in C_I$. Since the solution leaves $N(\mu, p)$ immediately in one or the other time direction, it leaves $N_2(\mu)$ (it can't enter another $N(\mu, p)$ since they are disjoint).

9. Suppose $p' \in \partial N_1$ but not in any of the $N(\mu, p)$, $p \in C_I$. Using Lemma B, it is either on an unbounded solution (so leaves $N_2(\mu)$) or leaves N_1 immediately in some direction (again leaving $N_2(\mu)$ also since it is in $\partial N_2(\mu)$; cf. 4.) or it is a critical point (so in $I(N_1)$) or the u-projection of the solution through p' lies in one or the other of the coordinate axis.

Suppose for example that it lies in the u_1-axis. If v_1 is never negative on the solution, it is in $I(N_1)$.

On the other hand if v_1 is sometimes negative then (since it is also sometimes positive) the solution must be an oval, or spiral down to a rest point in one or the other time direction (cf. Figure 5). Of course all the spiraling must take place near some critical point, p, in the u_1 axis since v_1 is not negative in N_1 . But solutions in the u_1 axis near $p \in C_I$ do not spiral, but rather look locally like hyperbolas (cf. Fig. 5).

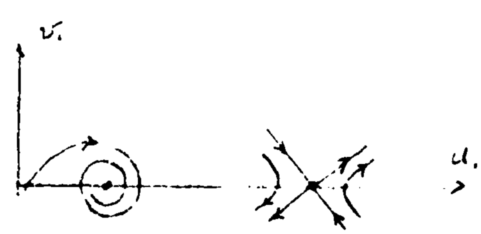

Fig 5

10. To finish the proof it must be shown that the points in $I(N_2(\mu)) \cap \partial N_1 \cap N(\mu, p)$ (for some $p \in C_I$) are also in $I(N_1)$.

In 9. it was seen that points in $I(N_2)$ with u_1 or u_2 equal to zero are also in $I(N_1)$. Therefore assume $u_1, u_2 > 0$.

Let $p = (\bar{u}_1, \bar{u}_2, 0, 0)$ be the critical point. If $p' \in \partial N_1 \cap \partial N(\mu, p)$ then one of the v_i is zero and one of the $|u_i - \bar{u}_i|$ is $\mu\Delta_1$. If, $v_1 = 0$ and $|u_1 - \bar{u}_1| = \mu\Delta_1$ then the point leaves N_1 and $N(\mu, p)$ in both time directions (by 4.) so the point isn't in $N_2(\mu)$. Similarly if $v_2 = 0$ and $|u_2 - \bar{u}_2| = \mu\Delta_1$, the orbit through p' is not in $N_2(\mu)$.

11. Suppose then that $u_1 - \bar{u}_1 = -\mu\Delta_1$ and $v_1 \neq 0$. Then $v_2 = 0$. There are two cases (indicated by the curved arcs in Figure 4). Either $u_2 f_2 \geq 0$ (the point lies on or below the curve $u_2 f_2 = 0$) or $u_2 f_2 < 0$.

In the first case v_2 becomes positive as time decreases, so the point leaves N_1 as well as $N(\mu, p)$ in negative time. (If $u_2 f_2 = 0$, $v_2'' = -u_2\, \partial f_2/\partial u_1 v_1 < 0$).

Suppose then that $u_2 f_2 < 0$. Then v_2 becomes positive as time increases. Now following the solution forward in time, it enters $N(\mu, p)$ and u_2 increases initially. Since the curve $u_2 f_2$ has non-positive slope in $N(\mu, p)$, the solution stays above this curve at least until v_2 becomes zero. Suppose v_2 does become zero at some point on the solution curve. Choose the first such point. At this point, $v_2' = -u_2 f_2$ is positive. But this contradicts the fact that v_2 is positive up to this point.

Therefore v_2 cannot become zero so long as the curve is in $N(\mu, p)$. This implies u_2 is increasing on the curve—so it cannot approach the critical point p, and from there that the curve must sometime leave $N(\mu, p)$. But then it leaves with v_2 positive so it also leaves $N_1 \cup N(\mu, p)$.

12. The other cases $(u_1 - \bar{u}_1 = \mu\Delta_1, \quad v_1 > 0$ and $|u_2 - \bar{u}_2| = \mu\Delta_2,$ $v_2 < 0)$ are similarly treated, and it is shown that points in $\partial N(\mu, p) \cap \partial N_1$ with $u_1, u_2 > 0$ are not contained in $N_2(\mu)$.

Together with 8. and 9. this completes the proof that $I(N_2(\mu)) \cap \partial N_2(\mu) \subset I(N_1)$, and by 6. that $I(N_2) = I(N_1)$.

D. Excision of Relative Attractors and Repellers.

Definition

Let S be a compact invariant set and let $\hat{S}$ be a subset of S. Then $\hat{S}$ is called an isolated invariant set relative to S if there is a compact relative neighborhood $\hat{N}$ of $\hat{S}$ in S such that $\hat{S} = I(\hat{N})$. If this is the case, let $A^+ = A^+(\hat{S}, S)$ (resp. $A^- = A^-(\hat{S}, S)$) be the set of points on solutions in $S \backslash \hat{S}$ that tend to $\hat{S}$ in forward time (resp. backward time).

Lemma: Suppose N is compact, let $S = I(N)$, and let $\hat{S}$ be isolated relative to S. Suppose either of $A^+ = A^+(\hat{S}, S)$ or $A^- = A^-(\hat{S}, S)$ is empty. Then for all small enough open neighborhoods, U, of $\hat{S}$, $I(N\backslash u) \cap \partial(N\backslash u) = I(N) \cap \partial N \backslash A^+ \cup A^- \cup \hat{S}$.

Proof. Suppose A^- is empty. Let $\hat{N}$ be an isolating neighborhood of $\hat{S}$ in S. Since $\hat{S}$ is the maximal invariant set in $\hat{N}$ and A^- is non-empty every point in the boundary of $S \cap \hat{N}$ (relative to S) must leave $\hat{N}$ in backward time. It follows (using [1], Ch. II, 5.10) that S can be written as a disjoint union $\tilde{S} \cup T \cup \hat{S}$ where $\tilde{S}$ is a compact invariant subset of S and solutions in T tend to $\tilde{S}$ in backward time and $\hat{S}$ in forward time.

Now let U be any neighborhood of $\hat{S}$ with closure disjoint from $\tilde{S}$. Then $I(N \setminus u) = \tilde{S}$ and the result follows. The argument when A^- is empty is the same after a time reversal.

E. Non-Removable Critical Points.

Lemma: The isolated rest points in $I(N_1)$ are also isolated as invariant sets relative to $S(N_1)$.

Let $p = (u_0, 0)$ be a rest point and let M be the Jacobean matrix of $(u_1 f_1, u_2 f_2)$ at u_0.

Assume $\det M \neq 0$ (then $(u_0, 0)$ is isolated).

Suppose that $A^+ \equiv A^+(p, I(N_1))$ and $A^- \equiv A^-(p, I(N_1))$ are both non-empty (so p cannot be "removed" from N_1 in the way described in D.). Then:

1. If either coordinate of u_0 is positive, the corresponding diagonal entry of M is negative.
2. If both are positive then $\det M > 0$.
3. If both coordinates of u_0 are zero then at least one of the diagonal entries is negative. Furthermore, if the first diagonal entry is positive, then the wave velocity θ is negative, while if the second diagonal entry is positive, θ is positive.

Proof. As seen in B., $g(u) = u_2 - u_1$ is strictly decreasing on non-constant solutions in $I(N_1)$. It follows (in general) that all solutions have their α- and ω-limit sets in different components of the rest point set and from there that isolated rest points are also isolated invariant sets relative to $I(N_1)$.

Now suppose $p = (u_0, 0)$ is a rest point and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \quad \Theta = \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{pmatrix} \quad \text{and} \quad \bar{M} = \begin{pmatrix} 0 & I \\ -M & \Theta \end{pmatrix} .$$

Then $\bar{M}$ is the matrix of the linearized equation at p and $\det \bar{M} = \det M$ which has been assumed to be non-zero. It follows that $(u_0, 0)$ is isolated.

Suppose that λ is an eigenvalue of M with eigenvector $(\xi_1, \xi_2, \eta_1, \eta_2)$. Let $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$; then $\eta = \lambda \xi$ and $-M\xi - \Theta\eta = \lambda\eta$. These equations are equivalent to:

$$(12) \qquad \begin{aligned} &(\lambda^2 + \theta_1\lambda + a)\xi_1 + b\xi_2 = 0 \\ &(\lambda^2 + \theta_2\lambda + d)\xi_2 + c\xi_1 = 0 \end{aligned}$$

If A^+ is non-empty there is a solution in $I(N_1)$ tending to u_0 in forward time. It must come in along some eigenspace of $\bar{M}$ corresponding to an eigenvalue with non-positive real part. But it cannot oscillate either, so the eigenvalue must be real and therefore negative (since it is not zero).

Furthermore, η_1 and η_2 cannot have the same sign since $v_1 \geq 0$ and $v_2 \leq 0$ on solutions in $S(N_1)$. Since $\xi = \lambda\eta$, ξ_1 and ξ_2 cannot have the same sign either so it can be assumed that $\xi_1 \geq 0$ and $\xi_2 \leq 0$.

Similarly, if A^- is non-empty, $\bar{M}$ admits a real positive eigenvalue whose corresponding eigenvector satisfies $\xi_1 \geq 0$ and $\xi_2 \leq 0$.

Now suppose the first coordinate of u_0 is positive. Then $b = \partial/\partial u_2 (u_1 f_1)|p$ must be negative. From the first equation in (12) it follows that for any eigenvector, ξ_1 cannot be zero and so the eigenvalue must have the form $\lambda = -\theta_1/2 \pm \sqrt{\theta_1^2/4 - (a + b\xi_2/\xi_1)}$.

Suppose $a + b\xi_2/\xi_1$ is positive. Then the real part of any eigenvalue has the same sign as $-\theta_1 = -\delta_1\theta$ or $-\theta$ *since* $\delta_1 > 0$. *Thus it is not* possible that $a + b\xi_2/\xi_1$ is positive for both of the eigenvectors. But $b\xi_2/\xi_1$ is non-negative; it follows that a is negative.

Similarly if the second coordinate of u_0 *is positive then* d *is* negative.

Furthermore, if both coordinates are positive, there must be an eigenvector such that both of $a + b\xi_1/\xi_2$ and $d + c\xi_2/\xi_1$ are *negative* (namely the one corresponding to the eigenvalue with sign different from $-\theta$). But this easily gives $ad - bc = \det M > 0$.

Finally, suppose both coordinates of u_0 are zero.

Then the solution in A^+ must lie in U_2^- while that in A^- must lie in U_1^+ . The eigenvalue for the first must be $\lambda = -\theta_2/2 \pm \sqrt{\theta_2^2/4 - d}$ and that for the second must be $-\theta_1/2 \pm \sqrt{\theta_1^2/4 - a}$. Since one of these cannot have the sign of $-\theta$, one of a or d must be negative. Also, if $d > 0$ then the negative eigenvalue (for the solution in U_2^-) has the sign of $-\theta$, so $-\theta$ must be negative. Similarly, if $a > 0$, $-\theta$ must be positive. This concludes the proof.

§ 5. Connection Triples For The Main Problem

A. Hypothesis on the zero sets of the F's.

It will be assumed that F_1 and F_2 (and therefore f_1 and f_2) have zero sets as pictured in Figure 6.

There are, therefore, seven critical points, of which three are attractors for the reaction equations, (i.e. in C_I), namely $(\bar{u}_1, 0)$, $(0, \bar{u}_2)$ and $(0, 0)$. It is assumed that the Jacobean matrix, M, is non-degenerate at each critical point.

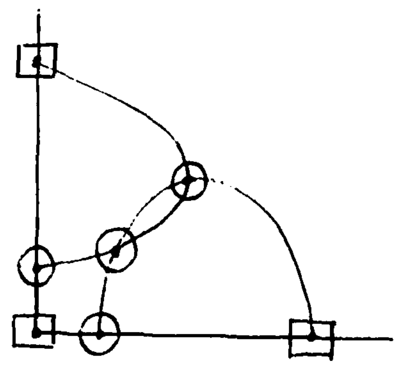

Figure 6

The squares are projections of the sets $N(1, p)$ in u-space as described in 4C.

Let N' be that about $(0, \bar{u}_2)$ and N'' that about $(\bar{u}_1, 0)$.

The union of the discs is the projection in u-space of a *neighborhood*, U of the remaining points which are to be deleted from N_2 (as in 4E.) An isolating neighborhood, N_3, is determined by $N_2 \setminus U$.

Now modify the δ's so that $\delta_1 = \delta_2$; of course N', N'' and N_3 remain isolating neighborhoods throughout.

B. A second hypothesis on the F's.

However, N_3 is not the desired neighborhood—in particular, in addition to $(\bar{u}_1, 0)$ and $(0, \bar{u}_2)$, it contains the critical point $(0, 0)$.

It is easy to see that for some (unique) value of θ_1, there is a solution of the equation

$$u_1' = v_1$$

(13)

$$v_1' = -\theta_1 v_1 - u_1 f_1(u_1, 0)$$

which connects $(0,0)$ to $(\bar{u}_1, 0)$. (For example, this could be done by continuing this system to that in §2). Furthermore, the sign of $-\theta_1$ is determined by that of $\int_0^{\bar{u}_1} f_1(s,0)\,ds$; namely, it has the same sign as this integral.

Similarly, there is a (unique) connection from $(0, \bar{u}_2)$ to $(0,0)$ for the equation

$$u_2' = v_2$$

(14)

$$v_2' = -\theta_2 v_2 - f_2(0, u_2) ,$$

and it has the same sign as $\int_0^{\bar{u}_2} f_2(0,s)\,ds$.

Impose the hypothesis now that both of the above mentioned integrals are positive. Then $\theta_2 > 0 > \theta_1$, so for any fixed values of the θ_i (with $\theta_1\theta_2 \geq 0$) the point $(0,0,0,0)$ is either an attractor or a repeller (or both) relative to $I(N_3)$.

Using the Lemma of 4.D, choose and delete from N_3 an appropriate neighborhood V of $(0,0,0,0)$ to obtain the last isolating neighborhood $N = N_3 \setminus V$ of the triple (N', N'', N) .

C. <u>Continuation</u>.

It is clear that the F's can now be modified, maintaining the hypothesis in A and B, to have the aspect in Figure 7 wherein three of the critical points have collapsed. In this situation it is again clear that any connection from $(0,\bar{u}_2)$ to $(0,0)$ must have $\theta_2 < 0$ and any one from $(0,0)$ to $(\bar{u}_1,0)$ must have $\theta_1 > 0$. So again an isolating neighborhood N is obtained on deleting a neighborhood of the origin. (N' , and N'' are always determined by the squares about $(\bar{u}_2,0)$ and $(0,\bar{u}_1)$).

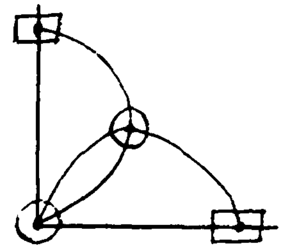

Figure 7

Now it is clear that the continuation can be carried through the degenerate situation in Figure 7 to the example of §3 with isolating neighborhoods N' , N'' and N as shown in Figure 8.

Now it must be shown that the neighborhoods N' , N'' and N determine the sets S', S'' and S of the connection triple of the problem in §3. It has already been seen that the sets N' and N'' determine the critical points $(0,\bar{u}_2,0,0)$ and $(\bar{u}_1,0,0,0)$ so it only remains to see that N determines the correct set.

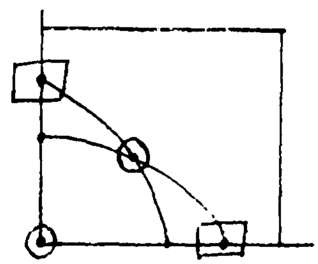

Figure 8

In order to do this it is sufficient to show that the maximal invariant set in the region indicated in Figure 7 is actually in the invariant manifold $w_1 = 1$. To do this consider the reaction vector field on the two solid lines in Figure 9 with slope -1. In the w coordinates of §3, the w_1 $(= u_1 + u_2)$ component of this field is

$4w_1 - 3w_1^2 - w_1^3 + 3w_2(1-w_1^2)$. In particular, it is positive if $w_1 < 1$ and negative if $w_1 > 1$. Thus the vector

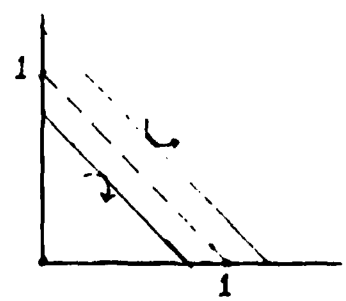

Figure 9

points into the region between the segments so long as this region contains the (dashed) line $w_1 = 1$. Since the negative of this reaction field acts as the "acceleration" in equations (6), it is clear that solutions must "bounce off" the lines as shown in Fig. 8. (At this stage $\theta_1 = \theta_2$ is used of course.) On collapsing the segments to the dashed line, it follows that the maximal invariant set in any such region is contained in the invariant manifold. Thus the continuation of the general problem to that of § 3 is completed.

D. Connection triples.

The last step of the proof of the theorem stated in the introduction is to show that the triple (N', N'', N) is always a connection triple provided θ is large enough. This will be true provided that for each fixed δ_1, δ_2 (> 0) values $\theta_0 < 0$ and $\theta_1 > 0$ can be chosen so that the equations (4) of the introduction do not admit connections from $(0, \bar{u}_2, 0, 0)$ and $(\bar{u}_1, 0, 0, 0)$ throughout the continuation.

The idea here is that if $|\theta|$ is large then bounded solutions of (4) with $0 \le u_i \le K$ have projections in u-space that are close to solutions of (1). To see this, make the transformation $w_i = -\theta v_i - u_i F_i$ and multiply the right hand side of the transformed equations by $-\theta$. The resulting equations are written more conveniently in the following notation: let $G = (u_1 F_1, u_2 F_2)$ and let dG be the Jacobean matrix with respect to G. Also let $\delta^{-1} = \text{diag}(\delta_1^{-1}, \delta_2^{-1})$ and $\varepsilon = \theta^{-2}$. The equations are then:

$$u' = \varepsilon G + \varepsilon W$$

$$w' = \delta w - \varepsilon dG(w + G) \quad .$$

Since dG and G are uniformly bounded for $0 \le u_1 \le K$, it is clear from these equations that $w \mid \{$bounded solutions with $0 \le u_1 \le K\}$ is uniformly bounded by $C\varepsilon$ for some constant C (cf. proof in 4A). Therefore, in the limit $\varepsilon \to 0$ $(|\theta| \to \infty)$ the u components of solutions satisfy (1) (with a time scale change). Of course this is true uniformly over the (compact) parameter range needed to deform the equation to the standard form in § 3.

Now the critical points $(0, \bar{u}_2)$ and $(\bar{u}_1, 0)$ are attractors so it is obvious that they cannot be connected by a solution. Therefore, for large enough $|\theta|$ the corresponding points of (4) cannot be connected and it follows that (N', N'', N) determines a connection triple provided $|\theta_0|$ and $|\theta_1|$ are large enough.

As a consequence, the theorem of the introduction is proved.

E. Concluding Remarks.

In view of the general lemmas in §4 other arrangements of the zero sets of the F's could be considered, but the case treated seems sufficient for the present.

More interesting is the situation where the integrals in B. above, and consequently the wave velocities θ_1 and θ_2 in that section, have the opposite signs from those chosen. In this case the deformation to the standard problem is not possible since the critical point at $u_1 = u_2 = 0$ is not a relative attractor or repeller in $I(N_3)$ for some parameter value.

Without a proof, it seems clear that the correct interpretation is the following. In the sense of D. of § 1, the attractor at $u_1 = u_2 = 0$ is now dominant for both of the equations (13) and (14) of § 5 B. If one imagines the population to be (initially) symmetrically distributed in space so that in half the space u_1 is at its maximal stable level $\bar{u}_1$ and u_2 is zero, and in the other half $u_2 = \bar{u}_2$ and $u_1 = 0$ then instead of either of them eventually dominating, one should expect that the competition on the mutual boundary drives both populations to such a low level that neither can survive. Thus the competition plus the weakness of the high level attractors relative to the zero level ones leads to mutual extinction.

In reference [4] more mathematical arguments are given bearing on this aspect of the problem.

Appendix

The indices $h(S')$, $h(S'')$ and $h(S',S'',S)$ for the example in §2 can be determined pictorially as follows (cf. also [2]).

First consider $S'(\theta)$ for, say, $\theta = 0$; this is the left hand (hyperbolic) rest point in Fig. A.1. The (solid) diamond shaped region about that point is N as well as N_1. The heavily shaded boundary segments are N_2. Then $[N_1/N_2]$ is homotopy equivalent to the pointed circle, Σ^1. (The deformation squeezes the diamond to an arc and N_2 to the end points. Pasting the end points together, one has the pointed circle).

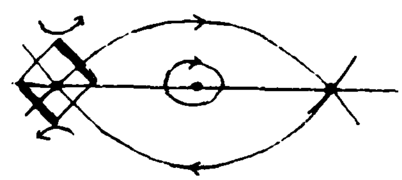

Figure A.1

The index of the right hand end point is seen to be Σ^1 in the same way.

To find $\overline{h}$ consider the region $N(\theta)$ in Figure A.2: the curved arcs in the boundary are orbits of the equation with $\theta = 0$. When θ is negative orbits cross out of the region on these arcs; when θ is positive orbits cross into the region across them. Otherwise, for all θ, orbits cross the straight boundary arcs as indicated. Those where they leave are heavily shaded; they are in N_2.

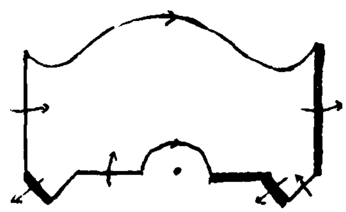

Figure A.2

If $\theta \le 0$, the top orbit segment is also in N_2; for $\theta \ge 0$ the bottom one is. In any case $N_2(\theta)$ has three components.

Suppose now that $\theta_1 > 0$. Then the unstable manifold from $S'(\theta_0)$ appears as in Fig. A.3. If $\theta_1 > 0$, the picture is as in Fig. A.4. In either case, the unstable manifold connects 2 components of N_2, but the pairs differ. Let $\tilde{N}_2 = N_2 \cup$ {unstable manifold of $S'(\theta_0)$ and $S'(\theta_1)$}. The way the components are connected is then measured by $[N/\tilde{N}_2]$. Up to homotopy N is a ball and $\tilde{N}_2$ consists of a set which can be deformed to a point. (This can be "seen" with a little patience). Thus $[N/\tilde{N}_2]$ is homotopy equivalent to the one-point space.

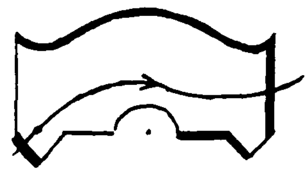

Fig A.3

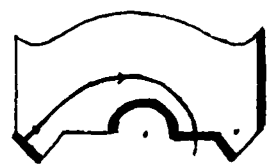

Fig A.4

In fact, $[N/\tilde{N}_2]$ is $\bar{h}$. (The definition of $\bar{h}$ could have been phrased in terms like the above—for "non-pathological" situations at least; however, the definition given in §2 is easier to work with).

If both θ_0 and θ_1 were negative (say) then Fig. A.3 would be the relevant one at both ends of $[\theta_0, \theta_1]$. In this case (defining $\tilde{N}_2$ similarly to the above) $\tilde{N}_2$ is homotopic to a point and a (disjoint) circle in the boundary of N. Then $\bar{h} = [N/\tilde{N}_2]$ is $\Sigma^2 \vee \Sigma^1$ as stated in §2.

Perhaps the most precise way to compute these indices is to continue the appended equation of the Lemma in 2.D. to an even easier form (as was done for the main problem) the detail would not be very interesting.

References

[1] C. Conley, Isolated Invariant Sets and the Generalized Morse Index; C.B.M.S. Regional Conference Series in Mathematics, No. 38, American Mathematical Society, Providence, Rhode Island.

[2] C. Conley and J. Smoller, Isolated invariant sets of parametrized systems of differential equations, The Structure of Attractors in Dynamical Systems (Eds. N. G. Markley, J. C. Martin and W. Perizzo) Lecture Notes in Mathematics, 668, Springer Verlag, Berlin (1978).

[3] C. Conley, A Qualitative Singular Perturbation Theorem, to appear in the proceedings of the conference on dynamical systems held at Northwestern University in June 1979.

[4] R. A. Gardner, Existence and stability of travelling wave solutions of competition models: a degree theoretic approach, to appear.

[5] R. L. Foy, Steady state solutions of hyperbolic systems of conservation laws with viscosity terms, Comm. Pure Appl. Math., Vol. 23, 1970, pp. 867-884.

[6] C. C. Conley and J. A. Smoller, Shock waves as limits of progressive wave solutions of higher order equations, Comm. Pure Appl. Math., Vol. 24, 1971, pp. 459-472.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) MRC-TSR 2144

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: TSR# 2144	2. GOVT ACCESSION NO.: AD-A096661
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): An Application of the Generalized Morse Index to Travelling Wave Solutions of a Competitive Reaction-Diffusion Model.	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): C. Conley [redacted] R. Gardner	8. CONTRACT OR GRANT NUMBER(s): DAAG29-80-C-0041, NSF-MCS80-1816
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 1 (Applied Analysis)
11. CONTROLLING OFFICE NAME AND ADDRESS: See Item 18 below.	12. REPORT DATE: Nov[redacted]80
	13. NUMBER OF PAGES: 38
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
Technical summary rept.	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, D.C. 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Competitive Systems; Travelling Waves; Diffusion-Reaction; Generalized Morse Index

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The existence of travelling wave solutions of a diffusion reaction-system is studied via the generalized Morse index of isolated invariant sets. This index theory is analogous to degree theory, and the method of proof follows lines familiar from the latter theory. The equations in question are "deformed" to a "standard" system where the index can be easily computed, and the existence theorem follows from the "non-triviality" of the index.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED 221200

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT, continued

The index theory has been described in other papers; here the main job is to construct "isolating neighborhoods" which are analogous (in the degree theory) to open sets with no critical points on the boundary. Some novel means of locating such neighborhoods are described.

The main theorem concerns a case where the reaction system is a competitive system: that is, the growth rate of one population decreases as the other population increases. In a plausible class of such models, each system admits three stable equilibria: the two at which one population is at a maximal stable level and the other is eliminated, and one at which both populations are at the zero level. In general there are (two) travelling waves connecting one or the other of the first two equilibria to the third. The theorem gives a criterion, in terms of the relative velocity of these two waves, in order that there be a third wave running between the first two equilibria.

DAT
FILM