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THE INVERSE OF A TOTALLY POSITIVE BIINFINITE BAND MATRIX.(U)
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THE INVERSE OF A TOTALLY POSITIVE BIINFINITE BAND MATRIX

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ABSTRACT

It is shown that a bounded biinfinite banded totally positive matrix A is boundedly invertible iff there is one and only one bounded sequence mapped by A to the sequence $((-)^i)$. The argument shows that such a matrix has a main diagonal, i.e., the inverse of A is the bounded pointwise limit of inverses of finite sections of A principal with respect to a particular diagonal, hence $((-)^{i+j}A^{-1}(i,j))$ or its negative is again totally positive.

AMS (MOS) Subject Classifications: 47B37 (primary), 15A09, 15A48

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SIGNIFICANCE AND EXPLANATION

Spline approximation is often most effective when the breakpoint (knot) sequence can be chosen suitably non-uniform. At the same time, the standard approximation schemes (such as least-squares approximation, or interpolation at suitable interpolation points by splines) are so far only known to be usable and bounded as long as the breakpoint sequence is almost uniform. The problem of showing existence and uniqueness of bounded spline approximants to bounded data boils down to showing invertibility of a certain infinite matrix A. The distinguished features of this matrix are its bandedness and its total positivity, i.e., all minors of A are nonnegative. In this paper we show that if there is exactly one bounded sequence mapped by a biinfinite totally positive banded matrix A to the particular sequence $(\cdots,-1,1,-1,1,-1,\cdots)$, then every bounded sequence is contained in the range of A. In spline terms, this result says, for example, that any bounded data sequence can be interpolated, and in exactly one way, with a bounded spline (with a given knot sequence, at a given interpolation point sequence) provided that the periodic data $(+1,-1)$ can be interpolated, and in exactly one way, by a bounded spline from that class. Further, our arguments show that such an interpolating spline can be constructed as the limit of splines which satisfy finitely many of the given interpolation conditions.

p-2

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE INVERSE OF A TOTALLY POSITIVE BIINFINITE BAND MATRIX

Carl de Boor

0. Introduction. This paper is a further step in a continuing effort to understand certain linear spline approximation schemes. Convergence of such processes is intimately tied to their stability, i.e., to their boundedness, as maps on C, say. Use of the B-spline basis shows this question to be equivalent to bounding the inverse of certain totally positive band matrices. The calculation of bounds on the inverse of a given matrix is in general a difficult task. It is hoped that the present investigation into the consequences of bandedness and total positivity for the structure of the inverse may ultimately prove helpful in obtaining such bounds.

The results in this paper were obtained in the study of a conjecture due to C. A. Micchelli [7]. In connection with his work on the specific approximation scheme of interpolation at a (strictly increasing) point sequence τ by elements of $\$_{m,t}$, i.e., by splines of some order m with some knot sequence $t = (t_i)$, Micchelli became convinced that every bounded function has one and only one bounded spline interpolant iff the particular function which satisfies $f(\tau_i) = (-)^i$, all i, has a bounded spline interpolant in $\$_{m,t}$. If $(N_i) = (N_{i,m,t})$ denotes the corresponding B-spline basis for $\$_{m,t}$, then Micchelli's conjecture can be phrased thus: The matrix $A := (N_j(\tau_i))$ is boundedly invertible iff the linear system

$$Ax = \left((-)^i\right) \tag{1}$$

has a bounded solution. Micchelli points out that, for a finite A, this conjecture is indeed true and can be established using the total positivity of A.

Whether $A = (N_j(\tau_i))$ is finite or not, it is not difficult to see that A fails to be invertible unless A is m-banded, i.e., unless τ and t so harmonize that at most $m+1$ consecutive bands of A are not identically zero. It is then a small step to the conjecture

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"A totally positive m-banded matrix A is boundedly invertible iff (1) has a bounded solution"

whose truth Micchelli demonstrated to me for the case that A is a (biinfinite) Toeplitz matrix.

As it turns out, this conjecture is incorrect; it fails unless one assumes, more strongly, that (1) has a <u>unique</u> bounded solution. This uniqueness plays a crucial role in the proof of the (corrected) conjecture given below. The proof is first given for strictly m-banded matrices and is then extended to a general m-banded matrix by a limit argument, using a 'smoothing' result from [4].

In outline, our argument is as follows: We show that the nullspace

$$\mathfrak{N} := \mathfrak{N}_A := \{ f \in \mathbb{R}^{\mathbb{Z}} : Af = 0 \}$$

of a strictly m-banded matrix A is Haar' in the sense that, for every $f \in \mathfrak{N} \setminus 0$, f' has less than $m = \dim \mathfrak{N}$ weak sign changes. Here, f' is obtained from f by changing the sign of every other entry,

$$f'(i) := (-)^i f(i) \text{ , all } i \text{ .}$$

This makes it possible to interpolate the bounded solution x to (1) at any m-set $I := \{i_1,\ldots,i_m\}$ by some $y_I \in \mathfrak{N}$. The next (and hardest) step consists in showing that, for $k = m^+ := \dim \mathfrak{N}^+$ with

$$\mathfrak{N}^+ := \{ f \in \mathfrak{N} : \lim_{i\to\infty} |f(i)| < \infty \} ,$$

y_I lies between x and 0 on the interval $I^{(k)} :=]i_k, i_{k+1}[$, if we assume that $i_1 < \ldots < i_m$. This implies that $x_I := x - y_I$ satisfies

$$\| x_I|_{I^{(k)}} \| \le \| x \| \text{ .}$$

We use this fact as follows. If J is any integer interval, and $A_{J,J+k}$ is the corresponding section of A having the k-th band as its main diagonal, then

$$((-)^i)_{|J} = (Ax_I)_{|J} = A_{J,J+k}(x_{I|J+k})$$

provided we choose $I = [J \cup (J+m)] \setminus (J+k)$. But then, because of the total positivity of A ,

$$\| (A_{J,J+k})^{-1} \| = \| x_{I|J+k} \|$$

and this is bounded by $\|x\|$ since $I^{(k)} = J+k$ in this case . This uniform boundedness of the inverses of all sections which are principal with respect to the k-th band is sufficient for the bounded invertibility of A itself.

In this way, we show not only that existence and uniqueness of a bounded solution for (1) implies bounded invertibility of A , but gain structure information about the inverse: The inverse is the pointwise bounded limit of the inverses of finite sections principal with respect to one particular band. In the terms of [1], [2], A has a main diagonal. This, in turn, permits the conclusion that the inverse of a totally positive band matrix is checkerboard, a statement conjectured in [5;p.319].

1. Preliminaries. In this section, we list certain notational conventions for easy reference.

We use lower case letters to denote elements of $\mathbb{R}^I$, i.e., real functions on some integer set I, with $f(i)$ the value at i of the function (or sequence) f. If f never vanishes, then $S(f)$ denotes the number of **sign changes** in f, i.e.,

$$S(f) := |\{ i \in I : f(i)f(s(i)) < 0 \}| ,$$

with $s(i) := \min\{j \in I : j > i\}$ the successor to i if we think of I as an ordered sequence. Here, $|J|$ denotes the cardinality of the set J. If f vanishes somewhere, then it is customary to distinguish between **strong** and **weak** sign changes. These are given by

$$S^-(f) := \inf\{S(v) : v \in \operatorname{sign} f\} , \quad S^+(f) := \sup\{S(v) : v \in \operatorname{sign} f\}$$

respectively, with $\operatorname{sign} f := \{v \in \{-1,1\}^I : v(i)f(i) = |f(i)| , \text{ all } i\}$. In the sequel, an unqualified "*sign change*" will always mean "*weak sign change*". It is convenient to supplement the definition of S^- and S^+ by setting

$$S^-(f) := -1 , \quad S^+(f) := |I| \quad \text{if } f = 0 .$$

It is then easy to check that

$$S^-(f) + S^+(f') = |I| - 1 , \text{ for all } f \in \mathbb{R}^I ,$$

with $I = \{i_1, \ldots, i_n\}$, $i_1 < \cdots < i_n$, and

$$f'(i_s) := (-)^s(i_s) , \quad s=1,\ldots,n .$$

We also employ the prime to indicate a signature change in every other entry in case $I = \mathbb{Z}$. To be definite, we set

$$f'(i) := (-)^i f(i) , \text{ all } i , \text{ all } f \in \mathbb{R}^{\mathbb{Z}} .$$

In particular, $1'$ denotes the biinfinite sequence given by

$$1'(i) := (-)^i , \quad \text{all } i .$$

If J is a subset of I , then $f_{|J}$ denotes the *restriction* of $f \in \mathbb{R}^I$ to J . In this connection, $\backslash J := I\backslash J$ denotes the complement of J in I . We write $\backslash j$ instead of $\backslash\{j\}$. Further,

$$J+k := \{j+k : j \in J\}$$

and

$$[i,j] \;:=\; \{k \in \mathbb{Z} : i \le k \le j\} .$$

More generally, $[J]$ denotes the smallest integer interval containing J.

Correspondingly (though not completely in the same way), we denote by $A_{K,L}$ the restriction of the "matrix" $A \in \mathbb{R}^{I\times J}$ to the subset $K\times L$ of $I\times J$. If $K = \{k_1,\ldots,k_p\}$ and $L = \{\ell_1,\ldots,\ell_p\}$, with $k_1 < \ldots < k_p$, $\ell_1 < \ldots < \ell_p$, then

$$\det A_{K,L} \;:=\; \det\left(A(k_i,\ell_j)\right)_{i,j=1}^{p} .$$

All norms are sup-norms. Explicitly,

$$\|f\| \;:=\; \sup_{i\in I}|f(i)| , \quad \text{all } f \in \mathbb{R}^I$$

and

$$\|A\| \;:=\; \sup\{\|Af\|/\|f\| : f \in \mathbb{R}^J, \|f\|<\infty\} \;=\; \sup_{i\in I} \sum_{j\in J} |A(i,j)| , \text{ for } A \in \mathbb{R}^{I\times J} .$$

Further, $\ell_\infty := \ell_\infty(\mathbb{Z}) := \{f \in \mathbb{R}^{\mathbb{Z}} : \|f\| < \infty\}$. We call $A \in \mathbb{R}^{\mathbb{Z}}$ **bounded** if A maps ℓ_∞ to itself, or, equivalently, if $\|A\| < \infty$. If need be, we distinguish between the bounded matrix A and the linear map induced by it on ℓ_∞ by calling the latter

$$A_{|\ell_\infty} .$$

The matrix $A \in \mathbb{R}^{I\times J}$ is **totally positive** (=: tp) if

$$\det A_{K,L} \;\ge\; 0 \quad \text{for all } K\times L \subseteq I\times J \text{ with } |K| = |L| .$$

See Karlin's comprehensive book [6] for details.

Finally, if $i_1, \ldots, i_p$ and $j_1, \ldots, j_p$ are sequences in I and J, respectively, and $A \in \mathbb{R}^{I\times J}$, then we use the customary abbreviations

$$A\begin{bmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{bmatrix} \;:=\; \left(A(i_s,j_t)\right)_{s,t=1}^{p}$$

and

$$A\begin{pmatrix} i_1,\ldots,i_p \\ j_1,\ldots,j_p \end{pmatrix} \;:=\; \det\left(A(i_s,j_t)\right)_{s,t=1}^{p} .$$

2. The nullspace of a strictly m-banded biinfinite matrix. The **r-th diagonal** or **band** of a matrix A is, by definition, the sequence $\big(A(i,i+r)\big)$. As in [1], we call a matrix A **m-banded** if all nonzero entries of A can be found in at most $m+1$ consecutive bands. Explicitly, the matrix A is m-banded if

$$\text{for some } \ell\ , \quad A(i+\ell,j) \neq 0 \quad \text{implies} \quad i \leq j \leq i+m\ .$$

Unless otherwise indicated (e.g., by context), we will always assume that $\ell = 0$. For a biinfinite matrix A, this is merely a normalization achieved by considering $E^{-\ell}A$ instead of A, with E the shift,

$$\big(Ef\big)(i) := f(i+1)\ , \text{ all } i\ , \text{ all } f \in \mathbb{R}^{\mathbb{Z}}\ ,$$

an invertible operator which preserves more or less all interesting structures in $\mathbb{R}^{\mathbb{Z}}$.

The m-banded matrix A is called **strictly** m-banded if

$$A(i,i)A(i,i+m) \neq 0\ , \text{ all } i\ ,$$

i.e., the first and last nontrivial band is never zero. In case of a biinfinite matrix A, this nontrivial assumption insures that, for every m-interval $I = \{i+1,\ldots,i+m\}$, every $a \in \mathbb{R}^I$ gives rise to one and only one sequence f with $Af = 0$ and $f_{|I} = a$. To put it differently, with

$$\mathfrak{N} := \mathfrak{N}_A := \{f \in \mathbb{R}^{\mathbb{Z}} : Af = 0\}$$

denoting the kernel or **nullspace** of A, strict m-bandedness insures that

$$\text{for every } I := [i+1,i+m]\ , \text{ the map } \mathfrak{N} \longrightarrow \mathbb{R}^m : f \longmapsto f_{|I} \text{ is 1-1 and onto.}$$

We now prove this statement to be true for <u>every</u> m-set I in case A is also tp. We begin with the following

Lemma 2.1 . <u>If</u> A <u>is strictly m-banded and tp, and</u> $i_1 < \ldots < i_p$, $j_1 < \ldots < j_p$, <u>then</u>

$$A\binom{i_1,\ldots,i_p}{j_1,\ldots,j_p} > 0 \quad \underline{\text{iff}} \quad i_r \leq j_r \leq i_r+m\ , \ \underline{\text{all}}\ r\ .$$

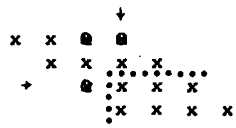
Proof. Proof of the 'if' part is by induction on p, it being true for $p = 0$ with the customary convention that $A\binom{\emptyset}{\emptyset} = 1$. If $i_1 = j_1$, then, A being m-banded, we have

$$A\binom{i_1,\dots,i_p}{j_1,\dots,j_p} = A\binom{i_1}{j_1}\, A\binom{i_2,\dots,i_p}{j_2,\dots,j_p}$$

and this is strictly positive, the first factor by the strict m-bandedness of A and the *second by induction*. The corresponding argument applies when $j_1 = i_1+m$.

Otherwise $i_1 < j_1 < i_1+m$. But then, A being m-banded and tp, we have

$$\begin{aligned} 0 &< A\binom{j_1-m, i_1,\dots,i_p}{i_1\ , j_1,\dots,j_p} \\ &= A\binom{j_1-m}{i_1} A\binom{i_1,\dots,i_p}{j_1,\dots,j_p} - A\binom{i_1}{i_1} A\binom{j_1-m}{j_1} A\binom{i_2,\dots,i_p}{j_2,\dots,j_p} \end{aligned}$$



with the subtrahend strictly positive by the strict m-bandedness of A and the induction hypothesis, and this implies that the factors of the minuend must be strictly positive, too.

As to the 'only if' part, $A\begin{bmatrix} i_1,\dots,i_p \\ j_1,\dots,j_p \end{bmatrix}$ has zeros in columns (rows) $1, \dots, r$ and rows (columns) $r, \dots, p$ in case $i_r > j_r$ $(j_r > i_r+m)$, *hence is then singular.* |||

Corollary. <u>If</u> A <u>is strictly m-banded and tp, and</u> J, K <u>are integer intervals</u> <u>with</u> $K = [J \cup (J+m)]$, <u>then</u> $x_{|K} \neq 0$ <u>and</u> $(Ax)_{|J} = 0$ <u>imply</u> $S^-(x_{|K}) > |J|$.

Proof. (An adaptation of the argument for Theorem 5.1.2 in [6;p.219f]). The assumption that $p := S^-(x_{|K}) < |J|$ leads to a contradiction as follows.

Let $K_0, \dots, K_p$ be a corresponding partition of K , i.e., (without loss of generality)

(2) $$0 \neq (-)^i x_{|K_i} > 0\ , \quad i=0,\dots,p\ .$$

Then

$$0 = \sum_{i=0}^{p} (-)^i v_i$$

with

(3) $$v_i := \sum_{k \in K_i} |x(k)|\ A(\cdot,k)_{|J}\ , \quad i=0,\dots,p\ ,$$

showing that $(v_i)_0^p$ is linearly dependent. This implies, with $V := [v_0 | \cdots | v_p]$, that, for any $i_0 < \cdots < i_p$ in J,

$$0 = V\binom{i_0, \ldots, i_p}{0, \ldots, p} = \sum_{k_0 \in K_0} \cdots \sum_{k_p \in K_p} |x(k_0)| \cdots |x(k_p)| \, A\binom{i_0, \ldots, i_p}{k_0, \ldots, k_p}$$

and all summands on the right are nonnegative by the tp of A. On the other hand, we can, by (2), choose $k_i \in K_i$, all i, so that $|x(k_0)| \cdots |x(k_p)| \neq 0$ and will therefore have reached a contradiction as soon as we exhibit a corresponding choice for $i_0 < \cdots < i_p$ in J for which

$$A\binom{i_0, \ldots, i_p}{k_0, \ldots, k_p} > 0 \quad . \tag{4}$$

This we can do as follows. Define $(i_r)_{-1}^{p+1}$ by

$$[i_{-1}, i_{p+1}] := J \ , \quad i_r := \max\{ i_{-1}+r \ , \ k_r - m \} \quad , \ r = 0, \ldots, p \ .$$

Then $i_0 < \cdots < i_p$ since both sequences $(i_{-1}+r)_r$ and $(k_r - m)_r$ are strictly increasing. Also, $i_0, \ldots, i_p \in J$ *since trivially* $i_{-1} \leq i_r$, while

$$i_r = \max\{ i_{-1}+r \ , \ k_r - m \} \leq i_{p+1}$$

since $r \leq p < |J|$ and $k_r \in K = [i_{-1}, i_{p+1}+m]$. Finally, $k_r - m \leq i_r \leq k_r$, since $i_{-1} \leq k_0 < \cdots < k_r$ implies that $i_{-1}+r \leq k_r$, hence

$$k_r - m \leq \max\{i_{-1}+r, k_r - m\} \leq k_r \ .$$

The lemma *now gives* (4) *and thereby the desired contradiction.* |||

Remark. We have proved here a particular instance of the statement: " *If* B is tp and of full rank, then $S^-(x) \geq S^+(Bx)$ ", provided we define $S^+(0) :=$ length of 0, as we did earlier.

We conclude that if $y \in \mathfrak{U}$ and $y_{|K} \neq 0$, then

$$S^+(y'_{|K}) = |K|-1 - S^-(y_{|K}) \leq |K|-1 - |J| = m-1$$

hence

Proposition 2.5. <u>If</u> A <u>is strictly m-banded and tp, then</u> $y \in \mathfrak{U} \setminus 0$ <u>implies</u> $S^+(y') \leq m-1$.

Since $S^+(f) \geq |\{ i : f(i) = 0 \}|$, this shows that $\mathfrak{N}$ is then a Haar space, i.e., $\mathfrak{N} \longrightarrow R^m : y \longmapsto y_{|I}$ is 1-1 and onto whenever $|I| = m$. Further, this shows that any nontrivial $y' \in \mathfrak{N}'$ with $m-1$ zeros changes sign across each of these zeros and nowhere else, i.e.,

Corollary. _If_ A _is strictly_ m_-banded and tp, and_ $y \in \mathfrak{N}\backslash 0$ _vanishes at_ $i_1 < \dots < i_{m-1}$, _then (with_ $i_0 := -\infty$, $i_m := \infty$_),_

(6) $$y'(i_1-1)(-)^r y'(i) > 0 \quad \underline{\text{for}} \quad i_r < i < i_{r+1} \ , \ r=0,\dots,m-1 \ .$$

We make use below of the two subspaces $\mathfrak{N}^+$ and $\mathfrak{N}^-$ of $\mathfrak{N}$. These are defined by

$$\mathfrak{N}^* := \{ f \in \mathfrak{N} : \overline{\lim_{i\to *\infty}} |f(i)| < \infty \} \quad , \ * = +,- \ .$$

Their intersection consists of all bounded solutions to the homogeneous problem $Ay = 0$. The intersection is therefore trivial iff (0.1) has at most one bounded solution. See [2] for conditions on A equivalent to having $\mathfrak{N} = \mathfrak{N}^+ \oplus \mathfrak{N}^-$.

3. The algorithms L and R . Let x be a bounded sequence satisfying $Ax = 1'$. In this section, we investigate y_I , the sequence in $\mathfrak{Y} = \mathfrak{N}_A$ which matches x at the m-set $I = \{i_1,\ldots,i_m\}$ with $i_1 < \ldots < i_m$. For $k = 0,\ldots,m$, the **interval** k of such an m-set is, by definition, the integer interval

$$I^{(k)} \quad := \quad]i_k, i_{k+1}[\ ,$$

with $i_0 := -\infty$, $i_{m+1} := \infty$.

Our ultimate goal is to show that, for $k = m^+ := \dim \mathfrak{N}^+$, y_I' on $I^{(k)}$ lies between x' and 0 . The proof of this fact involves certain manipulations which are conveniently described in terms of two algorithms, given below. For their analysis, the following fact is useful.

Proposition 3.1 . <u>Let</u> A <u>be strictly m-banded and tp, and let</u> $z = x - y_I$ <u>for some m-set</u> I . <u>Then</u>

(2) $$z(i) \neq 0 \quad \underline{\text{for all}} \quad i \notin I \ .$$

<u>More precisely,</u>

(3) $$(-)^k z'(i) > 0 \quad \underline{\text{for}} \quad i \in I^{(k)} \ , \ k=0,\ldots,m \ .$$

Proof. Let J be an integer interval containing I in its 'interior'. Since A is m-banded, we have

$$1'_{|J} \ = \ (Az)_{|J} \ = \ A_{J,L}(z_{|L}) \ , \quad \text{with} \quad L := [J \cup (J+m)] \setminus I \ ,$$

and, since A is strictly banded, $A_{J,L}$ is invertible, by Lemma 2.1. Thus $(A_{J,L})^{-1}$ is checkerboard, and

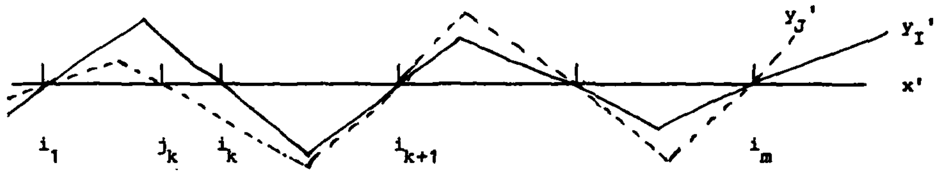
$$z_{|L} \ = \ (A_{J,L})^{-1}(1'_{|J}) \ .$$

It follows that $z_{|L}$ changes sign strongly at every slot, with $z(j)(-)^j > 0$ for all $j \in J$ and to the left of I . Since J is essentially arbitrary, we conclude that $z' > 0$ near $-\infty$ and that $z_{|\setminus I}$ strongly changes sign at every slot. |||

Corollary. *If the m-set $J = \{j_1,\dots,j_m\}$ is obtained from the m-set $I = \{i_1,\dots,i_m\}$ by moving the leftmost k points to the left, i.e., $j_r < i_r$, $r=1,\dots,k$, while $j_r = i_r$, $r=k+1,\dots,m$, then, on $[i_k,\infty[$, y_I' lies between x' and y_J', with equality only at the points $i_{k+1}, \dots, i_m$ (unless $J = I$).*

Proof. The assertion follows by repeated application of the special case

$$j_r = i_r \quad \text{for} \quad r \neq k\ , \quad i_{k-1} < j_k < i_k\ .$$



The proof for this special case goes as follows. By (3),

$$(-)^k(y_J' - y_I')(j_k) = (-)^k(x' - y_I')(j_k) < 0\ .$$

On the other hand, y_J and y_I agree at the $m-1$ points of $I \setminus i_k$, hence their difference changes sign strongly across each point of $I \setminus i_k$ and nowhere else, by the corollary to Proposition 2.1. Consequently,

(4a) $$(-)^r(y_I' - y_J')(i) > 0 \quad \text{for} \quad i_r < i < i_{r+1}\ , \quad r=k,\dots,m,$$

while, again by (3),

(4b) $$(-)^r(x' - y_J')(i) > 0 \quad \text{for} \quad i_r < i < i_{r+1}\ , \quad r=k,\dots,m\ .$$

|||

What is to follow is based on the proposition and its corollary and the following attempt at constructing a nontrivial element of $\mathfrak{N}^-$.

By the proposition, $S^+(x') < m$, hence x' has constant sign near $-\infty$. Let

(5) $$\varepsilon := \text{sign}\ x'(i) \quad \text{for} \quad i \text{ near } -\infty$$

and let $k \in [1,m]$ be even or odd depending on whether ε is 1 or -1. Choose an m-set I so that i_{k+1} lies to the left of all sign changes of x' (if any). Then, on

$I^{(k)}$, $\varepsilon y_I'$ lies below $\varepsilon x'$, by (3) . We then consider what happens to y_I on $I^{(k)}$ as we move $i_1, \ldots, i_k$ to the left. By the corollary, $\varepsilon y_I'$ must then decrease. There are two possibilities:

(i) No matter how far we move $i_1, \ldots, i_k$ to the left, $\varepsilon y_I'$ remains positive on $I^{(k)}$. Then we obtain as a limit point some $y \in \mathfrak{N}$ which agrees with x at $i_{k+1}, \ldots, i_m$, hence is not just 0 , and for which $\varepsilon y'$ lies between $\varepsilon x'$ and 0 on $]-\infty, i_{k+1}[$. But this implies that $y \in \mathfrak{N}^- \setminus 0$ since x is bounded by assumption. A refinement of the limit process actually gives

$$\dim \mathfrak{N}^- \geq m-k .$$

(ii) Eventually, $\varepsilon y_I'$ becomes nonpositive somewhere in $I^{(k)}$, hence has (at least) two sign changes there. We would then decrease k by 2 and try again. By the corollary, the two sign changes of y_I' just acquired would not be affected by subsequent moves.

In this way, we either obtain some nontrivial element of $\mathfrak{N}^-$ or else find ourselves once again at (ii) but with $k = 1$ or 2 , making further decreases in k impossible. The current y_I' must then have two sign changes for every time we passed through (ii). Since $S^+(y_I') < m$, this limits the number of times we can pass through (ii). In particular, if we start with $k = m$ or $m-1$, we must eventually reach (i).

This allows us to talk about the smallest k we manage to arrive with at (i) as we vary the initial k and I in the above procedure ; call it k_L . Analogously, we define k_R as the largest k we manage to arrive with at (i) as we play the game to the right rather than the left. The extremality of k_L and k_R and the fact that then

$$\dim \mathfrak{N}^- \geq m - k_L \ , \quad \dim \mathfrak{N}^+ \geq k_R$$

lead to the desired conclusion in ways to be made precise below.

We now give a formal description of the game just played.

Algorithm L . Input: the integer $k_{in} \in [0,m]$ and the m-set $I_{in} = \{i_1,\ldots,i_m\}$ with $i_1 < \ldots < i_m$. Also, recall $\varepsilon :=$ sign $x'(i)$ for i near $-\infty$.

<u>Step 0.</u> $k := k_{in}$, $I := I_{in}$.

Step 1. If $(-)^k\varepsilon < 0$, or if x' changes sign to the left of $I^{(k)}$, then **EXIT1** .

Step 2. Let $I_r := \{i_1-r, \ldots, i_k-r, i_{k+1}, \ldots, i_m\}$, $r=0,1,2,\ldots$.

Step 3. If, for all r , $\varepsilon y_{I_r}' > 0$ on $I_r{}^{(k)}$, then **EXIT2** .

Step 4. Pick an r for which $\varepsilon y_{I_r}' < 0$ somewhere in $I_r{}^{(k)}$ and replace I by I_r .

Step 5. If $k < 2$, then **EXIT3** .

Step 6. Decrease k by 2 and return to Step 2 .

We now analyse the output from this algorithm.

EXIT1 is a failure exit which allows us to be less careful about the input than we might otherwise have to be. In the applications of the algorithm, it will be obvious that we do not exit via EXIT1 .

EXIT2 is the most interesting of the three, because of the following

Lemma L . *If k is as on exit from Algorithm L via EXIT2, then*

$$\dim \mathfrak{N}^- > m - k .$$

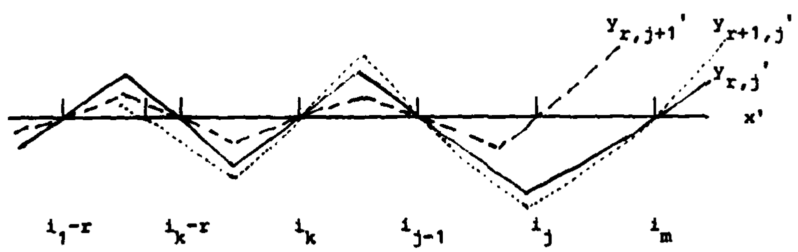
Proof. For $k = m$, there is nothing to prove, so assume $k < m$. Then we have in hand a sequence $(y_r) := (y_{I_r})$ in $\mathfrak{N}$ for which

$$0 < \varepsilon y_{r+1}'(i) < \varepsilon y_r'(i) < \varepsilon x'(i) \quad \text{for all } i \in]i_k-r, i_{k+1}[.$$

Now let $I_{r,k} := I_r$ and , for $r > 0$, $j > k$, consider

$$I_{r,j} := \{i_1-r, \ldots, i_k-r, i_k, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m\} .$$

Let $y_{r,j} := y_{I_{r,j}}$ be the corresponding interpolants to x from $\mathfrak{N}$; see Figure.



Then, by the corollary to the proposition,

$$0 < \varepsilon y_{r,k}'(i) < \varepsilon y_{r,k+1}'(i) < \dots < \varepsilon y_{r,m}'(i) < \varepsilon x'(i) \quad \text{for} \quad i_k - r < i < i_k$$

while $y_{r,j}(i) = x(i)$ for $i = i_{j+1}, \dots, i_m$, and $y_{r,j}(i_j)$ moves away from $x(i_j)$ as r increases. Since $\mathfrak{X}$ is finite dimensional and independent over any m-set (by Proposition 2.1), it now follows that $(y_{r,j})_r$ has limit points in $\mathfrak{X}$ and any such limit point y_j satisfies

$$0 \le \varepsilon y_j' < \varepsilon x' \quad \text{on} \quad]-\infty, i_k[\quad , \quad \text{and} \quad y_j(i) = x(i) \ , \ i = i_{j+1}, \dots, i_m \ .$$

Since x is bounded, this implies that $z_s := y_s - y_{s-1}$ is in $\mathfrak{X}^-$, vanishes at i_{s+1}, $\dots$, i_m , but does not vanish at i_s , i.e., the matrix $(z_s(i_t))_{s,t=k+1}^{m}$ is triangular with nonzero diagonal, hence invertible. This shows $(z_s)_{k+1}^{m}$ to be independent , hence $\dim \mathfrak{X}^- \ge m-k$. |||

Finally, if I is obtained via EXIT3 , then we are now certain that y_I has a weak sign change in each of the intervals 0 , 2, $\dots$, k_{in} of I in case $\varepsilon = 1$, or in each of the intervals $1, 3, \dots, k_{in}$ in case $\varepsilon = -1$. This is so because once a sign change is obtained, in Step 4 , in the current interval k , this sign change persists, by the corollary to the proposition. Further, since $\varepsilon x' > 0$ on that interval (we would have exited via EXIT1 otherwise), it follows that y_I' has two sign changes in each of the intervals $k_{in} - 2j$ with $k_{in} - 2j > 0$, and one in interval 0 if k_{in} is even, for a total of $k_{in}+1$ sign changes. Since a nontrivial $y \in \mathfrak{X}$ can have at most $m-1$ sign changes, an input of $k_{in} = m$ or $m-1$ (depending on the sign of ε) together with an I_{in} which lies to the left of all sign changes of x' (to avoid EXIT1) is guaranteed to bring us to EXIT2 . In particular, it makes sense to define

$$k_L := \min \{ k : k \text{ obtained as output via EXIT2 from Algorithm L} \}$$

and then

$$\dim \mathfrak{X}^- \ge m - k_L \tag{6}$$

follows.

Algorithm R is constructed just as Algorithm L , except that all moves are made toward the right rather than the left. For completeness, we give the full description.

Algorithm R . Input: the integer $k_{in} \in [0,m]$ and the m-set $I_{in} = \{i_1,\ldots,i_m\}$ with $i_1 < \ldots < i_m$. Also, set $\varepsilon :=$ sign $x'(i)$ for i near ∞ .

Step 0. $k := k_{in}$, $I := I_{in}$.

Step 1. If $(-)^k \varepsilon < 0$, or if x' changes sign to the right of $I^{(k)}$, then **EXIT1** .

Step 2. Let $I_r := \{i_1,\ldots,i_k, i_{k+1}+r,\ldots,i_m+r\}$, $r=0,1,2,\ldots$.

Step 3. If, for all r , $\varepsilon y_{I_r}' > 0$ on $I_r^{(k)}$, then **EXIT2** .

Step 4. Pick an r for which $\varepsilon y_{I_r}' \le 0$ somewhere in $I_r^{(k)}$ and replace I by I_r .

Step 5. If $k > m-2$, then **EXIT3** .

Step 6. Increase k by 2 and return to Step 2 .

A discussion very close to that following Algorithm L would establish the following facts.

Lemma R . *If* k *is as on exit from Algorithm R via EXIT2, then* $\dim \mathfrak{N}^+ \ge k$. *Further, the number*

$$k_R := \max\{ k : k \text{ obtained as output via EXIT2 from Algorithm R } \}$$

is well defined, and

(7) $$\dim \mathfrak{N}^+ \ge k_R$$

follows.

We are now ready to prove the main result of this section.

Theorem 3.8 . *If* x *is the unique bounded solution to the linear system* $Ax = 1'$, *with* A *a strictly m-banded biinfinite tp matrix, then, for* $k = m^+ := \dim \mathfrak{N}^+$ *and for any m-set* $I = \{i_1,\ldots,i_m\}$ *with* $i_1 < \ldots < i_m$, y_I' *lies between* 0 *and* x' *on the interval* $I^{(k)} =]i_k, i_{k+1}[$. *In particular, then*

(9) $$|x - y_I| \le |x| \text{ on } I^{(k)} .$$

Proof. Let

$$j_0 < \cdots < j_{S^+(x')}$$

be points on which x' alternates in sign, with $x(j_0) \neq 0$. Let I be any m-set with $i_{k_L+r} = j_r$, $r = 0,1,\ldots,s$, and

$$s := \min\{S^+(x'), m-k_L\} .$$

If $k_L > 1$, then, because of the minimality of k_L, an application of Algorithm L to the input k_L-2, I is bound to end via EXIT3, hence y_I' (with a possibly changed I) has two sign changes in each of the intervals k_L-2, k_L-4, ..., and one sign change in the interval 0 in case k_L is even, for a total of k_L-1 sign changes. In addition, y_I' alternates in sign on the points $i_{k_L}, \ldots, i_{k_L+s}$ since it agrees there with x' and x' does, giving an additional s sign changes. We conclude that

(10) $\quad y_I'$ has at least k_L-1+s sign changes to the left of i_{k_L+s},

and this conclusion holds trivially in case $k_L \leq 1$.

We now prove that

$$s = S^+(x') .$$

Suppose that $s < S^+(x')$. Then $s = m-k_L$, and we now know that y_I' has $m-1$ sign changes on $]-\infty, i_{k_L+s}[$, hence does not change sign on $[i_{k_L+s}, \infty[$, yet matches x' at the points $i_{k_L+s}, \ldots, i_{k_L+S^+(x')}$ on which x' alternates in sign, a contradiction.

We conclude that x' has no sign changes to the right of i_{k_L+s}, hence an application of Algorithm R to the input k_L+s, I is bound to terminate via EXIT2 (because of (10)) with some $k =: k_I$ which is at least as big as k_L+s, yet no bigger than k_R by the maximality of k_R. In symbols,

$$k_L+s \leq k_I \leq k_R \tag{11}$$

and therefore, with (6) and (7),

$$\dim \mathfrak{A}^- + \dim \mathfrak{A}^+ \geq (m-k_L) + k_R \geq m + s = \dim \mathfrak{A} + S^+(x') . \tag{12}$$

This proves that

$$\dim \mathfrak{A}^- \cap \mathfrak{A}^+ \geq S^+(x') .$$

Thus, if now x is the only bounded solution of $Ax = 1'$, then $\mathfrak{N}^- \cap \mathfrak{N}^+ = \{0\}$, hence then $s = S^+(x') = 0$, and

$$\dim \mathfrak{N} \geq \dim \mathfrak{N}^- + \dim \mathfrak{N}^+ .$$

This shows that there must be equality throughout (12). This implies equality in (7), i.e.,

$$m^+ = k_R$$

and equality in (11) (with $s = 0$), i.e.,

$$k_L = k_I = k_R ,$$

and thus shows (9) to hold for the original I . But now, since $S^+(x') = 0$, this could have been any m-set I . |||

Corollary 1. *If* A *is strictly m-banded and tp, and* x *is a bounded sequence satisfying* $Ax = 1'$, *then* $S^+(x') \leq \dim \mathfrak{N}^- \cap \mathfrak{N}^+$.

Corollary 2. *The conclusions of Theorem 3.8 and Corollary 1 remain valid if, in the hypotheses,* $1'$ *is replaced by any strictly alternating sequence* u .

Here, we call u **strictly alternating** if $u(i)u(i+1) < 0$ for all i .

4. The main result.

Theorem 4.1 . <u>If</u> A <u>is a bounded strictly m-banded biinfinite tp matrix, and</u> x <u>is the unique bounded sequence mapped by</u> A <u>to</u> $1'$, <u>then</u> A <u>is boundedly invertible on</u> ℓ_∞ <u>and</u> $\|A^{-1}\| = \|x\|$.

Proof. For any integer interval J , let $I := [J \quad (J+m)] \quad (J+k)$. Then I is an m-set, $I = \{i_1,\ldots,i_m\}$ with $i_1 < \ldots < i_m$, say. Let $z_J := x - y_I$, with $y_I \in \mathfrak{N} = \mathfrak{N}_A$ and $x = y_I$ on I . Then we conclude from Theorem 3.8 that

(2) $$|z_J(j)| \le |x(j)| \text{ , all } j \in J+k$$

while

$$1'_{|J} = (A(x-y_I))_{|J} = A_J(z_{J|J+k})$$

with

$$A_J := A_{J,J+k} .$$

Since A_J is tp, this implies that A_J is invertible (as a general result, though the invertibility of A_J could in the present circumstance be derived directly from Lemma 2.1), hence its inverse is checkerboard and so takes on its norm on the vector $1'_{|J}$, i.e.,

$$\|A_J^{-1}\| = \|A_J^{-1}(1'_{|J})\| = \|z_{J|J+k}\| .$$

Combine this with (2) to get

(3) $$\|A_J^{-1}\| \le \|x_{|J+k}\| \le \|x\| \text{ , for all intervals } J .$$

Since A is bounded and banded, it carries $c_0 := c_0(\mathbb{Z}) := \{f \in \mathbb{R}^{\mathbb{Z}} : \lim_{|i|\to\infty}|f(i)| = 0\}$ to itself, and the bounded invertibility of $A_{|c_0}$ follows now by a standard argument: Let P_J be the truncation projector,

$$(P_Jf)(i) := \begin{cases} f(i) \text{ , } i \in J \\ 0 \quad \text{otherwise} \end{cases} .$$

Then $P_J \longrightarrow 1$ pointwise on c_0 , therefore $P_JAP_{J+k} \longrightarrow A$ pointwise on c_0 as $J \longrightarrow \mathbb{Z}$. Now $A_J = A_{J,J+k}$ represents the interesting part of P_JAP_{J+k} , i.e., the map $P_JA_{|\mathrm{ran}P_{J+k}}$. Therefore, for $u \in c_0$ and $u_J := A_J^{-1}P_JAu \in \mathrm{ran}\, P_{J+k}$, we have

$$\|u_J - u\| = \|A_J^{-1}(P_J Au - P_J AP_{J+k}u)\|$$
$$\leq \|x\|\,\|A\|\,\|u - P_{J+k}u\| \underset{J\to Z}{\longrightarrow} 0$$

since $\|A_J^{-1}\| \leq \|x\|$ and $\|P_J A\| \leq \|A\|$. Thus $A_J^{-1}P_J$ converges pointwise on ran $A_{|c_0}$ to a left inverse of $A_{|c_0}$. Further,

$$\|u\| = \lim \|u_J\| \leq \overline{\lim}\,\|A_J^{-1}\|\|Au\| \leq \|x\|\,\|Au\| ,$$

i.e., $A_{|c_0}$ is bounded below, hence ran $A_{|c_0}$ is closed. The same argument shows that also $(A_{|c_0})^* = A^T{}_{|\ell_1}$ is bounded below, hence ran $A_{|c_0}$ is also dense. We conclude that $A_{|c_0}$ is 1-1 and onto c_0, hence boundedly invertible. Its inverse is therefore again (representable as) a matrix, i.e., $(A_{|c_0})^{-1} = A^{-1}{}_{|c_0}$ for some matrix A^{-1} whose rows are uniformly in ℓ_1, and $(A_{|c_0})^{-1} = \lim_{J\to Z}(P_J AP_{J+k})^{-1}P_J$ pointwise on c_0, hence

(4) $$A^{-1} = \lim_{J\to Z} A_J^{-1} \quad \text{entrywise.}$$

But then A^{-1} provides the inverse of A on $\ell_\infty = (c_0)^{**}$, and $\|A^{-1}\| = \|x\|$ since $\|A^{-1}\| \leq \|x\|$ from (3) and (4), while $A^{-1}(1') = x$. |||

The assumptions of Theorem 4.1 can be weakened in two ways.

As already pointed out earlier, the results of Section 3 do not depend on having a bounded sequence x satisfying $Ax = 1'$. It is sufficient to consider bounded sequences for which $u := Ax$ is strictly alternating, i.e., $u(i)u(i+1) < 0$, all i. For the results of Theorem 4.1, we need, more strongly, that u is uniformly alternating, i.e., strictly alternating and with $\inf |u(i)| > 0$. In that case, the diagonal matrix

$$D := \lceil \ldots , (-)^i u(i) , \ldots \rfloor$$

is bounded (since $u = Ax$ is) and boundedly invertible, while $D^{-1}A$ is still strictly m-banded and tp and carries the bounded sequence x to $D^{-1}u = 1'$, hence $D^{-1}A$ is invertible on ℓ_∞ and $\|(D^{-1}A)^{-1}\| = \|x\|$. Therefore $A = D(D^{-1}A)$ is invertible and $\|A^{-1}\| \leq \|(D^{-1}A)^{-1}\|\|D^{-1}\| = \sup_{i,j}|x(i)/u(j)|$.

Secondly, the assumption of <u>strict</u> m-bandedness, though essential for part of the argument, is not essential for the conclusion. For, according to [4], a bounded m-banded tp matrix A, whose rows and columns are linearly independent, is the uniform limit of strictly m-banded tp matrices A_ε (as $\varepsilon \longrightarrow 0$, say). In our case, the linear independence of the columns follows from the assumed uniqueness of the bounded solution to $Ax = u$, while the linear independence of the rows follows from the assumed total positivity of A and the assumed existence of x with Ax strictly alternating. Existence and uniqueness of a bounded solution to the equation $Ax = u$ (with a uniformly alternating u) therefore implies existence and uniqueness of a bounded solution x to the equation $A_\varepsilon x = u_\varepsilon$, with $u_\varepsilon := u - (A - A_\varepsilon)x$ again uniformly alternating for all sufficiently small ε. Consequently, A_ε is then boundedly invertible on ℓ_∞ and

$$\|A_\varepsilon^{-1}\| \le \sup_{i,j}|x(i)/u_\varepsilon(j)| \xrightarrow[\varepsilon\to 0]{} \sup_{i,j}|x(i)/u(j)| .$$

Thus A must be boundedly invertible on ℓ_∞ and $\|A^{-1}\| \le \sup_{i,j}|x(i)/u(j)|$.

Corollary. The conclusions of Theorem 4.1 remain true if A is only m-banded and $1'$ is replaced by a uniformly alternating sequence u.

The proof of Theorem 4.1 shows more than just the invertibility of A on ℓ_∞. It shows that A has its k-th diagonal as **main diagonal** in the sense introduced in [1]: The sections $A_J = A_{J,J+k}$ are invertible as $J \longrightarrow \mathbb{Z}$ and the corresponding set (A_J^{-1}) is bounded. Hence A^{-1} is the bounded entrywise limit of these finite matrices A_J^{-1}. Again, this conclusion persists if A is only m-banded since it is then the uniform limit of strictly m-banded tp matrices.

Theorem 4.5. Let A be an m-banded biinfinite tp matrix which is bounded and boundedly invertible. Then A has a main diagonal, i.e., for some k and all intervals J, $A_{J,J+k}$ is invertible and A^{-1} is the bounded entrywise limit of $(A_{J,J+K})^{-1}$.

Consequently, with D the diagonal matrix

$$D := \lceil \ldots, (-)^i, \ldots \rfloor ,$$

$(-)^k D^{-1} A^{-1} D$ is again tp. In particular, A^{-1} is checkerboard,

$$(-)^{i+j+k} A^{-1}(i,j) \ge 0 , \text{ all } i,j .$$

5. Concluding Remarks. S. Friedland, in reaction to a presentation of these results, suggested that a tp matrix, whether banded or not, must map ℓ_∞ onto itself if its range on ℓ_∞ contains 1' , since it is then possible to generate a pre-image for every $u \in \ell_\infty$ as a limit point of minimal solutions of $(P_J A)y = P_J u$, using the checkerboard nature of the inverses of finite sections of A . Further, A. Pinkus showed how to establish the sign regularity of $DA^{-1}D$, with A^{-1} the bounded inverse of a tp matrix A , without assuming that the inverse is the limit of inverses of finite sections. These matters are made precise in [3].

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It is shown that a bounded biinfinite banded totally positive matrix A is boundedly invertible iff there is one and only one bounded sequence mapped by A to the sequence $((-)^i)$. The argument shows that such a matrix has a main diagonal, i.e., the inverse of A is the bounded pointwise limit of inverses of finite sections of A principal with respect to a particular diagonal, hence $((-)^{i+j}A^{-1}(i,j))$ or its negative is again totally positive.

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