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(6) REGULARIZING EFFECTS FOR

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REGULARIZING EFFECTS FOR $u_t = \Delta\varphi(u)$

Michael G. Crandall and Michel Pierre

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ABSTRACT

One expression of the fact that the nonnegative solutions of the initial-value problems

$$\text{(IVP)} \qquad \begin{cases} u_t - \Delta u^m = 0 & t > 0,\ x \in R^N , \\ u(0,x) = u_0(x) , \end{cases}$$

where $m > 0$, are more regular for $t > 0$ than a rough initial data u_0 is the remarkable pointwise inequality $u_t = \Delta u^m \geq -(N/(N(m-1) + 2)t)u$ obtained by Aronson and Benilan for $t > 0$ and $m > \max((N-2)/N, 0)$. This inequality was used by Friedman and Caffarelli in proving that solutions of (IVP) are continuous for $t > 0$. The main results of this paper generalize the Aronson-Benilan inequality and show the extended inequality is valid for a much broader class of equations of the form $u_t = \Delta\varphi(u)$. In particular, the results apply to the Stefan problem which is modeled by $\varphi(r) = (r-1)^+$ and imply $(u-1)^+_t \geq -((u-1)^+ + N/2)/t$ in this case.

AMS(MOS) Subject Classifications: 35K15, 35K55

Key Words: regularizing effect, porous media equations, strong solution, degenerate parabolic equations, Stefan problem

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SIGNIFICANCE AND EXPLANATION

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Nonlinear diffusion equations of the form $u_t = \Delta\varphi(u)$ where φ is a given nondecreasing function occur in many situations. Existence and uniqueness of solutions of the initial-value problem for this type of equation have been studied by many authors. The regularity of the solutions, i.e. how smooth or continuous they are, is less well understood, although many results have recently been obtained in this direction. In this paper we contribute to the study of regularity by proving estimates of the general form $\varphi(u)_t \geq -c(\varphi(u) + a)/t$ on nonnegative solutions in all of space. That is, one can bound below the time derivative of $\varphi(u)$ in a pointwise fashion by $\varphi(u)$ itself. Those results generalize those for the special case $\varphi(r) = r^m$, obtained by Aronson and Bénilan.

REGULARIZING EFFECTS FOR $u_t = \Delta\varphi(u)$

Michael G. Crandall and Michel Pierre

This paper concerns nonnegative solutions of initial-value problems of the form

$$\text{(IVP)} \qquad \begin{cases} u_t - \Delta\varphi(u) = 0, & \text{for } t > 0,\ x \in R^N \\ u(0,x) = u_0(x), & x \in R^N \end{cases} .$$

Hereafter, it is assumed that

$$\text{(1)} \qquad \varphi : R \to R \text{ is continuous, nondecreasing and } \varphi(0) = 0 .$$

It is known that (see, e.g., [5] and its references) if $u_0 \in L^1(R^N) \cap L^\infty(R^N)$ then there is exactly one $u \in \underline{\underline{C}}([0,\infty) : L^1(R^N))$ which satisfies

$$\text{(2)} \qquad \text{ess inf } u_0 \le u \le \text{ess sup } u_0 ,$$

$u(0) = u_0$ and $u_t = \Delta\varphi(u)$ in $\mathcal{D}'((0,\infty) \times R^N)$ (i.e., in the sense of distributions). We refer to this u simply as the solution of (IVP). The mapping $u_0 \to u$ is nonexpansive from $L^1(R^N)$ in $\underline{\underline{C}}([0,\infty) : L^1(R^N))$. It follows from (2) that u is nonnegative and bounded whenever u_0 is nonnegative and bounded and it is hereafter assumed that $u_0 \ge 0$.

We will prove that if φ satisfies (1), is twice continuously differentiable and

$$\text{(3)} \qquad r(\varphi'(r))^2 \le K[\varphi(r) + a][r\varphi''(r) + \frac{2}{N}\varphi'(r)] \quad \text{for } 0 < r \le \sup u_0$$

("ess sup" is hereafter abbreviated to "sup") for some $K > 0$ and $a \ge 0$, then the solution u of (IVP) satisfies

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(4) $$\varphi(u)_t \geq \frac{K}{t}(\varphi(u) + a) \quad \text{in} \quad \mathcal{D}'((0,\infty) \times R^N) \quad .$$

In fact, we will establish (4) for nonlinearities φ which satisfy integrated forms of (3) and thus φ need not be twice differentiable.

The pointwise inequality (4) is a generalization of the result of Aronson and Bénilan [1] for the case $\varphi(r) = r^m$, $m > 0$. Indeed, for this φ, (3) with $a = 0$ becomes

$$m^2 \leq K(m(m-1) + m\frac{2}{N})$$

which is equivalent to $N(m-1) + 2 > 0$ and

$$K \geq m/((m-1) + 2/N) \quad .$$

Hence if $\varphi(r) = r^m$ and $N(m-1) + 2 > 0$, Theorem 1 implies

(5) $$(u^m)_t \geq -\frac{mN}{N(m-1)+2}\frac{u^m}{t} \quad .$$

Formally $(u^m)_t = mu^{m-1}u_t$ and (5) thus becomes

(6) $$u_t \geq -\frac{N}{N(m-1)+2}\frac{u}{t}$$

which is the main result of [1].

When φ satisfies (3) with $a = 0$, the estimate (4) formally yields

(7) $$u_t \geq -\frac{K}{t}\frac{\varphi(u)}{\varphi'(u)} \quad .$$

Although (7) is not an *immediate consequence* of (4) owing to regularity questions, the *possibility that* $\varphi'(u)$ may have zeros, etc. ..., we will show this inequality holds under assumptions that guarantee it is meaningful. If, moreover, $\varphi(u) \leq cu\varphi'(u)$ for some constant c (as is implied by (3) when $a = 0$, $N \geq 3$), an inequality of the type (6) can also be obtained.

These last considerations are clearly not relevant when $a \neq 0$ and $\varphi'(u)$ has zeros. In this case (4) is a different sort of estimate which can be obtained for a much *larger class of functions* φ than the $u_t \geq - cu/t$ estimates. In particular, if $\varphi(u) = \max(u-1,0)$ we have a model of the Stefan problem (see, e.g., [6], [9]) and the integrated form of (3) holds with $a = 1$, $K = N/2$.

The methods used in this work are similar to those of [1]. First (IVP) is approximated appropriately. Next it is shown that if u is a solution of the approximate problem then $p = \Delta\psi(u)$, where $\psi'(r) = \varphi'(r)/r$, satisfies a parabolic inequality which allows one to conclude that $\Delta\psi(u) \geq -h(v)/t$ for certain *functions* h *constructed* in the argument. The consequences for u are as stated above. Refinements and implications of (4) are also discussed. These primary arguments are given in Section 1. The question of "strong solutions" is addressed in Section 2.

Other works in which related estimates appear are [3] and [8] to which we refer for further references. In particular, [8] contains results which apply to (IVP) when the sign condition $u_0 \geq 0$ is dropped.

Section 1. The Main Results

Preparatory to formulating the main results, we restate the basic assumption

$$(1.1) \qquad r(\varphi'(r))^2 \le K(\varphi(r) + a)(r\varphi''(r) + \frac{2}{N}\varphi'(r)) \quad ,$$

in which $K > 0$ and $a \ge 0$, in integrated form. Assume that $\varphi(r) + a > 0$, and $\varphi'(r) > 0$ on some interval $(0,M)$ for the moment, so that (1.1) may be rewritten (on $(0,M)$) as

$$k\frac{\varphi'}{\varphi+a} \le \frac{\varphi''}{\varphi'} + \frac{2}{N}\frac{1}{r} \quad , \quad k = 1/K$$

or

$$0 \le (\ln \varphi')' - k(\ln(\varphi+a))' + \frac{2}{N}(\ln r)'$$

or

$$\ln\left(\frac{\varphi'}{(\varphi+a)^k} r^{2/N}\right) \text{ is nondecreasing}$$

or, finally, $\varphi'(r)r^{2/N}/(\varphi(r) + a)^k$ is nondecreasing. This final monotonicity property is conveniently regarded as the convexity of

$$(1.2) \qquad \begin{cases} \tau \to \frac{1}{1-k}[\varphi(\tau^{\frac{N}{N-2}}) + a]^{1-k} & N \ge 3 \quad , \\ \tau \to \frac{1}{1-k}[\varphi(e^{\tau}) + a]^{1-k} & N = 2 \quad , \\ \tau \to \frac{1}{1-k}[\varphi(-\frac{1}{\tau}) + a]^{1-k} & N = 1 \quad . \end{cases}$$

(If $k = 1$ above, $(\varphi+a)^{1-k}/1-k$ means $\ln(\varphi+a)$.)

A main result is:

<u>Theorem 1</u>. Let $0 \le u_0 \in L^1(R^N) \cap L^\infty(R^N)$ and assume that (1.2) is convex on

$$\begin{cases} 0 \le \tau \le (\sup u_0)^{(N-2)/N} & \text{if } N \ge 3 , \\ -\infty < \tau < \ell n(\sup u_0) & \text{if } N = 2 , \\ -\infty < \tau < -1/\sup u_0 & \text{if } N = 1 . \end{cases}$$

Let u be the solution of (IVP).

Then for $K = 1/k$

$$\varphi(u)_t \ge -\frac{K}{t}(\varphi(u) + a) \quad \text{in} \quad \mathcal{D}'((0,\infty) \times R^N) \quad . \tag{1.3}$$

The proof of Theorem 1 will be augmented to obtain the following refinement in the case in which $a = 0$ and $\varphi(r)/\varphi'(r)$ is a "good" function.

<u>Theorem 2</u>. In addition to the assumptions of Theorem 1, assume $a = 0$, $r_0 \ge 0$ and $\varphi(r) = 0$ for $0 \le r \le r_0$. Assume $\varphi \in \underset{=}{C}^1(0, \sup u_0)$, $\varphi'(r) > 0$ for $r > r_0$ and that $\lim_{r \downarrow r_0} \varphi(r)/\varphi'(r) = 0$. Then the solution u of (IVP) satisfies

$$u_t \ge -\frac{K}{t}\alpha(u) \quad \text{in} \quad \mathcal{D}'((0,\infty) \times R^N) \tag{1.4}$$

where

$$\alpha(r) = \begin{cases} \varphi(r)/\varphi'(r) & r > r_0 \\ 0 & r \le r_0 \quad . \end{cases}$$

We point out explicitly the special cases $\varphi(r) = (r-1)^{+m} = (\max(r-1,0))^m$ on $[0,\infty)$ in which we may take $r_0 = 1$. Now (1.1) will be satisfied with $a = 0$ if

$$rm \le K((m-1)r + \frac{2}{N}(r-1)) \quad \text{for} \quad 1 \le r \le \sup u_o \quad .$$

Thus if $m > 1$ we may use Theorem 2 and $K = m/(m-1)$ to conclude that

$$u_t \geq \frac{m}{m-1}\,\frac{1}{t}\,\frac{1}{m}\,(u-1)^+ = -\frac{1}{(m-1)t}\,(u-1)^+$$

no matter what u_0 is. If $m = 1$, corresponding to the Stefan problem, we cannot satisfy (1.1) with $a = 0$ and require instead

$$r \leq K((r-1)^+ + a)(\frac{2}{N}) \qquad r > 1$$

or $K \geq 1$ and $Ka\ 2/N \geq 1$. This is achieved by $K = 1$ and $a = N/2$. (The reader may check that the integrated condition holds in this case, erasing doubt as to the cavalier treatment of φ'' above.) Hence we conclude, by Theorem 1, only that

$$(u-1)^+_t \geq -\frac{1}{t}((u-1)^+ + \frac{N}{2}) \quad .$$

It is interesting to notice that if φ is convex and Lipschitz continuous on $(0, \sup u_0)$ and satisfies (1) then (1.1) (or its integrated form) holds with

$$Ka = \frac{N}{2} \sup_{r\varepsilon(0,\sup u_0)} (r\varphi'(r)) \quad .$$

Since K can be chosen as small as we desire, Theorem 1 yields $\varphi(u)_t \geq -Ka/t$.

We begin the proofs of Theorems 1 and 2. To this end we consider the problem (IVP) under the strong assumptions that

$$u_0 \in \underset{=}{C}{}_0^\infty(R^N),\ u_0 \geq 0 \tag{1.5}$$

and that φ satisfies (1) and for some $K, a, \delta \geq 0$

$$\left\{\begin{array}{lll} \text{(i)} & \varphi \in \underset{=}{C}{}^\infty(0,\infty) & \\ \text{(ii)} & r(\varphi'(r))^2 \leq K(\varphi(r)+a)(r\varphi''(r) + \frac{2}{N}\varphi'(r)) & \text{for } 0 < r \leq \sup u_0 + \delta \\ \text{(iii)} & \varphi'(r) > 0 & \text{for } 0 < r \leq \sup u_0 + \delta \quad . \end{array}\right. \tag{1.6}$$

Standard methods ([9], [10]) then guarantee that for $0 < \varepsilon < \delta$ the problem

$$(1.7) \qquad \begin{aligned} w_{\varepsilon t} - \Delta\varphi(w_\varepsilon) &= 0 \\ w_\varepsilon(0,x) &= u_0(x) + \varepsilon \end{aligned}$$

has a unique solution $w_\varepsilon \in C^\infty([0,\infty) \times R^N)$ satisfying

$$(1.8) \qquad \sup u_0 + \varepsilon > w_\varepsilon > \varepsilon \quad .$$

and the derivatives of w_ε are bounded for bounded $t > 0$. Moreover,

$$(1.9) \qquad w_\varepsilon < w_\eta \quad \text{for} \quad 0 < \varepsilon < \eta < \delta$$

and

$$(1.10) \qquad \int_{R^N} (w_\varepsilon - \varepsilon)dx = \int_{R^N} u_0 dx \quad .$$

It follows that $\lim\limits_{\varepsilon \downarrow 0}(w_\varepsilon - \varepsilon) = \lim\limits_{\varepsilon \downarrow 0} w_\varepsilon$ is the solution of *(1.1) and that in order to prove* $\varphi(u)_t \geq -(K/t)(\varphi(u) + a)$ it suffices to show $\varphi(w_\varepsilon)_t \geq -(K/t)(\varphi(w_\varepsilon) + a)$.

For convenience in writing we now drop the subscript ε *in* w_ε. Thus w is to solve $w_t - \Delta\varphi(w) = 0$, $w(0,x) = u_0(x) + \varepsilon$. The main steps in the proof involve the function

$$(1.11) \qquad \psi(\zeta) = \int_{\varepsilon/2}^{\zeta} \frac{\varphi'(s)}{s} ds \quad \text{for} \quad \zeta > 0 \quad ,$$

and the standard change of dependent variable

$$(1.12) \qquad v = \psi(w) \quad .$$

Lemma 1. Let (1.5), (1.6) hold, $0 < \varepsilon < \delta$, w be the classical solution of (1.7) and v be given by (1.12), (1.11). Then

$$v_t = g(v)\Delta v + |\nabla v|^2 \quad \text{for} \quad g(v) = \varphi'(\psi^{-1}(v)) \quad , \tag{1.13}$$

$$w_t = w\Delta v + (\varphi'(w)/w)|\nabla w|^2 \quad , \tag{1.14}$$

and $p = \Delta v$ satisfies

$$p_t \geq g(v)\Delta p + 2(\nabla g(v) + \nabla v)\cdot\nabla p + g''(v)|\nabla v|^2 p + (g'(v) + \frac{2}{N})p^2 \quad . \tag{1.15}$$

In the statement of Lemma 1, $\nabla = (\partial/\partial x_1,\ldots,\partial/\partial x_N)$ and $a\cdot b = a_1b_1+\cdots+a_Nb_N$. The proof consists of direct calculations. E.g., plugging the identities

$$\begin{aligned} v_t &= \psi'(w)w_t,\ \nabla v = \psi'(w)\nabla w \\ \Delta v &= \frac{\varphi'(w)}{w}\Delta w + \Big(\frac{\varphi''(w)}{w} - \frac{\varphi'(w)}{w^2}\Big)|\nabla w|^2 \\ &= \frac{\varphi'(w)}{w}\Delta w + \Big(\frac{\varphi''(w)w - \varphi'(w)}{(\varphi'(w))^2}\ |\nabla v|^2 \end{aligned}$$

into $w_t = \Delta\varphi(w) = \varphi'(w)\Delta w + \varphi''(w)|\nabla w|^2$ yields (1.13) and (1.14). To obtain (1.15) one applies Δ to the equation $v_t = g(v)\Delta v + |\nabla v|^2$ and uses

$$\Delta|\nabla v|^2 = 2\sum_{i,j=1}^{N}(v_{x_ix_j})^2 + 2\nabla v\cdot\nabla(\Delta v) \geq (2/N)(\Delta v)^2 + 2\nabla v\cdot\nabla(\Delta v).$$

To continue, let $\mathcal{L}$ be the nonlinear operator

$$\mathcal{L}(Z) = g(v)\Delta Z + 2(\nabla g(v) + \nabla v)\cdot\nabla Z + g''(v)|\nabla v|^2 Z + (g'(v) + \frac{2}{N})Z^2 \tag{1.16}$$

so the parabolic inequality (1.15) satisfied by $p = \Delta v$ can be rewritten $p_t \geq \mathcal{L}(p)$. We now seek a comparison function Z in the form $Z = -h(v)/t$ with the property $Z_t \leq \mathcal{L}Z$.

If $h(r) > 0$ for $r > 0$, we then have $Z \leq p$ for small $t > 0$ (since "$Z = -\infty$" at $t = 0$) and so $Z \leq p$ for all $t > 0$ by standard comparison results [9].

<u>Lemma 2</u>. Let w be the solution of (1.7) and $v = \psi(w)$ as above. Let $a \geq 0$, $K \geq 0$,

$$h(r) = K\,\frac{\varphi(\psi^{-1}(r)) + a}{\varphi'(\psi^{-1}(r))\psi^{-1}(r)} \tag{1.17}$$

and $Z = -h(v)/t$. Then

$$Z_t = \mathcal{L}(Z) + \left(\frac{1}{h(v)} - \left(g'(v) + \frac{2}{N}\right)\right)Z^2 \quad . \tag{1.18}$$

In particular, if

$$\frac{1}{h(v)} \leq g'(v) + \frac{2}{N} \tag{1.19}$$

then $Z \leq \Delta v$ for $t > 0$ and

$$\varphi(w)_t \geq -\frac{K}{t}(\varphi(w) + a) \quad \text{for} \quad t > 0 \quad . \tag{1.20}$$

<u>Proof</u>. We set aside the verification of (1.18) for the moment. Once this is established and (1.19) holds we have $Z_t \leq \mathcal{L}(Z)$ and then $Z \leq \Delta v$ by the maximum principle as remarked above. Returning to (1.14) we have

$$w_t = w\Delta v + \frac{\varphi'(w)}{w}|\nabla w|^2 \geq w\Delta v \geq -\frac{wh(v)}{t} = -K\,\frac{(\varphi(w) + a)}{\varphi'(w)t} \tag{1.21}$$

or $\varphi(w)_t = \varphi'(w)w_t \geq -K(\varphi(w) + a)/t$ as claimed in (1.20).

Rather than verify (1.18) by direct computation, we consider the result of computing $Z_t - \mathcal{L}(Z)$ when $Z = -h(v)/t$. The identities

$$Z_t = -\frac{h'(v)}{t} v_t + \frac{h(v)}{t^2} = -\frac{h'(v)}{t} v_t + \frac{1}{h(v)} Z^2$$

$$\nabla Z = -\frac{h'(v)}{t} \nabla v, \quad \Delta Z = -\frac{h'(v)}{t} \Delta v - \frac{h''(v)}{t} |\nabla v|^2$$

imply that $v_t = g(v)\Delta v + |\nabla v|^2$ becomes, upon multiplication by $-h'(v)/t$,

$$\begin{aligned}
Z_t - \frac{1}{h(v)} Z^2 = Z_t - \frac{h(v)}{t^2} &= g(v)(\Delta Z + \frac{h''(v)}{t} |\nabla v|^2) + \nabla v \cdot \nabla Z \\
&= g(v)\Delta Z + 2(\nabla g(v) + \nabla v)\cdot\nabla Z + g''(v)\, |\nabla v|^2 Z + (g'(v) + \frac{2}{N})Z^2 \\
&+ \frac{h''(v)}{t} g(v)\, |\nabla v|^2 - (2\nabla g(v) + \nabla v)\cdot\nabla Z - g''(v)\, |\nabla v|^2 Z \\
&- (g'(v) + \frac{2}{N})Z^2 \\
&= \mathcal{L}(Z) + (h''(v)g(v) + 2g'(v)h'(v) + h'(v) + g''(v)h(v)) \frac{|\nabla v|^2}{t} \\
&- (g'(v) + \frac{2}{N})Z^2 \quad .
\end{aligned}$$

Simplifying the above a bit, it follows that

$$(1.22) \qquad Z_t = \mathcal{L}(Z) + \left(\frac{1}{h(v)} - (g'(v) + \frac{2}{N})\right)Z^2 + ((gh)''(v) + h'(v)) \frac{|\nabla v|^2}{t} \quad .$$

The general solution h of $(gh)'' + h' = 0$ is

$$(1.23) \qquad h(r) = \frac{C_1}{g(r)} \exp(-\int \frac{1}{g}) + \frac{C_2}{g} (\exp(-\int \frac{1}{g}))(\int \exp \int \frac{1}{g})$$

where $\int f$ denotes a function whose derivative is f and C_1, C_2 are constants. Now recall that $g(r) = \varphi'(\psi^{-1}(r))$, $\psi'(r) = \varphi'(r)/r$ and so $\int 1/g = \ln \psi^{-1}$, $\int \psi^{-1} = \varphi(\psi^{-1})$. Thus (1.23) can be written

$$(1.24) \qquad h(r) = \frac{C_1}{\varphi'(\psi^{-1}(r))} \frac{1}{\psi^{-1}(r)} + \frac{C_2}{\varphi'(\psi^{-1}(r))} \frac{1}{\psi^{-1}(r)} \varphi(\psi^{-1}(r)) \quad .$$

Set $C_1 = Ka$ and $C_2 = K$ in (1.24) to find (1.17). Now (1.18) follows from (1.22) and $(gh)'' + h' = 0$. This completes the proof of Lemma 2.

We now can prove:

Lemma 3. In addition to the assumptions of Theorem 1, let φ satisfy (1.6) with $\delta = 0$. Then (1.3) holds.

Proof. If $u_0 \in \underline{\underline{C}}_0^\infty(\mathbb{R}^N)$, $u_0 \ge 0$, (1.6) is satisfied with $\delta > 0$ and w_ε solves (1.7), then by Lemma 2, (1.20) holds provided (1.19) is satisfied with h given by (1.17) and $v = \psi(w_\varepsilon)$. Recalling again that $g(r) = \varphi'(\psi^{-1}(r))$, etc., (1.19) reduces to $w_\varepsilon(\varphi'(w_\varepsilon))^2 \le K(\varphi(w_\varepsilon) + a)(w_\varepsilon\varphi''(w_\varepsilon) + (2/N)\varphi'(w_\varepsilon))$. This holds by (1.6), (1.9) and $0 < \varepsilon < \delta$. Since w_ε decreases to the solution u as ε decreases to zero, (1.20) will hold (in $\mathcal{D}'((0,\infty) \times \mathbb{R}^N)$) for u. The mapping $u_0 \to u$ is nonexpansive from $L^1(\mathbb{R})$ into $\underline{\underline{C}}([0,\infty) : L^1(\mathbb{R}^N))$ and hence (1.3) holds for general $0 \le u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying the hypotheses of the lemma when it holds for the special $u_0 \in \underline{\underline{C}}_0^\infty(\mathbb{R}^N)$ just treated. (To allow $\delta = 0$, approximate u_0 suitably from below.)

Proof of Theorem 1. According to Lemma 3, the assertion of the theorem is correct if also φ is smooth and $\varphi'(r) > 0$ on $r < 0$. Indeed, the convexity assumed in the theorem is equivalent to (1.6) (ii) (with $\delta = 0$) in this case. It is also known that if $\{\varphi_n\}$ is a sequence of *nondecreasing* continuous functions with $\varphi_n \to \varphi$ everywhere on $[0, \sup u_0]$, then the solutions u_n of

$$\begin{cases} u_{n_t} - \Delta\varphi_n(u_n) = 0 \\ u_n\big|_{t=0} = u_0 \end{cases}$$

satisfy $\lim\limits_{n\to\infty} u_n = u$ in $\underline{\underline{C}}([0,\infty) : L^1(B))$ for every compact $B \subset \mathbb{R}^N)$ ([4]). Thus the result will follow at once if we can approximate functions φ satisfying the hypotheses of Theorem 1 by sequences satisfying (1.6) with $K = 1/k$ in (1.6) (ii). This may be done as follows if $N \ge 3$, $k \neq 1$. Let $\left(\varphi(\tau^{\frac{N}{N-2}}) + a\right)^{1-k}/(1-k) = \Phi(\tau)$ be convex on $0 \le \tau \le (\sup u_0)^{\frac{N-2}{N}}$. Choose $\underline{\underline{C}}^\infty$ convex functions Φ_n on this interval with

$\Phi_n(0) = a^{1-k}/(1-k)$ and $\Phi_n(\tau) > 0$ for $0 < \tau$ so that $\Phi_n \to \Phi$ uniformly on $[0, \sup u_0]$. Set $\varphi_n(r) = \left((1-k)\Phi_n(r^{\frac{N-2}{N}})\right)^{\frac{1}{1-k}} - a$. Then $\varphi_n \to \varphi$ uniformly on $[0, \sup u_0]$ and φ satisfies (1.6) with $K = 1/k$. Obvious modifications need to be made for $k = 1$ or $N = 1,2$, but the arguments are basically the same and we omit them.

Proof of Theorem 2. Again appropriate approximations need be made and again we explicitly discuss only $N \geq 3$. Recall that now $a = 0$ and set $\Phi(\tau) = \frac{1}{1-k}\varphi\left(\tau^{\frac{N}{N-2}}\right)^{1-k}$. Define φ_λ by

$$\frac{1}{1-k}\varphi_\lambda\left(\tau^{\frac{N}{N-2}}\right)^{1-k} = \Phi_\lambda(\tau) = \Phi(\tau) + e^{-\lambda^2}(e^{\lambda\tau} - 1) .$$

Now $\varphi_\lambda \in \underset{\approx}{C}^1$ and $\varphi'_\lambda(r) > 0$ on $(0, \sup u_0]$. We may solve (1.7) with $\varepsilon > 0$ and φ replaced by φ_λ for $w = w_\lambda$ *(with the dependence on* ε *suppressed). If suffices to* assume that u_0, φ and ψ are smooth in the process and remove this subsequently by approximation. Then, by Lemma 2,

$$\varphi'_\lambda(w_\lambda)w_{\lambda t} \geq -\frac{K}{t}\varphi_\lambda(w_\lambda)$$

and, since w_λ is smooth and $\varphi'_\lambda(w_\lambda) > 0$,

$$w_{\lambda t} \geq -\frac{K}{t}\frac{\varphi_\lambda(w_\lambda)}{\varphi'_\lambda(w_\lambda)} . \tag{1.25}$$

But

$$\frac{\varphi_\lambda\left(\tau^{\frac{N}{N-2}}\right)}{\varphi'_\lambda\left(\tau^{N/N-2}\right)} = \frac{(1-k)N}{N-2}\,\tau^{\frac{2}{N-2}}\,\frac{\Phi_\lambda(\tau)}{\Phi'_\lambda(\tau)} = \frac{(1-k)N}{N-2}\,\tau^{\frac{2}{N-2}}\,\frac{[\Phi(\tau) + e^{-\lambda^2}(e^{\lambda\tau}-1)]}{(\Phi'(\tau) + e^{-\lambda^2}\lambda e^{\lambda\tau})} .$$

We have assumed that

$$\frac{\varphi(\tau^{\frac{N}{N-2}})}{\varphi'(\tau^{N/N-2})} = (1-k)\,\frac{N}{N-2}\,\tau^{\frac{2}{N-2}}\,\frac{\Phi(\tau)}{\Phi'(\tau)}, \quad \tau > r_0^{\frac{N-2}{N}}$$

extends continuously to $[0, r_o^{\frac{N-2}{N}}]$ as 0, or

$$\lim_{\tau \downarrow r_0^{\frac{N-2}{N}}} \frac{\Phi(\tau)}{\Phi'(\tau)} = 0 \quad .$$

From this it follows that

$$\lim_{\lambda\to\infty} \frac{\Phi(\tau) + e^{-\lambda^2}(e^{\lambda\tau} - 1)}{\Phi'(\tau) + e^{-\lambda^2}\lambda e^{\lambda\tau}} = \begin{cases} \dfrac{\Phi(\tau)}{\Phi'(\tau)}, & r_0^{\frac{N-2}{N}} < \tau \leq (\sup u_o)^{\frac{N-2}{N}} \\ 0 & 0 \leq \tau \leq r_o^{\frac{N-2}{N}} \end{cases}$$

uniformly on $[0, (\sup u_0)^{\frac{N-2}{N}}]$. The verification is left to the reader. Hence one may pass to the limit as $\varepsilon \downarrow 0$ in (1.25) and then as $\lambda \to \infty$ to complete the proof of Theorem 2.

<u>Corollary</u>. In addition to the assumptions of Theorem 2 let $N \geq 3$. Then

$$u_t \geq -\frac{N}{N-2}(K-1)\frac{u}{t} \quad . \tag{1.26}$$

<u>Proof</u>. The convexity of $\tau \to \varphi\left(\tau^{\frac{N}{N-2}}\right)^{1-k} = \Phi(\tau)$ implies $\Phi(\tau) \leq \tau\Phi'(\tau)$ or

$$\varphi\left(\tau^{\frac{N}{N-2}}\right) \leq \tau^{\frac{N}{N-2}}(1-k)\,\frac{N}{N-2}\,\varphi'\left(\tau^{\frac{N}{N-2}}\right)$$

or

$$(1.27) \qquad \frac{\varphi(r)}{\varphi'(r)} < \frac{N(1-k)}{N-2}\, r \quad ,$$

provided $\varphi'(r)$ exists and is positive. Since the estimate on φ/φ' depends only on k, it is shared by any smooth approximation $\tilde{\varphi}$ of φ with $\tilde{\varphi}'(r) > 0$ and the same k. We showed above how to make such approximations and then (1.26) holds for the solutions of these approximate problems by Theorem 2 (with $r_0 = 0$) and (1.27). Hence (1.26) follows.

Remark: (1.27) implies $k < 1$ (unless $\varphi \equiv 0$) since φ/φ' is somewhere positive. If $N = 1$ or 2 we cannot deduce an estimate on $\varphi(r)/r\varphi'(r)$ from our other assumptions. However, we may use Theorem 2 with $r_0 = 0$ if it applies. In particular, if $\varphi(r) = r^m$ with $m > 0$ and $N = 1$ or 2 we recover the estimate of [1]. For $N > 2$ and $m > (N-2)/N$, (1.26) with $K = mN/((m-1)N+2)$ also recovers the result of [1] for this case.

Section 2. Remarks on Strong Solutions

We discuss the question of when the results of the previous section imply that the solution u of (IVP) is a "strong solution" in $L^1(\mathbb{R}^N))$. By this we mean $u \in C([0,\infty) : L^1(\mathbb{R}^N))$, $\varphi(u) \in L^1_{\ell oc}((0,\infty) \times \mathbb{R}^N)$, $u_t \in L^1_{\ell oc}((0,\infty) : L^1(\mathbb{R}^N))$ and $u_t = \Delta\varphi(u)$.

Theorem 3. Let the hypotheses of Theorem 2 be satisfied with $r_0 = 0$. Assume, moreover, that there is a constant c such that $\varphi(r)/\varphi'(r) \le cr$ for $r \in (0, \sup u_0]$. Then the solution u of (IVP) is a strong solution. *Moreover,* $t \int |u_t(t,x)|dx \le 2CK \int u_0(x)dx$.

Proof. This result follows from the arguments given in [1]. By Theorem 2

$$u_t \ge -\frac{Kcu}{t}$$

and u_t is therefore a measure on $[0,\infty) \times \mathbb{R}^N)$. Since $\varphi(u)$ is continuous on $(0,\infty) \times \mathbb{R}^N$ by the results of [11], $u = \varphi^{-1}(\varphi(u))$ is also continuous and it follows as in [1], that $u_t \in L^1_{\ell oc}((0,\infty) \times \mathbb{R}^N)$. Since u_t is bounded below by an integrable function a.e. on $t > 0$, $\int u_t(t,x)dx$ is defined for a.e. $t > 0$ (but it might be ∞). However, $t \to \int_{\mathbb{R}^N} u(t,x)dx$ is nonincreasing. Thus

$$\int_{\mathbb{R}^N}(\{u_t(t,x)\}^+ - \{u_t(t,x)\}^-)dx = \int_{\mathbb{R}^N} u_t(t,x)dx \le 0$$

and so

$$\begin{aligned}\int|u_t(t,x)|dx &= \int(|u_t(t,x)^+| + |u_t(t,x)^-|)dx \\ &\le 2\int |u_t(t,x)^-|dx \\ &\le \frac{2cK}{t}\int u(t,x)dx \\ &\le \frac{2cK}{t}\int u_0(x)dx \quad .\end{aligned}$$

It follows that $tu_t \in L^\infty(0,\infty : L^1(\mathbb{R}^N)) \subseteq L^1_{\ell oc}(0,\infty : L^1(\mathbb{R}^N))$.

<u>Remark</u>: This discussion assumed (1.2) convex on the intervals specified in Theorem 3 and $u_0 \in L^\infty(\mathbb{R}^N)$. However, if the convexity is global, then $u_t > -Kcu/t$ holds for all $0 < u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. and hence for all $0 < u_0 \in L^1(\mathbb{R}^N)$. Moreover, the results *of* [2], [12] imply $u(t) \in L^\infty(\mathbb{R}^N)$ for $t > 0$ under certain conditions (which hold under our assumptions if $N > 3$). *Whenever* $u(t) \in L^\infty(\mathbb{R}^N)$ *for* $t > 0$, [11] *provides* a modulus of continuity on each set $[\tau,T] \times B$, $\tau > 0$, B a ball in $\mathbb{R}^N$. Thus the above arguments apply *and we have* strong solutions for all $u_0 \in L^1(\mathbb{R}^N)$, $u_0 > 0$.

The hypotheses $\varphi'(r) > 0$ for $r > 0$ appears *to be nearly necessary* for the existence of strong solutions. We remark that if $\varphi(r) = (r-1)^+$ and $N = 1$ then strong solutions need not exist. Indeed, in this model of the Stefan problem (see [6], [9]), $\varphi(u)$ is the temperature. If $\varphi(u_0)$ is continuous, smooth on $x > 0$ and $x < 0$ and $\varphi(u_0) > 0$ on $x > 0$, $\varphi(u_0) = 0$ on $x < 0$, then there is a smooth curve $x = s(t)$ such that $\varphi(u) > 0$ for $x > s(t)$ and $\varphi(u) = 0$ for $x < s(t)$ ([7]). Moreover, the derivative $\varphi(u)_x$ jumps across $x = s(t)$ and $\Delta\varphi(u) \in L^1$.

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One expression of the fact that the nonnegative solutions of the initial-value problem

$$\text{(IVP)}\qquad \begin{cases} u_t - \Delta u^m = 0 & t > 0,\ x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & \end{cases}$$

where $m > 0$, are more regular for $t > 0$ than a rough initial data u_0 is

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ABSTRACT (continued)

the remarkable pointwise inequality $u_t = \Delta u^m \geq -(N/(N(m-1) + 2)t)u$ obtained by Aronson and Benilan for $t > 0$ and $m > \max((N-2)/N,0)$. This inequality was used by Friedman and Caffarelli in proving that solutions of (IVP) are continuous for $t > 0$. The main results of this paper generalize the Aronson-Benilan inequality and show the extended inequality is valid for a much broader class of equations of the form $u_t = \Delta\varphi(u)$. In particular, the results apply to the Stefan problem which is modeled by $\varphi(r) = (r-1)^+$ and imply $(u-1)^+{}_t \geq -((u-1)^+ + N/2)/t$ in this case.

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